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FLUID MECHANICS

Ram Srivatsa

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FLUID MECHANICS

A Text Book in S.I. Units for Engineering Students

By

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He is currently practising as a consultant in Fluid Power Technology and is the author of the book on 'Basic Industrial Hydraulics' published by Standard Book House.

Preface

Fluid Mechanics, Hydraulic machines, Oil hydraulics and Pneumatics can be broadly classified under the general term ***Fluid Power Technology.*** Of these, Fluid Mechanics and Hydraulic Machines have been the core subjects in the curriculum of engineering colleges across the world, from the very beginning. This is because a study and understanding of the mechanics of fluid, in static and in dynamic condition is vital to students of all branches of engineering unlike many other subjects which are specific to a particular discipline. This book is therefore addressed to engineering students of all disciplines and meets the curriculum requirements of most universities.

Fluid Mechanics is basically a theoretical subject and therefore a conceptual understanding of the subject becomes vital. It is a subject that explains the principles of science and mathematics of any fluid in static and in dynamic state and how we can make use of these principles as a theoretical tool to solving practical problems. Keeping this fact in view the author has tried to explain at length the basic concepts in simple terms with supportive diagrams and objective type questions and answers, wherever called for.

Hydraulic Machines is a subject wherein we apply the basic concepts of 'Fluid Mechanics' to harness the natural energy contained in flowing water. This energy acts as the input energy required to drive prime movers such as a turbine for the sole purpose of generating electricity.

In *Oil Hydraulics and Pneumatics* we impart energy to a fluid such as oil or air, as the case may be before harnessing this energy. A book on 'Basic industrial hydraulics' has already been published and 'Hydraulic machines' would follow soon.

It has been the endeavor throughout to ensure that even complex problems are explained in simple terms so as to eliminate the tendency of the students to refer to more than one book to get a better comprehension of a particular topic. The problems are hand picked from the question papers of various universities in India and abroad and some of them are from the competitive examination papers such as, UPSC, IRSE, and IES and GATE. Incidentally the chapters also cover the entire syllabus for section B of the AMIE examination conducted by the Institution of Engineers.

A word on the S.I. (System International) units employed in this book is called for at this stage.

This system is a modern version of the metric system (MKS) extended to include seven basic units and is currently the system universally employed except in USA, Canada and Australia, where the FPS system of units is still in vogue.

The seven base units are, Length in metres, mass in kilograms (kg), time in seconds (s), electric current in amperes (A), thermodynamic temperature in Kelvin (K), substances in moles (mol) and luminous intensity in candela (Cd).

All other SI units are derived from these by forming products or quotients. Kelvin is simply written as K and not as °K.

In this book all forces are designated as 'N' which is an abbreviation for 'Newton'. This designation replaces the previously applied designation 'kilo pond' or 'kgf'. 1 Newton is the force required to accelerate a mass of 1kg @ 1 m/s^2. Since F = ma, it follows that 1 N = 0.102 kgf or 1 kgf = 9.81 N ≈ 10 N.

The unit of pressure is '**N/m^2**' or Pascal abbreviated as '**Pa**'. There is as yet no international agreement concerning units for gauge pressure and partial vacuum. So 'bar' and 'Pascal' are still acceptable. (1 bar = 10^5 Pa). The displaced units are 'at' for gauge pressure and 'atm' for atmospheric pressure.

Similarly 'J' (Joule) is the unit in SI system and this displaces cal (calorie). 1 cal = 4.186 J. Horse power or PS stands displaced. Instead kW (kilowatt) is the governing unit. 1 PS = 0.735 kW and 1 HP = 0.746 kW. The other displaced units are, 'Torr' and 'mm Hg' for pressure.

This book is primarily designed to cater to the needs of undergraduate students of Indian engineering colleges and the entire book is contained within twenty chapters. In addition there are seven appendices lending support with relevant information.

At the end of each chapter there is a section called Question Bank. ***Section A*** contains Theoretical Questions for which no answers are given. Students are expected to find the answers all by themselves or in discussion with others. ***Section B*** contains Objective Type of Questions for which answers are given. ***Section C*** contains Problems for the students to solve, for which answers are given.

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Chapter 1

Properties of Fluids

1.1 INTRODUCTION

Fundamental laws of Physics such as laws of conservation of energy, force momentum equations that are applicable to solids, are as well applicable to liquids with due consideration to the fact that unlike solids, liquids deform under the action of shearing forces without regard to the magnitude of the force (Fig. 1.1). Also when analyzing solids, we consider them as individual bodies whereas while dealing with liquids, we consider them merely as a stream of fluid with no

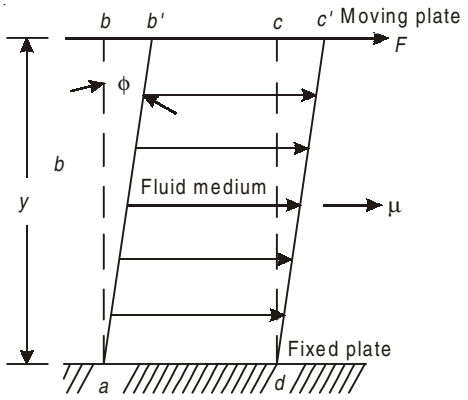


Fig. 1.1

definite longitudinal boundaries. Normally when analyzing the fluid properties such as viscosity, shear stress etc., the molecular structure of the fluid is considered, which means, they are assumed to be continuous substances not

withstanding the fact that the velocity is zero in a molecular medium, as such an assumption would not make a significant difference. But when considering the behaviour of a small element of the fluid, we assume the molecular structure of the fluid to be a continuous medium which is often referred to as the *continuum*. Under this assumption, velocity of the fluid at any point is the mean velocity of all molecules around it but within the limits of the mean distance between molecules. Fluids can either be liquids or gases depending upon how closely spaced the molecules are. In liquids the molecules are more closely spaced than in gases. This is one reason why liquids are considered incompressible whereas gases can be easily compressed. This means that the volume of a given mass of gas changes with pressure and temperature whereas, liquids exhibit no such characteristics. Liquids do have Bulk Modulus of elasticity like solids and store up energy but only when subjected to very high pressures. Liquids and gases have no shape of their own but take the shape of the container. Both liquids and gases cannot resist shear stresses however small the force is. Both offer a characteristic resistance to flow, but the extent of resistance offered, differ in each case. Since the inter-molecular force is higher in liquids as compared to gases, liquids can withstand higher tensile forces and it is this feature that causes them to be viscous. The viscosity of a liquid changes, with temperature; higher the temperature, lower the viscosity. The viscosity of a gas however, increases with temperature due to higher molecular activity.

An *Ideal fluid* is that which exhibits no viscosity, surface tension or compressibility. No such fluid exists in nature and water is the best that there is. However if we assume a fluid as 'ideal' it simplifies our mathematical analysis.

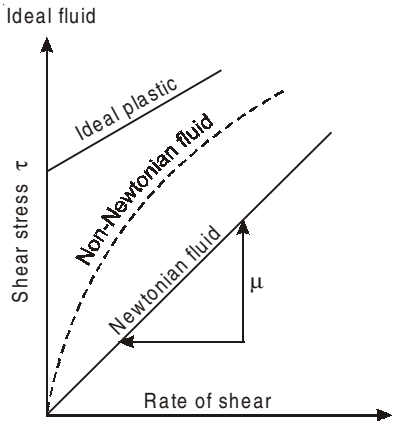


Fig. 1.2

The rest are all termed as *real fluids.* Real fluids are further classified as *Newtonian* fluids or as *non-Newtonian* fluids. In Newtonian fluids the rate of deformation is proportional to the applied force. A non-Newtonian fluid exhibits no such linear relationship. Figure 1.2 illustrates the linear and non-linear relationship that exists between the rate of deformation and the consequent

shear stress in a Newtonian fluid and a non-Newtonian fluid. It also illustrates the extent of resistance to shear that plastics exhibit compared to liquids.

Figure 1.1 illustrates the angular deformation (φ) of the fluid medium contained between two parallel plates due to force F applied on the top plate. The shape before deformation is the rectangle (a–b–c–d) and the shape after deformation is the Rhomboid ($a - b' - c' - d$). If 'y' is the thickness of the fluid film and 'μ' is the velocity of flow then it can be shown that $\left(\frac{dy}{du}\right)$ the rate of shear is proportional to the viscosity. A high viscosity causes higher resistance to the relative movement of two bodies separated by a fluid film. This is the essence of Newton's Law of Viscosity. Even solids deform under the action of large forces but in case of liquids the force component need be no bigger than what is required to slide the two plates relative to one another, to cause the liquid to deform.

1.2 PROPERTIES OF FLUIDS

Both *physical and thermodynamic* properties are considered here. The basic physical properties are, density, specific gravity, specific volume, viscosity, surface tension and compressibility. If the fluid we are considering happens to be either gas or steam, then we need to further consider how the above properties are influenced by a change in the temperature and this is dealt with under the nomenclature *Thermodynamic* properties. Students entering engineering colleges are already familiar with these properties and hence the attempt here is merely to refresh their memories.

Density (ρ) is mass per unit volume. Hence $\rho = \frac{m}{V}$ kg/cubic-metre. Since it is mass that we are considering here, it is called mass density. If weight is the criteria then it is called weight density or specific weight, which is weight per unit volume. Hence specific weight of a fluid $\gamma = \frac{w}{V}$. Since weight per unit volume is the product of mass per unit volume and the acceleration due to gravity (g), $\gamma = \rho g$ Newtons per cubic-metre. Specific volume (v) is volume per unit mass, which is inverse of density and hence can be written as, $v = \frac{1}{\rho}$ cubic metre/kg. Specific gravity (s) also called relative density is a ratio of the density of the fluid under consideration to the density of some other standard fluid. The standard fluid usually considered is water at 4 degrees centigrade and at atmospheric pressure. Hence $(s) = \frac{\rho \text{ of liquid}}{\rho \text{ of water}}$. Specific gravity is a pure number and has no units.

Viscosity in simple terms is a measure of the resistance to flow of any fluid. If a fluid flows easily it is less viscous and vice versa. As compared to air, water offers more resistance to flow and as compared to water, oil offers more resistance to flow. Molasses and tar offer the highest resistance to flow. A high viscosity leads to higher shear stresses and consequently affects the relative motion of two bodies separated by the fluid film.

The principle reason why a fluid flows is because it cannot resist shearing stresses as has already been made clear. The flow pattern is such that the layers in contact with the boundary have the same velocity as the boundary and layers below which are parallel to the boundary have velocities which are somewhat higher. The flow pattern is similar to the way individual playing cards slide one over the other; the top card slides first followed successively by cards down below whereas the bottom most card remains stationery. This in effect means that if a fluid flows through a stationery pipe the top layers of the fluid have zero velocity as compared to successive layers below resulting in a velocity profile such as the one shown below in Fig. 1.3. The equation for this velocity profile which is parabolic is given by $\mu = ay^2 + by + c$, where a, b, c are all constants whose value depends on the boundary condition. At $y = \frac{y}{2}$, that is at the vertex, u is maximum and at $y = 0$, $u = 0$.

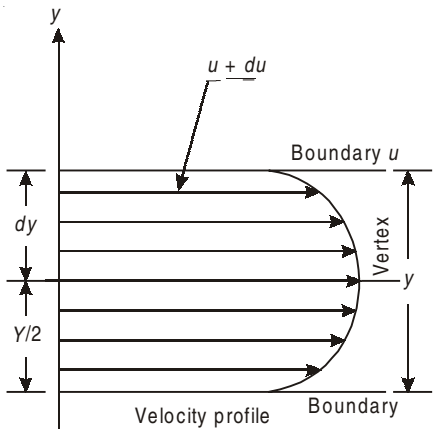


Fig. 1.3

When two layers of fluid at a distance dy flows with two different velocities say, u and $u + du$ the shear stress caused between adjacent layers is proportional to the rate of change of velocity. This is in conformity with the *Newton's Law of Viscosity* as applied to Newtonian fluids which states that the shear stress is proportional to the rate of angular deformation. This means $\tau \propto \frac{du}{dy}$ or $\tau = \mu \frac{du}{dy}$ where μ is the constant of proportionality and depends on the viscosity of

the fluid under consideration. Here the Greek alphabet τ refers to the shear stress and $\frac{du}{dy}$ indicates the change of velocity with respect to distance, in the differential form.

The effect of temperature and pressure on the viscosity of fluids is of considerable significance and hence calls for a deeper understanding. Ideally, the fluid should have the same viscosity at very low temperatures as at very high temperatures. But in reality this is not so. Viscosity of most fluids increases as pressure increases and decreases as temperature increases. The viscosity of most hydro-carbon fluids which are paraffin based increases appreciably at higher pressures whereas in case of water it merely doubles even when the pressure increases to 1000 atmospheres from its pressure at the mean seal level. The viscosity of a gas however, increases with temperature. It is important to understand why liquids behave differently from the gases as far as a change in viscosity is concerned.

Kinetic theory of gases states that the viscosity is proportional to the square root of the absolute temperature. This is because if the temperature increases the molecular mass interchange increases based on the Newton's law of viscosity. Viscosity of a gas therefore increases as the temperature increases.

In liquids however the cohesive forces between molecules are greater than in gases and the cohesive forces decrease with increase in temperature. Molecular mass interchange even here contributes to increase in viscosity but the net result contributes to a reduction in viscosity with increase in temperatures with $\mu \to 0$ as $T \to \infty$ as per the equation

$$\mu_1 = \frac{\mu_0}{[1 + \alpha T + \beta T^2]}$$

μ_1 is the viscosity at T° C, μ_0 is the viscosity at 0° C and α and β are the constants whose values depend on the fluid under consideration. If the fluid under consideration is water then the following values hold. $\mu_0 = 0.0179\ P$, $\alpha = 0.03368$ and $\beta = 0.000221$.

Viscosity index (VI) is a number through which we can identify the resistance offered by the fluid under consideration to changes in temperature. Higher the VI number, higher will be resistance offered by the fluid. Modern hydraulic oils, which have a paraffin base, have a viscosity index of 90 –105 as compared to oils with naphthenic base, which have an index of only 35. A high viscosity index becomes important in applications subjected to wide temperature range. Special additives are used to improve the viscosity index of most fluids.

Viscosity (μ) that we hither to discussed, is *absolute viscosity* also called dynamic viscosity. The unit of measurement is (N.sec/m^2). For practical purposes however we divide the value of absolute viscosity of a fluid by its mass density and consider only the resulting value which is termed as *Kinematic viscosity* (υ). Thus $\upsilon = \frac{\mu}{\rho}$. The unit of measurement is (m^2/sec).

Say bolt universal seconds (SUS) is a number which indicates the viscosity of a fluid in practical terms using an instrument called Viscometer. This is termed *Relative viscosity* and is determined by timing the flow of a given quantity of fluid through a standard orifice at a given temperature. Centistokes (cSt) and degrees angler are other units commercially employed to measure *kinematic viscosity*. However in both MKS and SI units it is in m^2/sec. (one m^2/sec. = 10^{-6} cSt.) This is a derived unit. The SI unit of absolute viscosity is Pascal seconds (*Pa.s*) which is one Newton-seconds/square metre. Under the now obsolete 'cgs' system the absolute viscosity is measured in, Poise or centipoises (cp). One cp = one mega Pascal.

Surface tension (σ) is another interesting property of a fluid especially of a liquid. It is this property that causes the liquid droplets to assume a near spherical form and also, the cause of capillary action in liquids. Due to molecular attraction beneath the surface of a liquid, a layer or a film that could be termed either protective or supportive is formed at the surface of any liquid that is in contact with a gaseous medium such as air. This film behaves as if it were an elastic membrane under tension. Surface tension is the resistance offered by this membrane and acts at right angles to the surface. Since this resisting force is always equal and opposite to the expanding force and is uniform throughout, the equilibrium in not disturbed. In case of a liquid droplet because a sphere has the least surface area for a given volume, it naturally assumes that shape. To expand any further, the bulk of the molecules within have to exert an outward force sufficient to overcome the resisting force exerted by the enveloping membrane.

Mathematically surface tension can be defined as the pressure energy per unit area required to bring the molecules within, to surface. It is expressed in Newton/metre (N/m).

For a spherical droplet of radius r, (Fig.1.4) the expanding force is $p \pi r^2$ where, p is the internal pressure. The enveloping force due to surface tension σ

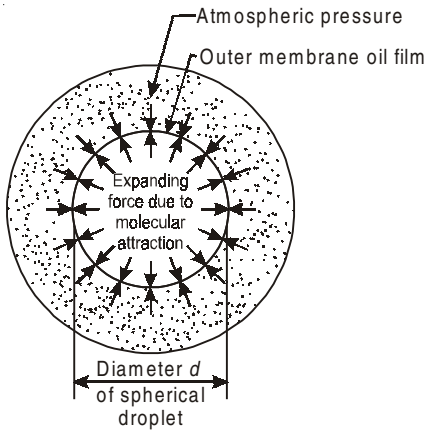


Fig. 1.4 *Forces on a droplet*

is $2\pi r \sigma$. In a spherical droplet these two opposing forces are in balance. By equating the two forces it gets simplified to form the equation $p = 2 \sigma/r$.

Even in case of a liquid jet (Fig. 1.5) the expanding and enveloping forces are in balance and the mathematical treatment is the same except for the fact that we have the length of the jet to consider.

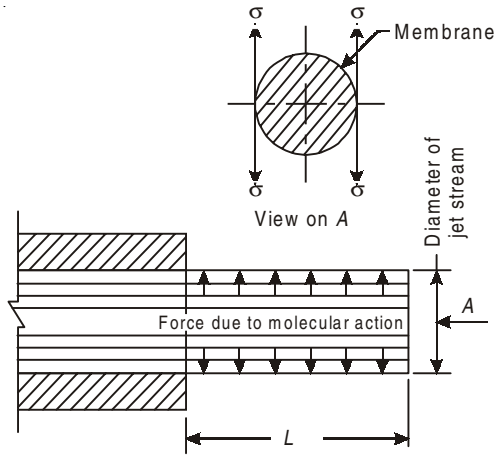


Fig. 1.5 *Forces on a liquid jet*

Consider a jet stream of length L. Force due to internal pressure on one half of the jet stream = $p \times L \times \frac{d}{2}$. Likewise the force due to surface tension on one half of the jet stream = $\sigma \times L$. Equating the two we obtain the simplified relation $p = \frac{2\sigma}{d}$.

In a soap bubble on the other hand we have two surfaces to consider, the outer and the inner, surrounding a thin membrane. Both are under the influence of air pressure. The force due to surface tension $\sigma = 2\ (2\pi r\ \sigma)$. The balancing force due to internal pressure = $p \times \pi \times r^2$. Equating the two we obtain the relation $p = \frac{4\sigma}{r}$.

In practice, the effect of surface tension is generally overlooked because the magnitude of the force is negligible compared to other forces the engineers are concerned about. Surface tension decreases with increasing temperatures.

1.3 CAPILLARITY

When a glass tube is immersed vertically into a water container as shown in Fig. 1.6, depending on the internal diameter of the tube immersed, the water that enters the tube raises to a level h above the level in the container. If this diameter is large, the height h will be smaller. This tendency of the liquids is

termed capillarity. In addition to surface tension, cohesive and adhesive forces in a liquid also contribute to this capillary action of liquids. Water wets the walls of the tube and hence the meniscus is depressed (Fig. 1.6) whereas in case of mercury which does not wet the walls of the tube the meniscus bulges out (Fig.1.7). However, the raise itself is lower than the surface level of mercury in the container for a tube of the same inside diameter. The familiar wicks stove used in house holds is a perfect example to appreciate the capillary raise of kerosene in a woven material partially dipped in. The height to which the liquid raises can be derived by computing the surface tension σ which is causing the liquid to raise and by equating it to the weight of the fluid that is resisting the pull.

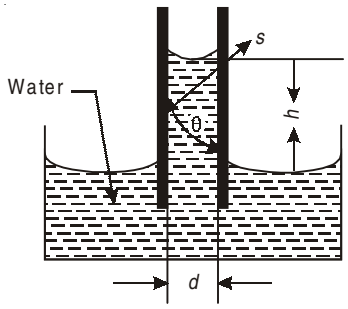


Fig. 1.6

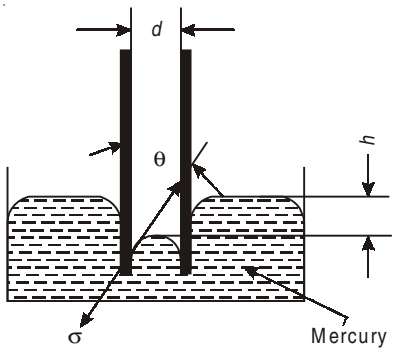


Fig. 1.7

Since only the vertical component of the surface tension need be considered the upward force due to surface tension is $\sigma \cos \theta \times \pi d$. The weight of the liquid which in this case is water and which is resisting the upward movement is $\rho g h$.

Equating the two forces which are in equilibrium, $\left(\frac{\pi}{4}\right)d^2 \cdot$

$\sigma \cos \theta \times \pi d = \rho g h \times \left(\frac{\pi}{4}\right)d^2$ or the capillary raise $h = \frac{4\sigma \cos \theta}{\rho g d} \cdot$ Here ρg constitutes the weight density of the fluid.

The angle θ depends on the liquid under consideration and the smoothness of the walls of the glass tube. If we assume the frictional force between the walls and the liquid is zero and the liquid is distilled water at 32° F, then an ideal condition would result. Under these conditions the raise h is maximum as $\theta = 0$ and hence $\cos \theta = 1$.

Liquids such as mercury because of its much higher density (value of ρg) as compared to water and also because adhesive forces are lower than cohesive forces (the value of cos θ will be lower) do not raise above its surface but falls below (Fig.1.7). Obviously the force due to surface tension is much lower than

the resisting force. Mathematical treatment is still the same and h even in this case = $\frac{4\sigma\cos\theta}{\rho g d}$.

1.3.1 Compressibility

A spring undergoes deflection upon application of load and this causes the spring to store up energy which can be used whenever required. The spring stiffness is indicated by the extent of deflection suffered due to the applied load. Compressibility in fluids is similar to spring stiffness and is indicative of the extent of energy absorbed by the fluid in a given time by suffering a change in volume. Solids, liquids and gases have this property to store up energy by suffering a change in volume on application of pressure. However, this property is of no great significance in solids and liquids because there is no appreciable change in volume unless the pressure applied is very high. Hydro-carbon fluids are practically incompressible at low pressures but do suffer a volume change at higher pressures, at the rate of 0.5% for every 10 MPa.

The compressibility of a liquid or solid will depend on its *Bulk modulus (K)* which indicates the extent of volume change that occurs.

Bulk modulus is a ratio of volumetric stress to volumetric strain. Since in liquids it is the applied pressure that results in volumetric stress, Bulk modulus as applied to liquids is stated as a ratio of change in pressure (δp) to volumetric strain $\left(\frac{\delta v}{V}\right)$.

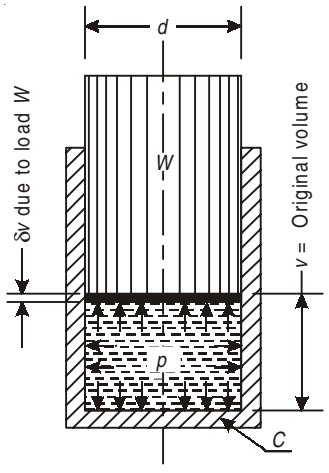


Fig. 1.8

Imagine a steel container C (Fig.1.8) which has an inside diameter d and contains a fluid such as hydraulic oil. Let the initial fluid volume inside the

container be V. Now if a cylindrical object of weight W, which has the same diameter as the container, is placed on the fluid in the container as shown, then this weight compresses the fluid molecules and due to this compression the initial fluid volume V suffers a reduction in volume of magnitude δv. This change in volume is possible because the fluid molecules come closer together due to the effect of weight W thus subjecting the fluid to a pressure p which according to Pascal's law is uniform and acts in all directions. The magnitude of this pressure is given by the relationship, $p = \dfrac{W}{0.7854d^2}$. Here $0.7854\, d^2$ = the cross sectional area of the container. Here it is important to appreciate that the pressure increase is responsible for the reduction in volume of the fluid. If we are considering the absolute pressure the pressure increase is (p + 1 atmosphere) and we may call it δp. If we are considering the gauge pressure then the pressure increase is p itself since the initial pressure is zero.

Bulk modulus $$K = \frac{\text{Change in pressure}}{\text{Volumetric strain}}.$$

Thus $$K = \frac{p}{\left[\dfrac{-\delta v}{V}\right]} \text{ N/m}^2.$$

The negative sign applied for δv is to signify the decrease in volume. Bulk modulus is a factor in the *velocity of propagation of sound* as given by the equation $C = \sqrt{\dfrac{K}{\rho}}$; where ρ is the mass density of the liquid and C is the velocity of sound in that particular fluid medium. (For a gas undergoing a status change under isentropic conditions $C = \sqrt{kRT}$, where k is the isentropic index, R is the universal gas constant and T is the temperature.) The reciprocal of K which is $\dfrac{1}{K}$ indicates compressibility of the fluid. Compressibility indicates how the resistance to further compression increases as the pressure is increased. In gases this resistance to further compression is far less than in liquids and the value of K therefore is not constant but depends on the applied pressure and changes rapidly. Also in gases when pressure is applied there is a simultaneous change in temperature too in addition to volume change. Bulk modulus is therefore not considered while dealing with gases.

1.4 VAPOR PRESSURE AND CAVITATION

Liquids normally evaporate to outer space due to continuous molecular agitation within. This is further aggravated by heat and consequent raise in molecular temperature. Some of these escaping molecules do condense and return, but if the space above the liquid surface is confined, a state of equilibrium will soon be

reached so as to cause the remaining molecules to exert a partial pressure on the surface of the liquid often referred to as the *vapor pressure*. When the vapor pressure becomes equal to the pressure above the liquid, the liquid begins to boil. It becomes evident therefore that by reducing the pressure above the liquid it is possible to decrease the temperature at which the liquids normally boil. This fact is taken advantage of in vacuum driers.

Vapor pressure is always measured as a pressure head and is stated in metres. For instance the vapor pressure of water at 10°C is 0.12 m and at 30°C it is 0.44 m indicating that the vapor pressure increases as the temperature increases. Vapor pressure in Pascal's can be obtained by multiplying the pressure head with the weight density of the liquid. Thus P_v = (pressure head × ρg). One of the reasons why mercury is considered an ideal fluid for Barometers is due its low vapor pressure.

Wherever liquid transfer between two points is caused by pressure difference it is necessary to maintain just the right difference sufficient to transfer all of the flow at the recommended velocity. Any increase in either the velocity or the pressure difference will lead to the formation of very low pressures, less than the vapor pressure of the liquid. This results in the formation of vapor pockets which are ultimately forced into high pressure areas even as they rapidly expand. Upon entering the high pressure area these vapor pockets suddenly collapse damaging the walls of the fluid conductors besides contributing to increased noise levels. This phenomenon is termed, *cavitation.* Cavitation is often the cause of erosion of metal parts in hydraulic pumps, turbines etc. Cavitation can also occur due to dissolved air in the liquid and the collapse of air bubbles has the same deleterious effect on the fluid conductors.

1.5 PROPERTIES OF GASES

1.5.1 Thermodynamic Properties

The effect of pressure, temperature and volume changes in gases and how these variables are interrelated is discussed under this caption.

When a gas is compressed or allowed to expand the process brings about a status change in the gas, which is governed by the perfect gas equation,

$$pV = mRT \qquad \text{... (1.1)}$$

where p is the absolute pressure in N/m^2, V is the gas volume in m^3, m is the mass of the gas in kg and R is the universal gas constant in N-m / kg- K.

If we consider mass-density instead of mass then the above equation can be expressed as,

$$p = \rho RT \qquad \text{... (1.2)}$$

where, ρ is density in kg/m^3.

If we consider the gas law on the basis of kg-mole instead of kg-mass then Eq 1.1 can be written as

$$pV = nMR_0 T \qquad \text{... (1.3)}$$

where, n is the number of moles in a certain volume of gas, and M is its molecular weight (here m is replaced by nM). The product of M and R_0 together constitutes

the universal gas constant R. The advantage is, Eq. 1.3 is now a general equation applicable to all gases because the value of universal gas constant (R) can now be expressed in terms of the molecular weight of the gas. It can be derived mathematically that R_0 has a value ≈ 8314 m-N/kg-mole-K. From this, the value of universal gas constant (R) can be obtained.

$$R = \frac{8312}{M} \text{ m-N / kg-K}$$

Since for a particular gas, the value of 'mR' is a constant Eq. 1.1 can written as, pV/T = constant

or
$$\frac{P_0 V_0}{T_0} = \frac{P_1 V_1}{T_1} \quad ...(1.4)$$

When the change takes place over a long period of time the temperature of the gas remains constant and such a change is called *isothermal*, resulting in the equation,

$$P_0 V_0 = P_1 V_1 = P_2 V_2 \quad ...(1.5)$$

When the change occurs instantaneously there is no time for heat transfer from the work to the environment; such a change is called isentropic or reversible adiabatic and is given by,

$$PV^k = \text{Constant}$$

or
$$P_0 V_0^k = P_1 V_1^k = P_2 V_2^k \quad ...(1.6)$$

All changes between isothermal and isentropic are called polytrophic.

Pressure-volume diagram Fig. 1.8 will help us to understand how the volume variation as a function of pressure is depending on the value of the polytrophic exponent k which for nitrogen is contained within the limits $1 \leq k \leq 1.4$. The value of k is taken to be equal to '1' if the compression and expansion process takes place under isothermal process. For adiabatic conditions the value of k is taken = 1.4.

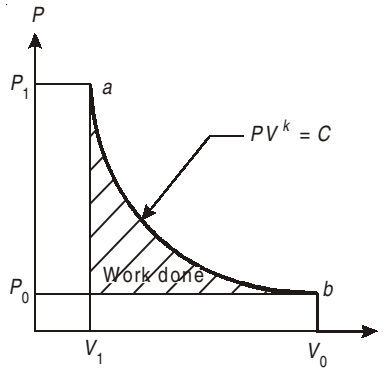


Fig. 1.8

1.5.2 Specific Heat

If a certain mass of gas is allowed to expand even while it is being heated then all of the heat energy is not contributing to a raise in the internal energy of the gas as the expansion process causes a change in volume and consequently some of the internal energy is lost to the environment as work done. The amount of heat energy required to raise the temperature of a gas from T_1 to T_2 under such a process will obviously be higher than that required when the heat energy is imparted keeping the volume constant. Temperature raise in a certain mass of gas therefore depends on the process adopted and the term *specific heat* is a constant for a unit mass of gas and is equal to the ratio of heat energy supplied and the rise in temperature. The value of this constant is specific to the process adopted. If the heat energy was imparted under constant volume conditions then this constant is denoted as C_v and if the heat energy was imparted keeping the pressure constant then it is denoted as C_p. Both are measured in J/kg-K.

Thus when the heat energy was imparted at constant volume $H = m\ C_v (T_2 - T_1)$ and when the heat energy was imparted at constant pressure $H = m\ C_p (T_2 - T_1)$ where m is the mass of gas. It can further be proved that $(C_p - C_v) = R$ and the ratio $\frac{C_p}{C_v}$ usually denoted as k has a value = 1.4 for most gases.

1.6 VARIATION OF PRESSURE WITH ALTITUDE

1.6.1 Compressible Fluids

Pressure increases with depth in liquids whereas in gases pressure decreases as the altitude increases. This decrease of pressure in gases is mainly due to a decrease in their mass densities. This decrease is appreciable as the altitude increases. The standard atmospheric pressure of air at sea level is 101.32 kPa (abs.), whereas, at an altitude 10,000 metres above sea level the absolute pressure is just around 26 kPa. The decrease in temperature too is equally appreciable. It is around 6.5 °C per 1000 metres. Since at altitudes less than 1000 metres the change in density is not considered significant we can determine the changes in pressure by assuming that the density is constant. Under such an assumption two conditions are possible.

1. Variation of pressure under *isothermal* condition.
2. Variation of pressure under *adiabatic* and isentropic conditions.

Variation of pressure: *Isothermal condition.*

Readers by now should be familiar with the basic equation $(p_2 - p_1) = -\gamma (z_2 - z_1)$. This equation in its differential form is strictly applicable for liquids but if we assume that the density remains constant and introduces a negative sign to indicate that the pressure decreases as the height increases, then the equation becomes applicable for gases too. The differential form of this equation is,

$$\frac{dp}{dz} = -\gamma = -\rho g \qquad ...(1.7)$$

Since $p = \rho RT$

or $\rho = \frac{p}{RT}$.

$$\frac{dp}{dz} = -\frac{pg}{RT}$$

$$\frac{dp}{p} = -\frac{g}{RT}\, dz$$

$$\int_{p1}^{p2} \frac{dp}{p} = -\frac{g}{RT} \int_{z1}^{z2} dz .$$

This upon integration yields the desired equation.

$$\frac{P_2}{P_1} = -\exp\left\{\frac{g}{RT}\right\}(Z_2 - Z_1) \qquad ...(1.8)$$

1.6.2 Variation of Pressure: Adiabatic Condition

The relationship between pressure and volume for a gas undergoing an adiabatic change is given by PV^k = Constant. If we consider specific volume, which is volume per unit mass then $V = \frac{1}{\rho}$

Therefore, $\frac{p}{\rho^k} = \frac{p_1}{\rho_1^{\,k}} = C$

or $\rho = \rho_1 \left(\frac{p}{p_1}\right)^{\frac{1}{k}}$

If we substitute this value for ρ in Eq. 1.8 above, then

$$\frac{dp}{dz} = -g\rho_1 \left(\frac{p}{p_1}\right)^{\frac{1}{k}}$$

or $$dz = \frac{dp}{-g\rho_1}\left(\frac{p}{p_1}\right)^{\frac{1}{k}}$$

Re-arranging the above term,

$$dz = -\frac{dp}{-g\rho_1}\left\{\frac{dp}{(p/p_1)^{\frac{1}{k}}}\right\}$$

or $$dz = \left(\frac{p_1^{\frac{1}{k}}}{g\rho_1}\right)\left\{p^{\frac{-1}{k}}dp\right\}$$

By integrating the expression within the flower bracket between limits

$p = p_1$

when $z = z_1$

and $p = p_2$

when $z = z_2$

$$z_2 - z_1 = -\left(\frac{p_1^{\frac{1}{k}}}{g\rho_1}\right)\left[\frac{p^{\frac{k-1}{k}}}{\frac{(k-1)}{k}}\right]$$

$$z_2 - z_1 = -\left(\frac{k}{k-1}\right)\left(\frac{p_1^{\frac{1}{k}}}{g\rho_1}\right)\left[p_2^{\frac{k-1}{k}} - p_1^{\frac{k-1}{k}}\right]$$

$$z_2 - z_1 = -\left(\frac{k}{k-1}\right)\left(\frac{p_1}{g\rho_1}\right)\left[\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} - 1\right]$$

$$z_2 - z_1 = -\left(\frac{k}{k-1}\right)\left(\frac{RT_1}{g}\right)\left[\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} - 1\right]$$

$\therefore$ $$\left(\frac{p_1}{\rho_1}\right) = RT_1$$

By Re-arranging,

$$\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} = g\,\frac{(z_2 - z_1)}{RT_1}\left[\frac{(k-1)}{k}\right] + 1$$

or $$\left(\frac{p_2}{p_1}\right) = \left[1 - \left\{g\frac{(z_2 - z_1)}{RT_1}\right\}\frac{(k-1)}{k}\right]^{\frac{k}{k-1}} \quad \text{... (1.9)}$$

The value of exponent k depends on the particular gas under going a status change under the adiabatic process. If the process is polytropic or reversible adiabatic where the exponent often used is n (Eq. 1.9) remains the same except for the value of n.

The expression for variation of temperature with altitude $\left(\frac{T_2}{T_1}\right)$ can be derived in a similar way from the basics and is given by,

$$\left(\frac{T_2}{T_1}\right) = \left[1 - \left\{g\frac{(z_2 - z_1)}{RT_1}\right\}\frac{(k-1)}{k}\right] \quad \text{... (1.10)}$$

The temperature gradient that is the rate at which the temperature changes with altitude can be obtained by differentiating T_2 in Eq. 1.10 with respect to Z. Since, T_1, R, k and g are constants, the equation after differentiation takes the form,

$$\frac{dT_2}{dZ} = -\left[\frac{(k-1)}{k}\right]\frac{g}{R} \quad \text{... (1.11)}$$

The negative sign indicates that the temperature decreases as the altitude increases for values of $k > 1$. Beyond a certain level, at around 35,000 metres above seal level, the temperature begins to rise which means the value of k beyond this point is < 1.

If we combine equation Eqs. 1.9 and 1.10 we can obtain an expression for T_2 in terms of the temperature gradient $\frac{dT_2}{dZ}$. This accomplished as follows.

$$\left(\frac{T_2}{T_1}\right) = \left[1 - \left\{g\frac{(z_2 - z_1)}{RT_1}\right\}\frac{(k-1)}{k}\right]$$

$$T_2 = T_1 - \left\{T_1 g\frac{(z_2 - z_1)}{RT_1}\right\}\frac{(k-1)}{k}$$

$$T_2 = T_1 - \left\{g\frac{(z_2 - z_1)}{R}\right\}\frac{(k-1)}{k}$$

$$T_2 = T_1 - \left\{\frac{g}{R}\frac{(k-1)}{k}\right\}(z_2 - z_1)$$

$$T_2 = T_1 - \left\{\frac{dT_2}{dz}\right\}(z_2 - z_1) \quad \text{... (1.12)}$$

TYPICAL PROBLEMS AND SOLUTIONS

PROPERTIES OF FLUIDS

Problem 1.1 *The specific weight of water is 9.81 × 103 N/m³. Calculate its density, relative density and specific volume.*

Solution Density ρ of any fluid = Specific weight w and acceleration due to gravity g. Thus, $\rho = 9.81 \times \frac{10^3}{9.81} = 1000$ kg/m³.

Relative density also called specific gravity of any fluid (sg) is a ratio of the density of fluid to the density of water at 4° C. Here since the fluid considered is water, sg. $= \frac{1000}{1000} = 1$.

Specific volume υ of any fluid is $\frac{1}{\rho}$. $\upsilon = \frac{1}{1000} = 0.001$ m³/kg

Problem 1.2 *The specific gravity of Benzene is 0.880. Calculate its specific weight and density.*

Solution Specific gravity is another name for relative density. Therefore specific gravity sg. of any fluid = Density of fluid/ Density of water at 4° C.

Therefore $0.880 = \frac{\rho}{1000}$

and hence, $\sigma = 0.880 \times 1000 = 880$ kg/m³.

Since specific weight $\gamma = \rho \times g$ we have, $\gamma = 880 \times 9.81 = 8632.8$ N/m³.

Problem 1.3 *A body weighs 15 N on earth. What will be its weight at some other location where the acceleration due to gravity 7.5 m /sec².*

Solution Weight of the body at the new location

$$= \frac{15 \times 7.5}{9.81} = 11.46 \text{ N}$$

Problem 1.4 *If an oil reservoir has a volume of one cubic metre and contains oil to $\frac{2}{3}$ of its capacity, calculate the weight of oil, its specific weight and specific gravity. Density of oil is 860 kg/m³.*

Solution

Volume of oil in the reservoir $= \left(1 - \frac{1}{3}\right) = 0.667$ m³.

Mass of oil = 860 × 0.667

= 573.62 kg

Weight of oil W = 573.62 × 9.81

= 5627 N (one Newton = 1 kg-m/s²).

Since specific weight $\gamma = \rho \times g$ we have, $\gamma = 860 \times 9.81 = 8436.6$ N/m.^{3} Specific gravity

$$\sigma = \frac{860}{1000} = 0.860.$$

Problem 1.5 *A cylindrical steel container of diameter 20 cm has a mass of 10 kg, when empty. When filled with fluid to a depth of 30 cm, the container and the fluid together weigh 1500 N. Calculate the specific gravity of the fluid in the container.*

Solution Weight of the container $w_1 = 10 \times 9.81 = 98.1$ kg-m/sec^2 or 98.1 N.

Weight of oil in the container

$$w_2 = 1500 - 98.1 = 1401.9 \text{ N}$$

Mass of oil in the container

$$(M) = \frac{1401.9}{9.81} = 142.9 \text{ kg}$$

Volume of oil in the container

$$(V) = \text{area of the container} \times \text{depth of fluid of fluid}$$

$$V = \{0.7854 \times (0.2)^2\} \times 0.3 \text{ m} = 0.0094 \text{ m}^3$$

$$\text{Density of the fluid } \rho = \frac{M}{V}$$

$$= \frac{142.9}{0.0094}$$

$$= 15{,}202 \text{ kg/m}^3.$$

$$\text{Specific gravity of the fluid } \gamma = \frac{15{,}200}{1000} = \mathbf{15.202}$$

Problem 1.6 *A fluid of certain viscosity is contained between two parallel plates. When a force of 10 N is exerted on the top plate the angular deformation is 60 rad/sec. Calculate the viscosity of the fluid. The area of the top plate is 0.75 m^2.*

Solution

$$\text{Shear stress } \tau = \frac{\text{Shear force}}{\text{Area}}$$

$$= \frac{10}{0.75} = 13.33 \text{ N/m}^2$$

$$\text{Shear strain} = 60 \text{ rad/sec}$$

$$\text{Viscosity} \qquad \mu = \frac{\text{Shear stress}}{\text{Shear strain}}$$

$$\mu = \frac{\tau}{du/dy} = \frac{13.33}{60}$$

$$= 0.22 \text{ Ns/m}^2$$

Problem 1.7 *A plate placed horizontal and parallel to a fixed plate has an oil layer of thickness 1.5 mm underneath. The plate moves with a velocity of 0.5 m/sec when a certain force is applied. The top surface of the moving plate is exposed to atmosphere and offers no resistance of its own. Determine the oil viscosity. Assume the value of shear stress as 2 N/m².*

Solution The value of shear stress τ is give by, $\tau = \mu \frac{du}{dy}$

Since the plate was initially at rest, the change in velocity du = 0.5 m/s. dy is change of distance = 1.5 mm or 1.5×10^{-3} m. τ is given as 2 N/ m². μ, the oil viscosity is required to be determined.

The above equation can be written as, $\mu = \frac{(\tau \times dy)}{du}$

$$= \frac{[2\,(1.5 \times 10^{-3})]}{0.5}$$

$$\mu = 6 \times 10^{-3} \text{ N.s/m}^2$$

Problem 1.8 *A certain fluid is flowing through a pipe of diameter 20 cm. The velocity of the fluid at the center of the pipe is 75 cm/sec. Determine the velocity gradient and shear stress. Assume a fluid viscosity = 1.2 N.s/m².*

Solution It has already been explained that when a fluid flows through a pipe the velocity profile formed is parabolic (Ref. Fig. 1.3) and the equation for such a flow is given by,

$$u = ay^2 + by + c$$

For a pipe of diameter 20 cm, the vertex is at a distance of $\frac{y}{2}$ which is 10 cm. Velocity at the vertex is given as 75 cm/ s. The value of the constants a, b, c is first determined from the boundary condition and $\left[\frac{du}{dy}\right]$, the velocity gradient is obtained by differentiating u with respect to y. The boundary conditions are;

(1) at y = 0, the velocity u = 0.

(2) at y = 10 cm u = 75 cm/s.

(3) at y = 10 cm, $\frac{du}{dy} = 0$.

If we substitute condition (1) in the equation $u = ay^2 + by + c$, the value of c = 0.

If we substitute condition (2) then the equation takes the form,

$$75 = a\,(10)^2 + 10b$$

or $$100a + 10b = 75.$$

Before we substitute condition (3) we have to differentiate the equation

$$u = ay^2 + by + c$$

which upon differentiation takes the form,

$$\frac{du}{dy} = 2ay + b$$

Since $$\frac{du}{dy} = 0 \text{ at } y = 10$$

We can re-write the above as $2a\,(10) + b = 0$ or $20a + b = 0$.
We have now two equations: $100a + 10b = 75$ and $20a + b = 0$.
Solving for a and b we get,

$$a = -0.75$$

and $$b = 15.$$

Substituting these values in the equation,

$$u = ay^2 + by + c, \text{ we get,}$$

$$u = -0.75y^2 + 15y + 0$$

$$\frac{du}{dy} = -0.75\,(2y) + 15$$

$$= -1.5y + 15$$

From this we can obtain the velocity gradient.

at $$y = 0 \text{ cm}$$

$$\frac{du}{dy} = 15/\text{sec}$$

at $$y = 5 \text{ cm}$$

$$\frac{du}{dy} = 7.5/\text{sec}$$

at $$y = 10 \text{ cm}$$

$$\frac{du}{dy} = 0$$

Shear stresses

at $y = 0$, $$\tau = \mu \frac{du}{dy} = 1.2 \times 15 = 18 \text{ N/m}^2$$

at $y = 5$ $$\tau = \mu \frac{du}{dy} = 1.2 \times 5 = 6 \text{ N/m}^2$$

at $y = 10$ $$\tau = \mu \frac{du}{dy} = 1.2 \times 0 = 0 \text{ N/m}^2$$

Problem 1.9 *A 75 mm diameter shaft is sliding in a sleeve 200 mm long @ 12 cm / sec due to a force = 5 N. Calculate the viscosity between the shaft and the sleeve if the diametric clearance between the shaft and the sleeve is 0.1 mm.*

Solution

Length of the sleeve L = 200 mm
= 0.2 m
Diameter of the shaft d = 75 mm
= 0.075 m
Linear velocity u = 12 cm/sec
= 0.12 m/s
Diametric clearance δ = 0.1 mm
= 0.0001 m
Radial clearance t = .0001/2
= 0.00005 m
Force F = 5 N

Shear stress $$\tau = \frac{F}{\text{Contact area}}$$

$$= \frac{F}{\pi \times d \times L}$$

$$= \frac{5}{\pi \times 0.075 \times 0.2}$$

$$= \mathbf{106\ N/m^2}$$

The value of shear stress μ is given by, $\tau = \mu \frac{du}{dy} = \mu \frac{[u]}{[t]}$

Since we are required to determine the value of viscosity μ we re-arrange the above equation suitably.

$$\mu = \frac{(\tau \times t)}{u}$$

$$= \frac{106 \times 0.00005}{0.12}$$

$$= \mathbf{0.044\ Ns/m^2}$$

Problem 1.10 *A 100 mm diameter steel shaft of length 350 mm slides down vertically @ 0.2 m/sec due to its own weight through a tube containing a certain fluid having a viscosity 0.2 N-s / m². A constant fluid film thickness is maintained throughout descend. Determine the clearance between the tube and the outer sleeve. Density of steel = 7. 85 gms / cc.*

Solution Since the shaft slides due to its own weight, we have to first determine its weight.

Weight of the shaft = cross sectional area of shaft × length × density.

Cross sectional area = $[100 \times 10^{-3}]^2 \times 0.7854$
= **0 007854 m²**

Length of the shaft = 350 mm = 0.35 m

Density = 7.85 gms /cc

= **7.850 × 1000**

= 7850 kg / m^3

Viscosity μ = 0.2 N-s / m^2

Velocity of descend u = 0.2 m/s

Diameter of the shaft d = 100 mm or 0.1 m

Force causing the weight to descend

= F = 0.007854 × 0.35 × 7850

= 21.5 kg

or 21.5×9.80 = 210 N. (1 kgf = 9.80 N)

$$\text{Shear stress } \tau = \frac{F}{\text{Contact area}} = \frac{F}{\pi \times d \times L}$$

$$= \frac{210}{\pi \times 0.1 \times 0.35}$$

= **1909 N/m²**

Since we are required to determine the oil film thickness t which incidentally is the shaft and the sleeve clearance, we can obtain t by rearranging the equation,

$$\tau = \frac{du}{dy} = \mu \frac{[u]}{[t]}$$

or $$\text{Clearance } t = \frac{[u]}{[\tau]} = 0.2 \frac{0.2}{1909} = 0.02 \text{ mm}$$

Problem 1.11 *An object of weight 2 N slides down an inclined plane which is at an angle, 30 degrees to the vertical. There is a fluid film separating the plane and the object of thickness 0.25 mm. Viscosity of the fluid is 0.07 N-s/m². Assuming the velocity distribution in the oil film is linear; determine the terminal velocity of the object. The contact area is 0.5 m². (Fig.1.9)*

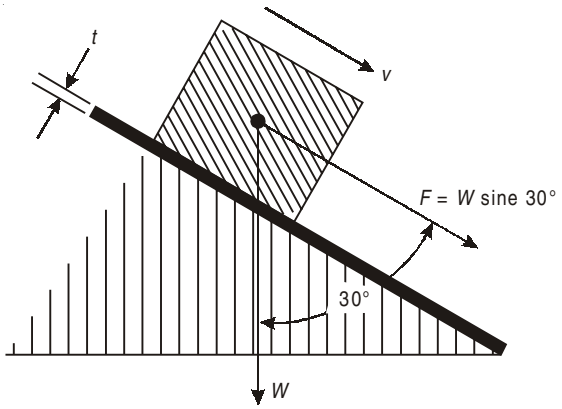


Fig. 1.9

Solution

Weight W = 2 N

Thickness of the fluid film = 0.25 mm

= 2.5×10^{-4} m

Viscosity μ = 0.07 N-s/m^2

Contact area A = 0.5 m^2

The sum of the forces acting in the direction of motion of the plate is zero as the velocity $\rightarrow 0$.

These forces are, the positive force pulling the plate forward which is a component of force W and negative or the resisting force due to the shear stress.

$$W \sin \theta - \tau A = 0 \qquad ...(1.13)$$

where $$\tau = \mu \frac{[du]}{[dy]} = \mu \frac{u}{t}$$

$$= 0.07 \frac{u}{2.5 \times 10^{-4}}$$

and from this we can obtain the value of τ in terms of the velocity u as $\tau = 70\ u$.

Substituting this value in Eq. 1.13 we get $W \sin \theta - 70\ u\ A = 0$ or $2 \sin 30° - 70\ u\ (0.5) = 0$ or $35\ u = 1$ or $u = .028$ m/sec.

Problem 1.12 *A shaft of diameter 50 mm rotates in a journal bearing of length 200 mm at 1500 RPM. The diametric clearance between the shaft and the bearing is 0.2 mm. The viscosity of the lubricant used is 65 × 10^{-6} m^2/s. The specific gravity of the lubricant is 0.86. Calculate power loss due to oil resistance.*

Solution

Diameter of shaft D = 50 mm or 0.05 m

Length of the Journal = 200 mm = 0.2 m

The moment arm equal to the radius of the shaft

$$R = \frac{0.05}{2} = 0.025 \text{ m.}$$

The diametric clearance = 0.2 mm or 0.2×10^{-3} m.

The radial clearance equal to the thickness of the oil film

$t = 0.1 \times 10^{-3}$ m

RPM of the shaft N = 1500.

Kinematic viscosity is given as 65×10^{-6} m^2/ sec and the specific gravity as 0.86. Therefore mass density = $0.86 \times 1000 = 860$ kg/m^3. From these values we can obtain the absolute viscosity μ of the lubricant.

$\mu = 860 \times (65 \times 10^{-6})$

= 0.0559 kg m^{-1} s^{-1} or N.s/m^2 or Pa.s

To obtain the value of power the basic equation is

$$P = 2\pi \frac{NT}{60} \qquad ... (1.14)$$

In Eq. 1.14 the only unknown is torque T. Since $T = F \times$ Lever arm R, we

have to obtain the value of F which depends on the contact area and the shear stress due to oil resistance τ.

$$\tau = \mu \frac{[du]}{[dy]} = \mu \frac{u}{t}$$

Here, u is the tangential velocity and t is the thickness of the oil film. Since the RPM is given as 1500, we can determine the tangential velocity u as,

$$\begin{aligned} u &= \pi DN \\ &= \pi (0.05)\ 1500 \\ &= 235.6 \text{ m/minute} \\ &= 3.9 \text{ m/sec.} \end{aligned}$$

Shear stress $\tau = 0.0559 \ \dfrac{3.9}{0{,}1 \times 10^{-3}} = 2180 \text{ N/m}^2$

Contact area of the lubricant $= \pi DL$
$= \pi \times 0.05 \times 0.2$
$= 0.0314 \text{ m}^2$

Force F = Shear stress × Contact area
$= 2180 \times 0.0314$
$= 68.452 \text{ N}$

Torque = $T = F \times$ Lever arm R,

we have $T = 68.452 \times 0.025$
$= 1.7 \text{ Nm}$

As per Eq. 1.14 $\text{Power} = (2\pi) \times 1500 \times \dfrac{1.7}{60} = 267 \text{ W}$

$$= \frac{267}{9.81} = 27.2 \text{ kgf-m /sec or } \frac{27.2}{75} = 0.36 \text{ HP}$$

(This is because 9.81 watts = 1 kgf-m/sec and 75 kgf-m/sec = 1 HP)

Problem 1.13 *A shaft slides horizontally in a sleeve due to force of 300 N applied on its axis and attains a linear velocity = 1 metre / s. The clearance between the shaft and the sleeve is filled with a lubricant. Prove that if the applied force increases by three times, the velocity increases by three times.*

Solution If we assume that the clearance between the shaft and the sleeve is very small and the fluid is Newtonian then the velocity distribution in the lubricant will be linear. Under such an assumption if the applied force is increased by three times, then the velocity should also increase by as many times.

We know $$\tau = \mu \frac{[du]}{[dy]} = \mu \frac{u}{t}$$

Here, u is the velocity, t is the thickness of the oil film or the clearance between the shaft and the sleeve. If we substitute $\dfrac{F}{A}$ in place of τ the equation can be re-written as follows.

$$\frac{F}{A} = \mu \frac{u}{t} \text{ or } \frac{F}{u} = \frac{\mu A}{t}$$

and since μA and t are all constants,

$$\frac{F}{u} = \frac{F_1}{u_1}$$

Here $F = 300$ N

$F_1 = 900$ N

$u = 1$ m/s

Therefore $u_1 = 3$ m/s

Problem 1.14 *A shaft of diameter 200 mm revolves concentrically in a stationery drum at 100 RPM as shown in Fig 1.10. The annular space between the drum and the shaft which is 1 mm is filled with oil to a length of 350 mm. In addition the bottom of the shaft is in contact with an oil film of thickness 1.5 mm. Assume a linear velocity variation in the oil film and determine the power loss due to viscous resistance. Assume the viscosity of the oil as* 3×10^{-3} *Pa.s.*

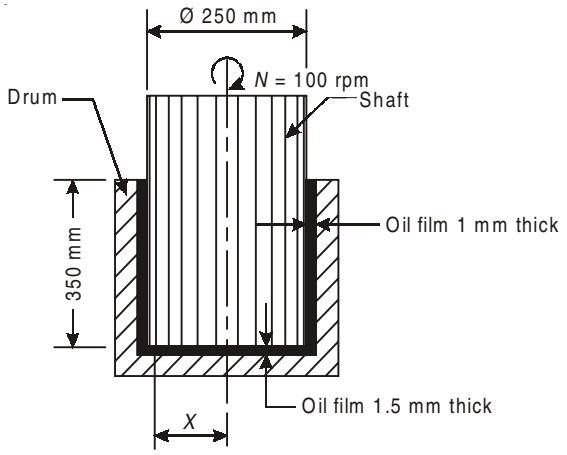


Fig. 1.10

Solution

The fluid in the annular space has a definite velocity gradient which is determined by its tangential velocity, whereas in case of the fluid at the bottom of the shaft the velocity varies from zero at the center to maximum at the periphery limited to its radius R. Hence the torque required to overcome viscous resistance has to be considered separately and is the sum of the torque due to viscous resistance offered by the fluid in the annular space and the fluid film at the bottom of the shaft.

Area of the annular space $A = \pi DL$

$= \pi \times 0.2 \times 0.35$

$= 0.20$ m^2

Tangential velocity $V = \pi DN$

$= \pi \times 0.2 \times 100$

$= 62.8$ m/minute or $\approx$ 1m /sec.

Thickness of oil film, $t = 1$ mm or 0.001 m.

Shear stress on the shaft due to viscous resistance $\tau = \mu \left[\frac{du}{dy}\right] = \mu\left[\frac{V}{t}\right]$

$$\tau = 3 \times 10^{-3}\left[\frac{1}{0.001}\right]$$

$= 3$ N/m^2

Shear force $F = \tau \times A = 3 \times 0.20$

$= 0.6$ N

Torque on the shaft $T_1 = F \times R$

$= 0.6 \times 0.1$

$= 0.06$ Nm

We now have to calculate the torque T_2 due viscous resistance in the bottom layer.

An equation for this can be derived by considering a small element of the shaft of width δr at a distance r from the axis.

Let the tangential velocity at this point

$$v = \omega r$$

The thickness of the oil film in this case is 1.5 mm

$= 0.0015\ m = t_1$

Shear stress $\tau = \mu \frac{du}{dy}$

$= \mu \frac{v}{t}$ (assuming a linear velocity variation)

or $\frac{F}{A} = \mu \frac{v}{t}$

or $F = \mu \frac{vA}{t}$

or $F = \frac{\mu(\omega r)(2\pi r.\delta r)}{t_1}$

Torque on this element $\lambda T_2 = F \times r$

$= \frac{\left[\mu(\omega r)(2\pi r.\delta r)\right]r}{t_1}$

Total torque $$T_2 = \int_0^R \lambda T_2 = \int_0^R \left\{\frac{[2\pi\mu(\omega)]}{t_1}\right\} r^3\, \delta r$$

$$T_2 = \left[\left\{\frac{[2\pi\mu(\omega)]}{t_1}\right\}\frac{r^4}{4}\right]_0^R = T_2$$

$$= \left\{\frac{[2\pi\mu(\omega)]}{2t_1}\right\} R^4$$

Substituting the values already obtained in the equation for T_2,

$$T_2 = \left\{\frac{\pi\,(3 \times 10^{-3})\,(1)}{2\,(0.0015)}\right\}\,(0.1)^4$$

$$T_2 = 3 \times 10^{-4}\ \text{Nm}$$

Total torque $$T_2 = T_1 + T_2$$
$$= 0.6 + 3 \times 10^{-4}$$
$$= 0.6003\ \text{Nm}$$

Power loss $$= \frac{(2\pi \times 100 \times 0.6003)}{60}$$

$$= 6.29\ \text{W}$$

Problem 1.15 *Two steel tubes are placed concentrically one inside the other as shown in Fig. 1.11. Determine the magnitude of the force required to lift the inner tube @ 0.20 m/sec. Density of steel = 7.85 gms. /cc. Assume μ = 0.1 Pa·s.*

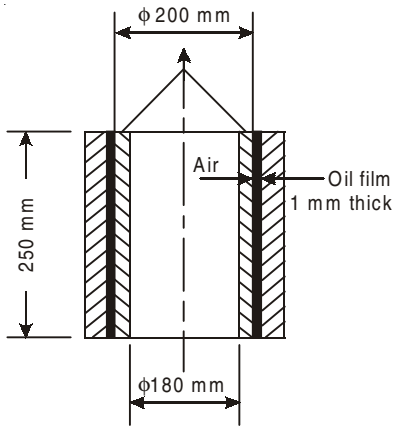


Fig. 1.11

Solution

$$\text{Density of steel} = \frac{(7.85 \times 10^6)}{10^3}$$
$$= 7850 \text{ kg/m}^3$$
$$= 7850 \times 9.81$$
$$= 77{,}000 \text{ N/m}^3.$$
$$\text{Volume of steel tube} = V = 0.7854 \{(0.20)^2 - (0.18)^2\} \times 0.250$$
$$= 1.5 \times 10^{-3} \text{ m}^3$$
$$\text{Weight of steel tube} = W = (1.5 \times 10^{-3}) \times 77{,}000$$
$$= 115.5 \text{ N}$$
$$\text{Surface area of the steel tube} A = \pi DL$$
$$= \pi \times 0.20 \times 0.25$$
$$= 0.157 \text{ m}^2$$
$$\text{Desired velocity of lift } v = 0.20 \text{ m/sec}$$
$$\text{Thickness of the oil film} = t = 1.5 \times 10^{-3} \text{ m}$$

Shear stress $\tau = \mu \left[\frac{du}{dy}\right]$. Here du refers to change of speed which in this case is the same as v because the initial velocity is zero and dy refers to the thickness t of the oil film if we assume a linear change in velocity profile.

$$\tau = 0.1 \left[\frac{0.20}{1.5 \times 10^{-3}}\right]$$
$$= 13.3 \text{ N/m}^2$$
$$\text{Force } F = \tau \times A = 13.3 \times 0.157$$
$$= 2 \text{ N}$$
$$\text{Total force required } F_T = W + F$$
$$= 115.5 + 2$$
$$= 117.5 \text{ N}$$

If the steel tube shown in Fig 1.11 instead of being lifted is made to rotate at 100 RPM along the same axis then to calculate the required torque, we consider the tangential velocity instead of linear velocity.

$$\text{Tangential velocity} = v = \pi \frac{DN}{60}$$
$$= \pi (0.2) \frac{100}{60} = 1 \text{ m/sec}$$

Since all other parameters remain the same, in the equation $\tau = \mu \left[\frac{du}{dy}\right]$ only the value of du changes. $du = v = (1 - 0) = 1$ since the initial RPM is zero.

Therefore, $$\tau = 0.1 \left[\frac{1}{1.5 \times 10^{-3}}\right] = 66.66 \text{ N/m}^2$$

Force in this case is $F_1 = \tau \times A$

$= 66.66 \times 0.157 = 10.46$ N

Total force required $F_T = W + F_1$

$= 115.5 + 10.46$

$= 126$ N

Torque $T = F_T \times R$

$= 126 \times 0.1$

$= 12.60$ Nm

Problem 1.16 *A thin steel plate is placed between two plane fixed surfaces containing a fluid of viscosity 0.105 N·s/m² as shown in Fig.1.12. The total surface area of the plate exposed to the fluid is 1.5 m². What force is required to pull the plate at a constant velocity of 1m/sec if,*

(*1*) *The plate is located at the center of the two planes.*

(*2*) *The plate is located at 2/3 the distance from any one plane.*

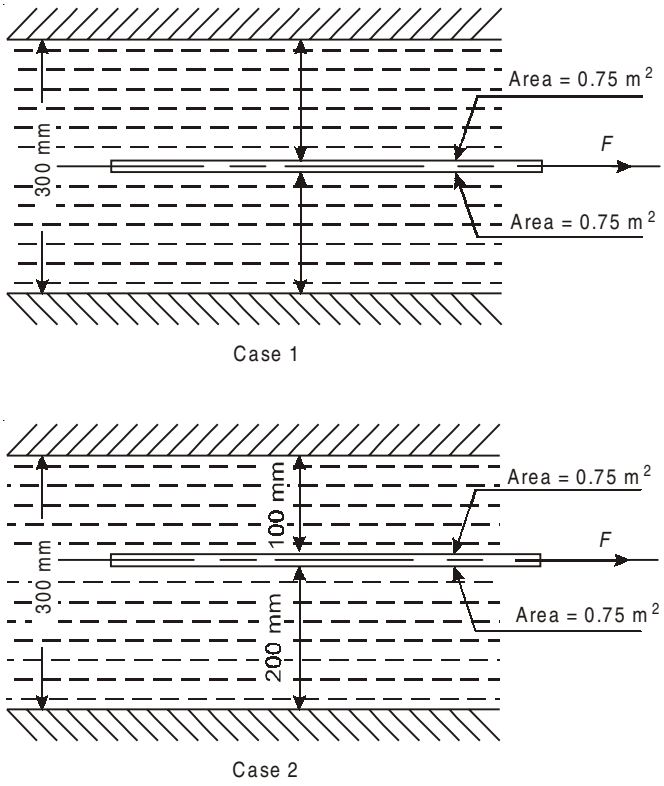


Fig. 1.12

Solution

(1) *When the plate is located at the center of the two planes.*

In this case when the plate is pulled the shear stress is equal on either side of the plate.

In the equation $\tau = \mu \left[\frac{du}{dy}\right]$ $\mu = 0.105$ N·s/m^2, du is the velocity of the plate = 1m/sec. dy, the distance from the fixed plane to the plate and can be taken = $\frac{300}{2}$ = 150 mm or 0.150 m if we neglect the thickness of the plate.

$$\text{Therefore } \tau = 0.105 \left[\frac{1}{0.150}\right]$$

$$= 0.7 \text{ N/m}^2$$

Force $F = \tau \times A$ where A is the area of one side of the plate

$$= \frac{1.5}{2}$$

$$= 0.75 \text{ m}^2$$

$$F = (0.7 \times 0.75)$$

$$= 0.525 \text{ N.}$$

Since the force is the same even on the other side, the total force required to pull the plate at the given velocity

$$= (0.525) \times 2$$

$$= 1.05 \text{ N}$$

(2) When the plate is located at 2/3 the distance, that is at 200 mm from the bottom plane.

In the equation $\tau = \mu \left[\frac{du}{dy}\right]$ $\mu = 0.105$ N·s/m^2, du is the velocity of the plate = 1m/sec. dy, the distance from the fixed plane to the plate is 200 mm or 0.200 m if we neglect the thickness of the plate.

Therefore, τ at the bottom layer of the plate

$$= 0.105 \left[\frac{1}{0.200}\right]$$

$$= 0.525 \text{ N/m}^2$$

Force $F_1 = \tau \times A$ where A is the area of one side of the plate

$$= \frac{1.5}{2}$$

$$= 0.75 \text{ m}^2$$

$$F_1 = (0.525 \times 0.75)$$
$$= 0.4 \text{ N.}$$

Similarly τ at the top layer of the plate $= 0.105 \left[\frac{1}{0.100}\right] = 1.05 \text{ N/m}^2$ because the top layer is at a distance of 100 mm or 0.1m from the top plate.

$$F_2 = (1.05 \times 0.75)$$
$$= 0.7875 \text{ N.}$$

Total force required to pull the plate when the plate is at 2/3 the distance from the bottom plane

$$= 0.4 + 0.7875$$
$$= 1.1875 \text{ N.}$$

SURFACE TENSION AND CAPILLARITY

Problem 1.17 *10 air bubbles of diameter 2.5 mm were found on the surface of water. Calculate the excess pressure over and above the atmospheric pressure that could have caused the bubbles to form and the work done.*

$$\sigma = 73 \times 10^{-3} N/m.$$

Solution The equation governing the pressure p surface tension σ and the radius r in an air bubble is given by,

$$p = 2\,\frac{\sigma}{r}$$

where,
$$r = \left(\frac{2.5}{2 \times 10^{-3}}\right)$$
$$= 1.25 \times 10^{-3} \text{ m}$$

Substituting the relevant values we have $p = \dfrac{2\,(73 \times 10^{-3})}{1.25 \times 10^{-3}}$

$$= 116.8 \text{ N/m}^2.$$
$$\text{Work done} = \sigma \times (\text{total surface area})$$
$$= (73 \times 10^{-3})\,\{10\,(\pi \times d^2)\}$$
$$\text{Work done} = (73 \times 10^{-3})\,\{10\,\pi\,(2.5 \times 10^{-3})^2\}$$
$$w = 1.4 \times 10^{-5} \text{ J}$$

Problem 1.18 *A nitrile rubber O ring of mean diameter = 25 mm was found floating on the surface of water. Calculate the force required to lift the ring. Assume surface tension of water as 73×10^{-3} N/m. Neglect the weight of the ring.*

Solution If we neglect the weight of the ring the only other force holding the ring in place is the surface tension σ. Surface tension is force/unit length and in this case it is acting on either side of the circumference of the ring.

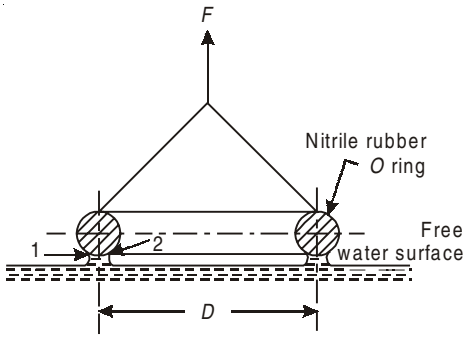


Fig. 1.13

Therefore, σ = F/unit length

$= F / 2\sigma D$

or $F = 2 \pi D \sigma$

$F = (2 \times \pi \times 0.025 \times 73 \times 10^{-3})$

$= 0.01$ N

Problem 1.19 *A jet of mercury cylindrical in shape oozes out of an opening of diameter 2 mm. Calculate the pressure difference inside and outside of the jet. Assume σ of mercury = 472 × 10^{-3} N/m.*

Solution In case of cylindrical jet the pressure difference $\delta p = \frac{\sigma}{r}$ where r is the radius of the cylindrical jet which in this case is $\frac{(2 \times 0.5)}{1000} = 0.001$ m

Therefore $\delta p = \frac{(472 \times 10^{-3})}{0.001} = 472$ N/m^2

Problem 1.20 *Calculate the pressure inside a water droplet of diameter = 5 mm, if the pressure out side is atmospheric. σ = 73 ×10^{-3}N/m.*

Solution We know that for a water droplet $p = 2\frac{\sigma}{r}$, where r is the radius of the droplet.

Therefore the pressure $p = 2\frac{(73 \times 10^{-3})}{(2.5 \times 10^{-3})} = 58.4$ N/m2

Problem 1.21 *How much pressure is required to form an air bubble inside a tube immersed vertically in water to a depth 30 mm? The inside diameter of the tube is 6 mm and σ = 73 ×10^{-3} N/m.*

Solution The pressure required to form the air bubble is the sum of pressures inside and outside of the bubble. Since in this case the bubble was formed at a depth of 30 mm below the surface of water,

Pressure outside of the bubble = p_1

$= \rho g h$

$= 1000 \times 9.81 \times 0.030$

$= 294.3 \text{ N/m}^2$

Assuming the diameter of the air bubble is the same as the inside diameter of the tube.

Pressure inside the bubble

$$= p_2 = 2 \frac{\sigma}{r}$$

$$= 2 \times \frac{(73 \times 10^{-3})}{(3 \times 10^{-3})} = 48.6 \text{ N/m}^2$$

The total pressure = $p_1 + p_2$

$= 294.3 + 48.6 = 343 \text{ N/m}^2$

Problem 1.22 *Two tubes of inside diameters 4 mm and 6 mm were immersed in water to a certain depth. Calculate the difference in capillary raise between the tubes. Assume* $\sigma = 73 \times 10^{-3}$ *N/m.*

Solution The capillary raise h for any liquid is given by the equation

$$h = \frac{4 \sigma \cos \theta}{\rho g d}$$

The contact angle θ may be assumed as zero for water. Hence in this case $\cos \theta = 1$. Since all other parameter is constant in either case except for the diameters of the two tubes, the difference in capillary raise is $(h_1 - h_2)$ where h_1 is the raise in the smaller tube and h_2 is the raise in the bigger tube.

$$(h_1 - h_2) = \frac{4 \sigma \cos \theta}{\rho g d_1} - \frac{4 \sigma \cos \theta}{\rho g d_2}$$

ρg together constitute the weight density of the water which is

$(1000 \times 9.81) = 9810 \text{ N/m}^3$.

$d_1 = 0.004$ m

and $d_2 = 0.006$ m.

Substituting these values,

$$(h_1 - h_2) = \frac{4 (73 \times 10^{-3})}{9810 \times 0.004} - \frac{4 (73 \times 10^{-3})}{9810 \times 0.006}$$

$= 2.4$ mm.

Problem 1.23 *A glass tube of inside diameter 6 mm was vertically introduced in to an oil tank till it touched the base and the oil level in the tube rose to 1100 mm mark. If the contact angle between the liquid and the tube was* 50° *determine the depth of oil in the tank. Assume* $\sigma = 30 \times 10^{-3}$ *N/m and density of oil as* 860 *kg/m*^{3}.

Solution Depth of oil in the tank = 1100 – capillary raise above the surface of oil (h).

$$\begin{aligned} \text{Here, } \sigma &= (30 \times 10^{-3}\ \text{N}), \\ (\rho \times g) &= (860 \times 9.81) \\ &= 8436.6\ \text{N/m}^3 \\ \cos\theta &= \cos 50^\circ \\ &= 0.6428 \\ d &= 6 \times 10^{-3}\ \text{m} \end{aligned}$$

Substituting these values in the equation for capillary raise we have,

$$h = \frac{4\sigma\cos\theta}{\rho g d}$$

$$= \frac{4\,(30 \times 10^{-3})\,(0.6428)}{8436.6 \times (6 \times 10^{-3})}$$

$$\begin{aligned} h &= 1.5\ \text{mm} \\ \text{Depth of oil in the tank} &= 1100 - 1.5 \\ &= 1098.5\ \text{mm}. \end{aligned}$$

Problem 1.24 *A U tube has limbs of different diameters. The diameter of the left limb is one half of the right limb. Determine the diameters if the difference in water column between the two limbs is 10 mm. Assume* $\sigma = 73 \times 10^{-3}$ *N/m.*

Solution The difference in the height of the water column between the two limbs is due to different surface tension forces acting at each limb due to a difference in their diameters. The weight of the water column in the smaller limb is balancing the upward forces due to surface tension.

Which means, $$\rho g h = \frac{4\sigma}{d_1} - \frac{4\sigma}{d_2} \text{ or } 4\sigma\left\{\frac{1}{d_1} - \frac{1}{d_2}\right\}.$$

If d_2 is the diameter of the right limb then $d_2 = 2d_1$

$$\rho g h = 4\sigma\left\{\frac{1}{d_1} - \frac{1}{d_2}\right\} = 4\sigma\left\{\frac{1}{2d_1}\right\}$$

$$\rho g h = 4\,(73 \times 10^{-3})\left(\frac{1}{2d_1}\right)$$

$$(1000 \times 9.81)\,(10^{-2}) = 4\,\frac{(73 \times 10^{-3})}{2d_1}$$

$$d_1 = \frac{2(73 \times 10^{-3})}{(1000 \times 9.81)\,(10^{-2})}$$

$$d_1 = 1.5\ \text{mm and } d_2 = 3\ \text{mm}$$

COMPRESSIBILITY AND BULK MODULUS OF LIQUIDS

Problem 1.25 *A hydraulic cylinder and piston arrangement shown in Fig. 1.14, can lift a load of 4000 kN through a distance of 20 cm when hydraulic oil is pumped through its port. Diameter of the ram is 30 cm. Determine the change in volume of the oil on application of maximum pressure. The Bulk modulus K is 1.9 ×10⁹Pa.*

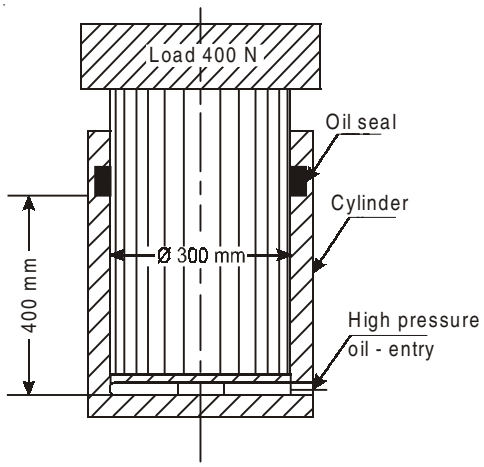


Fig. 1.14

Solution

$$\text{Bulk modulus } K = \frac{\text{Change in pressure}}{\text{Volumetric strain}}$$

$$\text{Thus } K = \frac{p}{\left[\frac{-\delta v}{V}\right]} \text{ N/m}^2.$$

Here the unknown values are, change in pressure p and the volume V. The pressure changes from 0 to maximum required to lift the load.

$$\text{Load to be lifted} = 4000 \text{ kN}$$

or $$4000 \times 1000 = 4 \times 10^6 \text{ N}.$$

$$\text{Area of the piston} = 0.7854 \times (0.3)^2$$

$$= 7 \times 10^{-2} \text{ m}^2$$

$$\text{Pressure } p = \frac{\text{Load}}{\text{Area}} = \frac{[4 \times 10^6]}{[7 \times 10^{-2}]}$$

$$= 5.7 \times 10^7 \text{ N/m}^2 \text{ or Pa}$$

The original volume V = product of the area of the piston and the distance travelled.

$$V = [7 \times 10^{-2}] \times [0.2]$$

$= [1.4 \times 10^{-2}]$ m^3.

Rearranging the equation for K the change in volume

$$\delta v = \frac{[p \times V]}{K}$$

$$\text{Substituting the values, } -\delta v = \frac{[5.7 \times 107] \times [1.4 \times 10^{-2}]}{1.9 \times 10\ 9}$$

$$= -4.2 \times 10^{-4} \text{ m}^3$$

Problem 1.26 *When the pressure was increased from 1 Mpa to* 2 *Mpa, the decrease in volume was observed to be 0.5% of the initial volume. Calculate the Bulk Modulus of the liquid.*

Solution Let the initial volume be x .

$$0.5\% \text{ of } x = 0.005$$

where, x = initial volume

$$\frac{-\delta v}{V} = \frac{0.005\ x}{x} = -0.005$$

$$\text{Change in pressure } \delta p = (1 - 2) = -1 \text{ Mpa}$$

$$\text{Bulk Modulus } K = \frac{-1}{-0.005} = 200 \text{ Mpa}$$

PROPERTIES OF GASES

Problem 1.27 *0.25 kg mass of oxygen is held in a container of volume one liter at* 30° *C. Determine the pressure. Molecular weight of oxygen = 32.*

Solution

$$V = 1 \text{ liter} = 0.1 \text{ m}^3$$

$$T = (30 + 273) = 303 \text{ K}$$

$$R = \frac{8312}{32}$$

$$= 259.75 \text{ Nm/ kg-K}$$

$$\rho = \frac{0.25}{0.1} = 2.5 \text{ kg/m}^3.$$

$$P = \rho RT$$

$$= 2.5 \times 259.75 \times 303$$

$$= 2 \times 10^5 \text{ Pa.}$$

Problem 1.28 *Air is contained in a cylinder at 30*°C. *Find the pressure exerted if its density is 14 kg/m*^{3}. *Molecular weight = 29.*

Solution

$$\rho = 14 \text{ kg/m}^3$$

$$T = (30 + 273) = 303 \text{ K}$$

$$R = \frac{8312}{29}$$

$$= 286.62 \text{ Nm/kg. K}$$

$$p = \rho RT$$

$$= 14 \times 286.62 \times 303$$

$$= 1.2 \times 106 \text{ N/m}^2 \text{ or } 1.2 \text{ MPa}$$

Problem 1.29 *2 kg mass of Nitrogen is contained in a steel cylinder at a pressure of 8 MPa and at 30° C. If an additional 3.5 kg of Nitrogen is pumped in to the cylinder, determine the final pressure. The consequent raise in temperature is 70° C. Molecular weight of Nitrogen = 28.*

Solution

Initial mass of Nitrogen $m_1 = 2$ kg.

Initial pressure $p_1 = 8 \times 10^6$ N/m^2

Initial temperature $T_1 = (30 + 273)$

$= 303$ K

$$R = \frac{8312}{28}$$

$$= 296.86 \text{ Nm/kg·K}$$

$$p_1 V_1 = m_1 R T_1$$

Therefore, $$V_1 = \frac{m_1 R T_1}{p_1}$$

$$= \frac{2 \times 296.86 \times 303}{8 \times 10^6}$$

$$= 0.0225 \text{ m}^3$$

$$T_2 = (70 + 273)$$

$$= 343 \text{ K, m}^2$$

$$= (2 + 3.5) = 5.5 \text{ kg}$$

$$V_2 = V_1 = 0.0225 \text{ m}^3$$

$$p_2 V_2 = m_2 R T_2$$

Therefore, $$p_2 = \frac{m_2 R T_2}{V_2}$$

$$= \frac{5.5 \times 296.86 \times 343}{0.0225}$$

$$= 24.9 \text{ Mpa.}$$

Problem 1.30 *A certain gas weighs 140 N/m^3 and is contained in a cylinder at a pressure of 1.2 MPa (abs) at 300 K. Determine the molecular weight of the gas.*

Solution To determine the molecular weight, we have to first determine the value of R from the equation $p = \rho RT$. Since weight density is given as 140 N/m^3, we convert it into mass density ρ.

$$\rho = \frac{w}{g} = \frac{140}{9.81}$$

$$= 14.27 \text{ kg/m}^3$$

$$T = 300 \text{ K}$$

$$p = 1.2 \times 106 \text{ N/ m}^2$$

$$R = \frac{p}{\rho T} = \frac{1.2 \times 10^6}{14.27 \times 300}$$

$$= 280.30 \text{ Nm/kg. K}$$

$$\text{Molecular weight} = \frac{8312}{R}$$

$$= \frac{8312}{280.30} \approx 29$$

Problem 1.31 *Air is compressed adiabatically to half its initial volume. The initial temperature was 30° C. Determine the final temperature.*

Solution

Since it is an adiabatic process

$$pV^k = \text{Constant}$$

and $$pV_2 = RT$$

or $$p = \frac{RT}{V}$$

pV^k = Constant can be written either as

$$\left(\frac{RT}{V}\right)V^k = \text{Constant or as } RT\,V^{k-1} = \text{constant.}$$

Since R is also a constant the equation gets further simplified to the form TV^{k-1} = constant.

or $$T_1\,V_1^{k-1} = T_2\,V_2^{\,k-1}$$

or $$\left(\frac{T_2}{T_1}\right) = \left(\frac{T_1}{V_2}\right)^{k-1}$$

or $$T_2 = T_1\left(\frac{T_1}{V_2}\right)^{k-1} \qquad ...(1.15)$$

Since the gas in this case is air, we may take the value $k = 1$.

$$T_1 = (30 + 273) = 303 \text{ K.}$$

$$\left(\frac{V_1}{V_2}\right) = \left(\frac{2}{1}\right) = 2$$

Therefore $T_2 = 303\ (2)^{1.4-1}$

$= 303\ (2)^{0.4}$

$= 400$ K

Problem 1.32 *Determine the pressure, temperature and density of air at an altitude of 1050 metres from sea-level: (1) Under isothermal conditions. (2) Under adiabatic conditions. Assume R = 287 J/ kg-K, atmospheric pressure at sea-level = 101.32 kPa (abs), the temperature as = 200 C and k = 1.4.*

Solution

1. *Isothermal condition*

The equation governing this condition is

$$\frac{P_2}{P_1} = -\exp\left\{\frac{g}{RT}\right\}(z_2 - z_1)$$

Substituting the relevant values,

$$\frac{P_2}{P_1} = -\exp\left\{\frac{9.81}{287}(273 + 20)\right\}(1050 - 0)$$

$= \exp(-0.1225)$

$P_2 = P_1\ (e^{-0.1225}) = (101.34)\ (e^{-0.1225})$

$= 89.6$ kPa

Density of air at 1050 metres

$$\rho_2 = \frac{P_2}{RT}$$

$$= 89.6 \times \frac{103}{287}(273 + 20)$$

$= 1.06$ kg/m^3

Since under the isothermal process, it is assumed that the temperature remains constant, there is no need to determine the change in temperature.

2. *Adiabatic condition*

The equation governing this condition is

$$\frac{P_2}{P_1} = \left[1 - \left\{g\frac{(z_2 - z_1)}{RT_1}\right\}\frac{(k-1)}{k}\right]^{\frac{k}{k-1}}$$

Substituting the relevant values in the above equation,

$$\frac{P_2}{P_1} = \left[1 - \left\{9.81\frac{(1050 - 0)}{287(273 + 20)}\right\}\frac{(1.4 - 1)}{1.4}\right]^{\frac{1.4}{1.4-1}}$$

$$\frac{P_2}{P_1} = \left[1 - \left\{\frac{10{,}300.5}{84{,}091}\right\}\frac{(0.4)}{1.4}\right]^{\frac{1.4}{0.4}}$$

$$\frac{P_2}{P_1} = [1 - \{0.122\}\ 0.285]^{3.5}$$

$$\frac{P_2}{P_1} = [0.965]^{3.5} = 0.882$$

$$P_2 = P_1 \times 0.882$$

$$P_2 = P_1 \times 0.992 = 101.34 \times 0.882$$

$$= 89.45 \text{ kPa}$$

The equation governing the change in temperature is,

$$\left(\frac{T_2}{T_1}\right) = \left[1 - \left\{g\frac{(z_2 - z_1)}{RT_1}\right\}\frac{(k-1)}{k}\right]$$

Since values within the RHS bracket remain the same as in Eq. 1.19 and have already been calculated, we can insert this value straight away.

$$\left(\frac{T_2}{T_1}\right) = [0.965]$$

Therefore

$$T_2 = T_1 \times [0.965]$$

$$= (273 + 20)\ (0.965)$$

$$= 282.74 \text{ K}$$

$$\rho_2 = \frac{P_2}{RT_2}$$

$$= 89.45 \times \frac{10^3}{287} \times 282.74$$

$$= 1.10 \text{ kg/m}^3$$

Problem 1.33 *Find the pressure and temperature at an altitude of 8000 metres if the temperature and pressure readings at 2000 metres above the sea level were 275 K and 79.5 kPa, respectively. Assume that the temperature decreases @ 6.5 °C/1000 metres and that the process is adiabatic. R = 287 J/kg-K.*

Solution Since the value of adiabatic index k is not given, in order to determine the temperature, we have to first determine the value of k by using Eq. 1.11 and use this value in Eq. 1.10 to obtain the value of T_2. The second option is to use Eq.1.12 and directly arrive at the value of T_2. So let us choose the second option.

$$T_2 = T_1 - \left\{\frac{dT_2}{dZ}\right\}(z_2 - z_1)$$

$$T_2 = 275 - \{0.0065\}\ (8000 - 2000)$$
$$T_2 = 275 - 39$$
$$= 236\ \text{K}$$

To determine the pressure at 8000 metres, we have to first determine the value of k using Eq. 1.13.

$$\frac{dT_2}{dZ} = -\left[\frac{(k-1)}{k}\right]\frac{g}{R}$$

$$-(0.0065) = -\left[\frac{(k-1)}{k}\right]\frac{9.81}{287}$$

$$-(0.0065) = -\left[\frac{(k-1)}{k}\right] 0.034$$

$$-\frac{(0.0065)}{0.034} = -\left[\frac{(k-1)}{k}\right]$$

$$-\frac{(0.0065)}{0.034} = -\left[\frac{(k-1)}{k}\right]$$

$$-0.19 = -\frac{(k-1)}{k}$$
$$0.19\,k = (k-1)$$
$$k = 1.23$$

$$\frac{P_2}{P_1} = P_1\left[1-\left\{g\frac{(z_2-z_1)}{RT_1}\right\}\frac{(k-1)}{k}\right]^{\frac{k}{k-1}}$$

Re-arranging,

$$P_2 = \left[1-\left\{g\frac{(z_2-z_1)}{RT_1}\right\}\frac{(k-1)}{k}\right]^{\frac{k}{k-1}}$$

Substitute relevant values,

$$P_2 = 79.5\left[1-\left\{9.81\frac{(8000-2000)}{287\ (275)}\right\}\frac{(1.23-1)}{1.23}\right]^{\frac{1.23}{1.23-1}}$$

$$P_2 = 79.5\left[1-\left\{\frac{58860}{78925}\right\}(0.187)\right]^{5.34}$$

$$P_2 = 79.5\ [1 - (0.139)]^{5.34}$$
$$P_2 = 79.5\ [0.86]^{5.34}$$
$$= 35.53 \text{ kPa}$$

Problem 1.34 *Determine the pressure at an altitude of 7500 metres from the ground level given that the temperature lapse rate is 0.0065 K/meter. Pressure and temperature at ground level is 10^5 Pa and 15°C respectively and $\gamma_{air} = 12$ N/m^3.*

Solution The relevant equation is,

$$\frac{P_2}{P_1} = \left[1 - \left\{g\frac{(z_2 - z_1)}{RT_1}\right\}\frac{(k-1)}{k}\right]^{\frac{k}{k-1}}$$

By rearranging,

$$P_2 = P_1\left[1 - \left\{g\frac{(z_2 - z_1)}{RT_1}\right\}\frac{(k-1)}{k}\right]^{\frac{k}{k-1}} \quad \ldots (1.16)$$

The value of universal gas constant R and the value of adiabatic index k have to be determined.

$$R = \frac{P_0}{\rho_0}(T_0) = \frac{10^5}{1.22}\ (273 + 15)$$

$$\left(\text{since } \rho_0 \frac{12}{9.81} = 1.22 \text{ kg/m}^3\right)$$

$$R = 284.6 \text{ J/kg-K}$$

The value of adiabatic index k is obtained from the following equation

$$\frac{dT_2}{dZ} = -\left[\frac{(k\ -\ 1)}{k}\right]\frac{g}{R}$$

$$-(0.0065) = -\left[\frac{(k-1)}{k}\right]\frac{9.81}{284.6}$$

$$-(0.0065) = -\left[\frac{(k-1)}{k}\right]$$

$$0.034 - \frac{(0.0065)}{0.034} = -\left[\frac{(k-1)}{k}\right]$$

$$-\frac{(0.0065)}{0.034} = -\left[\frac{(k-1)}{k}\right]$$

$$-0.19 = -\frac{(k-1)}{k}$$

$$0.19\,k = (k - 1) \text{ or } k = 1.23$$

We may now introduce the derived values and the given values in Eq. 1.16 to obtain the value of the pressure at the desired altitude.

$$P_2 = 10^5 \left[1 - \left\{9.81 \frac{(7500 - 0)}{284.6\,(288)}\right\} \frac{(1.23 - 1)}{1.23}\right]^{\frac{1.23}{1.23-1}}$$

$$P_2 = 10^5 \left[1 - \left\{\frac{73575}{81964.8}\right\} 0.187\right]^{5.34}$$

$$P_2 = 10^5\,[0.832]^{5.34}$$
$$= 3.745 \times 10^4 \text{ Pa}$$
$$= 37.45 \text{ kN/m}^2$$

QUESTIONS BANK

Section A

Review Questions

1. Distinguish clearly between liquids and gases.
2. What are the various forms in which a liquid possess energy?
3. What are the characteristics you would consider as important while choosing a liquid?
4. What does '*Compressibility*' indicate?
5. Apart from 'Newton' what is the other unit of measurement of force in SI system?
6. A negative sign follows the equation for bulk modulus in liquids. But this is not so when we consider solids. Explain clearly the reason?
7. How do you physically distinguish between a less viscous and a more viscous fluid?
8. What is the principal advantage one gains from a knowledge of viscosity?
9. State the nature of change in viscosity of liquids and gases when heated.
10. What is the unit of measurement of kinematic viscosity in S.I system?
11. What does Newton's law on viscosity indicate?
12. Define dynamic viscosity.
13. What does 'Viscosity index' indicate?
14. Define 'ideal fluid' and name a fluid that almost meets this definition?
15. Why is the meniscus curved inwards in case of mercury whereas in case of water it is curved upwards?
16. Explain surface tension and capillarity with examples.
17. Differentiate between vaporization and boiling.
18. Give two reasons why mercury is preferred as a barometer fluid?
19. Why engine oil is more viscous than other lubricating oils?
20. Explain how vaporization could lead to cavitation?
21. Define an ideal gas.

22. Explain the difference between the equations $p = \rho RT$, $PV = mRT$, and $pV = nMRT$.
23. What is polytrophic expansion of a gas and within what limits the Polytrophic index, lies?
24. Distinguish between a solid and a liquid in terms of the applied stress, strain and Bulk modulus.
25. Explain why $C_p > C_v$.
26. Draw the velocity and shear stress variation graphs for a liquid flowing through a pipe. Assume laminar flow.

Section B

Multiple Choice Questions

Choose the right answer: (in some cases more than one answer could be correct)

1. Which is the most precise method of defining viscosity;
   (a) dynamic- Viscosity (N.s/m^2)
   (b) Kinematic viscosity (m^2/sec)
   (c) Relative viscosity (SUS)
   (d) SAE numbers
2. Static deformation is a phenomenon attributable only to
   (a) Liquids (b) solids
   (c) gases (d) Pseudo plastics
3. Which among these exhibit a greater resistance to deformation?
   (a) Plastic (b) Pseudo plastic
   (c) Ideal fluid (d) Newtonian fluid
4. Clay is an example for
   (a) Bingham plastic (b) Thixotropic substance
   (c) Rheopectic material (d) Pseudo plastic
5. For a spherical water droplet the decrease in surface area is balanced by
   (a) decrease in pressure
   (b) increase in internal pressure
   (c) increase in the force at which it drops.
6. Compared to water, capillary raise in hydrocarbon fluid is,
   (a) lower
   (b) higher
   (c) remains the same
7. High Bulk modulus indicates
   (a) high compressibility
   (b) high rate of angular deformation
   (c) low compressibility

8. The speed of sound is higher in solids than in liquids because
   (a) density is higher
   (b) bulk modulus is higher
   (c) exhibit properties such as elastic limit
9. The value C_p at NTP for air is
   (a) 1050 J/kg-K (b) 834 J/kg-K
   (c) 993 J/kg-K
10. Which among these has a higher value of absolute viscosity
   (a) castor oil (b) crude oil
   (c) glycerin
11. The value of surface tension depends on
   (a) pressure (b) temperature
   (c) viscosity

Section C

Problems

1. Glycerin has a specific gravity of 1.250. Determine the mass of 0.75 m^3 of the fluid. (*Ans.* 937.50 kg)
2. Specific gravity of mercury is 13.54. Determine the specific volume, mass and weight of 10 m^3 of the fluid.
   (*Ans.* 7.4×10^{-5} m^3/ kg, 135,400 kg, 13.8 kN)
3. One liter of crude oil weighs 10 N. Calculate its specific weight, specific-gravity and density. (*Ans.* 10000 N/m^3, 1.019, 1019 kg/m^3)
4. A cubical block with sides measuring 55 mm, weighs 50 kg.m /sec^2 and is sliding down an inclined plane due to gravity. The inclined surface has a coating of lubricating oil of thickness 1.5 mm and is inclined at angle of 25° to the horizontal. If the density of the lubricating oil is 860 kg/m^3, and its kinematic viscosity is 6.5×10^{-5} m^2/ sec, determine the terminal velocity of the block. (*Ans.* 188.6 m/sec)
5. Surface tension of mercury is 5×10^{-1} N/m at atmospheric pressure and at 25°C. If the pressure outside a mercury droplet of diameter 1 mm is 101 kPa, find the pressure inside. (*Ans.* 103 kPa)
6. Pressure inside a water jet is 101.5 kPa (abs). Determine the surface tension if the diameter of the jet is 5 mm. Atmospheric pressure = 101 kPa. (*Ans.* 1.25 N/m)
7. One limb of a U tube has a diameter twice that of the other. When the tube was held upright and partially filled with water the difference in capillary raise between the two tubes was observed to be 2.5 mm. Calculate the inside diameter of the tubes if the surface tension is 7×10^{-2} N/m. (*Ans.* φ 2.8 mm and φ 5.6 mm)
8. An inverted cone with an included angle of 60° acts as a thrust bearing and revolves at 500 rpm over an identical conical cavity with a layer

of lubricating film of oil of thickness 1.5 mm. The base of the cone has a diameter 100 mm. Determine the torque required if $\mu = 3 \times 10^{-3}$ N.sec/m^2. (*Ans.* 10^{-4} N-m)

9. The Bulk modulus of elasticity of a certain hydrocarbon fluid is 1.4×10^9 N/m^2. The fluid is compressed from 300 kPa to 900×10^2 kPa. Determine the volume after compression if the initial volume was 0.04 m^3. (*Ans.* 0.039 m^3)
10. Soap solution was blown up to form a bubble. If the energy expended during the process is 0.005 N-m, determine the diameter of the bubble if the surface tension is 0.05 N-m. (*Ans.* φ 126 mm)
11. Determine the velocity of sound in air at 100° C undergoing an isentropic change. R = 287 J/kg-K. (*Ans.* 387 m/sec)
12. Derive an expression for the temperature gradient of a gas starting from the characteristic equation for status change of a gas in adiabatic condition.
13. Up to what range in the atmosphere the temperature and density is assumed to remain approximately constant? The atmospheric pressure at sea level is 101.3 kPa. The temperature at the same point is 15.15°C. What is the change in pressure and temperature at an altitude of 8300 m from seal level if the change is assumed to occur under

    (1) Isothermal condition

    (2) Adiabatic condition.

    Where appropriate assume a suitable value for the temperature lapse rate. γ_{Air} = 12 N/m^3. R = 287 J/ kg.K.

Answers to Multiple Choice Questions

1. (a) **2.** (b) **3.** (a) **4.** (d) **5.** (b) **6.** (b)
7. (c) **8.** (b) **9.** (c) **10.** (a) **11.** (a)

Chapter 2

Pressure and Pressure Gauges

2.1 INTRODUCTION

Everyone is aware of the pressure the earth's atmosphere exerts on the surface. Atmospheric pressure is the force exerted by a column of air of unit area as high as the atmosphere. If we say the atmospheric pressure at sea level is 1 kg/cm^2 ≈ 1 bar, then we are referring to the force exerted on the earth's surface by a column of air as high as the atmosphere of cross sectional area 1 cm^2. It is basically due to the mass of the column of air. The intensity of this pressure decreases as the height increases since air is less dense at higher altitudes, which is why we consider the mean sea level to determine its maximum intensity.

Pressure exerted by water bodies on the other hand, increases with depth as per the equation $p = \rho gh$. This means the pressure exerted by a column of water at a depth of 1 metre is $1000 \times 9.81 \times 1 = 9.81 \times 10^3$ N/m^2, here ρ is the density of water in kg/m^3, g is the acceleration due to gravity in appropriate units and h the height of the column of water in metres. So the weight of a fluid at a certain height or depth causes pressure. Pressure is also caused by the resistance to the flow of fluid. For instance, if the flow passage area is reduced at any point the pressure 'upstream' will be higher than the pressure 'down stream'. If the flow passage area is completely blocked then the full potential of the pressure intensity can be realized. It is important to note that a fluid chooses the path of least resistance and resistance in series adds to pressure. The above relationship ($p = \rho gh$) is valid for a homogeneous liquid at rest and is not applicable to gases as the specific weight of gas changes with pressure.

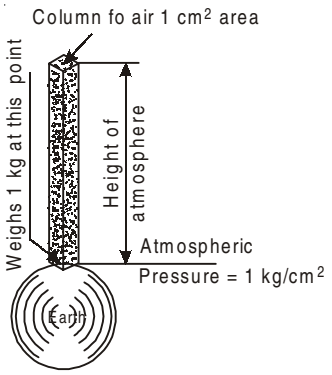


Fig. 2.1

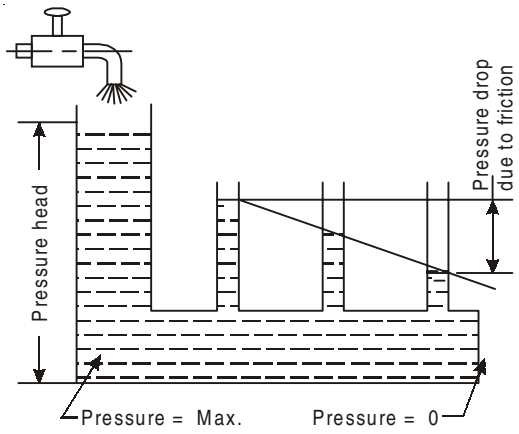


Fig. 2.2

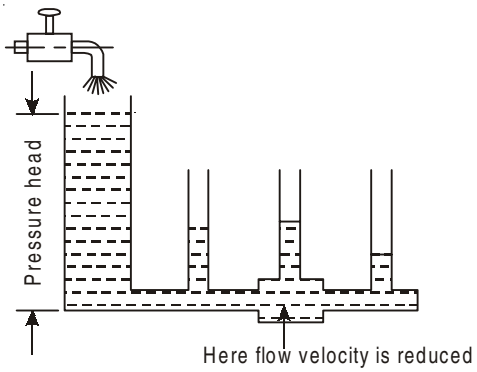


Fig. 2.3 *Friction and velocity affect pressure*

A fluid flows because there exists, an unbalanced force sufficient to cause motion. This unbalanced force must be higher than the frictional resistance offered by the fluid conductor of constant diameter. Sudden enlargements, sudden constraints, abrupt change in the direction of flow path or very high flow velocities causes 'turbulence' in the flow pattern and this could also contribute to resistance to flow and affects pressure. (Ref. Figs. 2.1 to 2.3).

Torricelli and his contemporary Blaise Pascal have together contributed immensely to our understanding of pressure. Torricelli proved how the head of water in a tank affects the flow velocity and Blaise Pascal proved that pressure applied on a confined fluid is transmitted equally in all directions and acts with equal force on equal areas. Pascal's principle should explain

why a glass bottle filled with water up to the brim should break, when the stopper is forced in (Fig. 2.4).

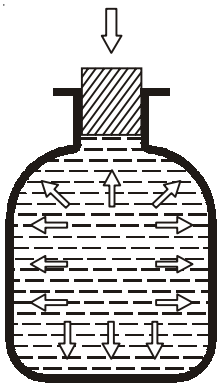


Fig. 2.4

During Torricelli's time pressure was expressed in terms of the height of a column of water. A head of one foot of water was equivalent to 0.434 lbf/inch^2 and a head of 5 feet of water was equivalent to 2.17 lbf/inch^2 based on the fact that a cubic foot of water weighs 62.4 lbs. But with the modern definition of pressure expressed in terms of the *force acting on a unit area,* we can measure pressure at any point in more convenient units like kgf/cm^2 as in MKS system of units, lbf/inch^2 as in the FPS system or N/m^2 or Pascal as in the now internationally accepted SI system of units. The term bar (1 bar ≈ 1 kgf /cm^2 or 14.7 lbf /inch^2) although is a part of the MKS system of units is even today the more widely accepted unit for measurement of fluid pressure since Pascal is relatively a small unit in comparison. (1 bar = 10^5 Pa). Pressure expressed in terms of the height of a column of fluid, such as 'inches of mercury' or 'feet of water' or 'mm of mercury' is still the desired method in some applications.

The equation $p = \rho gh$ is a simplified version of the generic term $(p_2 - p_1) = \gamma (z_2 - z_1)$. Here, γ is the specific weight of the liquid, $(p_2 - p_1)$ is the difference in pressure Δp as measured at the datum surfaces z_1 and z_2.

The above equation is developed by applying the Pascal's law and the law of static equilibrium which states that for a body at rest and in equilibrium the sum of the forces acting on it from all directions = 0.

Imagine a glass tube with ends open immersed in water to a depth z. The cross sectional area of the tube is constant at both ends and may be assumed as a. Let us consider the forces both horizontal and vertical acting on the element of water inside the tube of length δz. (Fig. 2.5).

According to Pascal's law the intensity of pressure is the same both along the axis and at right angle to the axis. The pressure inside the tube in the horizontal direction is the same as the pressure outside and therefore there is

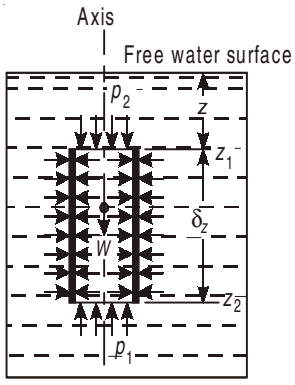


Fig. 2.5

no unbalanced force and the horizontal forces need not be considered in this case. The only force to be considered is the force along the axis, both upward and downward.

Let us consider the forces

1. Force due to the weight of the water $w = (\gamma \times a \times \delta z) \downarrow$
2. Force due to pressure at datum $z_1 = (p_1 \times a) \downarrow$
3. Force due to pressure at datum $z_2 = (p_1 + \delta p)\ a \uparrow$ δp is the change in pressure that occurs because the depth changes between points z_1 and z_2.

Equating these forces to zero,

$$(p_1 + \delta p)\ a - (p_1 \times a) - (\gamma \times a \times \delta z) = 0$$

$$(p_1 \times a) + \delta p\ (a) - (p_1 \times a) - (\gamma \times a \times \delta z) = 0$$

or $$\delta p\ (a) - (\gamma \times a \times \delta z) = 0$$

or $$\delta p - (\gamma \delta z) = 0$$

or $$\delta p = (\gamma \delta z)$$

or $$\int_{P_1}^{P_2} \delta p = \int_{Z_1}^{Z_2} \gamma \delta z$$

$$= \gamma \int_{Z_1}^{Z_2} \delta z$$

(since the specific weight of the fluid is a constant it can be taken out of the integral sign. So we have,

$$\int_{p_1}^{p_2} \delta p = \gamma \int_{Z_1}^{Z_2} \delta p$$

Upon integrating and applying the limits,

$$(p_2 - p_1) = \gamma\,(z_2 - z_1) \qquad \ldots (2.1)$$

Equation 2.1 forms the basis for measurement of pressure.

Since the change in pressure depends only on the variable z in the equation $(p_2 - p_1) = \gamma\,(z_2 - z_1)$, γ being a constant for the liquid being considered, the pressure at the bottom of any container, regardless of its shape or size depends merely on the elevation or the difference in height h. The pressure exerted at the bottom of any container is the same even if the areas are different so long as

the elevation is the same. In other words it means, whatever may be the shape of the container and whatever may be the weight of the liquid in the container the thrust exerted at the bottom merely depends on the cross sectional area at the bottom surface. This phenomenon of thrust exerted by the fluid at the bottom surface of any container being independent of the weight of the fluid is often referred to as *Pascal's paradox.*

It is this phenomenon that enables engineers to convey high pressure fluid through small fluid conductors to where the system needs it. The all too familiar over head water tanks not only acts as a reservoir but also ensures that sufficient pressure is maintained in the supply lines so that the water is supplied at the desired flow rates.

While there is no theoretical limit to the highest possible, pressure, the lowest possible pressure is the near perfect vacuum pressure $\approx 10^4$ Pa or N/m^2. This means the height to which water could be lifted by suction (negative pressure) is theoretically limited to 102 metres (based on the formula $p = \rho gh$) whereas, there is no such limitation to pumping the water (positive pressure) to any height that we want.

Absolute pressure is the pressure that includes the atmospheric pressure. *Gauge pressure* is the pressure indicated by the pressure gauge which excludes the atmospheric pressure. *Vacuum pressure* is any pressure below the atmospheric pressure. At sea level the standard atmospheric pressure is 101.3 kPa (abs). At different points on the earth's surface it could vary between 95 kPa – 105 kPa.

Absolute pressure = Atmospheric pressure + Gauge pressure.

Absolute pressure = Atmospheric pressure – Vacuum pressure

A comparative chart, depicting the pressure and vacuum scales as measured in different units is given below for reference. (Fig. 2.5).

2.2 PRESSURE AND VACUUM SCALE

Vacuum or the negative pressure is measured in the same units as positive pressure. However, most vacuum gauges are still calibrated in millimetres or

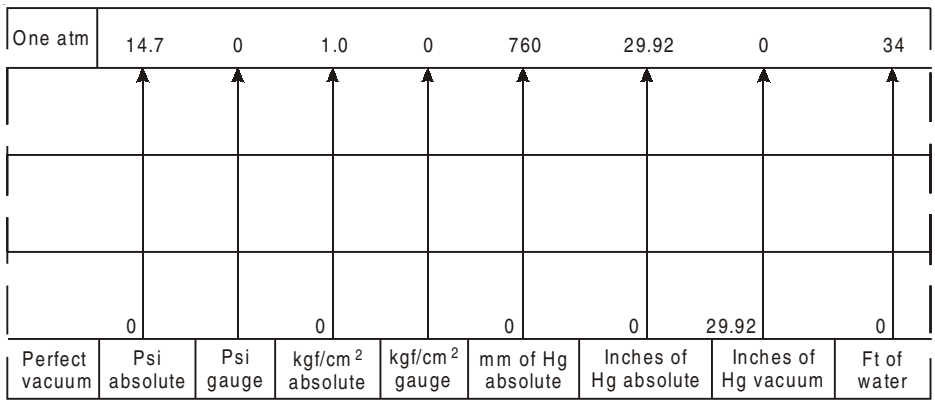


Fig. 2.6

inches of mercury. A perfect vacuum will support a column of mercury 763 mm or 29.92 inches high. (Fig 2.7).

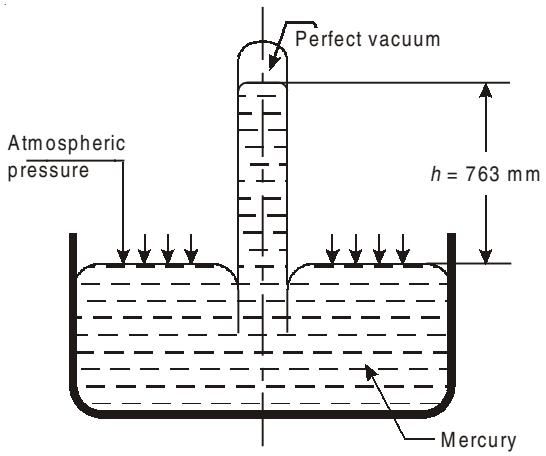


Fig. 2.7

The device shown in Fig. 2.7 called the *Barometer* is used to measure both vacuum pressure as well as the atmospheric pressure. While measuring atmospheric pressure however the height h should be multiplied by the specific weight of mercury which has a value = 133 kN/m^3.

Perhaps the simplest device for measurement of small pressures above the atmospheric pressure is the *Piezometer*. It is a tube one end of which connected directly to the point where the pressure measurement is required and the other end left open to atmosphere. Figure 2.8 shows unlike the manometer, the piezometer makes use of no other liquid.

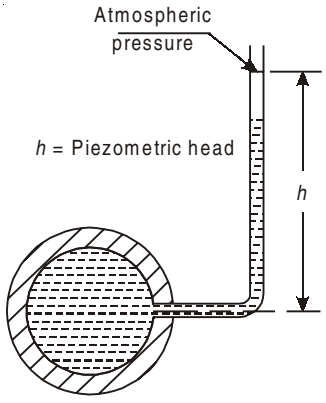


Fig. 2.8

Incompatibility of this device for measuring pressures above the atmospheric

pressure is obvious. Higher the pressure in the pipe line higher will be the height h to which the liquid will raise. Calculation of pressure involves direct application of the formula $p = \gamma h$.

Pressures above the atmospheric pressure is measured using a device called the *Manometer*. These are simple but accurate instruments but they have a limited range and hence are not suitable for measurement of very high pressure.

There are two types of manometers.

1. S*imple U tube* manometer that indicates the pressure at any one point. (Fig. 2.9). Can be used for measuring both positive and vacuum pressures.
2. *Differential U tube manometer* that indicates the difference in pressure between any two points without indicating the pressure specific to either of the points (Fig. 2.10).

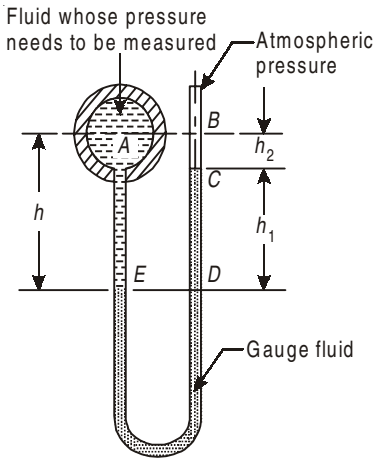


Fig. 2.9 *Simple U-tube monometer*

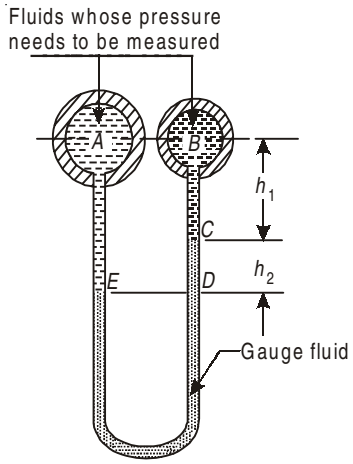


Fig. 2.10 *Differential u-tube monometer*

Another significant difference between the two is that in the simple U tube manometer one end is always left open to the atmosphere, whereas in the differential type both ends remain connected to the points where the pressure readings are required. The two tapping points may be in the same pipe or it may be in two different pipes carrying liquids of different specific gravity. Both simple and differential U-tube manometers contain a gauge fluid that does not mix with the fluid whose pressure is being measured. Typical gauge fluids are mercury and coloured oil. The specific gravity of the gauge fluid must always be higher than the specific gravity of the fluid whose pressure needs to be determined.

A variation to differential U-tube manometer is the *inverted U-tube manometer,* (Fig. 2.11) used for measuring minor variations in fluid pressure over a specific length, such as head loss due to friction in small pipes. A gauge fluid is normally avoided in an inverted U-tube manometer. The process fluid itself is used to measure the differential pressure. In such cases the space above

the fluid in the limbs contains air admitted through a valve which can be opened or closed at will. Whenever a gauge fluid is used it is necessary to ensure that the specific gravity of the gauge fluid is always less than the specific gravity of the fluid whose pressure needs to be measured.

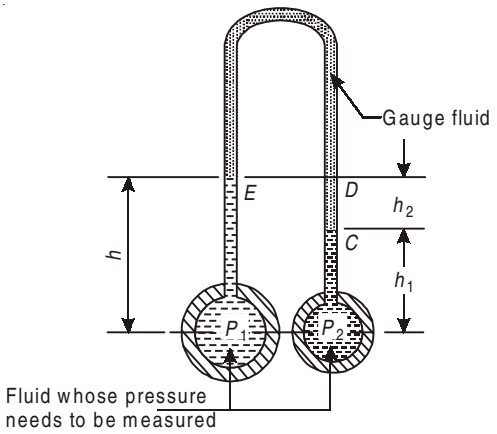


Fig. 2.11 *Inverted U-tube monometer*

There are several variations to the *simple U-tube* manometer. *Straight single limb and inclined single limb* are some of them. (Refer Figs. 2.12 and 2.13). These make use of the basic hydrostatic principle that a fluid finds its own level. Deflection of the fluid level in the left tube due to an increase in pressure causes the fluid in the right limb to rise until the weight of the fluid is sufficient to make up the difference in pressure.

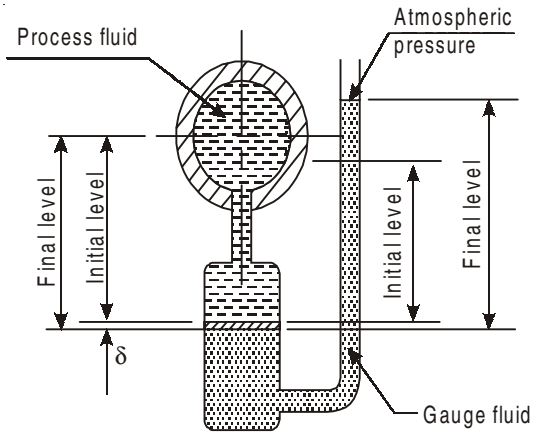


Fig. 2.12 *Straight single limb monometer*

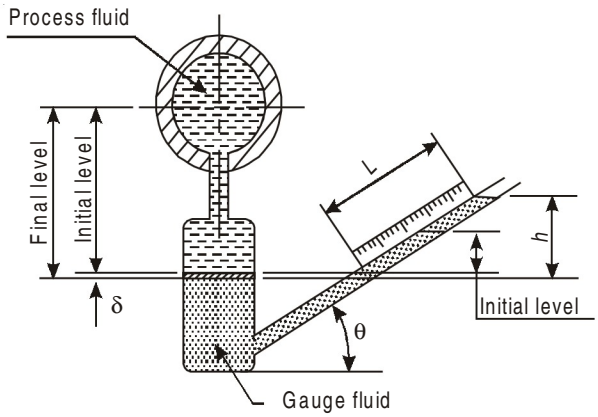


Fig. 2.13 *Inclined single limb monometer*

In either case the gauge fluid suffers a small deflection when exposed to the fluid whose pressure needs to be measured. This change in depth of the gauge fluid in the left tube, which is so small that it is often neglected. This results in a disproportionate change in the fluid level in the graduated limb due mainly to difference in their areas. This change in level of the limb is sufficient to indicate the pressure in the main line and this is their advantage over the simple U-tube manometers. The *inclined type* has an added advantage over the other type, in that its range is comparatively higher because of its longer scale length.

Hydrometer. It is a device used to measure the density or the specific gravity of a liquid. It is built to float on the liquid whose specific gravity is required to be measured by ensuring that the weight of the fluid it displaces is less than its

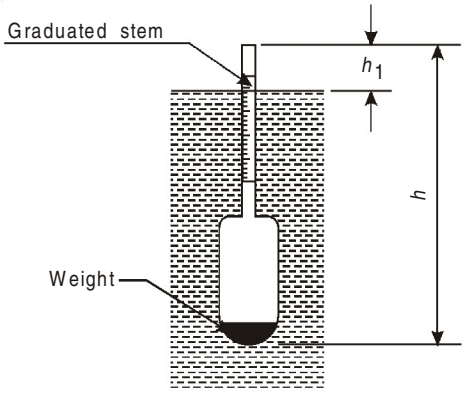


Fig. 2.14 *Hydrometer*

own weight. The hydrometer is a simple glass tube of varying cross section with a long graduated stem at one end and a relatively short but wider cylinder which is bottom heavy, at the other end. (Fig. 2.14) The portion of the graduated

stem above the liquid (L_1) directly indicates the specific gravity or the density of the liquid.

2.3 MECHANICAL ELECTROMECHANICAL AND ELECTRONIC PRESSURE GAUGES

Manometers as pressure measuring devices have their limitations. They are unsuitable for high pressure applications. They are also not acceptable where the process fluid is subject to temperature variations, which in turn affects its density. Instant pressure recordings are not possible since a certain time is required for the gauge fluid to settle down. Very accurate readings are not possible as surface tension and capillarity affects the readings. Above all they are bulky and fragile.

Mechanical, Electro-Mechanical and Electronic Pressure Gauges are designed to overcome these limitations and are today the most widely employed instruments for pressure measurement. Compared to manometers they are very compact and can be used to measure gauge pressures in excess of 10^8 Pa. The pressure gauge is a very familiar component of the fluid power control system. Pressure gauges are required to evaluate the performance of fluid power components and can be helpful in troubleshooting. But their most important function is to help set and monitor the system pressures.

Pressure gauges generally are of the Bourdon tube type. Metal tube is bent to form an arc or a segment of a circle or in some cases a spiral, one end of which is closed and the other end is connected by means of tap (1) to the point where the pressure is measured. When the high-pressure fluid is admitted via valve (1) to the Bourdon tube it tends to straighten out. This actuates the linkage to the pointer gear (2) and moves the pointer to indicate the pressure on the

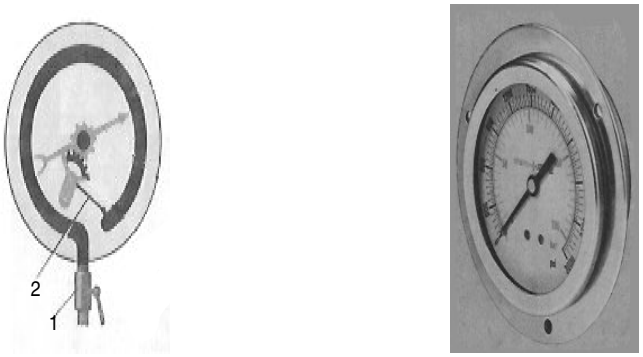


Fig. 2.15 *Pressure gauge*

calibrated dial. When the pressure is released the tube assumes its original position and the gauge reads zero. The gauges read zero at atmospheric pressure and are calibrated to read pressures in almost any of the familiar units. For low

and medium pressure applications normal arc type phosphor bronze Bourdon tube would be sufficient. For very high pressures spirally wound tubes are recommended.

To increase gauge life and reduce the effect of system pulsations and mechanical vibrations glycerin filled gauges are available. Glycerin, a stable viscous fluid dampens the vibrations of the Bourdon tube and the pointer to ensure positive gauge read out.

Pressure gauges are generally manufactured in 4 different dial sizes, 65 mm, 100 mm, 150 mm and 250 mm are standard sizes. There are models that can be mounted directly on the fluid line. Models, which are of three-hole panel mounting type, are also available. The maximum dial reading recommended is no more than 65% of the dial range for constant pressures and 60% for variable pressures.



Fig. 2.16 *Electrical contact gauge*

Magnetic electrical contact pressure gauges not only read system pressure but also help control the pressure at the set value by switching 'off' or 'on' electrical control valves thus eliminating the need for a separate switching control for these valves.

Vacuum gauges are almost similar in construction to bourdon tube pressure gauges. The only difference is that the phosphor bronze bourdon tube is bent inwards instead of outwards. They are calibrated to read pressure in mm of mercury.

Diaphragm gauges have mechanical linkages similar to bourdon tube gauges but instead of the bourdon tube a diaphragm acts like the sensor and converts the deflection it suffers upon exposure to pressure into mechanical upward thrust which in turn is transferred to the linkages to indicate pressure readings on a dial.

Strain gauges, the linear variable differential transformer or the LVDT and solid state sensors, generally termed as pressure transducers, are some of the common electronic pressure measuring devices. All of them accept a physical quantity such as fluid pressure as an input signal and generate an electronic signal which is proportional to the physical quantity. This signal is then passed on to a display unit which is a digital voltmeter calibrated to display pressure in appropriate units.

TYPICAL PROBLEMS AND SOLUTIONS

Problem 2.1 *A domestic water pump has a pressure limitation of 5×10^5 Pa. What is the maximum height to which it can deliver the water? Assume $\rho = 1000$ kg/m^3.*

Solution

$$\text{Height } h = \frac{5 \times 10^5}{1000 \times 9.81} = 51 \text{ metres.}$$

(Based on the formula $p = \rho gh$)

Problem 2.2 *A pump draws water from a sump placed 3 metres below the inlet to the pump. What is the pressure at the inlet? Express this pressure in mm of Hg. Assume $\gamma = 9810$ N/m^3.*

Solution

$$\text{Pressure } p = 9810 \times 3$$
$$= 2.90 \times 10^4 \text{ N/m}^2$$

(Based on the formula $p = \gamma \times h$)

$$1 \text{ atmosphere} \approx 10^5 \text{ Pa} = 763 \text{ mm Hg}$$

Therefore, $$2.9 \times 10^4 \text{ Pa} = (2.9 \times 10^4) \frac{763}{10^5}$$
$$= \mathbf{221 \text{ mm Hg.}}$$

Problem 2.3 *In a Barometer mercury rises only to a height of 700 mm from the free surface. State all the possible reasons and determine the absolute pressure above the column of Hg inside the tube. Take the specific weight of Hg = 133 kN/m^3.*

Solution

The possible reasons are,

1. The atmospheric pressure is less than 101.3 kPa.
2. Imperfect vacuum above the column of mercury in the tube.

Vacuum pressure at the top of the tube

$$p = (133 \times 10^3) \times 0.7 \text{ m}$$
$$= \mathbf{93 \times 10^3 \text{ Pa.}}$$

This value should ideally be = 101.3×10^3 Pa for perfect vacuum condition.

Absolute pressure = Atmospheric pressure – Vacuum pressure

$$(101.3 \times 10^3) - (93 \times 10^3) = \mathbf{8.3 \times 10^3 \text{ Pa.}}$$

Problem 2.4 *A pump supplies water through a series of resistances as shown in Fig. 2.17. What would be the gauge pressure at stations 4, 3, 2, 1, if the local resistances are, as indicated?*

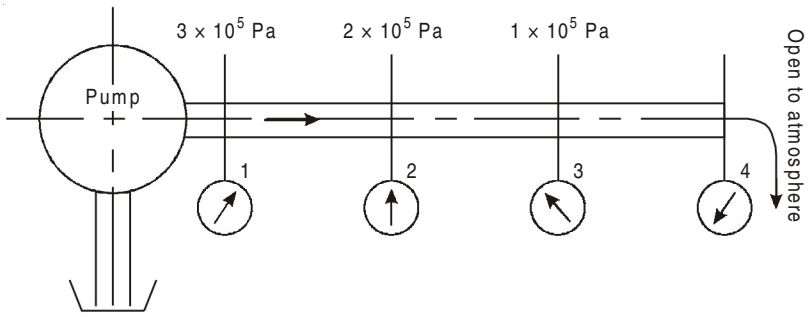


Fig. 2.17

Solution

1. There is no resistance to flow at station 4 and hence gauge pressure = 0.

2. The resistance to flow at station 3 = 10^5 Pa and hence gauge would show only 10^5 Pa.

3. The resistance to flow at station 2 = the local resistance (2×10^5 Pa) + resistance from station at 3 in the form of back pressure. Hence the gauge would show ($2 \times 10^5 + 10^5$) = 3×10^5 Pa.

4. The resistance to flow at station 1 = [the local resistance 3×10^5 + back pressures from stations 3 and 2]. Hence the gauge would show ($3 \times 10^5 + 3 \times 10^5$) = 6×10^5 Pa.

Problem 2.5 *Determine the fluid pressure at points 1, 2, 3, 4 and the air pressure at point 5 in the tank (Fig. 2.18) containing oil of specific weight 8.5 kN/m³.*

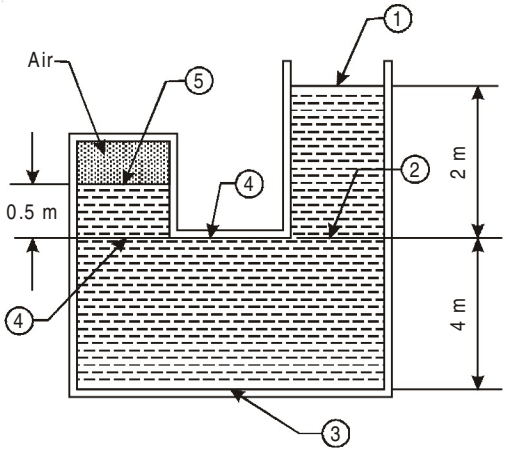


Fig. 2.18

Solution

At point (1), the fluid is exposed to the atmosphere and so

p_1 = 0 kPa (gauge).

At point (2), $p_2 = (8.5 \times 2) + p_1 = (17) + 0$

$= \mathbf{17\ kPa.\ (gauge)}$

At point (3), $p_3 = (8.5 \times 4) + p_2 + p_1$

$= (34) + 17 + 0$

$= \mathbf{51\ kPa\ (gauge)}$

Since point (4) is at the same height as point 2 the pressure is the same.

At point (5), $p_5 = p_2 - (8.5 \times 0.5)$

$= 17 - (4.25)$

$= \mathbf{12.75\ kPa\ (gauge)}$

This is also, the air pressure at point (5).

Problem 2.6 *If the plunger (Fig. 2.19) applies a pressure of 10 × 10⁵ Pa, what weight W can be lifted by the fluid underneath? Assume that the surfaces are leak proof.*

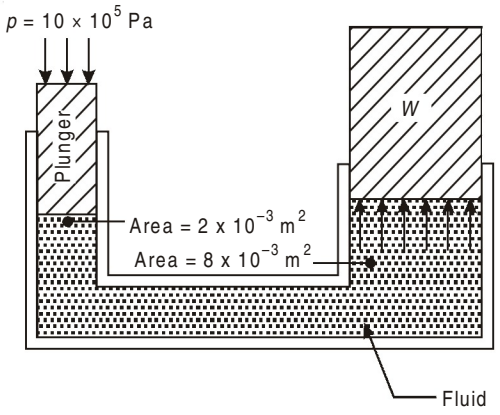


Fig. 2.19

Solution The hydraulic leverage here is similar to mechanical leverage and is directly proportional to the area ratios of the plunger and the weight W. Since the applied pressure is uniform throughout (Pascal's law),

$\frac{F_p}{A_p} = \frac{W}{A}$ here, F_p refers to the force generated by the plunger and A_p refers to the plunger area.

$$\frac{(10 \times 10^5) \times (2 \times 10^{-3})}{(2 \times 10^{-3})} = \frac{W}{(8 \times 10^{-3})}$$ upon simplifying,

$$\mathbf{W = 8000\ N}$$

Problem 2.7 *In the arrangement shown in Fig. 2.20, determine the minimum height h of the water column required to balance a load of 80 N if the area directly beneath the load is 0.008 m²? Assume the specific weight of the fluid as 9.810 × 10³ N/m³ and a leak proof interface.*

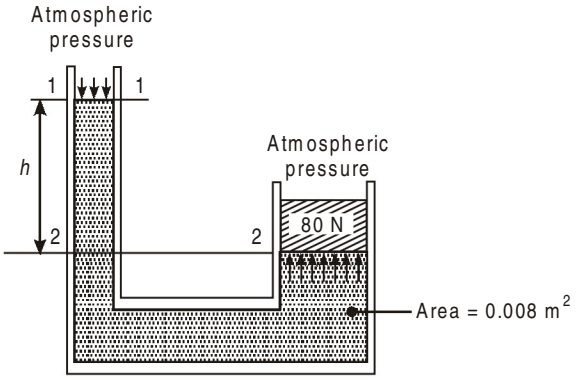


Fig. 2.20

Solution

Pressure required to lift the load = $\frac{80}{0.008} = 10 \times 10^3$ N/m^2

This must be equal to the pressure exerted by the water column at the interface 2–2

Therefore, $\gamma_w h = 10 \times 10^3$

$\gamma_w = 9.810 \times 10^3$ N/m^3

$h = \frac{10 \times 10^3}{9.810 \times 10^3}$

= **1.020 metres.**

Problem 2.8 *Water is being pumped through a steel pipe to which a simple U-tube manometer is connected. Determine the fluid pressure in the tube for the conditions shown in Fig 2.21. Assume the specific gravity of the gauge fluid as 13.6.*

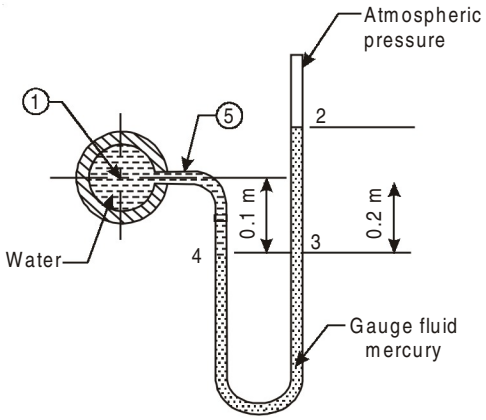


Fig. 2.21

Solution We have to determine the pressure at point (1). We know the pressure at point (2). Let us say it is p_2

The pressure at $p_3 = p_2 + \gamma_m (0.2)$.

The pressure at p_4 is also $= p_2 + \gamma_m (0.2)$.

Since point (4) is at the same elevation as point (3).

The pressure at $p_5 = p_2 + \gamma_m (0.2) - \gamma_w (0.1)$

Since p_5 is at the same elevation as p_1 we can re-write the expression for p_5 as

$$P_1 = p_2 + \gamma_m (0.2) - \gamma_w (0.1)$$
$$p_2 = p_{atm} = \mathbf{0\ Pa\ (Gauge)}$$
$$\gamma_m = (13.6) \times 9.81$$
$$= \mathbf{133.4\ kN/m^2}$$
$$\gamma_w = 1000 \times 9.81$$
$$= \mathbf{9.810\ kN/m^2}$$

Substituting these values,

$$P_1 = 0 + 133.4\ (0.2) - 9.810\ (0.1)$$
$$= 26.68 - 0.9981$$
$$= \mathbf{25.7\ kPa\ (Gauge)}$$

Problem 2.9 *A simple U-tube manometer is connected to the inlet of a vacuum pump being used to empty a drum containing oil of specific gravity 0.95. Find the gauge reading at the inlet to the pump from the values indicated in Fig. 2.22. The gauge fluid Hg has a specific gravity = 13.6.*

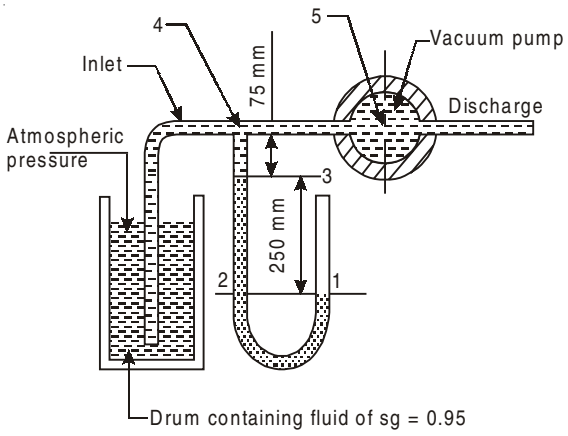


Fig. 2.22

Solution Equate the pressure head above the datum 2–1 in the left limb to the pressure head above the same datum in the right limb.

Pressure in the left limb above datum 2–1

$$= \gamma_m (0.250) + \gamma_w (0.075) + P_5$$

Pressure in the right limb above datum 2–1= 0

Therefore,

$$\gamma_m\,(0.250) + \gamma_o\,(0.075) + P_5 = 0$$

$$(13.6 \times 9.81)\,(0.250) + (0.95 \times 9.81)\,(0.075) = -P_5$$

or $$\boldsymbol{P_5 = -\ 34\ kPa}$$

Problem 2.10 *The readings on a straight single limb manometer are as shown in Fig. 2.23. The fluid in the pipe is water (sp. gravity = 0.95) and the gauge fluid is mercury (sp. gravity = 13.6). Determine the pressure in the tube at point (3). Take the cross sectional area ratios of the left tube and the right limb as 100:1. What would be the pressure if the deflection of the gauge fluid in the left tube is neglected?*

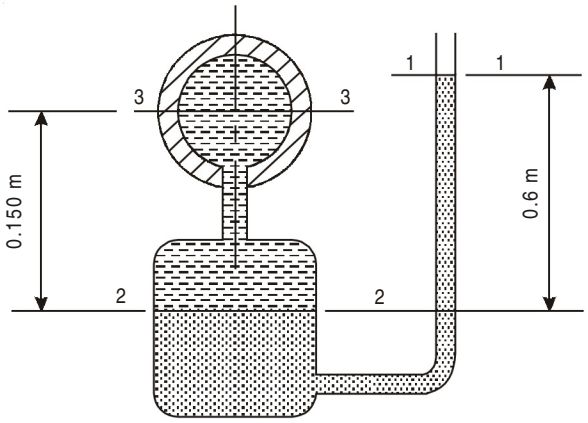


Fig. 2.23

Solution Let δ be the deflection of the gauge fluid in the left tube. Since this deflection is proportional to the area ratios, $\delta = 0.6\left(\frac{1}{100}\right) = 0.006$ m, where 0.6 m is the height of the fluid in the limb subsequent to the deflection of the gauge fluid in the left tube.

1. Pressure exerted by the fluid above the datum 2–2 in the left tube up to point (3).

$$p_3 + (0.150 + 0.006)\,\gamma_w$$

$$\gamma_w \text{ (sp. wt. of water)} = (0.95 \times 9.81)$$

$$= \mathbf{9.3\ kN/m^3}$$

2. Pressure exerted by the fluid in the right limb above the datum 2–2 .

$$(0.6 + 0.006)\,\gamma_m$$

$$\gamma_m \text{ (sp. wt. og Hg)} = (\ 13.6 \times 9.81)$$

$$= \mathbf{133.4\ kN/m^3}$$

Equating the two,

$$p_3 + (0.150 + 0.006)\,\gamma_w = (0.6 + 0.006)\,\gamma_m$$

$$p_3 = (0.6 + 0.006)\,\gamma_m - (0.150 + 0.006)\,\gamma_w$$

$$p_3 = (0.6 + 0.006)\ 133.4 - (0.150 + 0.006)\ 9.3$$

p_3 = (0.6 + 0.006) 133.4 – (0.150 + 0.006) 9.3

$\boldsymbol{p_3}$ **= 79.38 kN/m^2**

If the deflection of the gauge fluid in the left tube is neglected then the above equation would be

p_3 = (0.6) 133.4 – (0.150) 9.3

= **78.645 kN/m^2**

Problem 2.11 *The readings on an inclined single limb manometer are as shown in Fig. 2.24. The fluid in the pipe is water (sp. gravity = 0.95) and the gauge fluid is mercury (sp. gravity = 13.6). Determine the pressure in the pipe at point (3). Take the cross sectional area ratios of the left tube and the right limb as 100 :1.*

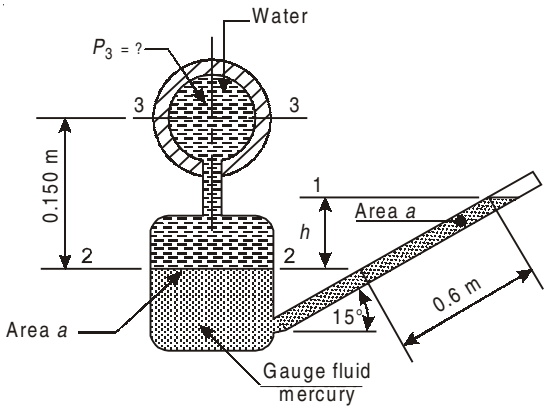


Fig. 2.24

Solution

Let δ be the deflection of the gauge fluid in the left tube. Since this deflection is proportional to the area ratios, $\delta = 0.6 \left(\frac{a}{A}\right) = 0.6\ (100) = 0.006$ m, where 0.6 m is the length of the fluid in the limb subsequent to the deflection of the gauge fluid in the left tube.

1. Pressure exerted by the fluid above the datum 2–2 in the left tube up to point (3).

$p_3 + (0.150 + 0.006)\ \gamma_w$

γ_w (sp. wt. of water) = (0.95 × 9.81)

= **9.3 kN/m^3**

2. Pressure exerted by the fluid in the right limb above the datum 2–2.

$(0.6 \sin 15^\circ + 0.006)\ \gamma_m$

γ_m (sp. wt. of Hg) = (13.6 × 9.81)

= **133.4 kN/m^3**

Since $\sin\theta = \frac{h}{L}$ (Ref. Fig. 2.13). Here only the vertical component of the pressure is considered.

Equating the two,

$$p_3 + (0.150 + 0.006)\,\gamma_w = (0.6 \sin 15^\circ + 0.006)\,\gamma_m$$

$$p_3 = (0.6 \sin 15^\circ + 0.006)\,\gamma_m - (0.150 + 0.006)\,\gamma_w$$

$$p_3 = \{0.6\,(0.259) + 0.006)\,133.4 - (0.150 + 0.006)\,9.3\}$$

$$p_3 = 21.5 - 1.45$$

$$= \mathbf{20\ kN/m^2}$$

Problem 2.12 *The readings on a differential manometer are as shown in Fig. 2.25. Calculate the pressure drop between points A and B. The gauge fluid is mercury (sg = 13.6) and the process fluid is oil (sg = 0.9).*

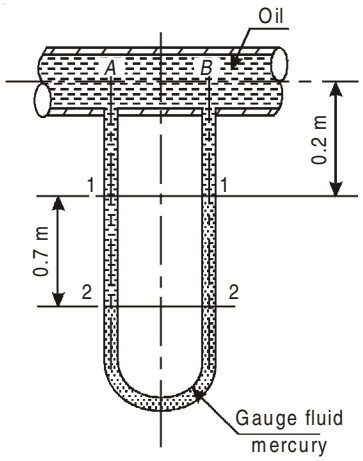


Fig. 2.25

Solution The pressure drop $\Delta_p = P_A - P_B$

$P_A + \gamma_o\,(0.9\text{ m})$ is the pressure at point 2–2

$P_A + \gamma_o\,(0.9\text{ m}) - \gamma_m\,(0.7)$ is the pressure at point 1–1

$P_A + \gamma_o\,(0.9\text{ m}) - \gamma_m\,(0.7) - \gamma_o\,(0.2\text{ m})$ is the pressure at point B

Therefore,

$P_A + \gamma_o\,(0.9\text{ m}) - \gamma_m\,(0.7) - \gamma_o\,(0.2\text{ m}) = P_B$

$P_A - P_B = \gamma_o\,(0.9\text{ m}) - \gamma_m\,(0.7) - \gamma_o\,(0.2\text{ m})$

Here, γ_o= sp. wt of oil $= (0.9) \times 9.81$

$= \mathbf{8.829\ kN/m^3}$

and γ_m = sp.wt of mercury) $= 13.6 \times 9.81$

$= \mathbf{133.416\ kN/m^3}$

substituting these values in equation,

$$P_A - P_B = 8.829\,(0.9\text{ m}) - 133.416\,(0.7) - 8.829\,(0.2\text{ m})$$

$$= \mathbf{-\ 87.2\ kN/m^2}$$

The negative sign indicates that the pressure at A is greater than the pressure at B.

Problem 2.13 *The readings on an inverted U-tube differential manometer are as shown in Fig. 2.26. The gauge fluid is air and the process fluid is oil (sg = 0.9). Determine the pressure drop across points A and B.* $\rho_{air} = 1.2\ kg/m^3$.

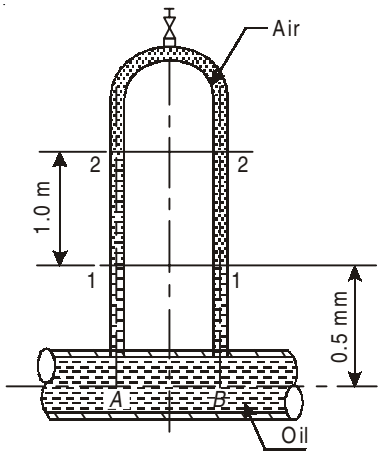


Fig. 2.26 *Inverted U-tube differential monometer*

Solution The pressure drop $\Delta_p = P_A - P_B$

$P_A - \gamma_0\,(1.5\text{ m})$ is the pressure at point 2–2 ... (2.5)

$P_B - \gamma_0\,(0.5\text{ m}) - \gamma_{air}\,(1.0\text{ m})$ is the pressure at point 1–1 ... (2.6)

Equating the expressions 2.5 and 2.6

$$P_A - \gamma_0\,(1.5\text{m}) = P_B - \gamma_0\,(0.5\text{ m}) - \gamma_{air}\,(1.0\text{ m})$$

$$P_A - P_B = \gamma_0\,(1.5\text{ m}) - \gamma_0\,(0.5\text{ m}) - \gamma_{air}\,(1.0\text{ m})$$

Here, γ_0 = sp. wt. of oil $= (0.9 \times 9.81)$

$= \mathbf{8.829\ kN/m^3}$

γ_{air} = sp.wt. of air $= (1.2) \times 9.81$

$= 11.77\ N/m^3$

$= \mathbf{0.0117\ kN/m^3}$

$$P_A - P_B = 8.829\,(1.5\text{ m}) - 8.829\,(0.5\text{ m}) - 0.0117\,(1.0\text{ m})$$

$$P_A - P_B = 13.24 - 4.41 - 0.0117$$

$= \mathbf{8.8\ kN/m^2\ or\ kPa}$

Problem 2.14 *Steel pipes A and B are set to transfer two different fluids. Each pipe is connected to one limb of a differential U-tube manometer. The readings are as shown in Fig. 2.27. Find the pressure drop across A and B. sg of gauge fluid = 13.6.*

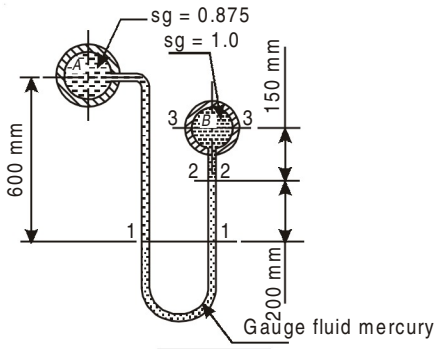


Fig. 2.27

Solution

$P_A + \gamma_A(0.6\text{ m})$ is the pressure at point 1–1
$P_A + \gamma_A(0.6\text{ m}) - \gamma_m(0.2\text{ m})$ is the pressure at point 2–2
$P_A + \gamma_A(0.6\text{ m}) - \gamma_m(0.2\text{ m}) - \gamma_B(0.150\text{ m})$ is the pressure at 3–3
Since the pressure at 3–3 is the same as the pressure at B,

$$P_A + \gamma_A(0.6\text{ m}) - \gamma_m(0.2\text{ m}) - \gamma_B(0.150\text{ m}) = P_B$$

$$P_A - P_B = \gamma_A(0.6\text{ m}) - \gamma_m(0.2\text{ m}) - \gamma_B(0.150\text{ m})$$

γ_A = Sp. wt of fluid in pipe A (0.875×9.81)
= **8.58 kN/m^3**

γ_m = Sp. wt of gauge fluid = (13.6×9.81)
= **133.4 kN/m^3**

γ_B = Sp. wt of fluid in pipe B (1.6×9.81)
= **15.7 kN/m^3**

$$P_A - P_B = 8.58\,(0.6\text{ m}) - 133.4(0.2\text{ m}) - 15.7(0.150\text{ m})$$
$$P_A - P_B = \mathbf{-\ 23.887\ kN/m^3}$$

Problem 2.15. *The readings on a differential U-tube manometer are as shown in Fig. 2.28. Determine the differential pressure across A and B in the inclined pipe transferring water (sg = 1). The gauge fluid is Hg (sg = 13.6).*

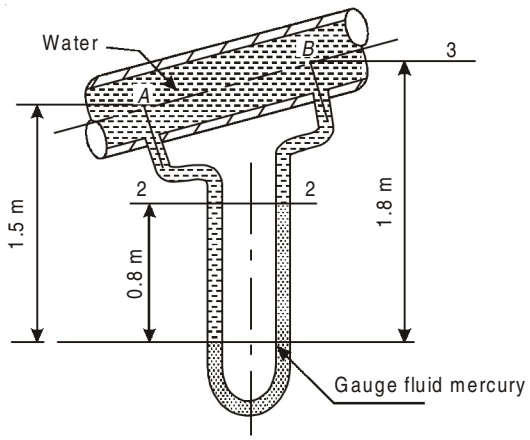


Fig. 2.28

Solution

$P_A + \gamma_w(1.5\text{ m})$ is the pressure at point 1–1

$P_A + \gamma_w(1.5\text{ m}) - \gamma_m(0.8\text{ m})$ is the pressure at point 2–2

$P_A + \gamma_w(1.5\text{ m}) - \gamma_m(0.8\text{ m}) - \gamma_w(1.8 - 0.8\text{ m})$ is the pressure at 3–3

Since the pressure at 3–3 is the same as the pressure at B,

$$P_A + \gamma_w(1.5\text{ m}) - \gamma_m(0.8\text{ m}) - \gamma_w(1.8 - 0.8\text{ m}) = P_B$$

$$P_A - P_B = \gamma_w(1.5\text{ m}) - \gamma_m(0.8\text{ m}) - \gamma_w(1.8 - 0.8\text{ m})$$

γ_w= Sp. wt of water = (1×9.81)

$= \mathbf{9.81\ kN/m^3}$

γ_m= Sp. wt of gauge fluid = (13.6×9.81)

$= \mathbf{133.4\ kN/m^3}$

$$P_A - P_B = 9.81(1.5\text{ m}) - 133.4(0.8\text{ m}) - 9.81(1\text{ m})$$

$$= -101.8\text{ kPa}$$

The negative sign indicates $P_B > P_A$

Problem 2.16 *Determine the differential pressure between points A and B for the conditions shown in Fig. 2.29. The gauge fluid is oil (sg = 0.8) and the process fluid has a value of* $\gamma = 14.7\ kN/m^3$.

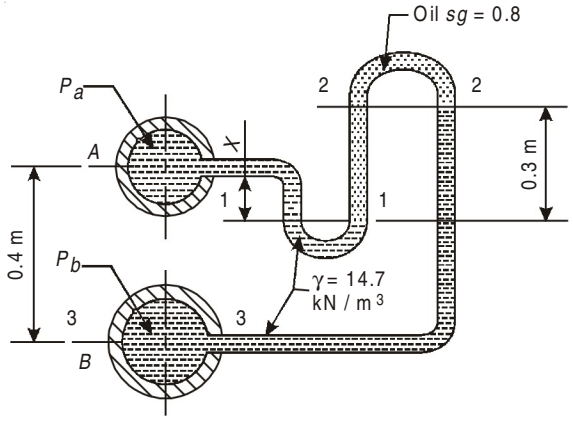


Fig. 2.29

Solution

$P_a + \gamma_p(x)$ is the pressure at 1–1

$P_a + \gamma_p(x) - \gamma_o(0.3\text{ m})$ is the pressure at 2–2

$P_a + \gamma_p(x) - \gamma_o(0.3\text{ m}) + \gamma_p(0.3 + 0.4 - x)$ is the pressure at 3–3

$P_a + \gamma_p(x) - \gamma_o(0.3\text{ m}) + \gamma_p(0.3 + 0.4 - \gamma) = P_b$. (Since pressure at 3–3 = P_b)

$$(P_a - P_b) = \gamma_p(x) - \gamma_o(0.3\text{ m}) + \gamma_p(0.3 + 0.4 - x)$$

$$(P_a - P_b) = \gamma_p(x) - \gamma_o(0.3\text{ m}) + \gamma_p(0.7) - \gamma_p(x)$$

$$(P_a - P_b) = -\gamma_o(0.3\text{ m}) + \gamma_p(0.7).$$

Substituting, for $\gamma_0 = 0.8\ (9.81) = 7.85\ \text{kN/m}^3$

and $\gamma_p = 14.7\ \text{kN/m}^3$

$$(P_a - P_b) = -7.85\ (0.3\ \text{m}) + 14.7\ (0.7)$$
$$= 7.93\ \text{kPa}$$

Problem 2.17 *The readings on a compound U-tube manometer are as shown in Fig.2.30. Determine the pressure at point A.*

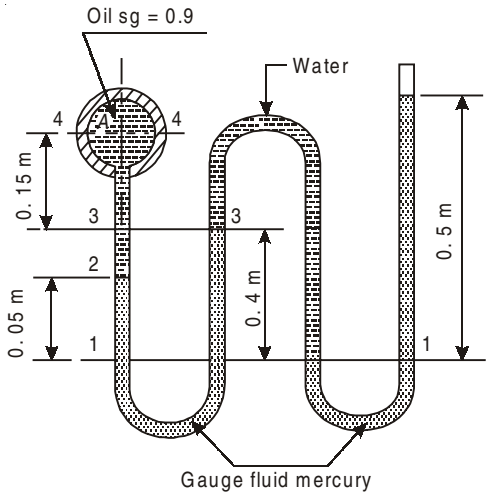


Fig. 2.30

Compound U-tube manometers are used to measure pressures higher than what is feasible for a simple U-tube manometer. They are also employed to measure differential pressures. There is no change however in the procedure followed to determine the gauge pressure of the process fluid. Since one end in this case is open to atmosphere, we start from that end for the sake of convenience and arrive at *A*.

Solution

$P_b + \gamma_m\ (0.5\ \text{m})$ is the pressure at 1–1

$P_b + \gamma_m\ (0.5\ \text{m}) - \gamma_w\ (0.4\ \text{m} + 0.050\ \text{m})$ is the pressure at 2–2

$P_b + \gamma_m\ (0.5\ \text{m}) - \gamma_w\ (0.4\ \text{m} + 0.050\ \text{m}) + \gamma_m\ (0.4\ \text{m})$ is the pressure at 3–5

$P_b + \gamma_m\ (0.5\ \text{m}) - \gamma_w\ (0.4\ \text{m} + 0.050\ \text{m}) + \gamma_m\ (0.4\ \text{m}) - \gamma_0\ (0.550\ \text{m})$ is the pressure at 4–4 or at point *A* and hence,

$$P_a = P_b + \gamma_m\ (0.5\ \text{m}) - \gamma_w\ (0.4\ \text{m} + 0.050\ \text{m}) + \gamma_m\ (0.4\ \text{m}) - \gamma_0\ (0.550\ \text{m})$$

$P_b = 0$ since the point is open to atmosphere.

$\gamma_m = 13.6 \times 9.81 = 133.4\ \text{kN/m}^3$

$\gamma_w = 9.77\ \text{kN/m}^3$

$$\gamma_0 = 0.9 \times 9.81$$
$$= 8.8 \text{ kN/m}^3$$
$$P_a = 0 + 133.4(0.5 \text{ m}) - 9.77 (0.4 \text{ m} + 0.050 \text{ m}) + 133.4 (0.4 \text{ m}) - 8.8 (0.550 \text{ m})$$
$$P_a = 110.8 \text{ kPa}$$

QUESTIONS BANK

Section A

Review Questions

1. Why is it that the equation $p = \rho gh$ is not applicable to gases?
2. Convert a pressure of one bar in to Pascal.
3. What is the atmospheric pressure if gauge pressure is zero and the absolute pressure is one bar?
4. Why is it that the absolute pressure is always positive?
5. State one reason for the difference in absolute pressure between two different locations on the surface of the earth?
6. What is the normal range of atmospheric pressure variation on the surface of the earth?
7. What is the function of a barometer?
8. What is the average height in metres of troposphere from sea level?
9. What is gauge pressure?
10. What is the difference between the force exerted by a fluid as compared to the force exerted by mechanical linkages?
11. How many Newton is one kilogram force?
12. What do you understand by the term *Pascal's paradox?*
13. Distinguish between Pascal's law and hydrostatic law.
14. What is the standard atmospheric pressure in terms of metres of Mercury?
15. What is Piezometric head?
16. What is the principle difference between a micro monometer and the differential U tube manometer?
17. A vacuum gauge reads 0. What does it indicate: absolute or gauge pressure?
18. What happens when a liquid is subjected to a pressure differential?

Section B

Multiple Choice Questions

Choose the right answer:

1. Standard atmospheric pressure in metres f Hg of sea water is
   (a) 0.760 m (b) 0.9 m
   (c) 10.32 m (d) 10.02 m

2. A cone resting on its sides is in equilibrium
   (a) unstable (b) stable
   (c) neutral
3. In liquids change in pressure is directly proportional to specific weight
   (a) height
   (b) acceleration due to gravity
   (c) specific weight
4. Force exerted by a fluid at rest depends on its
   (a) elevation (b) weight
   (c) pressure
5. To measure small variations in gauge pressure the best option is
   (a) Piezometer
   (b) single U tube manometer
   (c) Inclined limb manometer
6. Which is the best instrument for measuring partial vacuum?
   (a) Inclined limb manometer
   (b) U tube manometer
   (c) Piezometer
7. Gauge fluid must be lighter than the process fluid in straight
   (a) U-Tube manometer
   (b) inclined limb manometer
   (c) inverted U-tube manometer.
8. Inverted U-tube manometer is more sensitive than upright U-tube manometers because it is
   (a) inverted
   (b) gauge fluid is lighter than process fluid
   (c) no gauge fluid is required
9. In case of fluid at rest pressure varies
   (a) horizontally (b) vertically downwards
   (c) in the upward direction.
10. Multi U-tube manometers are employed to measure
    (a) differential pressure (b) low pressure
    (c) very high pressure
11. 772 mm Hg is equal
    (a) 102.5 kPa (b) 104 kPa
    (c) 106 kPa
12. One atmosphere in feet of oil is
    (a) 34 (b) 37
    (c) 39

13. Sensitivity of the U tube manometer with inclined leg depends on
    (a) Angle of inclination of the leg
    (b) specific weight of the gauge fluid
    (c) both

Section C

Problems

1. What is the water pressure on a submarine floating 80 metres below the surface of the sea? Assume a suitable value for the density of sea water. (*Ans.* 804.4 kPa)
2. The pressure on an object at a depth of 5 metres from the surface of the fluid in which it is immersed is 42.2 kPa. What is its specific gravity? (*Ans.* 0.86)
3. A hydraulic jack of piston diameter = 0.05 metres lifts a load of 10,000 N. Determine the applied pressure. (*Ans.* 5×10^3 kPa)
4. A hydraulic cylinder has two pistons placed one inside the other concentrically. The outer piston has an annulus area = 0.006 m^2 and weighs 1000 N. The inner piston has an area = 0.002 m^2 and weighs 1500 N. When a pressure of 200 kPa is applied to the bottom port of the cylinder which, of the two pistons, is expected to move first?
5. A vessel of the shape of a truncated cone rests on its smaller diameter = 0.5 m, which is the closed end. The bigger diameter is 0.75 m. What is the force at the bottom surface if the vessel holds water up to a height of 0.6 m? (*Ans.* 1155 N)
6. The pressure difference in a piezometer between two points *A* and *B* measured vertically is equivalent to 4 inches of water. If the distance from the free surface exposed to atmospheric pressure to point *B* is 0.3 m, determine the distance to point *A*. Assume the fluid as water. (*Ans.* 0.4 m)
7. What would be the pressure in kPa if the pressure head is
   (1) 300 mm of Hg of sg? = 13.6
   (2) 300 mm of water
   (3) 300 mm of oil of density 860 kg/m^3
   (*Ans.* (1) 39.84 kPa, (2) 2.94 kPa, (3) 2.53 kPa)
8. If the vacuum pressure in a pipe is 60 mm of Hg, what would be the absolute pressure in kN/sq metre if the atmospheric pressure is 10^5 Pa? (*Ans.* 92.03 kN/m^2)
9. A tank is filled with water to a depth of 500 mm. 400 mm of oil of sg 0.86 is added to the tank and is allowed to settle down. Determine the gauge pressure at the water and oil inter face and also at the base. (*Ans.* 3.37 kPa and 8.2 kPa)

10. A sealed tank contains air at a pressure of 10^3 N/m^2 over oil of sg 0.85. What is the pressure at a depth of 1.5 m from the surface of the oil? (*Ans.* 11.50 kPa)

11. The readings on an inclined limb manometer are as shown in Fig. P-11. Determine the pressure differential ($P_b - P_a$). The gauge fluid is oil of sg = 0.86. (*Ans.* 0. 873 kPa)

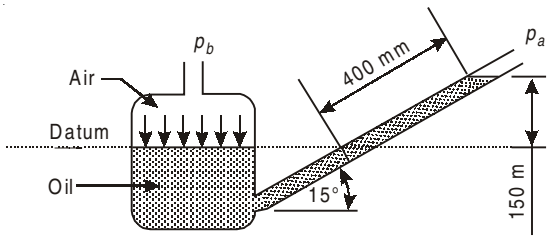


Fig. P. 11

Answers to Multiple Choice Questions

1. 0.760 m
2. Neutral equilibrium
3. Specific weight
4. Weight
5. Inclined Limb manometer
6. U-tube manometer
7. Inverted U-tube manometer
8. Gauge fluid is lighter than the process fluid
9. Vertically downwards
10. Very high pressure
11. 102.5 kPa
12. 37
13. both

Chapter 3

Hydrostatic Forces on Surfaces

Under properties of fluids, we defined pressure and in the subsequent chapter, we dealt with pressure measurement. In this chapter we are more concerned with the effect of this static pressure on rigid surfaces that creates a force which the surfaces must resist.

Determining the force that acts on surfaces subjected to uniform pressure is relatively easy and direct (force = pressure × cross sectional area). Since the pressure is uniform the force remains uniform and depends mainly on the surface area exposed to the fluid. On the other hand computing the net force acting on a dam built to contain a river is relatively more complicated. It becomes even more so when we try to compute the resultant force on submerged curved surfaces and the direction in which it acts or place the vector representing the resultant force in its proper location.

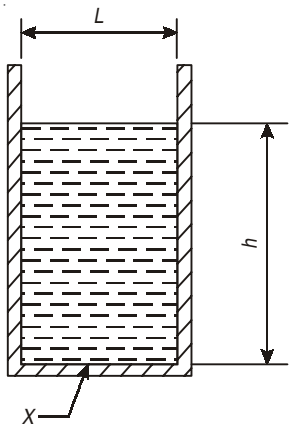


Fig. 3.1

Consider a square container as shown in Fig. 3.1. The container holds water and the water pressure on the bottom plate X is uniform and therefore $= \gamma h$. The force on the plate due to this pressure is a function of the area and $= \gamma h \times A$, where A is the area of the bottom plate.

The force exerted on the side walls of the container however has no such direct relation ship Fig. 3.2. The pressure distribution diagram (abc) is triangular, the maximum pressure acting at the interface of base (bc) and the minimum pressure, at the vertex in conformity with the law that the pressure exerted by a static fluid is proportional to the depth. Since the pressure is not uniform, the force at every point varies and therefore it becomes important to determine the magnitude, position and direction of the resultant force F_R.

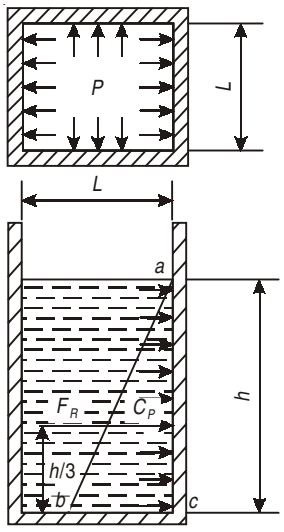


Fig. 3.2

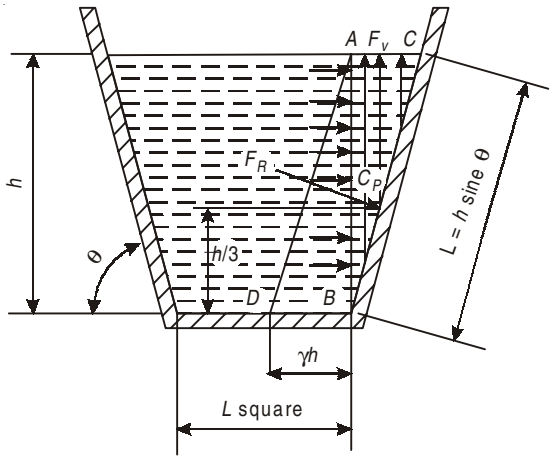


Fig. 3.3

The resultant force is the true force for which a structure must be designed and is computed by assuming that the pressure at every point could be summed up as one single entity acting through one single point. This entity is called *total pressure* and the point at which it acts is called the *centre of pressure* (C_p).

For the particular case considered in Fig. 3.2 since the pressure distribution diagram is a true triangle the area of the triangle represents the force per unit width.

$$\text{Therefore, force/unit width} = \frac{1}{2} \times bc \times h$$

$$\text{Since } bc = \gamma h$$

$$\text{Force/unit width} = \frac{1}{2} \times \gamma h \times h$$

$$\text{The resultant force } F_R = \left(\frac{1}{2} \times \gamma h \times h\right) \times (\text{width } L) \qquad \ldots (3.1)$$

The center of pressure is located at $\frac{1}{3}$ the distance from the base, $C_P = \frac{1}{3}h$.

The container in Fig. 3.3 which could be visualized as an inverted frustum of a pyramid has inclined walls making an angle θ to the horizontal. Even in this case the center of pressure remains at a distance $\frac{1}{3}$, from the base but the resultant force (F_R) which acts at right angles to the plate has two components to it.

F_V = the vertical component

and F_H = the horizontal component.

$$F_R \text{ is therefore } = \sqrt{F_v^2 + F_H^2} \qquad \ldots (3.2)$$

F_v = (Volume of fluid within the area of the triangle $ACB \times \gamma$).

$$F_H = \frac{1}{2} (\gamma \times h)\, hL,$$

where L is the breadth.

We may now develop a general equation for the resultant force (F_R) and center of pressure (C_P) that applies to any *submerged surface*, whether placed horizontally, vertically or makes an angle with the free surface of the liquid.

3.1 MAGNITUDE OF THE RESULTANT FORCE

3.1.1 Plane Submerged Surface

Consider a plane figure *ABCD* of area A as shown in Fig. 3.4, immersed in a liquid of specific weight γ and inclined at an angle θ to the free surface of the fluid. Imagine an elemental area δA within the plane figure and let its position be at a distance y measured vertically from the free surface of the fluid. Let $\bar{y}$ be the distance to the center of gravity and let h be its distance to the point where the center of pressure C_P lies.

Force on the elemental area = $p \times \delta A = (\gamma)\, y \times \delta A$

Since $y = x \sin \theta$

Force on the elemental area = $(\gamma) \times (x \sin \theta) \times \delta A$

Total force on the whole surface or the resultant force $F_R = (\gamma) \int x \sin \theta$ or $(\gamma) \sin \theta \int x\, \delta A$. But the quantity $\int (x)\, \delta A$ is the first moment of area of the surface *ABCD* about the free surface of the liquid and hence $= AZ_{cg}$

Therefore, $F_R = (\gamma) \text{ sine } \theta \ AZ_{cg}$ ... (3.3)

But $Z_{cg} \text{ sine } \theta = \bar{y}$

Therefore, $F_R = (\gamma) A \bar{y}$ ... (3.4)

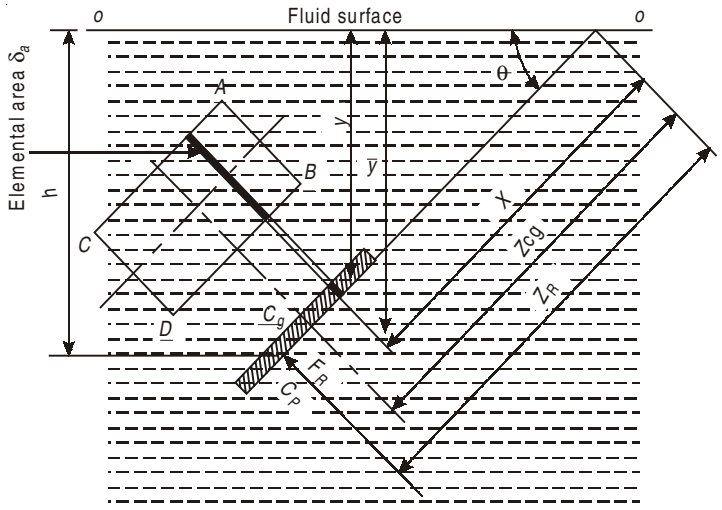


Fig. 3.4

3.1.2 Location of Center of Pressure

($F_R \times Z_R$) is the moment of the resultant force about the free surface of the liquid, that is, moment of F_R about z_R. According to principle of moments, this moment must be the same as the sum of the individual moments about the same axis.

That is, $(\gamma)\, y \times \delta A$ or $(\gamma)(x \sin \theta)\, \delta A$ is the force on the elemental area δA. The moment of this force about 0–0 = $(\gamma)(x \sin \theta)\, \delta A\, (x) = (\gamma)(\sin \theta)\, x^2\, \delta A$. The sum of the moments of all these forces about 0–0

$$(\gamma) \sin \theta \int x^2 \ \delta A$$

Therefore $F_R \times Z_R = (\gamma) \sin \theta \int x^2 \ \delta A$

But, $\left(\int x^2 \, \delta A\right)$ is the second moment of area of surface *ABCD* with respect to 0–0 and hence = I_{0-0}

Therefore, $F_R \times Z_R = (\gamma) \sin \theta \ (I_{0-0})$

or $$Z_R = \frac{\{(\gamma) \text{ sin } \theta \ (I_{0-0})\}}{F_R}$$

But $F_R = (\gamma) \text{ sine } AZ_{cg}$ from Eq. (3.3)

Therefore $$Z_R = \frac{(\gamma)\ \sin\ \theta\ (I_{0-0})}{(\gamma)\ \text{sine}\ AZ_{cg}}$$

$$Z_R = \frac{I_{0-0}}{AZ_{cg}} \qquad \text{... (3.5)}$$

But $I_{0\text{-}0} = I_{CG} + (AZ_{cg})^2$ as per the theorem of parallel axes. Substituting this result in Eq. (3.5),

$$Z_R = \frac{I_{CG} + (AZ_{cg})^2}{AZ_{cg}}$$

$$Z_R = \frac{I_{CG}}{AZ_{cg}} + (AZ_{cg})^2$$

$$Z_R = \frac{I_{CG}}{AZ_{cg}} + Z_{cg} \qquad \text{... (3.6)}$$

$$Z_R - Z_{cg} = \frac{I_{CG}}{AZ_{cg}} \qquad \text{... (3.7)}$$

The same way an equation can be derived for the location of center of pressure in terms of the vertical distance h as,

$$h = \frac{I_{CG}\ \text{sine}^2\ \theta}{A\,\bar{y}} + \bar{y} \qquad \text{... (3.7a)}$$

To summarize, the point of location of center of pressure C_p is given by the ratio of second moment of inertia to first moment of inertia about the free surface of the fluid. It is always located below the axis of CG or the centroidal axis except where the surface considered is horizontal to the free surface of the liquid. In such cases $Z_R = Z_{cg}$ because $\theta = 0$. Also if the area considered is symmetrical about a vertical plane then the center of pressure will coincide with the centroidal axis. As the depth of the immersed surface increases Z_R moves closer to Z_{cg}. In case of a rectangular surface placed parallel to the free surface of the liquid, $Z_R = \frac{h}{3}$, where h is the submerged height of the surface Fig. 3.2.

3.2 MAGNITUDE OF THE RESULTANT FORCE

3.2.1 Curved Submerged Surface

This section deals with submerged surfaces curved in a single plane, that is in the x and y planes only.

In case of a curved surface the elemental force $p\delta A$ acts perpendicular to the curved surface at all points and therefore is not parallel and cannot be

summed up in the usual way. (Fig. 3.5) The force has to be resolved in to its components in the x (F_H) and y (F_V) directions and then combined vectorially.

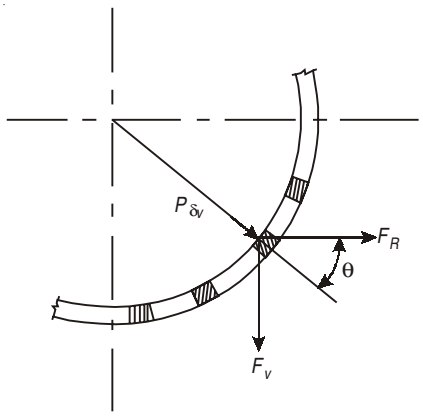


Fig. 3.5

Consider a curved surface AB placed at a distance h from the free surface of the liquid as shown in Fig. 3.6. Let x be the projected length of AB on the vertical plane. Let w be the width of the curved surface AB. Let $DABC$ be the liquid column directly above AB.

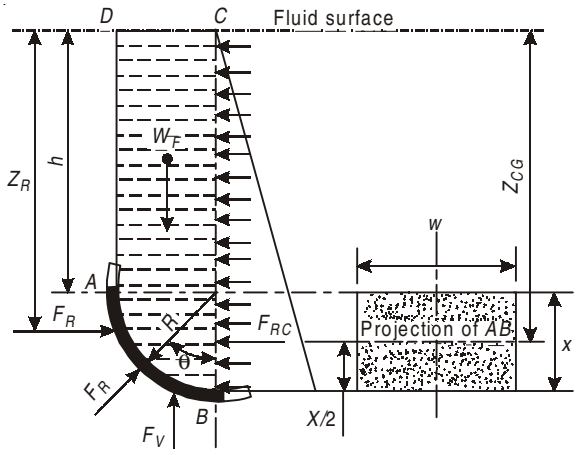


Fig. 3.6

Horizontal component : (F_H)

Assuming $DABC$ as a container by itself, the resultant force $F_{BC} = \gamma (Z_{CG}) A$. From considerations of equilibrium, $F_{BC} = F_H$ because there are no other forces influencing in the horizontal direction.

Therefore, $F_H = \gamma (Z_{CG}) A$

where $A = wx$ ... (3.8)

Location of F_H with respect to free surface of the liquid.
From Eq. 3.7, we have

$$Z_R - Z_{cg} = \frac{I_{CG}}{AZ_{cg}}$$

$$Z_R = \left(\frac{I_{CG}}{AZ_{cg}}\right) + Z_{cg} \qquad \text{... (3.8a)}$$

Vertical component : (F_V)
In the vertical direction there are only two forces which are equal in magnitude but opposite in direction. These are, the weight of the fluid W_F acting downwards and the reaction force F_V acting directly upwards. It is therefore evident that $F_V = W_F$.

$F_V = \gamma \times$ (volume of liquid directly above the curved surface AB up to the free surface of the fluid).

$$F_V = \gamma \{(h \times R \times w) + (\frac{1}{4} \pi R^2 w)\}.$$

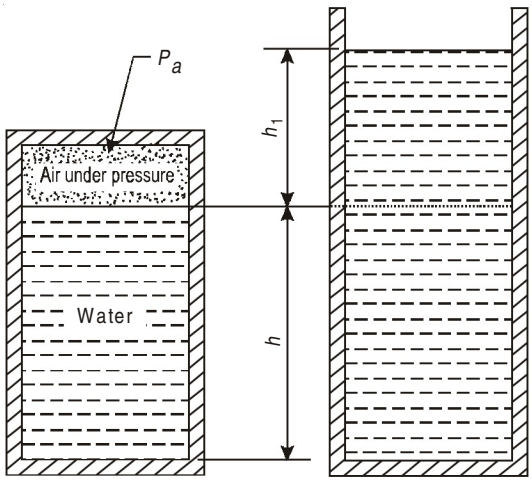
F_V acts through the center of gravity G of $ABCD$

Therefore $$F_R = \sqrt{(F_H)^2 + (F_V)^2} \qquad \text{... (3.9)}$$

F_R acts through the center of curvature of arc AB at an angle

$$\theta = \tan^{-1}\left(\frac{F_V}{F_H}\right) \qquad \text{... (3.10)}$$

The instances discussed above are simple and straight forward. In actual practice however there could be many variations to these instances. Although



h_1 P_a/y = head of water equivalent to air pressure

Fig. 3.7

conceptually the analysis doesn't change, it is still considered as important to discuss these variations and highlight the differences in their mathematical treatment.

We have all along discussed instances where the fluid is exposed to atmospheric pressure. If the surfaces under consideration are further subjected to externally induced pressure, then the net force acting on the surface would have to be computed accordingly. The simplest way of dealing with such situations is by converting the additional pressure in to an equivalent head of the liquid. This is called the *piezometric* head.

In Fig. 3.7 water in a sealed container is subjected to an additional air pressure (p_a). This is converted into an equivalent head of water $h_1 = \frac{p_a}{\gamma}$, where γ is the specific weight of water.

The depth that should be considered now for calculating the pressure at the bottom of the container is $(h + h_1)$.

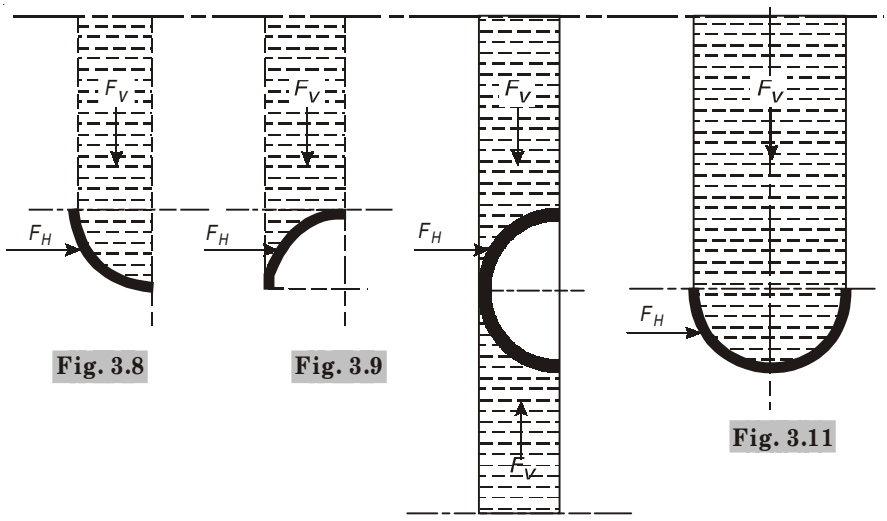


Fig. 3.8

Fig. 3.9

Fig. 3.10

Fig. 3.11

Submerged curved surfaces might support a liquid on either side of the surface or on one or the other side only. Figures 3.8, 3.9, 3.10 and 3.11 depict several variations. Figure 3.8 shows the fluid as being supported above the bottom surface. Figure 3.10 shows the fluid as being supported both from above and from below. In all of these cases $F_V = \gamma \times$ (volume of fluid directly above the surface). It is important that this volume is first isolated and its quantity determined. In case of Fig. 3.10 because there exists both upward and downward forces, F_V is the net downward force. The horizontal component is determined in the manner already explained.

TYPICAL PROBLEMS AND SOLUTIONS

Problem 3.1 *Wall AO, Fig. 3.12 is designed to block water which rises to a level of 4 m. Determine the magnitude of force F with respect to the hinge O that can effectively balance the force due to the pressure exerted by water. The wall is 6 m wide.*

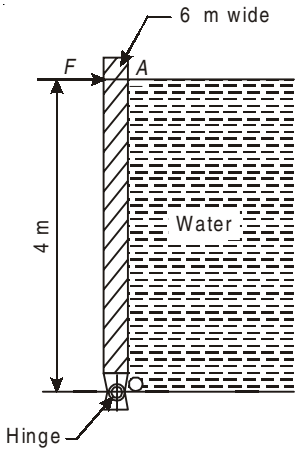


Fig. 3.12

Solution From Eq. 3.1

$$\text{The resultant force } F_R = \left(\frac{1}{2}\times\gamma h\times h\right)\times(\text{width})$$

$$F_R = \left(\frac{1}{2}\times 9.81\times 4\times 4\right)\times(6)$$

$$= 470.88 \text{ kN}$$

The center of pressure is located at $\frac{1}{3}$ the distance from the base,

$$C_P = \frac{1}{3}h$$

$$C_P = \frac{4}{3}$$

$$= 1.33 \text{ m or } (4–1.33)$$

$$= 2.66 \text{ m from the top.}$$

Therefore, $(470.88)\times 2.66 = F\times 4.$

$$F = 313.13 \text{ kN}$$

Problem 3.2 *The tank shown in Fig. 3.13 holds Glycerin. Calculate the magnitude and position of the resultant force on the inclined surface of the container.*

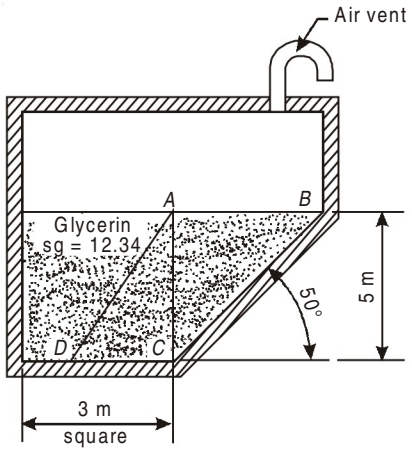


Fig. 3.13

Solution As per Eq. 3.2

$$F_R = \sqrt{F_v^2 + F_H^2}$$

F_v = (Volume of fluid within the area of the triangle) × γ

$$F_v = \left(\frac{1}{2} AB \times 5 \times 3\right) \times \gamma$$

Here the factor 3 m represents the width of the container.

$$AB = \frac{AC}{\tan 50^\circ} = \frac{5}{1.19} = 4.20,$$

$$\gamma = 9.81 \times 12.34 = 121$$

$$F_v = \left(\frac{1}{2} \times 4.20 \times 5 \times 3\right) \times 121$$

$$= 3811.5 \text{ kN}$$

$$F_H = \frac{1}{2} (\gamma \times h)\, h\, L, \text{ where } L \text{ is the breadth.}$$

$$F_H = \frac{1}{2} (9.81 \times 5)\, 5 \times 3 = 735.75 \text{ kN}$$

$$F_R = \sqrt{3811.5^2 + 735.75^2}$$

$$= \mathbf{3882\ kN}$$

The center of pressure is located at $\frac{1}{3}$ the distance from the base,

$C_P = \frac{1}{3}$ (5) = 1.66 m from the base and acts perpendicular to the inclined surface.

Problem 3.3 *A dam built across a river has a profile as shown in Fig. 3.14. The maximum water level is 20 m from the base level. Determine the resultant force, its position h and its angle of inclination θ to the horizontal. The dam is 50 m wide.*

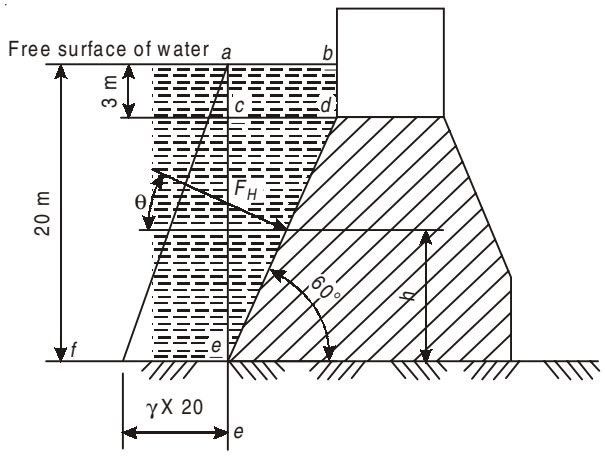


Fig. 3.14

Solution As per Eq. (3.2)

$$F_R = \sqrt{F_v^2 + F_H^2}$$

F_v = (Volume of water within the area of the triangle *dce* + volume of water within the area *abdc*) × γ

$$F_v = \left\{\frac{1}{2} cd\ (20-3)\ 50 + (3 \times cd) \times 50\right\} \gamma$$

Here the factor 50 m represents the width of the dam.

cd = ce tan 30°

= 17 × 0.577 = 9.8 m

γ = 9.81 kN/m^3

$$F_v = \left\{\left(\frac{1}{2} \times 9.8 \times 17 \times 50\right) + (3 \times 9.8) \times 50\right\} 9.81$$

= 55,279 kN

$F_H = \frac{1}{2}$ (γ × h) h L, where L is the breadth.

$$F_H = \frac{1}{2}(9.81 \times 20)\, 20 \times 50$$
$$= 98{,}100 \text{ kN}$$
$$F_R = \sqrt{55279^2 + 98{,}100^2}$$
$$= \mathbf{112602\ kN}$$

The center of pressure is located at $\frac{1}{3}$ the distance from the base,

$$C_P = \frac{1}{3}(20) = 6.66 \text{ m}$$

from the base and acts perpendicular to the inclined surface.
From Eq. 3.10

$$\theta = \tan^{-1}\left(\frac{F_V}{F_H}\right)$$
$$= \tan^{-1}\left(\frac{55279}{98100}\right)$$
$$= \tan^{-1} 0.563 = 29.4°$$

Problem 3.4 *A square plate of surface area 9 square metres is held vertically in water. Determine the total pressure on the surface and the center of pressure if its upper edge is 3 m below the surface of water.*

Solution The length of the square plate must be 3 m since the area is 9 m.

Figure 3.15 shows the position and distances from the free surface of the liquid.

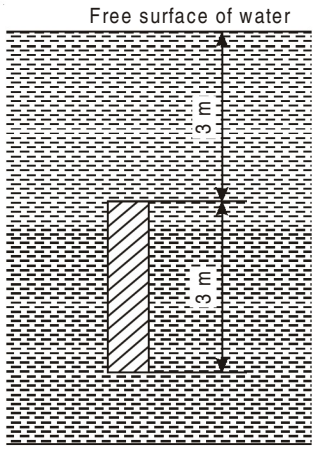


Fig. 3.15

Solution From Eq. 3.4

$$F_R = (\gamma) A \bar{y}$$

A = area of the plate and

$\bar{y}$ = distance from the free surface of water to C_G axis

$$F_R = 9.81 \times 9 \times 4.5$$

$$= \mathbf{3973\ kN}$$

From Eq. 3.7

$$Z_R - Z_{cg} = \frac{I_{CG}}{AZ_{cg}}$$

$$Z_{cg} = \bar{y} = 4.5 \text{ m}$$

$$I_{CG} = \frac{bd^3}{12} = \frac{(3 \times 3^3)}{12}$$

$$= 6.75 \text{ m}^4$$

$$Z_R - 4.5 = \frac{6.75}{9} \times 4.5$$

$$Z_R = 4.66 \text{ m}$$

Problem 3.5 *A gate controlling a channel is of trapezoidal section and has dimensions as shown in Fig. 3.16. Water flows through the channel always up to its full height. Gate A supported on hinge R is used to close the supply from the channel whenever required. What torque is required to raise the gate from its horizontal position and hold it in the vertical position against the water pressure?*

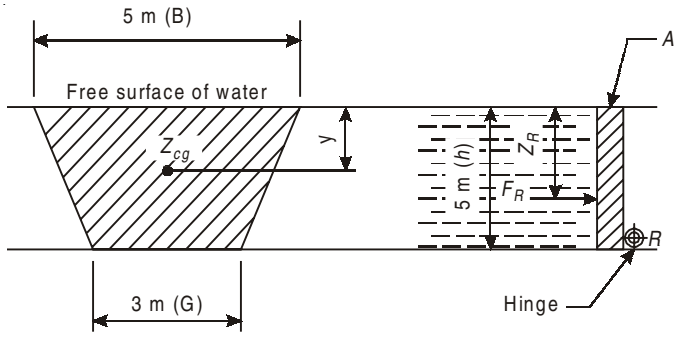


Fig. 3.16

Solution From Eq. 3.4

$$F_R = (\gamma) A \bar{y}$$

$$(\gamma) = 9.81 \text{ kN/m}^3$$

A = area of trapezium $= \dfrac{\{H \times (G+B)\}}{2} = \dfrac{\{1.5 \times (3+5)\}}{2} = 6 \text{ m}^2$

$$\bar{y} = \frac{\{H \times (G+2B)\}}{3(G+B)} = \frac{\{1.5 \times (3+2\times 5)\}}{3(3+5)}$$

$= 0.812$ m

Substituting, $F_R = (\gamma)\, A\, \bar{y} = (9.81)\, 6 \times 0.812$

$= \mathbf{47.8\ kN}$

From Eq. 3.8

$$Z_R = \left(\frac{I_{CG}}{AZ_{cg}}\right) + Z_{cg}$$

$A = 6 \text{ m}^2$

and $Z_{cg} = \bar{y} = 0.812$ m

I_{CG} for a trapezium is given by,

$$I_{CG} = \frac{H^3\ (G^2 + 4GB + B^2)}{36(G+B)}$$

$$= \frac{1.5^3\ \{32 + (4\times 3\times 5) + 5^2\}}{36\,(3+5)} = 1.10 \text{ m}^4$$

Substituting, $Z_R = \left(\dfrac{I_{CG}}{AZ_{cg}}\right) + Z_{cg}$

$$= \left\{\frac{1.10}{6\ (0.812)}\right\} + 0.812 = 1.03 \text{ m}$$

Moment about the hinge $= F_R \times (1.55 - Z_R)$

$= 47.8 \times (\ 1.55 - 1.03)$

$= 24.8$ kN-m

Torque required $= \mathbf{24.8\ kN\text{-}m}$

Problem 3.6 *The gate Fig. 3.17 is hinged at O as shown in the plan view. A hydraulic cylinder hinged at M is free to move with the gate and constantly applies a torque = 450 kN-m, at the center of pressure, to keep the door closed. Determine the limiting height h for the door to remain closed if the lateral distance x to the hinge = 1.27 m.*

Solution

$F_R \times (x) \leq 450$ kNm for the door to remain closed.

Therefore $$F_R = \frac{450}{x} = \frac{450}{1.27}$$

$$= 354 \text{ kN}$$

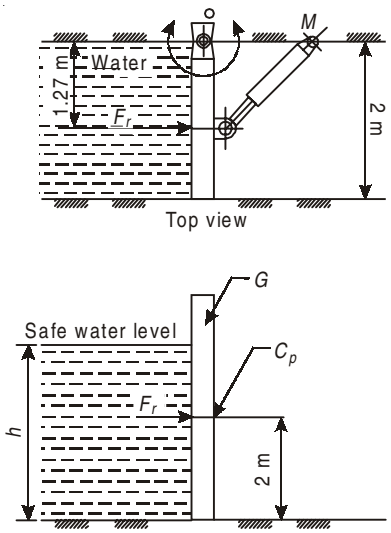


Fig. 3.17

From Eq. 3.1

The resultant force $$F_R = \left(\frac{1}{2} \times \gamma h \times h\right) \times (\text{width } L)$$

Therefore, $$\left(\frac{1}{2} \times \gamma h \times h\right) \times (\text{width } L) = 354$$

$$\left(\frac{1}{2} \times 9.81 \times h \times h\right) \times (2) = 354$$

Therefore $$h^2 = \frac{354}{9.81}$$

$$= 36.08$$

or $$\boldsymbol{h = 6 \text{ m}}$$

Problem 3.7 *A pipe of diameter 0.5 m interferes with the bottom surface of a reservoir as shown in Fig. 3.18. Determine the net force on the submerged surface of the pipe if the depth of water is 4 m and the length of the pipe is 5 m.*

Solution The submerged surface is a curved surface and therefore there should be two components to the force. However the horizontal component due to symmetry gets nullified and only the vertical component exists. The vertical component is merely the net weight of the water above the pipe.

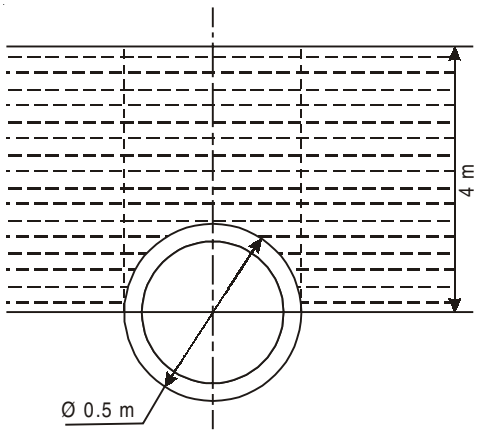


Fig. 3.18

$$F_V = \gamma \left\{ (D \times h \times L) - \left(\frac{\pi D^2}{8 \times L} \right) \right\}$$

$$F_V = 9.81 \left\{ (0.5 \times 4 \times 5) - \left(\frac{\pi\, 0.5^2}{8 \times 5} \right) \right\}$$

$$F_V = 9.81 \{(10) - (0.49)\}$$
$$= \mathbf{93.28\ kN}$$

$$F_R = \sqrt{F_v^2 + F_H^2} = \sqrt{93.28^2 + 0^2}$$
$$= \mathbf{93.28\ kN}$$

Problem 3.8 *A certain facility is provided with two lock gates as shown in Fig. 3.19. Each lock gate operates on two hinges placed 5 m apart with respect to the vertical center line of the gate. Each gate is 7 m in height and 2 m in width. Determine the net force on the top and bottom hinges from the conditions given.*

Solution Lock gates help the ship navigate through shallow waters by permitting transfer of water from one location to the desired location. Normally in such cases two pairs of lock gates work in unison each a certain distance apart and the space in between serves as a temporary transfer chamber. For the purposes of our calculations however only one pair is considered. Because lock gates are huge in size the magnitude of the resultant force is also equally significant and this force has to be borne entirely by the hinges. The gate has its own self weight which again has to be borne by the hinges, but the self weight is not considered here.

Let the net force on the top hinge and the bottom be designated as F_T and F_B respectively. We have to first determine the resultant force on the gate in order

to find the net force on the hinges. The resultant force F_R on the gate is the difference between the upstream (F_U) and the down stream (F_D) forces.

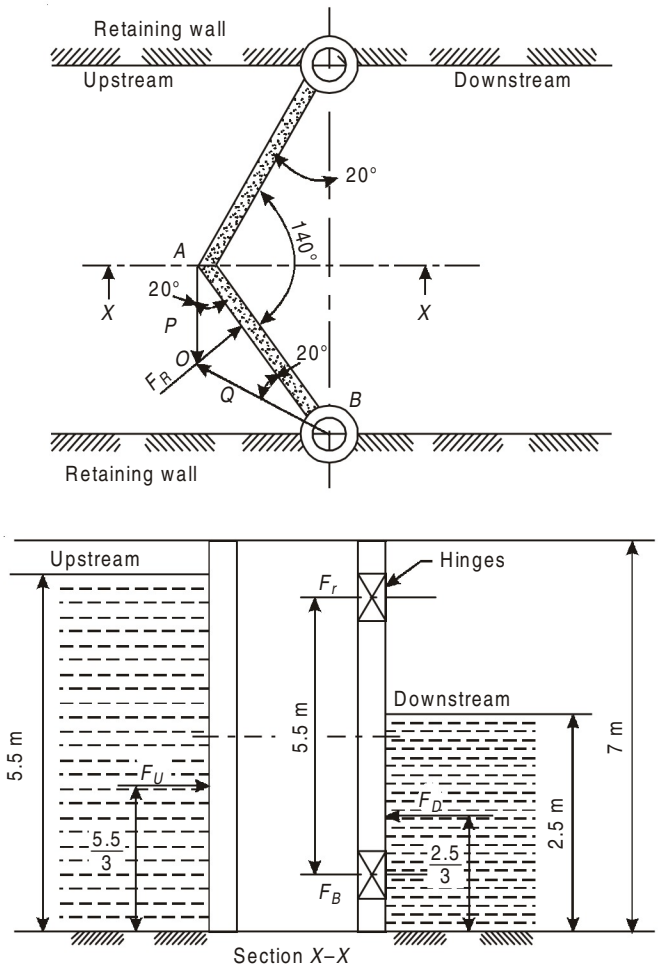


Fig. 3.19

$$F_U = (\gamma) A \bar{y}$$

$$F_D = (\gamma) A \bar{y}$$

$$F_U = 9.81\ (5.5 \times 2)\ \frac{5.5}{2}$$

$$= 296.75 \text{ kN}$$

$$F_D = 9.81\ (2.5 \times 2)\ \frac{2.5}{2} = 61.30 \text{ kN}$$

$$= 61.30 \text{ kN}$$

$$F_U - F_D = 296.75 - 61.30$$
$$= \mathbf{235.45\ kN}$$

The location of F_U is at a distance of $\frac{5.5}{3} = 1.83$ m and the location of F_D is at a distance $\frac{2.5}{3} = 0.833$ m from base B of the reservoir.

In order to find the location of F_R we have to take moments about B assuming F_R is located at a distance z from B

$$F_R \times z = F_U \times 1.83 - F_D \times 0.833$$
$$F_R \times z = 148.4 \times 1.83 - 30.65 \times 0.833$$
$$117.75 \times z = 246$$
$$\boldsymbol{z = 2.08\ m}$$

Let P be the reaction force offered by one gate upon the other which is perpendicular to the contact point at A. Let Q be the reaction offered by the hinges.

$$\text{Angles } OBA = OAB = \frac{(180 - 140)}{2} = 20°$$

If we resolve P and Q in a direction parallel to AB,

Then the magnitude of these forces are, P cos 20° and Q cos 20°

Since these two forces are equal in magnitude and opposite in direction,

$$P \cos 20° = Q \cos 20°$$

or $$P = Q$$

If we resolve P and Q in a direction normal to the gate,

$$P \text{ sine } 20° + Q \text{ sine } 20° = F_R$$

Since $$P = Q$$

$$Q \text{ sine } 20° + \text{Q sine } 20° = F_R$$

or $$Q = \frac{F_R}{2 \text{ sine } 20°}$$

Therefore, $$P = Q = \frac{235.45}{0.6840}$$
$$= \mathbf{344.20\ kN}$$

As the gate is in equilibrium under the action of the forces F_R, Q and P it may be assumed that they are co-planar, i.e., they intersect at the same point. With this assumption, we may now determine the reactions F_T and F_B by taking moments about the hinge.

$$F_T \times 5 + F_B \times 0 = 344.20\ (z - 1)$$

Here the value of z is already determined as = 2.08 m from the base.

$$F_T \times 5 = 344.20\ (2.08 - 1)$$
$$F_T = 74.35 \text{ kN and } F_B$$
$$= (344.20 - 74.35)$$
$$= \mathbf{270\ kN}$$

Problem 3.9 *The gate shown in Fig. 3.20 is hinged at O contains water on one side and oil on the other. What force F is required to hold the gate vertical? The width of the gate is 0.7 m. Also determine the reaction at the hinge.*

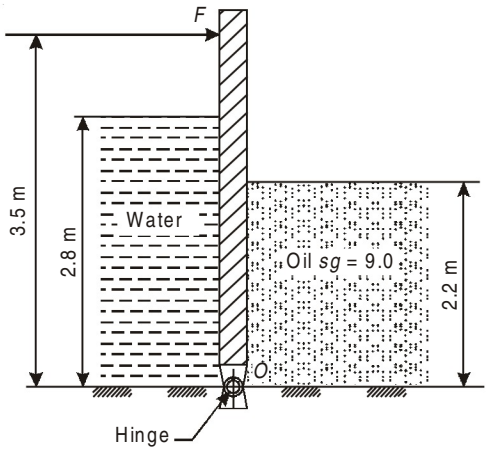


Fig. 3.20

Solution From Eq. 3.1

The force due to water column to the left of the gate,

$$F_L = \left(\frac{1}{2}\times\gamma h\times h\right) \times (\text{width})$$

$$F_L = \left(\frac{1}{2}\times 9.81\times 2.8\times 2.8\right) \times (0.7)$$

$$= 26.9 \text{ kN}$$

The force due to oil column to the right of the gate,

$$F_R = \left(\frac{1}{2}\times\gamma h\times h\right) \times (\text{width})$$

$$F_R = \left(\frac{1}{2}\times(0.9\times 9.81)2.2\times 2.2\right)\times (0.7)$$

$$= \mathbf{14.95\ kN}$$

The net force $F_N = 26.9 - 14.95$

$$= \mathbf{11.9\ kN}$$

F_L acts at distance $= \dfrac{2.8}{3} =$ **0.93 m** from the hinge.

F_R acts at distance $= \dfrac{2.2}{3} =$ **0.73 m** from the hinge

Therefore, the net force F_N acts at distance

$$= \frac{F_L \times 0.93 - F_R \times 0.73}{F_N}$$

$$F_N \text{ acts at distance} = \frac{26.9 \times 0.93 - 14.95 \times 0.73}{11.9}$$

$$= \mathbf{1{,}185\ m}$$

Taking moments about the hinge,

$$F \times 3.5 = 11.9 \times 1.185$$

$$\boldsymbol{F} = \mathbf{4\ kN}$$

$$\text{The reaction at the hinge} = 11.9 - 4$$

$$= \mathbf{7.9\ kN}$$

Problem 3.10 *A plate lies submerged in water as shown in Fig. 3.21. Determine the magnitude of the resultant force on the plate and its center of pressure.*

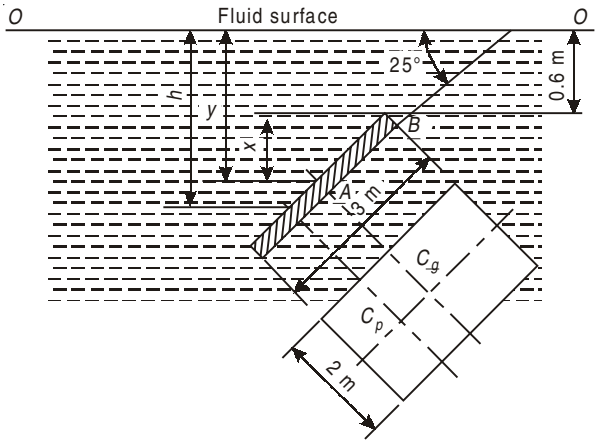


Fig. 3.21

Solution Let $\bar{y}$ be the vertical distance from free surface of water to the axis of C_G of the plate.

Let h be the vertical distance from free surface of water to the center of pressure C_P of the plate.

Let Z_{CG} and Z_R be the corresponding distances measured parallel to the plate. (Ref Fig. 3.4)

From Eq. 3.4

$$F_R = (\gamma)\, A\, \bar{y}$$

$$\bar{y} = (0.6 + X)$$

$$= (0.6 + AB \text{ sine } 25^\circ)$$

$$= (0.6 + 1.5 \times 0.4226)$$

$$= \mathbf{1.23\ m}$$

A = area of the plate
= $3 \times 2 = 6\ m^2$
(γ) = **9.81 kN**
F_R = $(9.81) \times 6 \times 1.23$
= **72.4 kN**

From Eq. 3.7 (a)

$$h = \frac{I_{CG}\ \text{sine}^2\ \theta}{A\bar{y}} + \bar{y}$$

Here, $$I_{CG} = \frac{(2 \times 3^3)}{12} = 4.5\ m^4$$

$$h = \frac{4.5\ (\text{sine}\ 25°)^2}{(6 \times 1.23)} + 1.23$$

h = **1.34 m**

Problem 3.11 *Gate AB is hinged at A and is inclined at 60° to the horizontal as shown in Fig. 3.22. The gate is 3 m long and 1.5 m wide. The gate has a hook at the point of center of pressure to which a steel wire rope is tied. The rope passes over a grooved pulley. Determine the force F required to lift the gate.*

Solution Let $\bar{y}$ be the vertical distance from free surface of water to the axis of C_G of the plate.

Let h be the vertical distance from free surface of water to the center of pressure C_P of the plate.

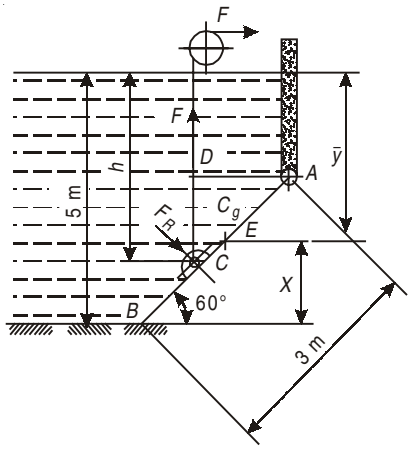


Fig. 3.22

From Eq. 3.4

F_R = $(\gamma)\ A\bar{y}$
$\bar{y}$ = $(5 - X) = (5 - BE\ \text{sine}\ 60°)$

$= (5 - 1.5 \times 0.8660) = \mathbf{3.7\ m}$

A = area of the plate $= 3 \times 1.5 = \mathbf{4.5\ m^2}$

$(\gamma) = \mathbf{9.81\ kN}$

$F_R = (9.81) \times 4.5 \times 3.7$

$= \mathbf{163.3\ kN}$

From Eq. 3.7 (a)

$$h = \frac{I_{CG}\ \sin^2\theta}{A\bar{y}} + \bar{y}$$

Here, $$I_{CG} = \frac{(1.5 \times 3^3)}{12} = 3.375\ \text{m}^4$$

$$h = \frac{3.375\ (\sin 60°)^2}{(4.5 \times 3.7)} + 3.7 = \mathbf{3.85\ m}$$

Taking moments about the hinge,

$$F \times AD = F_R \times CA$$

$$\text{Sine } 60° = \frac{(5-h)}{BC}$$

or $$BC = \frac{(5-h)}{\sin 60°}$$

$$= \frac{(5-3.375)}{0.8660} = \mathbf{1.876\ m}$$

$$CA = (BA - BC)$$

$$= (2 - 1.876) = 0.123\ \text{m}$$

$$\sin 30° = \frac{AD}{CA}$$

or $$AD = CA\ (\sin 30°) = 0.123 \times 0.5 = \mathbf{0.0617\ m}$$

By substituting the values of CA, F_R and AD in the above equation we can obtain the value of F.

$$F = \frac{(F_R \times CA)}{AD}$$

$$= \frac{(163.3 \times 0.123)}{0.0617} = \mathbf{325.54\ kN}$$

Problem 3.12 *A tank shown in Fig.3.23 contains oil of sg = 0.9 and has an opening of 0.1 m diameter which is closed by a sphere of diameter 0.120 m. The point of contact of the sphere with the circular opening is at a distance of 0.035 m from the center of the sphere. Determine (1) the net resultant force on the curved surface ABC due to oil pressure, its location and direction (2) the weight*

of the sphere that would just balance the force due to fluid pressure. The width of the tank = 2.5 m.

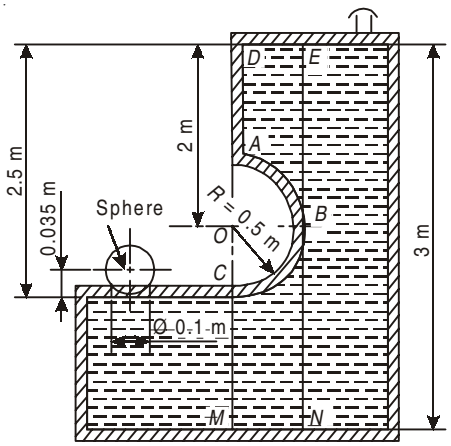


Fig. 3.23

Part 1

Sg of oil = 0.9

Therefore $\gamma = 0.9 \times 9.81$

$= \mathbf{8.83\ kN/m^3}$

Width of the tank = 2.5 m

Let us first compute the vertical component of the force acting ↓ on the curved surface *AB*. This force is the weight of the fluid column directly above *AB*.

$F\downarrow = \gamma$ {Volume of fluid within *EBOD* – $\frac{1}{4}$ the volume of the cylinder of 0.5 m radius and 2.5 m long}

$$F\downarrow = \gamma\{(2 \times 0.5 \times 2.5) - \frac{1}{4}(\pi \times 0.5^2 \times 2.5)\}$$

$= 8.83\ \{2.009\}$

$= 17.74$ kN

Let us now compute the vertical component of the force acting ↑ on the curved surface *BC*. This force is the weight of the fluid column that *BC* supports.

$F\uparrow = \gamma$ {Volume of fluid within *OBNM* – $\frac{1}{4}$ the volume of the cylinder of 0.5 m radius and 2.5 m long}

$$F\uparrow = \gamma\{(3 - 2) \times 0.5 \times 2.5) - \frac{1}{4}(\pi \times 0.5^2 \times 2.5)\}$$

$= 8.83\ \{0.76\}$

$= 6.7$ kN

Therefore, $F_V = F\downarrow - F\uparrow$
$= 17.74 - 6.7$
$= \mathbf{11.04\ kN}$

From Eq. 3.8 we have,
The horizontal force is given by $F_H = \gamma (Z_{CG})\, A$
Here A = the projected area of the curved surface ABC
$= 2(0.5) \times 2.5$
$= \mathbf{2.5\ m^2}$
Z_{CG} = distance to the axis of center of gravity
$= \mathbf{2\ m}$
$F_H = \gamma (Z_{CG})\, A$
$= 8.83 \times (2) \times 2.5$
$= \mathbf{44.15\ kN}$

Therefore the resultant force $F_R = \sqrt{F_v^{\,2} + F_H^{\,2}}$

$= \sqrt{11.04^2 + 44.15^2}$
$= \mathbf{45.50\ kN}$

Location of F_H is given by, (Ref. Eq. 3.8a)

$$Z_R = \left(\frac{I_{CG}}{AZ_{cg}}\right) + Z_{cg}$$

I_{CG} = the second moment of area of the projected surface ABC.
Since the projected area is a rectangle of breadth
$= 2.5$ m and depth $= 2(0.5) = 1$ m,

$$I_{CG} = \frac{(2.5 \times 1^3)}{12} = 0.208\ \text{m}^4$$

Therefore, $$Z_R = \left\{\frac{0.208}{(2.5)\,2}\right\} + 2 = 2.042\ \text{m}$$

Direction is given by Eq. 3.10
F_R acts through the center of curvature of arc ABC at an angle

$$\theta = \tan^{-1}\left(\frac{F_V}{F_H}\right)$$

$$= \tan^{-1}\left(\frac{11.04}{44.15}\right) = 14°$$

Part 2

The net horizontal force on the sphere is zero because the forces are equal and opposite and hence cancel each other out. The vertical force is in the upward direction and is due to fluid pressure acting on the surface area of the sphere which is in contact with oil.

$\downarrow F = \gamma \times h = 8.83 \times 2.5$
$= \mathbf{22.07\ kN/m^2.}$

Lateral area of the spherical segment is given by,

$$A = \frac{\pi}{4}(s^2 + 4h^2)$$

$$s = 0.1 \text{ m}$$

$$h = \left(\frac{0.120}{2 - 0.035}\right) = 0.025 \text{ m}.$$

Therefore $$A = \frac{\pi}{4}(0.1^2 + 4 \times 0.025^2) = \mathbf{0.098\ m^2}$$

Upward force on the sphere due to oil pressure

$$= (22.07 \times 0.098) = \mathbf{0.21\ kN}$$

A sphere of weight 0.21 kN will just balance the upward thrust due to oil pressure.

Problem 3.13 *The gate shown in Fig. 3.24 is hinged at O. Its profile is partly circular and partly straight as shown. If the depth of water is 2 m from the free surface of the fluid and the width is 4 m, determine the counter force F required to keep the gate closed against water pressure.*

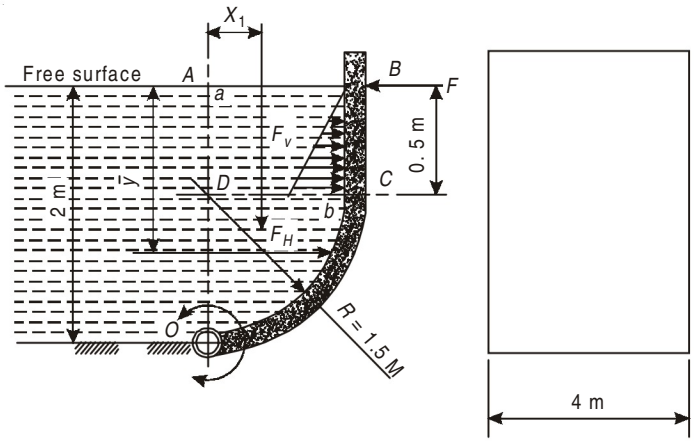


Fig. 3.24

Solution Vertical component of the force is equal to the weight of water contained within the area *OAB*.

$$F_V = \gamma\{(0.5 \times 1.5 \times 4) + \left(\frac{\pi R^2}{4} \times 4\right)$$

$$F_V = 9.81\{(0.5 \times 1.5 \times 4) + \left(\frac{\pi \times 1.5^2}{4} \times 4\right)$$

$$F_V = 98.77 \text{ kN}$$

F_V = 98.77 kN acting ↑ through the centroid of this volume. We will designate the centroid for this composite section as x_1 and the centroid of *ABCD* as x. Let x_2 be the centroid of the quadrant *ODC*.

$$x = \frac{1.5}{2} = 0.75 \text{ m}$$

$$x_2 = 0.424\, R$$
$$= (0.424 \times 1.5) \text{ (Ref. Appendix A)}$$
$$= 0.636 \text{ m}$$

Therefore $$x_1 = \frac{(ax) + (bx_2)}{(a + b)}$$

Here, a and b represent the areas *ABCD* and *ODC* respectively.

$$(ax) = \{(0.5 \times 1.5)\, 0.75\}$$
$$= (0.75 \times 0.75) = 0.5625$$

$$(bx_2) = \left(\frac{\pi R^2}{4}\right)(0.636)$$

$$= \left(\frac{\pi \times 1.5^2}{4}\right)(0.636)$$

$$= (1.767)\,(0.636) = 1.12$$

$$a = 0.75 \text{ m}^2$$

and $$b = 1.767 \text{ m}^2$$

Therefore $$x_1 = \frac{(ax) + (bx_2)}{(a + b)}$$

Therefore $$x_1 = \frac{(0.5625) + (1.12)}{(0.75 + 1.76)} = 1.114 \text{ m}$$

The magnitude of the horizontal force = $F_H = (\gamma)\, A\, \bar{y}$

$$A = \text{Projected area of } OCD$$
$$= (1.5 \times 4) = 6 \text{ m}^2$$

$$\bar{y} = \left(\frac{0.5 + 1.5}{2}\right) = 1.25 \text{ m}$$

$$F_H = (9.81) \times 6 \times 1.25$$
$$= 73.6 \text{ kN}$$

Therefore counter force F required is determined by taking moments about the hinge.

$$F \times 2 = F_V\,(x_1) + F_H\,(2 - \bar{y}\,)$$
$$F \times 2 = 98.77\,(1.114) + 73.6\,(2 - 1.25)$$
$$F \times 2 = 65.15$$

Therefore $$F = 82.5 \text{ kN}$$

Problem 3.14 *A radial gate in form of a sector of a circle of radius 6 m is installed at the crest of a spillway to contain water. For the conditions shown in Fig. 3.25, determine the magnitude and direction of the resultant force on the curved surface and its angle of inclination. Width of the gate is 5 m.*

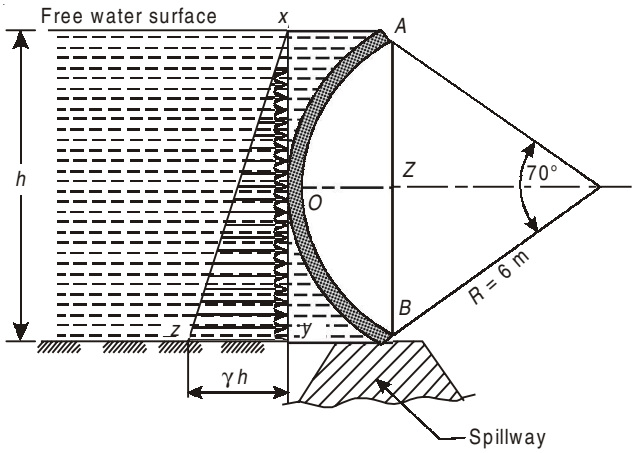


Fig. 3.25

Considering AB as the segment of a circle of radius 6 m and assuming that the water level reaches up to the top edge of the gate, the depth of water,

$$h = 2R\left(\text{sine}\ \frac{70}{2}\right)$$
$$= 2R\ (\text{sine}\ 35)$$
$$= (2 \times 6 \times 0.573)$$
$$= 6.88\ \text{m}$$

The resultant horizontal force on the projected area of the curved surface F_H per unit width is given by the area of the triangle $xyz = \left\{\left(\gamma h \times \frac{h}{2}\right)\right\}$

The magnitude of F_H on the whole surface of the gate $= \left\{\left(\gamma h \times \frac{h}{2}\right)\right\} \times$ width of the gate.

$$F_H = \left\{\left(\frac{9.81 \times 6.88 \times 6.88}{2}\right)\right\} \times 5$$
$$= \mathbf{1160.8\ kN.}$$

The vertical component of the force $F_V = (\gamma)$ {volume of water displaced sector ACB}.

$$F_V = 9.81\{(\text{area of } OAB) - 2\ (\text{area } OZA) \times 5\}$$

$$\text{Area of } OAB = \left(\frac{\pi R^2 (70)}{360}\right)$$

$$= \left(\frac{\pi 6^2 (70)}{360}\right) = 22 \text{ m}^2$$

$$\text{Area of } OZA = \frac{1}{2} (OZ) \times AZ$$

$$OZ = \frac{1}{2}\left(\frac{AZ}{\tan 35°}\right)$$

$$\therefore \quad \text{Area of } OZA = \frac{1}{2}\left(\frac{3.44}{\tan 35°}\right) 3.44$$

$$= 8.45 \text{ m}^2$$

$$F_V = 9.81\{(22) - 2\,(8.45) \times 5\}$$

$$= \mathbf{241.3\ kN}$$

Therefore the resultant force

$$F_R = \sqrt{F_v^2 + F_H^2}$$

$$= \sqrt{241.3^2 + 1160.8^2}$$

$$= \mathbf{1185.6\ kN}$$

Angle of inclination of F_R to the horizontal,

$$\theta = \tan^{-1}\left(\frac{F_V}{F_H}\right)$$

$$= \tan^{-1}\left(\frac{241.3}{1160.8}\right)$$

$$\theta = 11.7°$$

Problem 3.15 *A right circular cylinder blocks water on either side as shown in Fig. 3.26. Determine the net vertical and horizontal forces, the resultant force and its line of action. The cylinder is 2 m long.*

Solution

Forces to the left of CA

The horizontal force on *CDA* = the force on the projected area (*CA* × length of cylinder) in the vertical plane due to water pressure to the left of *CA*.

$$F_{HL} = \left(\gamma \times h \times \frac{h}{2}\right) \times 2$$

$$= \left(9.81 \times 2 \times \frac{2}{2}\right) \times 2$$

$$= 39.24 \text{ kN} \rightarrow$$

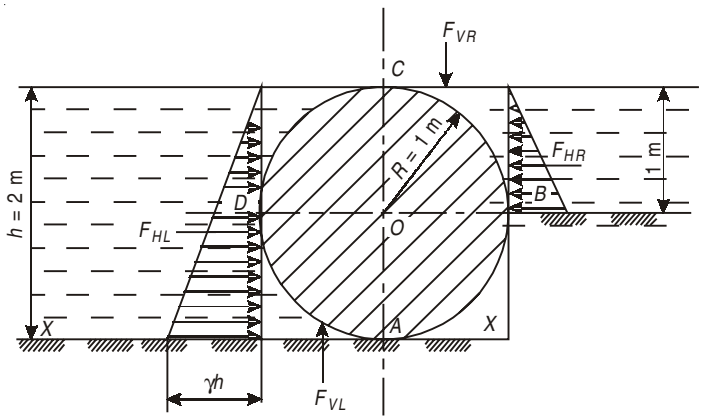


Fig. 3.26

F_{HL} acts at a distance $\frac{2}{3}$ = 0.66 m from the base.

F_{VL} = γ (volume of water displaced by the curved surface *ADC*)

$$F_{VL} = \gamma \left\{ \frac{1}{2} \left(\frac{\pi D^2}{4} \right) 2 \right\}$$

$$= 9.81 \left\{ \frac{1}{2} \left(\frac{\pi 2^2}{4} \right) 2 \right\}$$

$$= 30.8 \text{ kN} \uparrow$$

F_{VL} acts at a distance $\frac{\left\{4\left(\frac{D}{2}\right)\right\}}{3\pi} = \frac{\left\{4\left(\frac{2}{2}\right)\right\}}{3\pi}$ = 0.42 m to the left from *CA*

Forces to right of CA

The horizontal force on *BC* = the force on the projected area (*CO* × length of cylinder) in the vertical plane due to water pressure to the right of *CA*.

$$F_{HR} = \left(\gamma \times h \times \frac{h}{2} \right) \times 2$$

$$= \left(9.81 \times 1 \times \frac{1}{2} \right) \times 2$$

$$= 9.81 \text{ kN} \leftarrow$$

This acts at a distance $\left(\frac{1}{3}\right)$ = 0.33 m from base *zz*

F_{VR} = γ (volume of water displaced by the curved surface *BC*)

$$F_{VR} = \gamma \left\{\frac{1}{4}\left(\frac{\pi D^2}{4}\right)2\right\}$$

$$= 9.81\left\{\frac{1}{4}\left(\frac{\pi 2^2}{4}\right)2\right\}$$

$$= 15.4 \text{ kN} \downarrow$$

F_{VR} acts at a distance $\frac{\left\{4\left(\frac{D}{2}\right)\right\}}{3\pi} = \frac{\left\{4\left(\frac{2}{2}\right)\right\}}{3\pi}$ = 0.42 m to the right from *CA*.

The net force in the horizontal direction,

$$F_H = F_{HL} - F_{HR}$$
$$= 39.24 - 9.81$$
$$= 29.43 \text{ kN} \rightarrow$$

The net force in the vertical direction,

$$F_V = F_{VL} - F_{VR}$$
$$= 30.8 - 15.4$$
$$= 15.4 \text{ kN} \uparrow$$

The resultant force $F_R = \sqrt{F_v^2 + F_H^2}$

$$= \sqrt{15.4^2 + 29.43^2}$$
$$= 33.2 \text{ kN}$$

Direction of the resultant force

$$\theta = \tan^{-1}\left(\frac{F_V}{F_H}\right)$$

$$= \tan^{-1}\left(\frac{15.4}{29.43}\right)$$

$$= 27.6°$$

Location of the resultant force

Assuming F_H acts at a distance *y* from the base *xx* we take moments about *xx*, to determine its distance.

$$F_{HL} \times 0.66 - F_{HR} \times 1.33 = (F_H)\, y$$
$$39.24 \times 0.66 - 9.81 \times 1.33 = (29.43)\, y$$
$$y = 0.436 \text{ m from } xx$$

Problem 3.16 *The retaining wall of a masonry dam shown in Fig. 3.27 is a semi-elliptical curve satisfying the equation $y = x^2/2$. It retains water up to height of 16 m from the base. Determine the magnitude and direction of the resultant force per metre length of the dam. Also determine the point at which the resultant force cuts the base from A.*

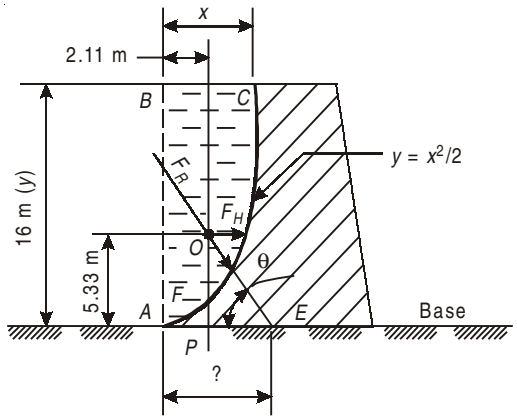


Fig. 3.27

Solution

Since $y = \dfrac{x^2}{2}$

$X = \sqrt{2y} = \sqrt{(2 \times 16)}$

$= 5.65$ m

Horizontal force $F_H = \left(\gamma \times h \times \dfrac{h}{2}\right) \times 1$

$= \left(9.81 \times 16 \times \dfrac{16}{2}\right) \times 1$

$= 1255.6$ kN

F_H acts at a distance $\dfrac{1}{3}(16) = 5.33$ m from base.

Vertical force $F_V = \gamma \left\{\dfrac{2}{3}(5.65 \times 16) \times 1\right\}$

$= 591.2$ kN

F_V acts through the center of gravity of the area $ABC = \left(\dfrac{3}{8}\right)X = \left(\dfrac{3}{8}\right) 5.65 =$ 2.11 m from AB.

The resultant force $F_R = \sqrt{F_v^2 + F_H^2}$

$$= \sqrt{591.2^2 + 1255.6^2}$$

$$= 1387.8 \text{ kN}$$

Angle of inclination of F_R to the horizontal,

$$\theta = \tan^{-1}\left(\frac{F_V}{F_H}\right)$$

$$= \tan^{-1}\left(\frac{591.2}{1255.6}\right)$$

$$= 25°$$

The point of intersection of F_R with the base is obtained by considering the triangle *PEO*.

$$\tan 25° = \frac{PO}{PE}$$

Therefore $$PE = \frac{PO}{\tan 25°}$$

$$= \frac{5.33}{0.466}$$

$$= 11.43 \text{ m}$$

$$AE = AP + PE$$

$$= 11.43 + 2.11$$

$$= 13.54 \text{ m.}$$

F_R intersects the base at a distance of 13.54 m from *A*

Problem 3.17 *The dam shown in Fig. 3.28 has a curved profile which satisfies the relation $y = x^2/4$. It retains water to a depth of 9 m. Considering the length of the dam as unity determine the magnitude and direction of the resultant force.*

Solution

Since $$y = \frac{x^2}{4}$$

$$X = \sqrt{4y} = 2\sqrt{y}$$

Horizontal force $$F_H = \left(\gamma \times h \times \frac{h}{2}\right) \times 1$$

$$= \left(9.81 \times 9 \times \frac{9}{2}\right) \times 1$$

$$= 397.3 \text{ kN}$$

F_H acts at a distance $\frac{1}{3}(9)$ = 3 m from base.

Vertical force F_V = γ {volume of water contained in ABC}

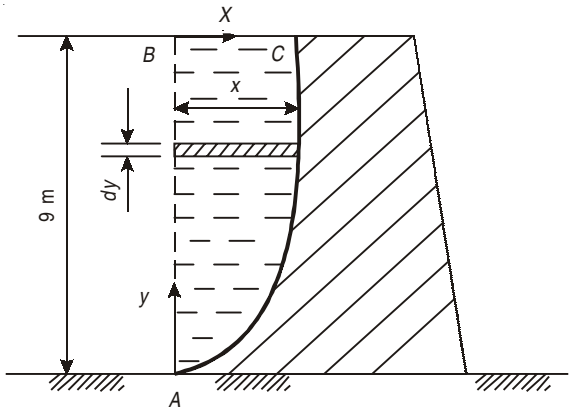


Fig. 3.28

Since the width of the dam is taken as unity, the volume of a small strip of the dam of width x and thickness $dy = x\,(dy)$. If we integrate this volume between limits 0 to 9 we can obtain the total volume of water within the area ABC.

$$F_V = \gamma \int_o^9 x\ (dy)$$

Since $x = 2\sqrt{y} = 2\ (y)^{\frac{1}{2}}$

$$F_V = \gamma \int_0^9 2\sqrt{y}\ (dy)$$

$$F_V = \gamma\,(2) \int_0^9 (y)^{\frac{1}{2}}\ (dy)$$

$$= \gamma\,(2) \left\{\frac{(y)^{1/2}}{3/2}\right\}_0^9$$

$$F_V = (9.81)\,(2) \left(\frac{2}{3}\right)\left[9^{3/2}\right] = 353 \text{ kN}$$

The resultant force $F_R = \sqrt{F_v^2 + F_H^2}$

$= \sqrt{353^2 + 397.3^2}$

$= 531$ kN

Angle of inclination of F_R to the horizontal,

$$\theta = \tan^{-1}\left(\frac{F_V}{F_H}\right)$$

$$= \tan^{-1}\left(\frac{353}{397.3}\right)$$

$= 41.6°$

QUESTIONS BANK

Section A
Review Questions

1. Define resultant force and center of pressure.
2. State the conditions necessary and sufficient for the center of pressure to coincide with the axis center of gravity.
3. How do you determine the inclination of the resultant force w.r.t. the horizontal?
4. Show that in case of static fluids the intensity of pressure is proportional to height as long as the plane remains horizontal.
5. Derive an expression for total pressure and center of pressure for an inclined plane surface submerged in a liquid.
6. Prove that for lock gates the reaction force = $\frac{F}{2}$ sine θ where F is the resultant force due water pressure and θ is the angle of inclination of the gate to the line passing through the hinges in the horizontal plane.
7. Differentiate between sluice gates and lock gates.

Section B
Multiple Choice Questions

Choose the right answer

1. If you are asked to determine the hydrostatic force on one face of a plane submerged surface which formula you would choose,

   (a) γAh (b) $\gamma A\bar{y}$

   (c) $\gamma A/h$ (d) γAC_g

2. The resultant force always acts through the point of
   (a) center of gravity (b) centroidal axis
   (c) center of pressure (d) None of these
3. Hydrostatic law is not applicable to compressible fluids because
   (a) their density is lower (b) their density varies
   (c) their viscosity varies with temperature
   (d) None of these
4. The specific weight of water at 15 °C is 9.81 kN / m^3. The specific weight of air at the same temperature is
   (a) 6 kN/m^3 (b) 8 kN/m^3
   (c) 12 kN/m^3 (d) None of these
5. 10^5 Pascal is equivalent to
   (a) 15 m of water (b) 12.2 meters of water
   (c) 10.36 metres of water (d) None of these
6. The pressure at the bottom of a 1 meter column of water is
   (a) 9.81 N/m^2 (b) 9.91 N/m^2
   (c) 8.81 N/m^2 (d) None of these
7. Piezometric head in metres of water for a confined gas at a pressure of 3×10^5 N/m^2 is
   (a) 29 (b) 30.58
   (c) 22.58 (d) None of these
8. The area of the force distribution diagram of a vertical submerged plane surface is 122.625 kN. The height of water contained by the submerged surface is
   (a) 7 m (b) 5.5 m
   (c) 5 m (d) None of these

Section C
Problems

1. A gate valve controls the flow of oil through a pipe of inside diameter 0.5 m. Oil exerts a pressure 3×10^5 Pa at the center of the gate. If the specific weight of oil is 8.4 kN/m^3 determine resultant force on the gate valve and the center of pressure.
   (*Ans.* 58.9 kN and C_p coincides with the center of the pipe)
2. Gate *AB* (Fig P-2) remains closed against the resultant force due to the combined effect of its self weight and an additional weight together equal to 500 kN acting through its axis of center of gravity. The length *AB* of the gate = 5 m and the width 7 m. Find the height h at which the gate begins to open. (*Ans.* 7.64 m)

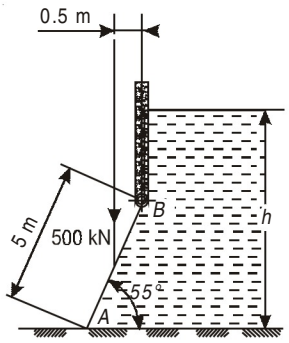


Fig. P-2

3. Determine the upward thrust F on the semi-cylindrical top of the container shown in Fig. P-3. The container is full of oil under a pressure of 2×10^5 Pa. (*Ans.* 1339 kN)

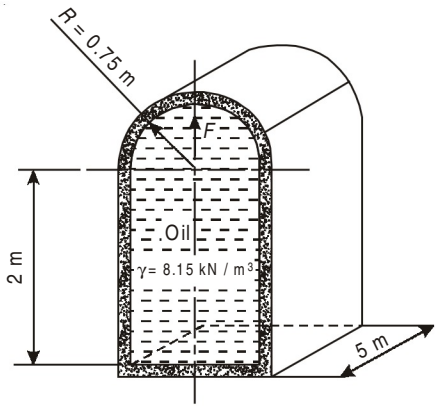


Fig. P-3

4. A container 2 m wide × 4 m long holds oil (γ = 8.5 kN/m^3) to a depth of 1.5 m. Determine the force on the bottom plate if the container is (1) accelerated vertically upwards @ 6 m/sec, (2) when the acceleration ceases and container has attained uniform velocity, (3) when the container is accelerated at the same rate but in the vertically downward direction. (*Ans.* 164 kN, 102 kN, 39.6 kN)
5. A rectangular closed container 2 m in length × 0.5 m wide × 1 m deep holds a certain liquid (sg = 0.85) to a depth 0.7 metres. It is carried in the cargo bay of an airplane. Determine the angle of inclination of the free liquid surface in the container and the pressure difference between the trailing edge and the leading edge when the plane accelerates on the ground @ 6 m/sec^2 to take off speed. Also determine the inclination

of the free liquid surface to the horizontal after the airplane takes off at an angle 40° to the horizontal. The acceleration during this phase is 10m/sec^2. Assume that the acceleration is along the length of the container. (*Ans.* $\theta = 31.45°$, 10.17 kN/m^2, $\theta = 25°$)

6. An open cylindrical container of $\varphi = 30$ cm and length 50 cm holds water to a depth of 40 cm. Determine the RPM of the container at which the water just touches the top edge of the container and the difference in forces due to rotation on the bottom plate.

(*Ans.* RPM = 126, $F = 0.20$ kN)

Answers to Multiple Choice Questions

1. (b) **2.** (c) **3.** (b) **4.** (c)
5. (c) **6.** (d) **7.** (b) **8.** (c)

Chapter 4

Buoyancy and Floatation

In Chapter 3, we discussed the forces acting on an object fully or partially submerged in any liquid. We didn't worry about whether the object was freely floating on its own while it lay submerged or was held in place by an external force. In this chapter we are more concerned with the forces that keep the bodies floating when they are partially or fully submerged. Obviously therefore we would be interested in the equilibrium of floating bodies and consequently their metacenter and the metacentric height which determines the degree of their stability.

The first principle of this chapter is the Archimedes' principle about which the readers are already familiar. Let us refresh our memories by recalling the principle here. '*A body whether floating or submerged, experiences an upward thrust which is proportional to the volume of the fluid displaced*'. This upward thrust is more familiarly known as '*Buoyancy*'. Mathematically speaking, Archimedes' principle simply means,

$$F_b = (\gamma \times v) \qquad ...(4.1)$$

Here γ the proportionality constant represents the specific weight of fluid, v is volume of fluid displaced and F_b is the buoyant force or the upward thrust. This upward thrust is experienced by both floating and sinking bodies.

4.1 STABILITY OF FLOATING AND SUBMERGED BODIES

The discussions hitherto were confined to the forces that keep a body floating or submerged. We now go further to understand the conditions governing their stability which is characterized by their ability to return to their original position when subjected to a turning moment about their horizontal axis. Based on this a floating body can be described as being in a state of *stable, unstable* or *neutral equilibrium.* If a body moves further away from the displaced position and

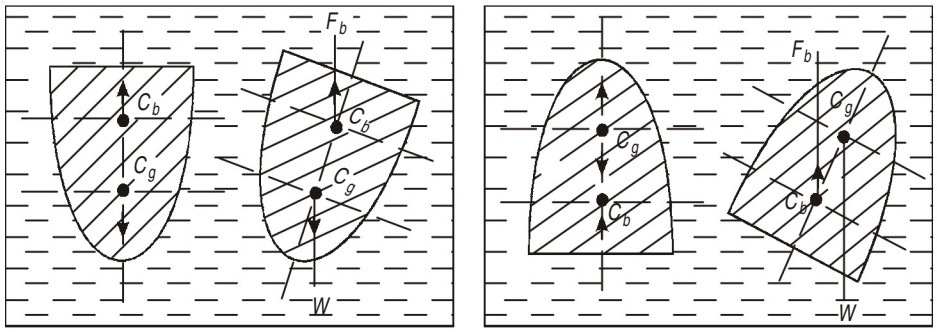
exhibits a tendency to topple it is said to be unstable whereas if it remains in the displaced position unable to return to its original position it is said to be *neutrally stable. A neutrally stable condition exists in a body when its center of buoyancy coincides with its point of center of gravity.* Whereas buoyant forces keep the body floating, a body in stable equilibrium will resist destabilizing forces such as high wind, ocean current and manouvering forces that occur while in motion.

Generally, a body too heavy to float sinks. A body can be considered too heavy if the weight of the liquid it displaces is less than its own weight. If the center of buoyancy is above the center of gravity it may safely be assumed that the body floats. If it is otherwise, we may have to apply other criteria to determine the stability especially in case of floating bodies.

Different criteria for stability apply for floating and fully submerged bodies. This is because in case of fully immersed bodies the volume of fluid displaced and the shape of this volume both remain unaltered even when the orientation of the body changes with respect to the horizontal axis. The position of center of buoyancy therefore remains unaltered. In case of partially submerged or floating bodies though the volume of fluid displaced remains the same, (F_B= W) when the orientation of the body changes the shape of this volume changes and hence the center of buoyancy shifts. Nevertheless both require that the turning moments due to external forces are adequately balanced by internal counter moments.

The condition for stability of *a fully submerged body is that its point of center of gravity must be below the center of buoyancy.*

Figure 4.1 (a) illustrates how when this condition is met the internal counter moments due to the weight of the object and the force of buoyancy together act as complimentary forces and resist the external turning moment. Figure 4.1 (b) illustrates how these forces aid the external force by producing moments in the same direction as the external force, if this condition is reversed.



(a) Condition for stability (b) Condition for instability

Fig. 4.1 *Fully submerged bodies*

The condition for stability of a partially submerged or floating *body is that its point of center of gravity must be below the metacenter.*

We have already stated that when a floating body tilts due to a turning moment the geometry of the displaced volume changes and hence the center of buoyancy shifts to a new position away from the axis of center of gravity of the body. *Metacenter (M) is a point on the centroidal axis of the body through which the force of buoyancy acts from its new location.*

As long as a floating body remains horizontal all of these points, the metacenter, the point of center of gravity and the center of buoyancy remain on the centroidal axis. Refer Fig. 4.2 (a, b) below. *Metacentric height* (*Z*) is the distance from the point of center of gravity to metacenter and can be determined either analytically or experimentally. *Metacentric height* enables us to determine the relative stability of one floating body over the other. The metacentric height should not be too large as this would increase frequency of rocking of the body whereas if it is too small the factor of safety required for stability would decrease.

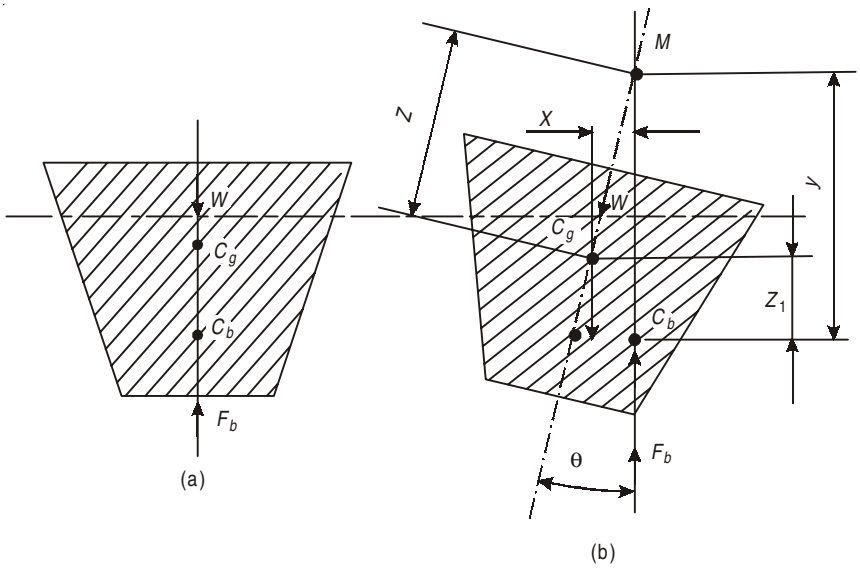


Fig. 4.2

The metacentric height $Z = (y \pm Z_1)$ where y is the distance from the center of buoyancy to the metacenter and is given by $y = \frac{I}{V_D}$, where I is the second moment of area and V_D is the displaced volume. Negative sign for Z_1 applies if C_G lies above C_B and the positive sign if C_G lies below C_B.

While deriving the above equation analytically it will be assumed that the angle of tilt is small so that the chord length x instead of the arc length is considered. Hence the distance $x = M\,C_G \times \theta$ radians.

Determination of Metacentric height (*Analytical method*)

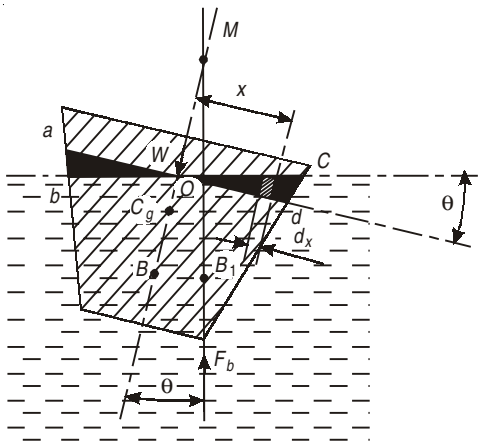


Fig. 4.3

When a body is subjected to a turning moment a counter moment is required to restore the body to its previous position. In the example shown above the center of buoyancy shifts from position B to B_1 due to the angular displacement and consequently the line of action of the force of buoyancy F_B forms a counter-clockwise moment with the centroidal axis of the body to restore the body to its equilibrium position.

When the body was in the equilibrium position F_B was acting through the centroidal axis and hence the fluid volume was equal on either side. Now due to the rotational motion there is loss of fluid volume equal to the volume of the triangular prism (*OAB*) to the left of the centroidal axis and a gain in fluid volume equal to the volume of the triangular prism (*OCD*) to the right. This displacement by itself acts as a balancing moment to the moment generated due to the shift of the C_B.

Consider a small area δA on the triangular prism at a distance x from o. The volume displaced $\delta V = (x \times \theta \times \delta A)$.

The upward force δF due to buoyancy corresponding to this swept volume,

$$\delta F = (w \times \delta V)$$
$$= w\,(x \times \theta \times \delta A).$$

The moment of this buoyant force about o $x\ \delta F = w\,(x^2 \times \theta \times \delta A)$.

The moment caused by the entire prism *OCD* about *O*

$$\int x\ \delta F = \int w\,(x^2 \times \theta \times \delta A)$$
$$= w\theta \int (x^2 \times \delta A)$$

But $\int (x^2 \times \delta A) = I$, the second moment of area or the moment of inertia about o

Hence, $\int x\ \delta F = (w\theta)\ I$ ...(4.2)

Now the moment due to shift of C_B from initial position B to $B_1 = F_B \times BB_1$. But

$$BB_1 = (BM)\ \theta.$$

Therefore, the moment due to shift of $C_B = F_B \times (BM)\theta$ ...(4.3)

This is equal to the balancing moment $\int x\ \delta F$

Therefore, $\int x\ \delta F = F_B\ (\text{BM})\ \theta$

or $\int x\ \delta F = (w \times v)\ (\text{BM})\ \theta$

or $(w\theta)\ I = (w \times v)\ (\text{BM})\ \theta$ from Eq. 4.4

or $$\text{BM} = \frac{I}{V} \text{ or Metacentric height}$$

$$M = \frac{I}{V} - BC_G \qquad ...(4.4)$$

I is the least area moment of inertia of the horizontal section at the surface of the liquid.

4.1.2 Determination of Metacentric height (Experimental Method)

In this method (Fig 4.4) an object of known weight F is placed on the centroidal axis of a floating body and is gradually moved from its initial position through a distance x This shift creates an unbalance in the body which hither-

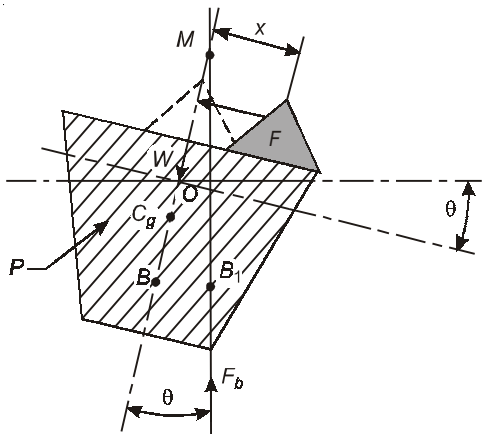


Fig. 4.4

to was under stable equilibrium. Consequently the body is displaced through an angle θ. In effect an overturning clockwise moment = ($F \times x$) is produced which must be balanced by a counter clockwise moment. Thus,

$$(F \times x) = W \times C_G M \tan \theta \qquad ...(4.5)$$

But it has already been mentioned that for small angles of θ, tan θ = sine θ = θ radians.

Thus metacentric height $C_G M = \dfrac{(F \times x)}{W \theta C_G}$ ...(4.6)

Oscillation of a floating body

For a body undergoing a simple harmonic motion the time period is given by,

$$t = 2\pi \sqrt{\frac{x}{a}}$$

where, x is the displacement and a is the acceleration. If it is assumed that the body is rolling on the horizontal axis then the displacement of the floating body is through an angle θ. Let the angular acceleration due to the displacement be a.

This displacement produces a counterclockwise torque $W \times (GM)\ \theta$ which acts as the restoring moment. W is the weight of the body, and GM is the distance from the center of gravity to metacenter.

We know that, mass moment of inertia (I) × angular acceleration = torque (T).

Therefore, $a = \dfrac{T}{I}$

but $T = W \times (GM)\ \theta$

and $I = \left(\dfrac{W}{g}\right) k^2$

Hence, $a = \dfrac{W \times (GM)\ \theta}{\left(\dfrac{W}{g}\right) k^2} = \dfrac{(GM)\ \theta g}{k^2}$.

Therefore, the time period $t = 2\pi \sqrt{\dfrac{\theta}{\dfrac{(GM)\ \theta g}{k^2}}}$

or $t = 2\pi \sqrt{\dfrac{k^2}{(GM)g}}$ ...(4.7)

TYPICAL PROBLEMS AND SOLUTIONS

Problem 4.1 *A steel ball 5 mm in diameter falls freely in water. Calculate the net downward force acting on the steel ball. Steel weighs = 76.9 × 10 ^{3} N/m^3. Neglect the viscous drag.*

Solution It is assumed that the fall is entirely due to gravity since no other forces are indicated. The forces resisting the free fall are the force due to buoyancy and the force due to viscous drag, both of which are acting in a direction which opposes the downward force due to gravity. The net downward force is the force due to gravity minus the magnitude of the two opposing forces.

The downward force F_w = [Volume of the steel ball (V) × specific weight of the steel ball γ)

$$V = \left(\frac{\pi}{6} \times D^3\right)$$

where,

D is the diameter of the ball = (5×10^{-3}) m

$V = \{0.523\ (5 \times 10^{-3})^3\} = 6.5 \times 10^{-8}$ m^3

$\gamma = 76.9 \times 10^3$ N/m^3

$F_w = (6.5 \times 10^{-8})\ (76.9 \times 10^3)$

$= 4.99 \times 10^{-3}$ N ↓

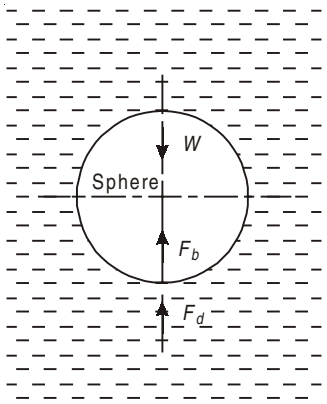


Fig. 4.5

F_D the viscous drag is assumed equal to zero.

Force due to buoyancy is lifting the ball up and is equal to F_B = (Specific weight of water) × (Volume of fluid displaced).

Specific weight = 9.81 kN/m^3 or, 9810 N/m^3

Volume of fluid displaced = the volume of the sphere

$= (6.5 \times 10^{-8})$ m^3

$F_B = 9810 \times (6.5 \times 10^{-8})$

$= 6.4 \times 10^{-4}$ N ↑

$$\text{The net force} = F_w - F_D - F_B$$
$$(4.99 \times 10^{-3}) - 0 - (6.4 \times 10^{-4}) = -4.35 \times 10^{-3}\ \text{N} \downarrow$$

In the above example by neglecting the viscous drag, we have assumed that the body is at rest in the liquid and thus were able to apply the principle of static equilibrium, $\Sigma F_V = 0$. (F_V is the force in the vertical direction both upward and downward.) Upward forces are considered as positive and downward forces as negative. Forces in the horizontal direction normally cancel each other and hence are not considered. While considering the vertical forces the possibility that there could be an external force in the vertical direction should not be lost sight of and must be eliminated from the equation $\Sigma F_V = 0$ only after verification.

Primary condition for floating is that the specific weight of the object must be less than the specific weight of the liquid. If this condition is reversed the object will tend to sink. An object will neither sink nor float but lay submerged in a certain position if the specific weights are equal. This is termed *neutral buoyancy*. A submarine is an example of neutral buoyancy.

A body when immersed in a liquid suffers an apparent loss of weight equal to the volume of the liquid it displaces. This is direct interpretation of the Archimedes' principle which has enabled us to determine the volume of even irregularly shaped objects for which no geometrical solution is possible. By immersing such objects in a liquid we can determine their apparent weight loss and compare it with the weight measured before immersion. The principle of buoyancy will now enable us to determine the volume of the irregular body.

Measure and record the weight of the irregular body using a spring balance. Assume it to be $F_W N \downarrow$. Now weigh the same body, using the same balance while holding it immersed in any liquid, say water. Let it be $F_X N \uparrow$. Let $F_B \uparrow$ be the buoyant force.

Let us now apply the principle of static equilibrium, $\Sigma F_V = 0$.

$$F_B + F_X - F_W = 0$$
$$(\gamma V) + F_X - F_W = 0.$$

Hence V the volume of the irregular body is given by,

$$V = \frac{(F_W - F_X)}{\gamma} \qquad \ldots (4.8)$$

Buoyant force acts through the *center of buoyancy* the same way the resultant force acts through the center of pressure. Buoyant force is a theoretical or imaginary force. What is more practical is the amount of fluid that an immersed body displaces and the force concentrated in that fluid due to its weight. Therefore the *center of buoyancy is always the point of center of gravity of the displaced fluid*.

Problem 4.2 *A wooden 10 cm wide cubical block floats in water. Determine the quantity of water displaced and the point of center of buoyancy. Specific gravity of wood = 0.6.*

Solution

$$\text{Volume of the cubical block} = 10 \times 10 \times 10$$
$$= 1000\ \text{cm}^3 = 0.001\ \text{m}^3$$

Weight of the cubical block F_W = 0.6 (9810) × 0.001
= 5.886 N ↓

Since the cubical block floats in water, it is only partially submerged. Let the submerged height be h. This means (10 –h) cm is above the water surface.

Volume of water displaced V_D = (h^3) m^3
Weight of water displaced = Buoyant force F_B
= $(h^3) \times \gamma_w$
For stability F_W = F_B
5.886 = $(h^3) \times \gamma_w$
= $(h^3) \times 9810$

Therefore $h = 3\sqrt{\frac{(5886)}{9810}} = 0.6$ m

Volume of water displaced = $(h^3) = (0.6)^3$
= 0.216 m^3

Center of buoyancy = $\frac{h}{2}$

= $\frac{0.216}{2} = 0.108$ m

Problem 4.3 *Determine the force required to hold the cube in Problem (2) completely submerged in glycerine, sg = 1.25.*

Solution

Apply the principle of static equilibrium, $\Sigma F_V = 0$.

$\uparrow F_B - \downarrow F_X - \downarrow F_W = 0$

Since F_X is the external force required to be determined,

$\downarrow F_X = \uparrow F_B - \downarrow F_W$ …(4.9)

F_W = 5.886 N

$F_B = (\gamma \times V)$
= (1.25 × 9810) × 0.001
= 12.26 N

(Here V is the total volume of the cube)

$\downarrow F_X$ = 12.26 – 5.886
= 6.37 N is required to hold the block completely submerged in glycerine.

Problem 4.4 *A log of wood 0.5 m in diameter and 6 m long floats in water (Ref. Fig. 4.5). Determine the number of men the log can carry, each weighing 700 N, with the log immersed in water up to its diameter. The density of wood is 6570 N/m^3.*

Solution

Weight of the log $W_L = \{0.7854\ (0.5)^2 \times 6\ \text{m}\} \times 6570$
= 7740 N↓

Total weight on the log including its self weight

$W_T = 7740 + (n \times 700)\downarrow$

where n is the number men the log can carry.

Weight of water displaced $W_W = \gamma_w (V_W)$ where, γ_w and V_W are weight density of water and volume of water displaced respectively.

$$W_W = 1000 \times 9.81\{0.7854\ (0.5)^2 \times 6\ \text{m}\}$$
$$= 11{,}557\ \text{N}\uparrow$$

Therefore, $11557\uparrow - [7740 + (n \times 700)]\downarrow = 0$

$$11557 - 7740 - (n \times 700) = 0$$
$$3817 = (n \times 700)$$

or $$\text{Number of men } (n) = \frac{3817}{700} = 5.45.\ \text{Say 5 men.}$$

Problem 4.5 *It is required to hold a steel rod of ϕ 15 mm and length 300 mm in equilibrium in water while it lay submerged horizontally by attaching a light weight foam through strings. Determine the volume of the foam required. Take* $\gamma_{Foam} = 700\ N/m^3$ *and* $\gamma_{Steel} = 76.9 \times 10^3\ N/m^3$.

Solution

Weight of steel $W_S = 76.9 \times 10^3\ \{0.7854 \times (0.015)^2 \times 0.3\}$
$= 4\text{N}\downarrow$

Weight of foam $W_F = \gamma_{\text{Foam}} \times V_F$
$= 700 \times V_F\downarrow$

Buoyant force due to steel $F_{BS} = 1000 \times 9.81\ \{0.7854 \times (0.015)^2 \times 0.3\}$
$= 0.52\ \text{N}\uparrow$

Buoyant force due to foam $F_{BF} = 1000 \times 9.81\ \{V_F\} = 9810\ V_F\ \text{N}\uparrow$

For equilibrium $\Sigma F_V = 0$

Therefore,

$$9810\ V_F + 0.52 - 700 \times V_F - 4 = 0$$
$$9810\ V_F - 700 \times V_F = 4 - 0.52$$
$$V_F(9810 - 700) = 3.48$$
$$V_F(9110) = 3.48$$

or $$V_F = \frac{3.48}{9110} = 3.8 \times 10^{-4}\ \text{m}^3$$

Problem 4.6 *An iceberg floats in sea water having a density of 1025 kg* m^{-3}. *Determine the density of the iceberg if the volume of the iceberg is 6000* m^3 *and if 30% of this volume is above the surface of water.*

Solution

Weight of the iceberg $W_I = (\rho \times 9.81) \times 6000 = \rho\ (58860)\downarrow$

Buoyant force F_B = weight of water displaced
$= (\rho_W \times 9.81)\ \{6000 - 6000 \times 0.3\}$
$F_B = (1025 \times 9.81)\ \{6000 - 6000 \times 0.3\}$
$= 42232050\ \text{N}\uparrow$

For equilibrium $W_I = F_B$

Therefore, $\rho\ (58860) = 42232050$

Density of Iceberg $\rho = 717.5\ \text{kg m}^3$

Problem 4.7 *A float shown in Fig. 4.6 regulates the flow of water to a cistern. The float is a spherical ball valve of diameter 0.160 m. The float and its linkages are so designed as to stop the flow completely by closing the water-inlet with a force of 12 N as soon as the water reaches a level of 0.25 m from the hinge o. Determine the weight of the float required. Neglect friction and weight of the linkages.*

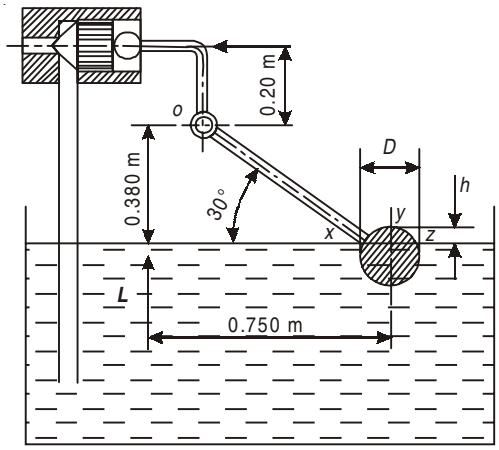


Fig. 4.6

Solution The center of the spherical float is at a distance of $(0.380 + L)$ meters from the hinge o.

Therefore, $$\tan 30° = \frac{(0.380 + L)}{0.750}$$

$$0.577\ (0.750) = 0.380 + L$$

$$L = 0.053 \text{ m}$$

The height h of the spherical segment xyz projecting above the water surface,

$$h = D - \left(\frac{L + D}{2}\right)$$

where D is the diameter of the float.

$$h = 0.160 - 0.053 - \frac{0.160}{2} = 0.027 \text{ m}$$

Volume of the spherical segment $xyz = V_1$

$$= \pi h^2 \left\{\frac{r - h}{3}\right\}$$

where $$r = \frac{D}{2}$$

Therefore, $$V_1 = \pi\, 0.027^2 \left\{\frac{0.080 - 0.027}{3}\right\}$$

$$= 1.62 \times 10^{-4} \text{ m}^3$$

Volume of the spherical float $V = \dfrac{\pi D^3}{6}$

$$= \frac{\pi(0.160^3)}{6}$$

$$= 2.14 \times 10^{-3}\ \text{m}^3$$

Volume of the float immersed in water

$$V_2 = V - V_1$$

$$= (2.14 \times 10^{-3}) - (1.62 \times 10^{-4})$$

$$V_2 = 1.97 \times 10^{-3}\ \text{m}^3$$

Buoyant force $F_B = \gamma_W \times V_2$

$$= 9810 \times 1.97 \times 10^{-3}$$

$$= 19.32\ \text{N}\uparrow$$

Downward vertical force = weight of the float itself = $W\downarrow$

Therefore, $(19.32\uparrow - W\downarrow)$ is the net upward force

Now taking moment about the hinge *o*

$$(19.32\uparrow - W\downarrow) \times 0.750 = 12 \times 0.2$$

$$(19.32\uparrow - W\downarrow) = \frac{(12 \times 0.2)}{0.750} = 3.2$$

$$W = 19.32 - 3.2$$

$$= 16.12\ \text{N} = \text{Weight of the float.}$$

Problem 4.8 *A ball float valve Fig 4.7 regulates the flow to tank T through linkages by opening or closing the inlet port which has a diameter = 15 mm. Water is admitted from an overhead tank situated 6 m above the inlet port. Determine the volume of the ball that must be submerged to keep the port closed.*

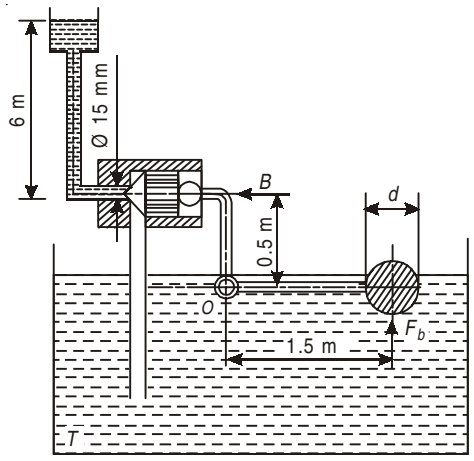


Fig. 4.7

Solution Diameter of the port that needs to be closed or opened

$$= 15 \text{ mm} = 0.015 \text{ m}$$

Force on link at point A, due to pressure head $= \gamma \times h \times A$ where A is the area of the port.

Therefore, $$F = 9810 \times 0.6 \times 0.7854\ (015)^2$$
$$= 1.040 \text{ N}$$

Now we take moments about O

$$F_B \times 1.5 = 1.040 \times 0.5$$
$$F_B = \frac{(1.040 \times 0.5)}{1.5}$$
$$= 0.346 \text{ N}\uparrow$$
$$\gamma \times V = 0.346$$

or $$V = \frac{0.346}{9810}$$
$$= 3.52 \times 10^{-5} \text{ m}^3$$

Problem 4.9 *A metallic object floats at the water and oil interface, water being on top of oil. If 40 percent of the object is submerged in oil (sg = 0.9) and the rest in water, determine the density of the metallic object.*

Solution

When an object is submerged at the interface of two immiscible liquids the force due to buoyancy is the sum of the buoyant forces individually exerted by each liquid.

$$F_B = F_{BW} + F_{BO} = (\gamma_W \times V_W) + (\gamma_O \times V_O)$$
$$F_B = (9810 \times V_W) + \{(9810 \times 0.9) \times V_O\}$$
$$F_B = (9810 \times 0.6\ V) + (8829 \times 0.4\ V)$$

(Note the object is submerged 40 % in oil and 60 % in water)

$$F_B = (5886\ V) + (3531.6\ V)$$
$$F_B = 9417.6\ V$$

F_B must be equal to the weight of the object $W = \gamma \times V$

$$\gamma \times V = 9417.6 \text{ V}$$

or $$\rho \times g = 9417.6$$

or $$\rho = \frac{9417.6}{9.81}$$
$$= 960 \text{ kg/m}^3$$

Problem 4.10 *A rectangular container of length 5 m, breadth 3 m and height 3.5 m floats on fresh water. The total weight of the container is 200 kN. Determine the stability of the container. Assume that the axis of center of gravity is at the center of the cross section on the vertical axis.*

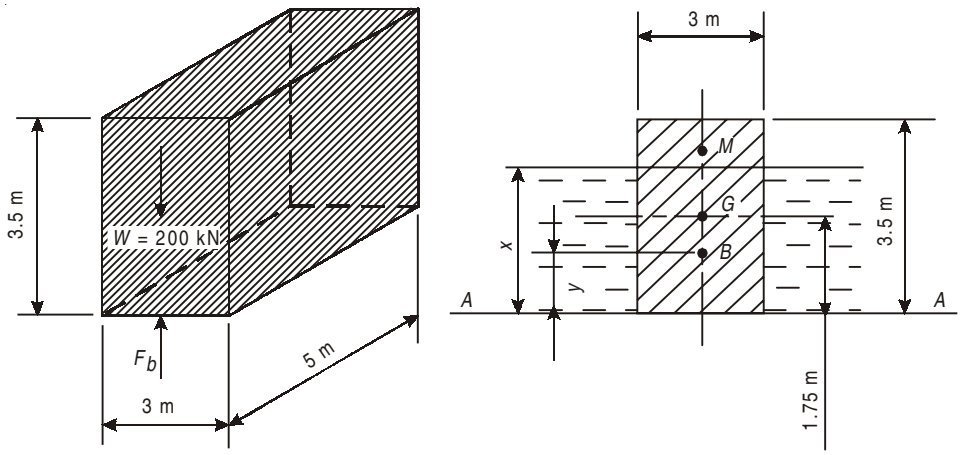


Fig. 4.8

Solution Apply the principle of static equilibrium, $\Sigma F_V = 0$ and determine how deep the container is submerged; the distance h

$$\uparrow F_B - \downarrow F_W = 0$$

or
$$\uparrow F_B = \downarrow F_W$$
$$\uparrow F_B = \gamma V_D = \gamma (L \times B \times h)$$
$$\downarrow F_W = \gamma (L \times B \times h)$$
$$200 = 9.81 (5 \times 3 \times h)$$

Therefore
$$h = 1.36 \text{ m}$$

and
$$y = \frac{1.36}{2} = 0.68 \text{ m}$$

For stability we must confirm that C_G lies below the metacenter M
From Eq. 4.2

$$BM = \frac{I}{V}$$

I is the least second moment of area of the section which is obtained by considering breadth 5 m and depth as 3 m.

Thus
$$I = \frac{(5 \times 3^3)}{12} = 11.25 \text{ m}^4$$
$$V = \text{the displaced volume}$$
$$= (L \times B \times h)$$
$$= (5 \times 3 \times 1.36) = 20.4 \text{ m}^3$$

Therefore,
$$BM = \frac{11.25}{20.4} = 0.55 \text{ m}$$
$$BM + y = 0.55 + 0.68 = 1.23 \text{ m}$$

Since the value of $BM + Y$ (1.23 m) is < than the value of G (1.75 m) as measured from reference axis AA the container is ***unstable*** because the meta-center lies below the center of gravity.

Problem 4.11 *A rectangular block 5 m wide and 25 m long weighs 2000 kN. Its center of gravity is 0.3 m above the water surface. If the block tilts by an angle $\theta = 12°$, find the metacentric height and the restoring couple.*

Solution First determine the depth of immersed surface (h) of the block.

The buoyant force = $F_B = (\gamma_W \times V_W) = 9.81 \times L \times B \times h$

$$F_B = 9.81 \times 25 \times 5 \times h$$

$$h = \frac{FB}{9.81} \times 25 \times 5$$

$$h = \frac{2000}{9.81} \times 25 \times 5 = 1.63 \text{ m}$$

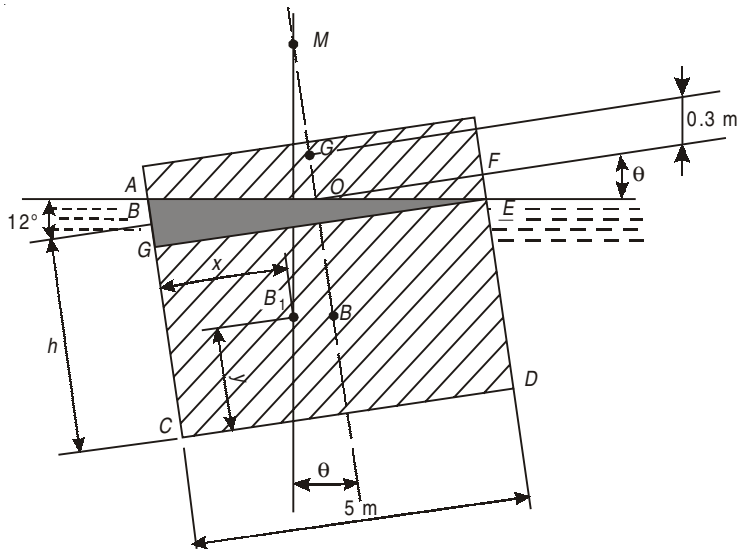


Fig. 4.9

$$\tan \theta = \frac{AB}{BO}$$

$$= \frac{AB}{2.5}$$

Therefore $$AB = 2.5 \tan 12°$$

$$= 0.53 \text{ m}$$

We have to determine the coordinates x and y for the new position of center of buoyancy. This is carried out by taking moments AC and CD

$$x = \frac{\text{Area of rectangle } GCDE \times \text{its moment about } AC + \text{Area of triangle } AGE \times \text{its moment about } AC}{\text{Area of rectangle } BFCD}$$

$$x = \frac{(ED \times CD)\frac{CD}{2} + \frac{1}{2}(AG \times GE)\left(\frac{GE}{3}\right)}{(h \times W)}$$

here
$$ED = h - BG$$
$$= 1.63 - 0.53$$
$$= 1.10 \text{ m}$$

$$x = \frac{(1.10 \times 5)\frac{5}{2} + \frac{1}{2}\left\{(2 \times 0.53)\,5\left(\frac{5}{3}\right)\right\}}{(1.63 \times 5)}$$

$$= 2.17 \text{ m}$$

$$y = \frac{\text{Area of rectangle } GCDE \times \text{its moment about } CD + \text{Area of triangle } AGE \times \text{its moment about } CD}{\text{Area of rectangle } BFCD}$$

$$y = \frac{(ED \times CD)\,\frac{ED}{2} + \frac{1}{2}(GE \times AG)\left(GC + \frac{AG}{3}\right)}{(h \times W)}$$

here
$$ED = h - BG$$
$$= 1.63 - 0.53$$
$$= 1.10 \text{ m}$$

$$y = \frac{(1.10 \times 5)\frac{1.10}{2} + \frac{1}{2}\left\{5 \times 1.06\left(1.63 + \frac{1.06}{3}\right)\right\}}{(1.63 \times 5)}$$

$$= 1.014 \text{ m}$$

Triangles ABO and MB_1B are similar.

Therefore,
$$\frac{AB}{BO} = \frac{B_1B}{MB}$$

$$\frac{0.53}{2.5} = \frac{(2.5 - 2.17)}{MB}$$

Therefore,
$$MB = 2.08 \text{ m}$$

G from base
$$G = h + 0.3 \text{ (from base)}$$
$$= 1.63 + 0.3$$
$$= 1.93 \text{ m}$$
$$GB = (1.93 - y)$$
$$= 1.93 - 1.014$$
$$= 0.916 \text{ m}$$

Metacentric height $MG = MB - GB$
$= 2.08 - 0.916$
$= 1.164$ m

Distance to metacenter from base
$= h + GO + MG$
$= 1.63 + 0.3 + 1.164$
$= 3.094$ m

Distance to center of gravity G from base
$= h + OG$
$= 1.63 + 0.3$
$= 1.93$ m

Since the metacenter is above G as measured from the base and Metacentric height is positive, the floating object is stable.

The restoring couple is $W \times C_G M \tan \theta$ from Eq. 4.3

For the particular case we can modify it as,

Restoring couple $= W \times GM \tan \theta$
$= 2000\ (1.164) \tan 12°$
$= 494.8$ kN-m

In the above example the value of θ *has been considered as significant and so it became necessary that we find the coordinates x and y to determine the actual position of* B_1 .

Problem 4.12 *A pontoon 10 m long and 7.5 m broad and 3 m deep floats in the sea. Its self weight is 800 kN and in addition it is carrying a triangular block of base 2 m and height 3 m which weighs 600 kN. The* C_G *of the pontoon coincides with the free water surface, while the* C_G *of the triangular block is at its geometrical location. Both however lie on the same vertical axis. Determine the stability or otherwise of the pontoon. Take* γ *for sea water as 10 kN/m*^{3}.

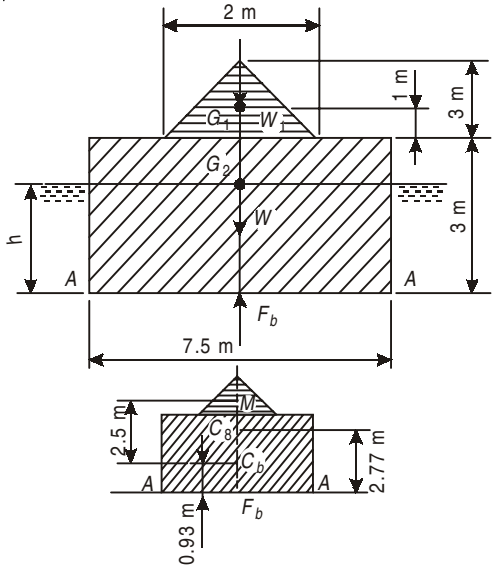


Fig. 4.10

Solution The condition for stability in this case is that the metacentric height must be above the combined C_G of both the pontoon and the triangular block.

First determine the depth of immersed surface (h) of the block.

The buoyant force = $F_B = (\gamma_W \times V_W)$

$= 10 \times L \times B \times h$

$F_B = 10 \times 10 \times 7.5 \times h$

$F_B = 750\ h$

or $$h = \frac{F_B}{750}$$

$$h = \frac{(600 + 800)}{750} = 1.86 \text{ m}$$

Now find the combined position of C_G for the entire floating body by taking moments about AA

$(600 + 800) \times C_G = [600 \times (1 + 3) + 800 \times (1.86)$

$C_G = 2.77\text{m}$

The center of Buoyancy $$C_B = \frac{h}{2}$$

$$= \frac{1.86}{2} = 0.93 \text{ m}$$

Now let us find the distance from the center of buoyancy to metacenter using the relationship,

$$\text{BM} = \frac{I}{V_D}$$

where, BM is the distance from center of buoyancy to meta-center, I is the least area moment of inertia of the immersed section, and V_D is the displaced volume.

$$I = \frac{(10 \times 7.5^3)}{12}$$

$= 351 \text{ m}^4$

$V_D = L \times B \times h$

$= 10 \times 7.5 \times 1.86$

$= 139.5 \text{ m}^3$

$$\text{BM} = \frac{351}{139.5} = 2.5 \text{ m}$$

From Base AA

$C_B = 0.93$ m

$C_G = 2.77$ m

$M = C_B + \text{BM}$

$= 0.93 + 2.5$

$= 3.43$ m

Since M lies above C_G the pontoon is in stable equilibrium.

Problem 4.13 *A wooden cylinder of diameter 300 mm and length 1500 mm has an additional steel round of the same diameter and 50 mm thick attached to its base. Determine the vertical stability or otherwise of the combined unit. Assume sg. (Steel) = 7.83 and sg. (Wood) = 0.6.*

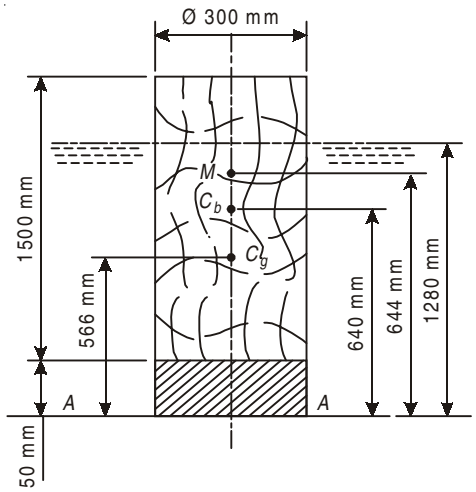


Fig. 4.11

Solution

Volume of wooden cylinder $V_w = 0.7854\ (0.3)^2 \times 1.5$

$= 0.106\ m^3$

Weight of wooden cylinder $= W_w = \gamma_w \times V_w$

$= (0.6 \times 9810) \times 0.106$

$= 624\ N$

Volume of steel round $V_S = 0.7854\ (0.3)^2 \times 0.05$

$= 3.5 \times 10^{-3}\ m^3$

Weight of steel round $W_S = \gamma_S \times V_S$

$= (7.83 \times 9810) \times (3.5 \times 10^{-3})$

$= 268.8\ N$

Center of gravity of the combined unit (G_C) is obtained by taking moments about *AA*

$$G_C = \frac{W_w \times \left\{\left(\frac{1.5}{2}\right) + 0.050\right\} + W_s \times \frac{0.050}{2}}{W_w + W_s}$$

$$G_C = \frac{624 \times \left\{\left(\frac{1.5}{2}\right) + 0.050\right\} + 268.8 \times \frac{0.050}{2}}{624 + 268.8}$$

$$G_C = 0.566\ m$$

Weight of water displaced = weight of the combined unit.

$\gamma_w \times V_w = 624 + 268.8$

$= 892.8$ N

$9810 \times (\text{Area} \times h) = 892.8$

$9810 \times 0.7854\,(0.3)^2 \times h = 892.8$

Therefore depth of immersion $h = 1.28$ m

$$\text{Center of buoyancy } C_B = \frac{1.28}{2} = 0.64 \text{ m}$$

The distance from center of buoyancy to metacenter BM = $\frac{I}{V}$

$$I = \frac{\pi(0.3)^4}{64}$$

$$= 3.97 \times 10^{-4}\,\text{m}^4$$

$$\text{BM} = \frac{(3.97 \times 10^{-4})}{(0.7854)(0.3^2)(1.28)}$$

$$= 0.0044 \text{ m}$$

From Base AA

$C_B = 0.64$ m $= 640$ mm

$C_G = 0.566$ m $= 566$ mm

$M = C_B + BM = 0.64 + 0.0044$

$M = 0.644$ m $= 644$ mm

Since M lies above C_G the cylinder is in stable equilibrium.

Problem 4.14 *A cone of base 300 mm diameter and height 180 mm floats in stable equilibrium in fresh water to a depth of 155 mm with its base up as shown in Fig. 4.12. Determine the specific gravity of the material of the cone.*

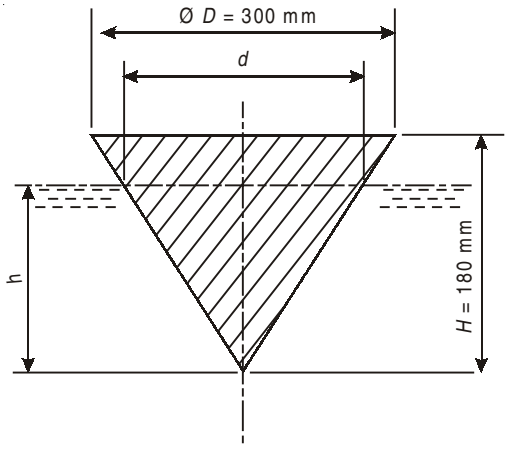


Fig. 4.12

Weight of the cone $W_C = \left(\frac{\pi D^2}{4} \times \frac{H}{3}\right) \times S_C \times \gamma_W$ ...(a)

Weight of the liquid displaced

$$W_L = \left(\frac{\pi d^2}{4} \times \frac{h}{3}\right) \times S_W \times \gamma_W$$

Here d is the diameter of the cone at the contact point with water, h is the depth of immersion and d can be expressed in terms of D and H thus, $d = \left(\frac{D}{H}\right) h$.

γ_W = specific weight of water, S_W = specific gravity of water, S_C = specific gravity of cone.

$$W_L = \pi \left\{ \frac{\left(\frac{D}{H} \times h\right)^2}{4} \times \left(\frac{h}{3}\right) \times S_w \times \gamma_W \right\} \quad \text{...(b)}$$

Equate (a) and (b)

$$\left(\frac{\pi D^2}{4} \times \frac{H}{3}\right) \times S_C \times \gamma_W = \pi \left\{ \frac{\left(\frac{D}{H} \times h\right)^2}{4} \times \left(\frac{h}{3}\right) \times S_w \times \gamma_W \right\}$$

Upon simplifying,

$$H \times S_C = \left(\frac{1}{H^2}\right) \times h^3 \times S_W$$

$$S_C = \frac{\left(h^3 \times S_W\right)}{H^3}$$

$$= \frac{\left(155^3 \times 1\right)}{180^3}$$

= 0.64 = Sp. gravity of cone.

Problem 4.15 *A vessel whose configuration is as shown in Fig. 4.13 (a) has to float in sea water. Determine the weight of the vessel and its stability. sg. of sea water = 1.03.*

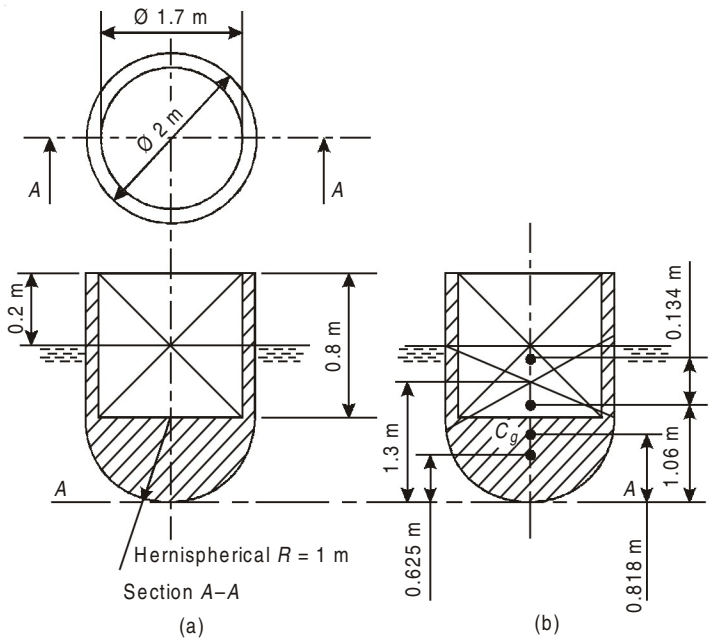


Fig. 4.13

Weight of the vessel W_V = (volume of water displaced) $sg \times \gamma_W$

W_V = {Volume of the cylindrical portion + volume of the hemisphere} $sg \times \gamma_W$

$$W_V = \left\{0.7854\,(2^2) \times (0.8 - 0.2) + \frac{1}{2}\left(\frac{\pi}{6}\right) \times 2^3\right\} \times (1.03) \times (9.81)$$

$$W_V = (5.86) \times (1.03) \times (9.81)$$

$$= 59.2 \text{ kN}$$

Specific weight of the material of the vessel

$$\gamma_v = \frac{W_V}{\left\{0.7854\,(2^2 - 1.7^2) \times 0.8 + \frac{1}{2}\left(\frac{\pi}{6}\right) \times 2^3\right\}}$$

$$= \frac{59.2}{2.79} = 21.2 \text{ kN/m}^3$$

Center of gravity of the vessel is obtained by taking moments about AA

$$C_G = \frac{\left\{0.7854\,(2^2 - 1.7^2) \times 0.8 \times 21.2\right\} \times 1.4 + \left\{\frac{1}{2}\left(\frac{\pi}{6}\right) \times 2^3 \times 21.2\right\} \times \left\{1 - \frac{(3 \times 2)}{16}\right\}}{59.2}$$

$$Y_{CG} = 0.818 \text{ m}$$

The center of buoyancy of the displaced volume from base AA

$$Y_{CB} = \frac{\left\{0.7854\,(2^2)\times 0.6\times 1.3\right\} + \left\{\frac{1}{2}\left(\frac{\pi}{6}\right)\times 2^3\times 0.625\right\}}{\left\{0.7854\,(2^2)\times 0.6\right\}\left\{\frac{1}{2}\left(\frac{\pi}{6}\right)\times 2^3\right\}}$$

$$= \frac{6.20}{5.86} = 1.06 \text{ m}$$

Distance to metacenter M from center of buoyancy C_B

$$C_BM = \frac{I}{V_D} = \frac{(\pi 2^4)}{64}$$

$$= 0.134 \text{ m}$$

Since the metacenter is above the point of CG the vessel is stable.

Problem 4.16 *A ship weighs 5 × 10^4 kN and measures 70 m × 12.7 m. The center of buoyancy is 2 m below the center of gravity and the radius of gyration is 3.7 m. Determine (1) the angle of tilt when a weight of 700 kN is moved across the deck through a distance of 10 m, (2) the time period of oscillation. Specific weight of sea water is 10.10 kN/m^3.*

Solution

$$\text{Volume of water displaced} = 5 \times \frac{10^4}{10.10} = 4950.5 \text{ m}^3$$

$$\text{Depth of immersion } h = \frac{4950.5}{70\times 12.7} = 5.56 \text{ m}$$

Distance of center of buoyancy from base of the ship

$$OB = \frac{5.56}{2} = 2.78 \text{ m}$$

If M is the metacenter then $BM = \frac{I}{V}$

$$I = \frac{(70\times 12.7^3)}{12} = 11,950 \text{ m}^4$$

$$\text{BM} = \frac{11{,}950}{4950.5} = 2.4 \text{ m}$$

$$\text{OM} = OB + BM = 2.78 + 2.4 = 5.18 \text{ m}$$

(a) From Eq. 4.6

The metacentric height $GM = \dfrac{(F \times x)}{W\theta}$

GM = 0.4 m (Ref Fig. 4.14), F is the load moved across the deck = 700 kN.

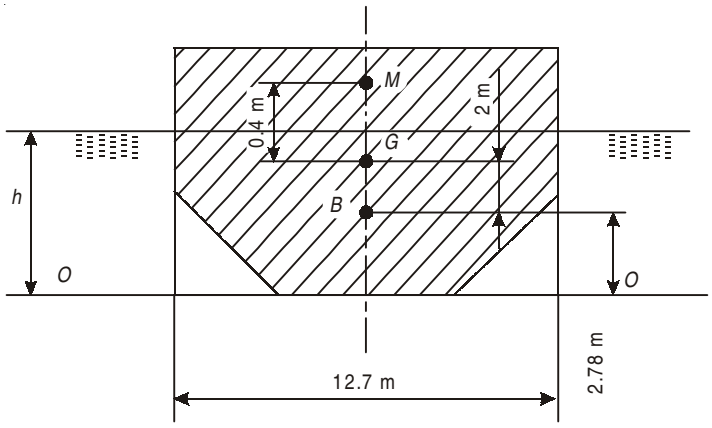


Fig. 4.14

X is the distance through which the load moves = 10 m,

W is the weight of the ship = 5×10^4 kN and θ is the angle of tilt.

Therefore $\theta = \dfrac{(700 \times 10)}{(5 \times 10^4)\, 0.4} = 0.35$ radians or 19°

(b) From Eq. (4.7) the time period of oscillation is given by

$$t = \frac{2\pi\sqrt{k^2}}{(GM)\, g}$$

K is the radius of gyration = 3.7 m,

Metacentric height GM = 0.4m (Ref. Fig. 4.14)

$$t = \frac{2\pi\sqrt{3.7^2}}{(0.4)\, 9.81} = 11.73 \text{ sec.}$$

Problem 4.17 *A cone of base φ = 0.3 m and height = 0.5 m floats in water as shown in Fig. 4.15. Determine the stability or otherwise of the cone. Specific gravity of the material of the cone = 0.7 and specific gravity of water = 1.*

Solution

Weight of the cone $W_C = 0.7854\,(D^2)\left(\dfrac{H}{3}\right)\gamma_W\, S_C$

Weight of water displaced $W_w = 0.7854\,(d^2)\left(\dfrac{h}{3}\right)\gamma_W\, S_W$

But $W_C = W_w$

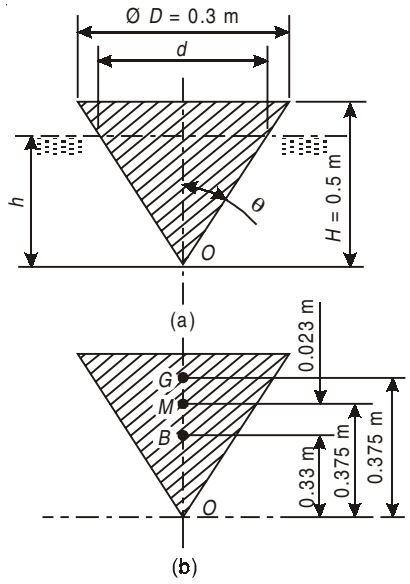


Fig. 4.15

Therefore, $0.7854\,(D^2)\left(\frac{H}{3}\right)\gamma_W S_C = 0.7854\,(d^2)\left(\frac{h}{3}\right)\gamma_W S_W$.

This upon simplification yields,

$$(D^2)\,(H)S_C = (d^2)\,(h) \times 1 \qquad \text{...(a)}$$

$$\tan\theta = \left(\frac{d}{2h}\right)$$

or $$d = 2h \tan\theta,$$

substituting this value of d in Eq. (a) yields,

$$(D^2)\,(H)\,S_C = (2h\tan\theta)^2\,(h)$$

$$(D^2)\,(H)\,S_C = 4\,(h^3)\tan^2\theta$$

$$(D^2)\,(H)\,S_C = 4\,(h^3)\left(\frac{D}{2H}\right)^2$$

$$(D^2)\,(H)\,S_C = 4\,(h^3)\left(\frac{D^2}{4H^2}\right)$$

$$(H)\,S_C = \left(\frac{4\,(h^3)}{4H^2}\right)$$

$$H^3 S_C = (h^3)$$

or $h = H(S_c)^{\frac{1}{3}}$

$= 0.5(0.7)^{\frac{1}{3}}$

$= 0.44$ m

For a cone the distance to the centroid is at a distance of $\frac{H}{4}$ from base or at a distance of $\frac{3}{4H}$ from the apex – O

If B is the center of buoyancy, then the center of buoyancy is at a distance of $(0.75 \times h)$ from O

$= (0.75 \times 0.44) = 0.33$ m

The center of gravity G is at a distance of $(0.75 \times H)$

$= (0.75 \times 0.5)$

$= 0.375$ m from O

The diameter at the surface of water

$$d = \left(\frac{D}{H}\right)h$$

$$= \left(\frac{0.3}{0.5}\right)0.44$$

$= 0.264$ m

$$\text{BM} = \frac{I}{V}$$

$$I = \frac{\pi d^4}{64}$$

$$= \frac{\pi(0.264)^4}{64}$$

$= (2.38 \times 10^{-4})$ m^4

V = volume of water displaced

$$= (0.7854\, d^2) \times \frac{h}{3}$$

$$= 0.7854\,(0.264^{\,2}) \times \frac{0.44}{3}$$

$V = (8 \times 10^{-3})$ m^3

Therefore $\text{BM} = \frac{(2.38 \times 10^{-4})}{(8 \times 10^{-3})} = 0.023$ m

$$OM = OB + BM$$
$$= 0.33 + 0.023 = 0.353 \text{ m}$$

Since the metacenter M is below the point of center of gravity G the cone is in unstable equilibrium.

Problem 4.18 *In order to keep a cylinder φ 4 m × 5 m long weighing 70 kN stable on its vertical axis an additional weight of 50 kN was attached through a chain hooked to the center of the cylinder. Determine the stability or otherwise of the combined unit. Take* $\gamma_W = 10.20 \text{ kN/m}^3$ *and neglect the self weight of the chain.*

Solution

Volume of water displaced by the combined unit

$$V_w = \frac{(70 + 50)}{10.20} = 11.76 \text{ m}^3$$

Depth of immersion $$h = \frac{(11.76)}{0.7854 \times 4^2} = 0.935 \text{ m}$$

Center of buoyancy from the base O of the cylinder,

$$B = \frac{0.935}{2} = 0.468 \text{ m}$$

Center of gravity G of the combined unit is obtained by taking moments about the base O

$$OG \times (70 + 50) = 70 \times 2.5$$

Therefore
$$OG = 1.458 \text{ m}$$
$$BG = 1.458 - 0.468$$
$$= 0.99 \text{ m}$$

$$= \frac{I}{V} = \text{BM } \frac{\left(\frac{\pi}{64}\right) \times 4^4}{11.76} = 1.068 \text{ m}$$

Since BM is > BG the unit is stable.

Problem 4.19 *A cargo vessel weighs 55 MN and has a value of* $I = 6000 \text{ m}^4$. *When a mass of 30000 kg initially placed on the vertical axis of the vessel is displaced through a distance of 6 m, the ship rolls about its vertical axis through 5°. Find (1) Metacentric height (2) the distance between the center of buoyancy and the center of gravity of the vessel. Specific gravity of water =1.025.*

Solution

Weight of the vessel $W = 55 \times 10^6$ N

Weight of the shifting mass $w = 30000 \times 9.81$
$$= 294300 \text{ N}$$

Distance through which the mass is displaced = 6 m

The angle of roll $\theta = 5°$
$$= 0.087 \text{ radians.}$$

$$\text{Metacentric height } GM = \frac{(294300 \times 6)}{(55 \times 10^6 \times 0.087)}$$

$$= 0.37 \text{ m}$$

$$\text{Displaced volume } V = \frac{55 \times 10^6}{9810 \times 1.025}$$

$$= 5649.7 \text{ m}^3$$

$$\text{BM} = \frac{I}{V}$$

$$= \frac{5840}{5649.7} = 1.03 \text{ m}$$

The distance from center of buoyancy (B) to center of gravity (G)

$$BG = BM - GM$$

$$= 1.03 - 0.37$$

$$= 0.660 \text{ m}$$

Problem 4.20 *A rectangular pontoon 20 m long × 7 m wide × 4 m deep has a displacement of 2000 kN in fresh water. Determine the time period of oscillation about its pitching axis.*

Solution

From Eq. 4.7 the time period of oscillation is given by

$$t = \frac{2\pi\sqrt{k^2}}{(GM)g}$$

There are two unknowns, the radius of gyration k and the metacentric height

$$I \text{ about the pitching axis} = \frac{(7 \times 20^3)}{12}$$

$$= 4666 \text{ m}^4$$

$$I = \frac{(W \times k^2)}{g}$$

$$4666 = \frac{k^2}{g} = \frac{4666}{W}$$

$$\frac{k^2}{g} = \frac{4666}{2000 \times 1000}$$

$$= (2.33 \times 10^{-3}) \text{ m-sec}^2$$

$$GM = MB - GB$$

$$MB = \frac{I}{V} \quad \frac{4666}{2000/9.81} = 22.88 \text{ m}$$

$$GB = OG - OB$$

$$OG = \left(\frac{4}{2}\right) = 2 \text{ m}$$

$$OB = \frac{h}{2}$$

$$h = \frac{2000}{9.81/20} \times 7 = 1.45 \text{ m}$$

Therefore

$$OB = \frac{1.45}{2} = 0.73 \text{ m}$$

$$GB = 2 - 0.73 = 1.27 \text{ m}$$

$$GM = 22.88 - 1.27 = 21.6 \text{ m}$$

$$t = 2\pi \sqrt{\frac{\left(2.33 \times 10^{-3}\right)}{21.6}}$$

$$= 0.065 \text{ sec.}$$

QUESTIONS BANK

Section A
Review Questions

1. What is the basic condition for a body to sink?
2. Where does the point of application of the buoyant force lie?
3. What is the significance of metacentric height?
4. A body weighs 20 kN in air and 10.19 kN in fresh water. What is its volume?
5. Define stable, unstable and neutral equilibrium.
6. Which principle you would adopt to compute the magnitude of the buoyant force?
7. What is the condition for stable equilibrium of a submerged body?
8. What effect the metacentric height has on the time period of oscillations of a floating body?
9. Define the terms 'rolling and pitching' and explain the criteria you would adopt in each case while determining the second moment of area of a floating body.
10. Define meta-centric radius.

Section B

Multiple Choice Questions

Choose the right answer:

1. A floating body is considered to be in stable equilibrium if
   (a) the weight of the fluid it displaces is equal to its own weight.
   (b) the center of gravity is below the center of buoyancy.
   (c) its metacenter is above the center of gravity
   (d) None of these
2. A completely submerged body is considered to be in stable equilibrium if
   (a) the metacenter is above the center of gravity
   (b) the center of gravity lies below the center of buoyancy
   (c) the meta-center coincides with the center of gravity
   (d) None of these
3. The center of buoyancy is located
   (a) at the center of the displaced volume of liquid
   (b) at the centroid of the floating body
   (c) at the metacenter
   (d) None of these
4. The buoyant force is directly proportional to
   (a) specific gravity of the fluid (b) viscosity
   (c) specific volume (d) None of these
5. The approximate range of meta-centric height for ships is between
   (a) 1–3 m (b) 2–5 m
   (c) 0.5–1 m (d) None of these
6. One of the causes of sea-sickness in ocean-liners is because
   (a) the buoyant force is high
   (b) metacenter is above the center of buoyancy
   (c) metacentric height is too large
   (d) None of these
7. A floating object is more stable about its
   (a) rolling axis (b) pitching axis
   (c) yawing axis (d) None of these

Section C

Problems

1. Two cubes have a volume 1 m^3 each and float in fresh water together suspended by a wire of negligible weight at mid point. If their specific gravities are 0.75 and 1.2, determine the portion of the lighter tube above the water and the tension in the wire. (*Ans.* 0.05 m and 1.96 kN)

2. Prove that the ratio of radius to height for a solid cylinder of *sg* = 0.7 must be φ 0.648 to attain stability while floating on its vertical axis.
3. A sphere of φ 300 mm floats in water half submerged. Determine its weight. (*Ans.* 69.3 N)
4. A ship weighing 0.5 GN rolls through an angle 2° when a load of 750 kN on its deck is displaced through a distance of 12 metres from its centroidal axis. Find the metacentric height of the ship. (*Ans.* 0.515 m)
5. A solid cone floats in water with the base upwards. If the specific gravity of the material of the cone is 0.6, determine its least apex angle for stable equilibrium. (*Ans.* 46° 30°)
6. A conical buoy of base diameter 2 m and height 3 m weighing 2 kN floats in fresh water. Determine its stability. If unstable determine its minimum weight for stability.
   (*Ans.* Unstable. Minimum Wt. for stability = 22.4 kN)
7. A wooden log of diameter 0.5 m × 6 m long is floating on fresh water. If *sg*. of wood is 0.7 determine the depth *h* to which the log lay immersed in water. (*Ans.* *h* = 0.329 m)

Answers to Multiple Choice Questions

1. (c) **2.** (b) **3.** (a) **4.** (a)
5. (c) **6.** (c) **7.** (b)

Chapter 5

Liquids in Relative Equilibrium

Hitherto we dealt with pressures and forces within a body of fluid at rest or in static condition. Computation of pressure and forces for fluids under static conditions is easy because there is no relative movement between adjacent layers of fluid and hence no shear stresses are involved. Even when the body of fluid is subjected to motion in a steady state, the principles applicable to fluids in static condition would still apply as long as the velocity is constant because it does not result in relative motion of fluid particles within the boundary. But where the body of fluid is subjected to uniform acceleration, the motion would get transferred to the fluid particles within the boundary which are now subjected to an accelerating force F_a thus affecting the force/pressure criteria normally applicable to static fluids. Under such conditions it is assumed that the fluid behaves as if it were a solid and thus, has attained a state of *relative equilibrium* with the container.

The motion of liquid masses can be linear or rotational. Linear acceleration could be either horizontal or up or down an inclined plane or in the vertical direction. In all of these cases no shear stresses are involved because there is no relative movement of particles of fluid with respect to the container and also because fluid pressure is normal to the surface at all points.

5.1 HYDRO-DYNAMIC FORCES IN LIQUIDS SUBJECTED TO UNIFORM ACCELERATION

5.1.1 Acceleration – Along the Horizontal Plane

When a body of fluid is subjected to a horizontal acceleration the free surface of the fluid does not remain parallel to the base of the container but acquires an inclination with respect to the horizontal and in the direction of the accelerating

force due to the fluid attaining a higher level in one plane as compared to the other. (Fig. 5.1)

However a particle of fluid of a certain mass on the free surface acquires the acceleration of the tank in which it is contained. Therefore, $F = ma$ applies with the only difference that a in this case $= g \tan \theta$, where θ is the angle of inclination of the fluid with respect to the horizontal and g is the acceleration that the particle attains. But $mg \tan \theta = ma$, because the water particle and the container are subjected to the same accelerating force. From this it emerges that in case of a fluid subjected to uniform horizontal acceleration, the accelerating force,

$$F_H = mg \tan \theta \qquad ...(5.1)$$

where,

$$\tan \theta = \frac{a_h}{g} \qquad ... (5.2)$$

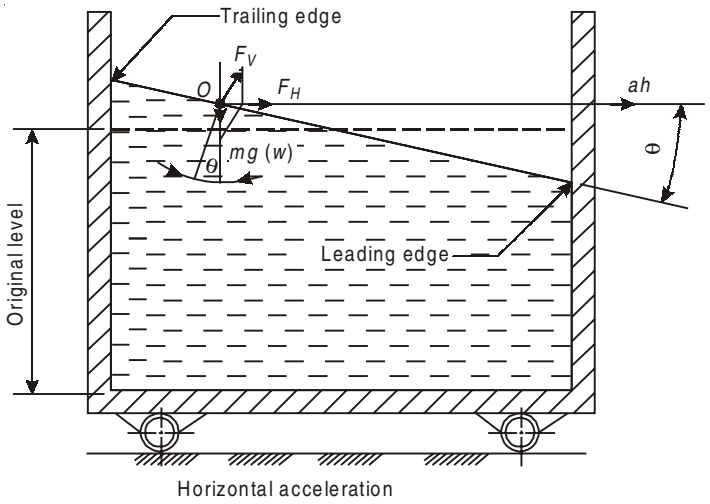


Fig. 5.1

mg is the weight of the fluid acting vertically downwards, F_P is the force due to fluid pressure which acts normal to the inclined plane of the fluid and F_H is the resultant of these two forces. Pressure at any depth below the free water surface would still be γh since the horizontal acceleration has no effect on forces in the vertical direction.

5.1.2 Acceleration—Along an Inclined Plane

Consider the forces acting on an element of mass δm at O as it is accelerated up the ramp of slope α as shown in Fig. 5.2.

The accelerating force F_R has components F_x and F_y. Force F_x acts horizontally and F_y vertically and θ is the angle of inclination of the fluid due to accelerating forces. F_1 is the force due to fluid pressure acting normal to CD and F_2 is the weight (mg) of the fluid acting vertically downwards.

Resolving the forces in the horizontal and vertical direction,

$$F_1 \text{ sine } \theta = F_x$$
$$F_1 \cos \theta = F_y + F_2$$

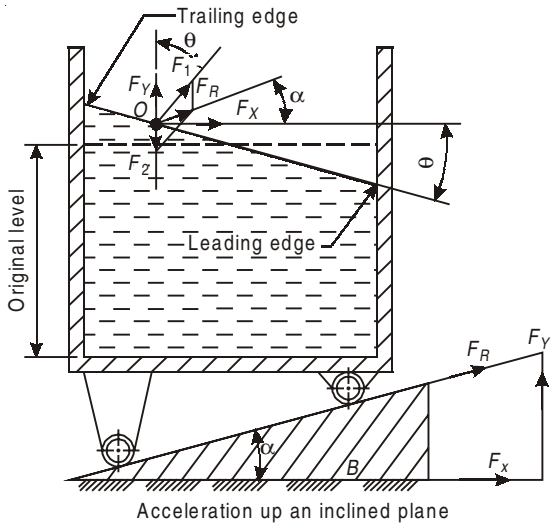


Fig. 5.2

or
$$\frac{\text{sine } \theta}{\cos \theta} = \frac{F_x}{F_y + F_2} = \tan \theta \quad \text{... (5.3)}$$

or, in terms of acceleration, $\tan \theta = \dfrac{a_x}{a_y + g}$ ... (5.4)

If the acceleration is in any random direction R within the x, y planes

$$\tan \theta = \frac{a_R \cos \alpha}{a_R \text{ sine } \alpha + g} \quad \text{... (5.5)}$$

If the acceleration is down the slope,

$$\tan \theta = \frac{a_x}{a_y - g} \quad \text{... (5.1)}$$

For horizontal acceleration, $\alpha = 0$, therefore Eq. (5.5) reduces to $\tan \theta = \dfrac{a_R}{mg}$, which confirms Eq. (5.2). If the acceleration is in the vertical direction then the free surface of the fluid remains horizontal, therefore $\alpha = 90°$, which means $\tan \theta = 0$.

5.1.3 Vertical Acceleration of Fluid Mass

In this case we are not concerned with determining the slope since tan θ = 0. On the other hand we would be concerned with the variation of pressure. The equation $p = \gamma h$, applicable for static fluids is not valid for fluids experiencing vertical acceleration. In fact the fluid mass affects accelerating forces to the extent of γhA where A is the area of the fluid mass.

Therefore the net accelerating force is the difference between the forces due to pressure below the column of fluid considered and the weight of the column of fluid. Thus the net upward accelerating force $a = (pA - \gamma hA)$. If we consider γhA as the mass of the column of fluid and a as its acceleration then $F = (\gamma hA \times a)$.

Thus, $\gamma hA \times a = pA - \gamma hA$

or $\gamma hA \times a = pA - \rho ghA$

which upon simplification can be written as,

$$p = \gamma h \left(1 + \frac{a}{g}\right) \quad \text{... (5.6)}$$

If the acceleration is in the downward direction then,

$$p = \gamma h \left(1 - \frac{a}{g}\right) \quad \text{... (5.7)}$$

Equations (5.6) and (5.7) could be further simplified as $p = \rho h\,(g + a) \uparrow$ and as $p = \rho h\,(g - a) \downarrow$ respectively. It should be noted that during downward acceleration at the instant the value of g becomes equal to a, $p = 0$, which means that in a freely falling mass of liquid, γh has no effect whatsoever and that the only influence on the liquid is the atmospheric pressure.

5.1.4 The Effect of Uniform Angular Acceleration on Liquids

When a container is rotated about its vertical axis with a uniform angular velocity (ω) any liquid in the container, in due course will attain relative equilibrium with the container. This means that the fluid in the container too will eventually rotate with the same angular velocity. However since the acceleration of the particles of fluid depends on their radii with respect to the axis of rotation it results in the formation of a *vortex* in the nature of a paraboloid, which is often referred to as the *forced vortex* because an externally induced torque is causing the formation of vortex. No such external influences play a part in case of *free vortex*. A whirl pool in a river is an example of free vortex. Since the fluid has attained a state of equilibrium with the container no shear stresses are involved and fluid pressures/forces are governed by the same hydrostatic laws that applies to static fluids. The net effect of rotation is a reduction in the hydrostatic force acting on the bottom plate and an increase in the force on the side walls.

Assume that a mass of fluid attains the acceleration of its container (Fig. 5.3) which is rotating at a certain velocity ω. Consider an elemental mass of

fluid on the paraboloid at a distance r from its axis. The forces acting on this mass (δM) in the x, y and z directions are the resultant force (F_R) due to acceleration acting inwards and perpendicular to the axis of rotation, the accelerating force ($F_y = 0$) and the force (F_z) due to the mass of fluid acting vertically downwards respectively. The fluid pressure force (F_P) acts perpendicular to the inclined face of the paraboloid. Let θ be the inclination of the free surface of the liquid.

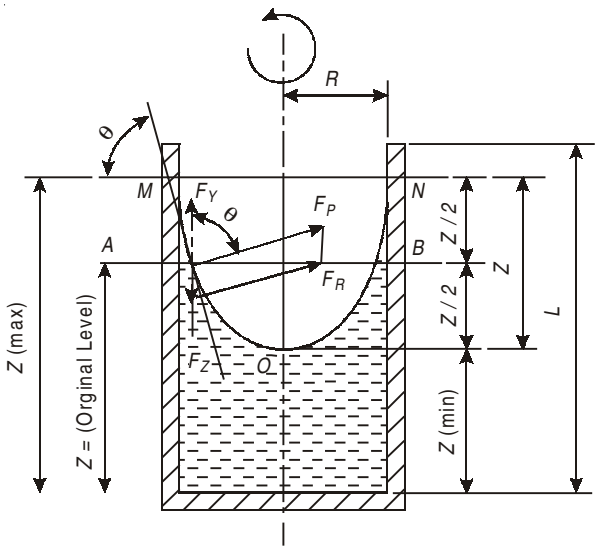


Fig. 5.3

Since the accelerating force is the same as the resultant F_R applying Newton's second law we obtain,

$$F_R = \left(\frac{W}{g}\right) \omega^2 r = F_P \sin\theta$$

Also, $$F_P \cos\theta = \delta m$$

since the accelerating forces in the y direction = 0

Therefore, $$F_P \sin\theta = \left(\frac{W}{g}\right) \omega^2 r$$

or $$F_P \sin\theta = \left(\frac{F_P \cos\theta}{g}\right) \omega^2 r$$

or $$\frac{\sin\theta}{\cos\theta} = \frac{\omega^2 r}{g}$$

or $$\tan\theta = \frac{\omega^2 r}{g} \quad \text{... (5.8)}$$

or $$\frac{dz}{dr} = \frac{\omega^2 r}{g}$$

or $$Z = \int_0^r \left(\frac{\omega^2 r}{g}\right) dr.$$

Therefore, $$Z = \frac{\omega^2 r^2}{2g} + C \quad \text{... (5.9)}$$

Here $\omega^2 r$ is the angular acceleration of the elemental mass of fluid at a radius r from the axis and the value of the constant of integration C is determined from the boundary conditions. For instance when $r = 0$, $z = 0$ and hence $C = 0$. Equation 5.9 further confirms that the profile is a paraboloid.

The pressure p at the base of the rotating drum is

$$= \gamma (Z_{min} + Z) \quad \text{... (5.10)}$$

Since the free surface of the liquid AB (the height h when the container was stationary) is equidistant from either end of the paraboloid, the values of Z_{max} and Z_{min} can be determined based on the knowledge that the volume of a paraboloid is one half of the circumscribing cylinder

Thus $$Z_1 = \frac{1}{2} (Z_{max} + Z_{min}) \quad \text{... (5.11)}$$

When Z_{max} becomes $\geq L$ the liquid begins to spill over the edge. In case of closed containers since no spilling is possible the profile of the curve changes as RPM increases until finally it assumes a cylindrical shape.

5.1.5 Closed Cylindrical Containers

In open cylindrical containers the radius R in the equation $Z = \frac{\omega^2 R^2}{2g}$ is always assumed equal to the radius of the container indicating that the paraboloid fills the container completely. In case of closed containers partially filled with the liquid the radius of the paraboloid R_1depends on the speed of rotation. It can be less than the radius of the container and so remains an unknown quantity and consequently the value Z in this case cannot be determined directly from the equation (Fig 5.4). However we can obtain this value of R_1 by equating the volume of air in the container to the volume of the paraboloid formed. In case of closed containers completely filled with the liquid no vortex is formed and

therefore only the force exerted by the liquid on the top and bottom plates needs to be determined.

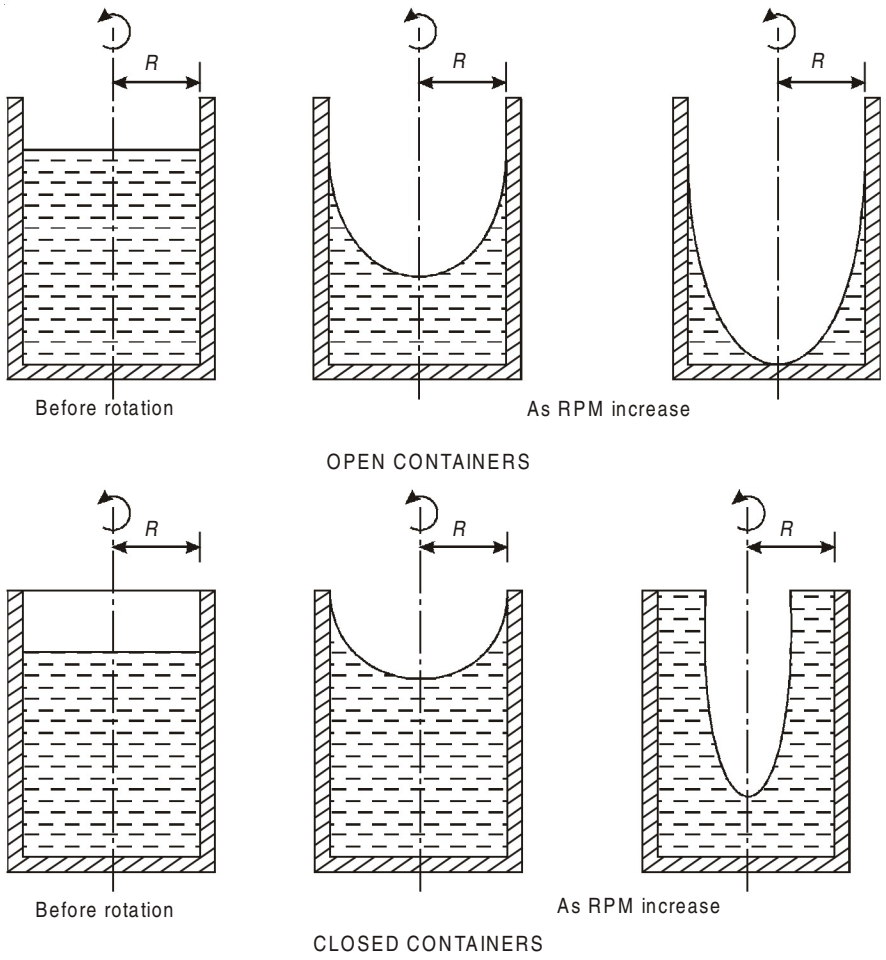


Fig. 5.4

TYPICAL PROBLEMS AND SOLUTIONS

Problem 5.1 *A container 2.5 m wide and 3 m long is filled to a depth of 1.8 metres with a liquid of density 840 kg/m^3. What will be force at the bottom of the container (1) when the container is accelerated vertically upwards @ 7 m/ sec^2. (2) When the acceleration phase ends but the container keeps moving in the same direction at constant velocity of 9 m/sec. (3) when the container is accelerated vertically downwards @ 8 m/sec^2.*

Solution

(a) *When the container is accelerated*

From Eq. 5.6

$$p = \gamma h\left(1+\frac{a}{g}\right)$$

$$\gamma = (840 \times 9.81) = 8240.4 \text{ N/m}^3$$

$$h = 1.8 \text{ m}$$

Area of the container bottom $A = 2.5 \times 3 = 7.5 \text{ m}^2$

$$a = 7 \text{ m/sec}^2$$

Substituting these values in the equation,

$$p = (8240.4)(1.8)\left(1+\frac{7}{9.81}\right) = 25416 \text{ N/m}^2$$

Force on the bottom $F = 25416 \times 7.5 = 190625$ N

(b) *When the container is travelling at uniform velocity.* There is no accelerating force. Therefore the pressure at the bottom is merely,

$$p = \gamma h = 8240.4 \times 1.8 = 14832 \text{ N/m}^2.$$

Therefore the force $F = 14832 \times 7.5 = 111240$ N.

(c) *When the container is accelerated @ 8 m/sec^2 vertically downwards*

From Eq. 5.7

If the acceleration is in the downward direction then,

$$p = \gamma h\left(1-\frac{a}{g}\right)$$

Therefore force on the base $F = p \times A$

$$= \gamma h\left(1-\frac{a}{g}\right) \times A$$

$$F = 8240.4 \times 1.8\left(1-\frac{8}{9.81}\right) \times 7.5$$

$$= \mathbf{20525 \text{ N}}$$

From this it is clear that as $a \rightarrow 9.81 \text{m/sec}^2$, $F \rightarrow 0$ in case of containers accelerated vertically downwards.

Problem 5.2 *A tank is 6 m long, 5 m wide and 4 m in depth and was accelerated along its length with its top open (Fig. 5.5). Determine the maximum acceleration that can be imparted to the tank without the water spilling over the edges if the initial water level was 3 m.*

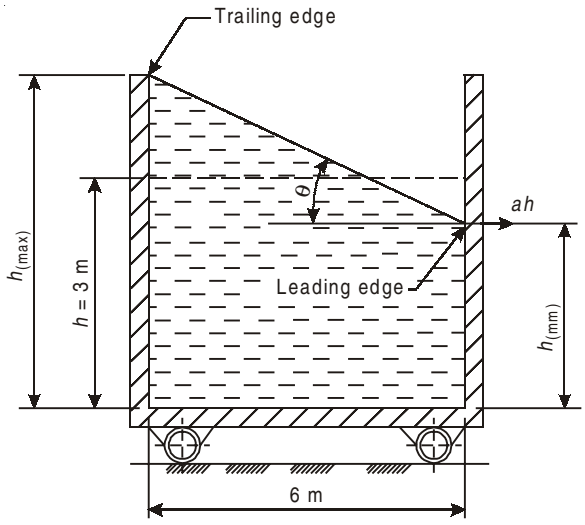


Fig. 5.5

Solution

The necessary condition to prevent spilling is $h_{\max} \leq h + \frac{1}{2}(L)\tan\theta$.

This means $4 = 3 + \frac{1}{2}(6)\tan\theta$

or $\theta = 18.43°$

From Eq. 5.2

$$\tan\theta = \frac{a_h}{g}$$

or $a_h = g\tan\theta$

$= 9.81\times \tan 18.43°$

$= 9.81 \times 0.333$

$= 3.27$ m/sec^2

Problem 5.3 *An open square tank 5 m wide (Fig.5.6) holds water to a depth of 3 m. If it is imparted an acceleration of 3 m/sec^2 up a slope inclined at 30° to the horizontal, calculate the inclination of water surface, the resultant force on the rear and bottom plate and front plates.*

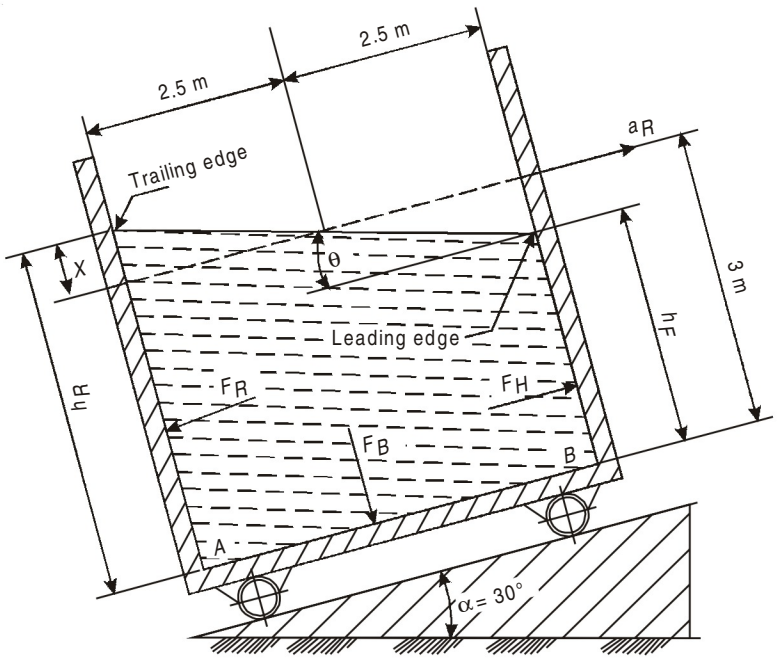


Fig. 5.6

Solution From Eq. 5.5

$$\tan \theta = \frac{a_R \cos \alpha}{a_R \sin \alpha + g}$$

$$\tan \theta = \frac{3 (\cos 30°)}{3 (\sin 30°) + 9.81}$$

$$\tan \theta = \frac{2.6}{1.5 + 9.81} = 0.229$$

or $$\theta = 13°$$

Force on the base plate

$$F_b = \rho \times L \times A_b (a_y + g) \qquad \text{... (5.12)}$$

$$\rho = 1000 \text{ kg/m}^3$$

From Eq. 5.11

$$L = \frac{1}{2} (h_R + h_F)$$

$$h_R = \text{initial water level} + x = 3 + 2.5 (\tan \theta)$$

$= 3 + 2.5 (\tan 13)$

$= 3.57$ m

h_F = initial water level $- x = 3 - 2.5 (\tan \theta)$

$= 3 - 2.5 (\tan 13)$

$= 2.42$ m

Therefore $L = \frac{1}{2} (3.57 + 2.42)$

$= 3$ m

$A_b = 5 \text{ m} \times 5 \text{ m}$

$= 25 \text{ m}^2$

$(a_y + g) = (1.5 + 9.81)$

$= 11.3 \text{ m/sec}^2$

Substituting these values in Eq. 5.12

$F_b = 1000 \times 3 \times 25 (11.3)$

$= 847500 \text{ kg-m/sec}^2$

Since $1 \text{ N} \approx 1 \text{ kg-m/sec}^2$

$F_b = 847500$ N

$=$ **847.5 kN**

Resultant force on the rear plate

$$F_R = (a_y + g) \left(\rho \times h_R \times \frac{h_R}{2} \right) \times \text{width} \quad \text{... (5.13)}$$

$$F_R = (1.5 + 9.81) \left(1000 \times 3.57 \times \frac{3.57}{2} \right) \times 5$$

$F_R = 360362 \text{ kg-m/sec}^2$

$= 360362$ N

$= 360.36$ kN

Resultant force on the front plate

$$F_F = (a_y + g) \left(\rho \times h_F \times \frac{h_F}{2} \right) \times \text{width} \quad \text{... (5.14)}$$

$$F_F = (1.5 + 9.81) \left(1000 \times 2.42 \times \frac{2.42}{2} \right) \times 5$$

$F_R = 165589.7 \text{ kg-m/sec}^2$

$= 165589.7$ N

$= 165.59$ kN

Problem 5.4 *A rectangular tank, 5 m long × 2 m wide × 1.5 m deep is fully filled with oil of sg = 0.85. A cover plate which is hinged at mid point as shown in Fig. 5.7, keeps the tank closed. What force F is required to keep the plate closed if the tank is accelerated along its length @ 4 m/sec² ?*

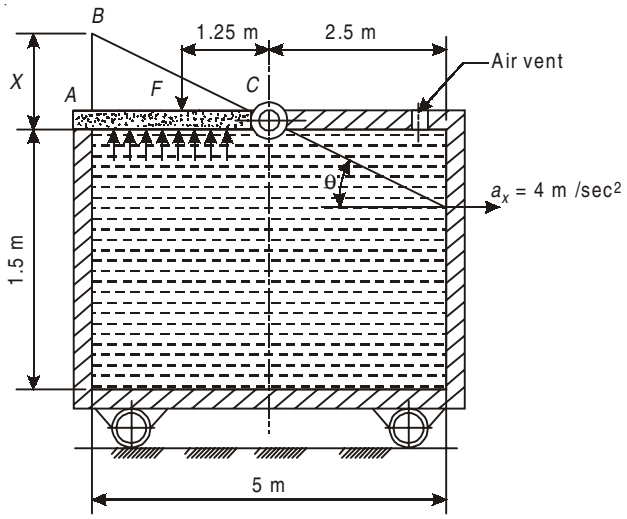


Fig. 5.7

Solution

$$\tan\theta = \frac{a_X}{g}$$

$$= \frac{4}{9.81} = 0.407$$

Therefore **θ = 22°**

The pressure distribution diagram relevant to the hinged plate is defined by the triangle *ABC*.

$$\text{The resultant force } F_R = \gamma\left(\frac{1}{2}\times AB\times AC\right)W \quad \ldots (5.15)$$

$$AB = AC\tan\theta$$
$$= 2.5\ (0.407)$$
$$= 1.01 \text{ m}$$
$$AC = 2.5 \text{ m}$$
$$\gamma = 9.81 \times 0.85$$
$$= 8.33 \text{ kN/m}^3$$
$$W = \text{width of the hinged plate} = 2 \text{ m}$$

Substituting these values in Eq. 5.15

$$F_R = 8.33 \left(\frac{1}{2} \times 1.01 \times 2.5\right) 2$$

$$= 21 \text{ kN}$$

F_R acts upwards at a distance of $\frac{2.5}{3} = 0.833$ m from AB.

Therefore taking moments about the hinge,

$$F_R (2.5 - 0.833) = F \times 1.25$$

$$21(1.66) = F \times 1.25$$

Therefore $F =$ **27.88 kN**

Problem 5.5 *A U-tube (Fig. 5.8) filled with a suitable gauge fluid has its limbs 0.05 m apart. The tube is held firmly in a fixture and is subjected to horizontal acceleration @ 30 m/sec². Determine the minimum length L of the limbs and the ideal height h of the fluid to prevent spilling.*

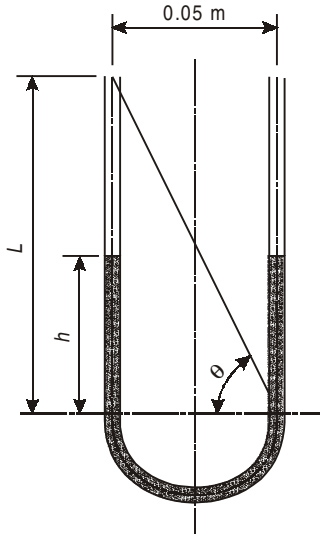


Fig. 5.8

Solution

$$\tan \theta = \frac{a_x}{g}$$

$$= \frac{30}{9.81} = 3.05.$$

Therefore $\theta = 71.8°$

$$\tan\theta \equiv \frac{L}{0.05}$$

Therefore $\dfrac{L}{0.05} = 3.05$

or $L = 3.05 \times 0.05$

$= 0.1525$ m or **152.5 mm**

Ideally height $h = \dfrac{L}{2}$

$= \dfrac{152.5}{2} =$ **76.25 mm**

Problem 5.6 *An open cylindrical container of diameter 0.5 m and depth 0.850 m is completely filled with oil. If this container with its top open is rotated on its vertical axis at 150 RPM determine (1) The height of oil from the vertex (2) the percentage of oil spill and the RPM at which this height is reduced to zero.*

Solution Referring to Fig. 5.8 and Eq. 5.9

(1) $Z = \left(\dfrac{\omega^2 r^2}{2g}\right)$

Angular velocity $\omega = \dfrac{2\pi N}{60}$

$= \dfrac{2\pi \times 150}{60}$

$=$ **15.7 rad/sec**

$r = 0.5/2$

$=$ **0.25 m**

Substituting these values in the above equation,

$$Z = \frac{\{(15.7)^2\ (0.25)^2\}}{2 \times 9.81}$$

$=$ **0.785 m**

The height of oil from the vertex

$Z_{min} = Z_{max} - Z$

$= 0.850 - 0.785$

$=$ **0.065 m**

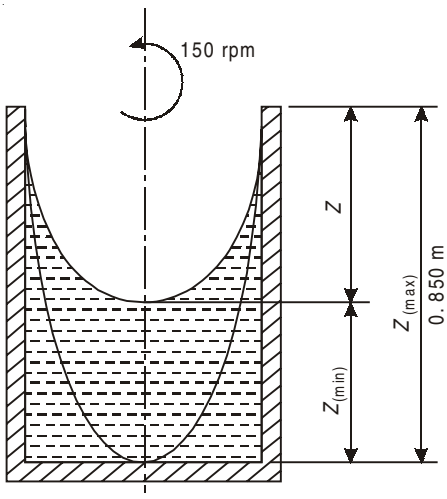


Fig. 5.9

Quantity of oil spill = Volume of the paraboloid

$$= \frac{1}{2} \text{ (volume of the circumscribing cylinder)}$$

$$= \frac{1}{2} (0.7854\, D^2 \times 0.785)$$

$$= \frac{1}{2} (0.7854 \times 0.5^2 \times 0.785)$$

$$= \mathbf{0.077\ m^3}$$

(2). Percentage of oil spill $= \dfrac{\text{Volume of the paraboloid}}{\text{Original volume}} \times 100$

Percentage of oil spill $= \dfrac{0.077}{(0.7854 \times 0.5^2 \times 0.850)} \times 100$

Percentage of oil spill = **46 %**

(3). Z_{min} becomes = 0 when Z_{max} becomes

$$= \left(\frac{\omega^2 r^2}{2g}\right)$$

Therefore $Z_{max} = \left(\dfrac{\omega^2 r^2}{2g}\right)$

$$0.850 = \frac{\omega^2\ (0.25)^2}{2 \times 9.81}$$

$$\omega^2 = 267$$

$$\omega = \sqrt{267} = \textbf{16.3 rad/sec}$$

Therefore RPM at which $Z_{min} = 0$ is given by

$$N = \frac{(\omega \times 60)}{2\pi}$$

$$N = \frac{(16.3 \times 60)}{2\pi}$$

$$= 156 \text{ RPM}$$

Problem 5.7 *Determine the quantity of water that spills over when an open tank of* $\varphi = 0.250$ *m and depth = 0.6 m is imparted an angular velocity = 30 rad/ sec.*

Solution

$$Z = \left(\frac{\omega^2 r^2}{2g}\right)$$

$$\omega = 30 \text{ rad /sec}$$

$$r = \frac{0.250}{2}$$

$$= 0.125 \text{ m}$$

Therefore $$Z = \frac{(30^2 \times 0.125^2)}{2 \times 9.81} = \textbf{0.716 m}$$

Since $Z >$ the depth of the tank, the paraboloid formed will be as shown in Fig. 5.10.

Quantity of water spill = the difference in the volumes of paraboloids *omn* and *oxy*.

Therefore $$Q = \left(\frac{\pi}{8} \times 0.25^2\right) \times 0.716 - \left[\frac{\pi}{8} \times (xy)^2\right] \times 0.116$$

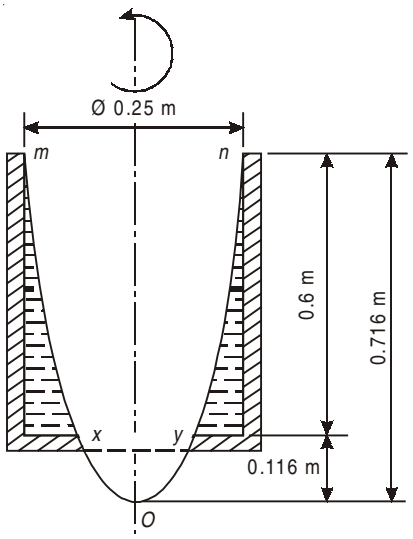


Fig. 5.10

$$\frac{0.25^2}{0.716} = \frac{(xy)^2}{0.116}$$

Therefore $$(xy)^2 = 0.116 \left(\frac{0.25^2}{0.716}\right) = 0.010$$

Now, $$Q = \left(\frac{\pi}{8} \times 0.25^2\right) \times 0.716 - \left[\frac{\pi}{8} \times 0.010\right] \times 0.116$$

$$= 0.017 \text{ m}^3 = 17 \text{ } L$$

Problem 5.8 *A closed cylindrical container (Fig.5.11) is 1.5 m long and has a diameter = 0.40 m. It contains water to* $\frac{3}{4}$ *of its depth. The rest of the space is filled with air. If the container is imparted an angular velocity = 30 rad/sec determine the volume of the paraboloid. At what RPM will the length of paraboloid becomes equal to the length of the container?*

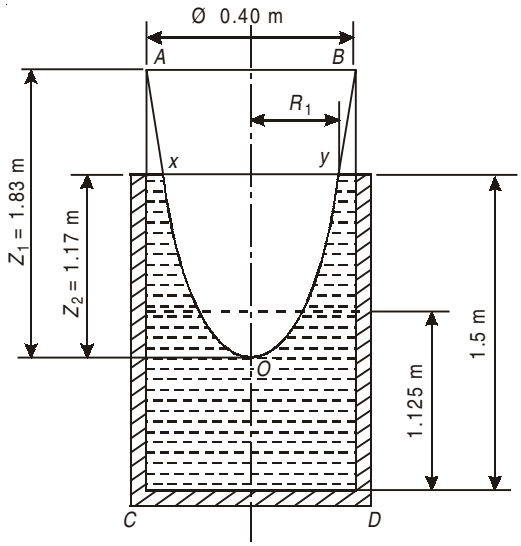


Fig. 5.11

Solution Because it is a closed container, the paraboloid is contained within the limits of the container. But let us assume it is an open container and that it extends to a length Z_1. This assumption will help us determine the profile of the paraboloid *OXY*.

$$Z_1 = \frac{\omega^2 R^2}{2g}$$

$$Z_1 = \frac{30^2\ (0.2^2)}{2\times 9.81}$$

$$= \mathbf{1.83\ m}$$

$$Z_2 = \frac{\omega^2 R_1^{\ 2}}{2g}$$

where R_1 is the radius of the paraboloid as it forms within the container.

$$Z_2 = \frac{30^2 R_1^{\ 2}}{2 \times 9.81}$$

$$= 45.87\ R_1^{\ 2}$$

Volume of air before rotation $V_1 = 0.7854\ (0.400)^2\ (1.5 - 1.125)$

$$= 0.047\ \text{m}^3$$

Volume of air after rotation $V_2 = \frac{1}{2}\ (\pi)\ R_1^{\ 2} \times Z_2$

= (volume of the paraboloid *OXY*)

But $V_1 = V_2$

Therefore $0.047\ \text{m}^3 = \frac{1}{2}\ (\pi)\ R_1^{\ 2} \times Z_2$ ... (5.16)

$$0.047\ \text{m}^3 = \frac{1}{2}\ (\pi)\ R_1^2 \times 45.87\ R_1^2$$

(because $Z_2 = 45.87\ R_1^2$)

$$0.047\ \text{m}^3 = 72\ R_1^4$$

Therefore, $R_1 = 0.160$ m

$$Z_2 = 45.87\ R_1^2$$

$$= 45.87 \times 0.160^2$$

$$= 1.17\ \text{m}$$

Volume of the paraboloid $OXY = \frac{1}{2}\,(\pi)\ R_1^2 \times Z_2$

$$= \frac{1}{2}\,(\pi)\ 0.160^{\ 2} \times 1.17$$

$$= \mathbf{0.047\ m^3}$$

To determine the RPM at which Z_2 becomes = 1.5 m, we have to determine the value of radius R_1 all over again because it is a closed container and as the RPM changes the value of R_1 changes.

Equating the volumes as before we have from Eq. 5.16

$0.047\ \text{m}^3 = \frac{1}{2}\ (\pi)\ R_1^2 \times Z_2$ (Here $Z_2 = 1.5$ m)

$$0.047\text{m}^3 = \frac{1}{2}(\pi) R_1^2 \times 1.5$$

Therefore $R_1^2 = 0.0199$

$R_1 = 0.14$ m

$$Z_2 = \frac{\omega^2 R_1^2}{2g}$$

$$\omega^2 = \frac{(Z_2 \times 2g)}{R_1^2}$$

$$= \frac{(1.5 \times 2 \times 9.81)}{0.0199}$$

$$= 1478.9$$

Therefore $\omega = \sqrt{1478.9}$

$= 38.5$ rad/sec

but $N = \frac{(60 \times \omega)}{2\pi}$

$$N = \frac{(60 \times 38.5)}{2\pi}$$

$= 367$ RPM

Problem 5.9 *If the container shown in Fig. 5.12 is subjected to an uniform angular velocity = 73 rad/sec, determine the upward thrust due to water pressure on the face BB.*

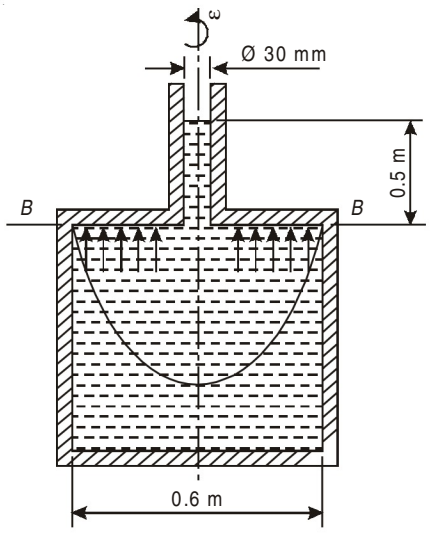


Fig. 5.12

Solution This can be considered as an example of closed containers completely filled with oil.

The upward thrust at any radius r on an annular ring of thickness dr is $t = p \times 2\pi\, r\, dr$ where p is the pressure on the ring.

The total thrust $$T = \int_0^R p \times 2\pi\ r\, dr$$

Since $$\frac{p}{\gamma} = \frac{(\omega^2 r^2)}{2g}$$

The total thrust $$T = \int_0^R \gamma \left\{\frac{(\omega^2 r^2)}{2g}\right\} \times 2\pi\, r\, dr$$

The total thrust $$T = \int_0^R \left\{\frac{(\omega^2 r^2)}{2g}\right\} \times 2\pi\, r\, dr$$

The total thrust $$T = \gamma\pi \frac{(\omega^2)}{g} \int_0^R r^3\, dr$$

The total thrust $$T = \gamma\pi \frac{(\omega^2)}{g} \left[\frac{r^4}{4}\right]_0^R$$

The total thrust $$T = \gamma\pi \frac{(\omega^2 R^4)}{4g} \quad \text{... (5.17)}$$

Thrust on face $$BB = \gamma\pi \frac{(\omega^2 R^2)}{4g} + (\gamma \times h \times A) \quad \text{... (5.18)}$$

$\gamma = 9.81$ kN/m^3

$\omega = 73$ rad/sec

$R = 0.3$ m

$h = 0.5$ m

$A = (0.7854 \times 0.6^2)$

$= 0.28$ m^2

Substitute these values in Eq. 5.18

Thrust on face $$BB = \frac{9.81\ \pi\ (73^2 \times 0.3^4)}{4 \times 9.81} + (9.81 \times 0.5 \times 0.28)$$

Thrust on face $$BB = \mathbf{35.27\ kN}$$

Problem 5.10 *A closed cylindrical container of length 0.5 m and diameter 0.25 m is full of oil of sg = 0.9. It is rotated on its vertical axis at 300 RPM. Determine the force exerted on the bottom plate.*

Solution Force on the bottom plate is the sum of the reaction force due to upward thrust (T) on the top plate exerted by the rotating mass of fluid plus the hydrostatic force (F).

From Eq. 5.17

$$T = \gamma \pi \frac{(\omega^2 R^4)}{4g}$$

$$\omega = 2\pi \frac{N}{60}$$

$$= \frac{2 \times \pi \times 300}{60}$$

$$= 31.4 \text{ rad/sec}$$

$$R = \frac{0.25}{2} = 0.125 \text{ m}$$

$$\gamma = 0.9 \times 9.81$$

$$= 8.83 \text{ kN/m}^3$$

$$T = \frac{8.83\ \pi\ (31.4^2 \times 0.125^4)}{4 \times 9.81}$$

$$= 0.170 \text{ kN}$$

$$F = \gamma h \times 0.7854\ (D^2)$$

$$= 8.83 \times 0.5 \times 0.7854\ (0.25^2)$$

$$F = 0.866 \text{ kN}$$

Total force on the bottom plate $= T + F$

$$= 0.170 + 0.866$$

$$= \mathbf{1.03\ kN}$$

QUESTIONS BANK

Section A

Review Questions

1. Differentiate between forced vortex and free vortex with examples.
2. Prove that in case of forced vortex the initial water level acts as the center of the paraboloid in the direction perpendicular to the axis of rotation.
3. Derive an expression for the total force exerted by the liquid on the top plate of a closed container completely filled with the liquid and subjected to an angular velocity ω.

Section B

Multiple Choice Questions

Choose the right answer

1. When an open container is accelerated horizontally @ 4 m/sec^2 the inclination of free water surface to the horizontal is
   (a) 33.2° (b) 22.18°
   (c) 15.3° (d) None of these
2. If the distance from the bottom plate of a container to the vertex of a paraboloid is 2 m and paraboloid is 3 m long, the initial water level was
   (a) 2 m (b) 2.5 m
   (c) 3.5 m (d) None of these
3. A closed cubical box of side 1.5 m is half filled with water and the other half with oil of sg. 0.85. If the box is accelerated upwards @ 5 m/sec^2 the pressure at the bottom of the box is
   (a) 18.5 kPa (b) 20.55 kPa
   (c) 19 kpa (d) None of these

Section C

Problems

1. A container 2 m wide × 4 m long holds oil (γ = 8.5 kN/m^3) to a depth of 1.5m. Determine the force on the bottom plate if the container is (1) accelerated vertically upwards @ 6 m/sec (2) when the acceleration ceases and container has attained uniform velocity. (3) When the container is accelerated at the same rate but in the vertically downward direction.
   (*Ans.* 164 kN, 102 kN, 39.6 kN)
2. A rectangular closed container 2 m in length × 0.5 m wide × 1 m deep holds a certain liquid (*sg* = 0.85) to a depth 0.7 metres. It is carried in the cargo bay of an airplane. Determine the angle of inclination of the free liquid surface in the container and the pressure difference between the trailing edge and the leading edge when the plane accelerates on the ground @ 6 m/sec^2 to take off speed. Also determine the inclination of the free liquid surface to the horizontal after the airplane takes off at an angle 40° to the horizontal. The acceleration during this phase is 10 m/sec^2. Assume that the acceleration is along the length of the container.
   (*Ans.* θ = 31.45°, 10.17 kN/m^2, θ = 25°).
3. An open cylindrical container of φ = 30 cm and length 50 cm holds water to a depth of 40 cm. Determine the RPM of the container at which the water just touches the top edge of the container and the difference in forces due to rotation on the bottom plate. (*Ans.* RPM = 126, F = 0.20 kN)

Answers to Multiple Choice Questions

1. (b) **2.** (c) **3.** (b)

Chapter 6

Fundamentals of Fluid Flow

In the previous chapters we discussed the effect of force and force per unit area on bodies fully or partially immersed in static fluids. Under 'relative equilibrium of floating bodies' we did mention terms like 'angular velocity' but we were merely referring to the angular velocity of the container and its effect on the fluid inside. The fluid on its own had no motion and consequently no velocity of its own.

In this chapter we are more concerned with *Fluid kinematics,* that is, the flow of fluid particles and consequently its velocity and acceleration and the nature of its distribution without considering the associated force or energy.

Since it is an endless stream of fluid that we are investigating we have to have a frame of reference within which is contained the system to be studied. Such a frame of reference is called the *control volume.* Forces that act on this system are called surroundings. This approach is called the Eulerian method and is used in fluid mechanics in preference to the Lagrangian method employed in solid mechanics. This is because, here we study the behaviour of the entire fluid in motion contained within the *control volume.* The shape and size of control volume is arbitrary and may at times coincide with solid boundaries. Where there are no such boundaries the 'stream tube' as such could be considered as the control volume. The Eulerian approach is mathematically simpler compared to the Lagrangian approach wherein the behaviour of a single particle in motion is analyzed.

Before we proceed further let us understand some commonly encountered terms used while analyzing fluid flow.

6.1 STEADY FLOW AND UN-STEADY FLOW

The flow of a fluid can be described as *steady* if the flow parameters such as velocity, pressure and density remain constant over a period of time. These

parameters may however vary between two points. Water flowing through a pipe of uniform cross section at constant velocity is an example for steady flow. If these parameters change with time the flow is considered as *unsteady*. A flow can be steady but *non-uniform.* In such cases the velocity varies due to varying cross sectional areas between two points of the solid boundary. A flow can be *unsteady* and yet be designated as *uniform.* Here the velocity remains constant between two points at any instant but may vary a moment later.

Mathematically, if $\frac{\delta v}{\delta t} = 0$

$\frac{\delta p}{\delta t} = 0$

and $\frac{\delta \rho}{\delta t} = 0$

then the flow is said to be steady.

If $\frac{\delta v}{\delta t} \neq 0$ then the flow is considered unsteady. If $\frac{\delta V}{\delta t} = 0$, where, V is the velocity vector then the flow may be regarded as uniform. In non-uniform flows $\frac{\delta V}{\delta t} \neq 0$.

6.2 LAMINAR AND TURBULENT FLOW

In laminar flow layers of fluid glide past each other smoothly without any lateral movement and without exhibiting any tendency to swirl. Water flowing through a faucet at low velocity wherein the stream appears uniform and circular indicates laminar flow. If the faucet is opened fully the velocity of flow increases and the stream is no longer circular or uniform. This can be considered as turbulent flow. Turbulent flow is not entirely without its advantages. Flow sometimes is intentionally made turbulent to achieve a better blend between two or more fluids or to increase the rate of heat transfer.

The velocity profile for a turbulent flow appears more uniform when compared to laminar flow as can be seen from Fig. 6.1. It belies the theory that

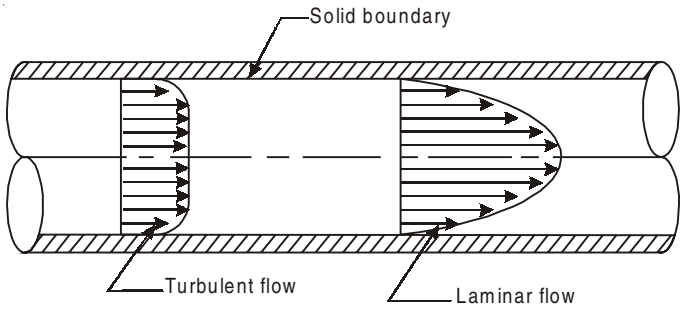


Fig. 6.1

the velocity of a fluid near a solid boundary is always zero and that the maximum velocity occurs at the center of the pipe. The main reason for this is that momentum transfer between molecules is higher during turbulent flows which results in a more uniform velocity profile.

All fluids seen in nature are compressible and exhibit frictional characteristics to varying degrees and therefore could be called *Real fluids*. However, while dealing with huge quantities of fluids or in cases where it simplifies our analysis we may assume that the fluid we are dealing with is incompressible and non-viscous. Such a fluid is termed an *Ideal fluid*. Such an assumption during the very early periods could have lead to purely hypothetical solutions. But since the introduction of the *Boundary layer theory* wherein the fluid within this layer is considered as *real fluid* and fluid outside as *ideal fluid* the margin of error has been significantly reduced. The velocity of the fluid particle is assumed to be zero at the boundary and 99 percent of the free stream velocity at the limiting line. The theory enables us to ignore the effect of shear stresses on fluids beyond the layer and treat them as ideal fluids. The thickness of the boundary layer depends on the velocity and the viscosity of the fluid; where these parameters have high value the thickness of the layer is further reduced.

Reynolds number is the yardstick that is used to categorize a flow as either laminar or turbulent. Reynolds number depends on the velocity of flow (V), flow pipe inside diameter (D) and the kinematic viscosity (υ) of the fluid.

For a fluid flowing through a circular pipe

$$N_R = \left(\frac{VD}{\nu}\right) \qquad \ldots (6.1)$$

If N_R is < 2000, then the flow is described as laminar. N_R > 4000 indicates turbulent flow. Since Reynolds number varies inversely as the kinematic-viscosity, more viscous fluids yield lower values of Reynolds number for the same values of V and D.

For a fluid flowing through a non-circular cross section such as a square or a rectangle,

$$N_R = \left[\frac{V(4R)}{\nu}\right] \qquad \ldots (6.2)$$

where R is the hydraulic radius, which is a ratio of net cross sectional area of the flow stream to the wetted perimeter. Wetted perimeter is the perimeter that the fluid is in contact with. As long as ratio of width to height of the section considered remains close to 1, which is the factor for a circular section, Eq. 6.2 gives reasonable results.

The flow may be characterized as *one, two or three dimensional flow*. In one dimensional flow the velocity varies along the direction of flow and not across.

Thus for one dimensional flow $u = f(x)$ where x represents the flow along only the x-axis. In two dimensional flow $u = f(xy)$. Since in this case the velocity varies both along and across the stream, the stream lines appear as curves. In three dimensional flow $u = f(x, y, z)$.

Generally most flows can be classified as three dimensional but in instances where the other two dimensions could be neglected, flow is assumed as one dimensional as it would greatly simplify our analysis. For instance laminar flow through a pipe can be considered as two dimensional as the velocity varies across the section but invariably it is treated as one dimensional flow as the margin of error is negligible.

Flow through two parallel plates of infinite width and flow over a weir of infinite width could be taken as examples for a two dimensional flow whereas flow through a converging or diverging duct could be considered as an example of three dimensional flow.

A flow is termed *compressible* if the density varies between two points and *incompressible* if the density remains constant. Liquids are generally considered in-compressible. Flow of gases through orifices or nozzles invariably causes density to vary and hence could termed as compressible flow.

The flow rate or the amount of fluid passing through a certain section at any instant can be quantified based on its volume, mass or weight and accordingly they are designated as *volume flow rate* (Q)*, mass flow rate* (M) *or weight flow rate* (W).

The volume flow rate which is based on the *continuity equation* remains the most fundamental of the three and is given by,

$$Q = Av \qquad \text{... (6.3)}$$

Here, A = area of cross section and v = average velocity.

The mass flow rate $M = \rho Q$ and weight flow rate $W = \gamma Q$

The units for Q, M and W are m^3/sec, kg/sec and N/sec respectively.

In case liquids it may safely be assumed that where the flow is *steady* the *volume flow rate* remains constant.

The *continuity equation* for any fluid is written as

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \qquad \text{... (6.4)}$$

If the fluid happens to be a liquid, since a liquid is considered in-compressible the above equation gets reduced to the form

$$A_1 V_1 = A_2 V_2 \qquad \text{... (6.5)}$$

Stream lines are imaginary but considered as continuous lines, the tangent to which at any point defines the velocity vector at that point of fluid flow along the stream. Obviously there can be no flow across the stream because the fluid particles are assumed to flow in the direction of the streamline. If the velocity vector is denoted as V then the velocity vector can be expressed in terms of time-dependent three dimensional-flows, thus

$$V = iu + jv + kw \quad \ldots (6.6)$$

Here u, v and w are the velocities of the fluid particle in the x, y and z directions respectively. The differential equation governing the stream line can be expressed as

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \ldots (6.7)$$

and any continuous line that satisfies the equation is a stream line.

Stream lines define the flow pattern at any given moment. The length of the stream line can extend to infinity in either direction. Space between stream lines varies inversely as the velocity and converging stream lines indicate accelerated flow.

Stream tubes (Fig. 6.2) are merely a series of stream lines drawn from every cross section and taken together over a finite length. Stream lines within a stream tube behave as if they were contained within a steel tube or a bend where no flow through the walls is possible if the flow is *steady*. The profile of a stream tube may change from one moment to the other depending upon changes in the position of streamlines.

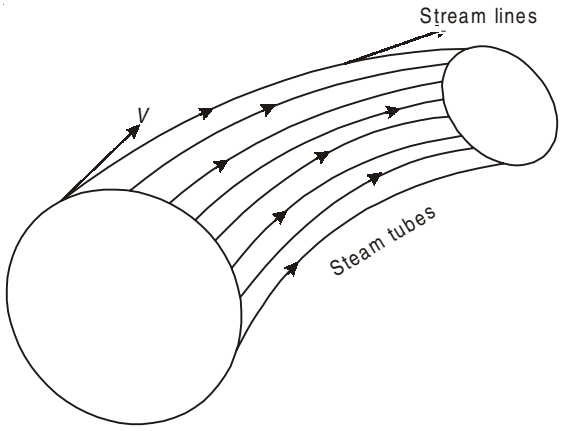


Fig. 6.2

While analyzing *unsteady* flows it becomes important to visually trace the position a fluid particle at any given instant. A dye or smoke is often injected in to the fluid stream so that a better comprehension of the flow pattern emerges. *Streak lines are trials of smoke or dye that becomes apparent* consequently. Where the flow is *steady* stream lines and streak lines follow the same path and hence there exists no distinction between the two.

The path of a fluid particle need not necessarily be in one direction. It could be in x, y, z directions within the stream. The path traced out by a fluid particle can be a two or three dimensional curve which is termed as the *path line.*

6.2.1 Continuity Equation for Three Dimensional Flow of any Fluid

In the beginning of this chapter the need for a specific frame of reference called the *control volume* for evaluating flow parameters such as velocity, density etc. was emphasized. This concept now forms the basis for obtaining the differential form of the continuity equation for three dimensional flows.

The continuity Eq. (6.4) for one dimensional flow, $\rho_1 A_1 V_1 = \rho_2 A_2 V_2$ is recalled here. If the fluid happens to be a liquid the equation gets reduced to the form $A_1 V_1 = A_2 V_2$. If the velocity variation across a section of the tube is considered then Eq. (6.4) could be written as,

$$\int_{A1} V \, \delta A = \int_{A2} V \, \delta A \qquad \ldots (6.8)$$

In the above case the stream tube itself may be considered as equivalent to a control volume with the only difference that there exists, no solid boundary.

While deriving the differential form of continuity equation for three dimensional flows we consider the mass flow rate out of the control volume per second in x, y, z directions separately and the sum of these individual flow rates is taken as the net out flow through the control volume per second.

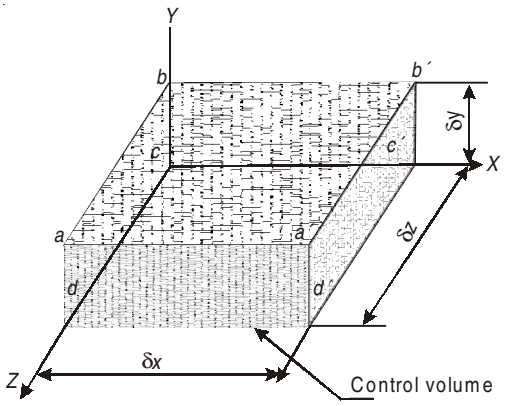


Fig. 6.3

Control volume considered here is a parallelepiped and it is assumed the fluid entry is through (a, b, c, d) and exit is through $(a' b' c' d')$

Mass flow rate into the system through (a, b, c, d) per second

$$\begin{aligned} M_E &= \rho Q \\ &= \rho A v \\ &= \rho V_X \delta y \, \delta z. \end{aligned}$$

Here M_E is the mass flow rate of the fluid entering the system, ρ is mass density, V_X is velocity in the x direction and $\delta y \, \delta z$ is area of face (a, b, c, d).

Mass flow rate out of the system through $(a' b' c' d')$ per second.

$$M_0 = \frac{(\rho V_x\ \delta y\ \delta z)+\delta}{\delta x\ (\rho V_x\ \delta y\ \delta z)\ \delta x}$$

Net flow rate out of the system or the gain in mass in the x direction per second is given by the difference between fluid influx and efflux.

$$(\rho V_X\ \delta y\ \delta z) - \left\{\frac{(\rho V_x\ \delta y\ \delta z)+\delta}{\delta x\ (\rho V_x\ \delta y\ \delta z)}\right\}\delta x$$

$$= -\frac{\delta}{\delta x(\rho V_x\ \delta x\ \delta y\ \delta z)} \quad \text{.....}x \text{ direction}$$

Likewise, net flow rate out of the system in the y and z directions,

$$= -\frac{\delta}{\delta y\ (\rho V_Y\ \delta x\ \delta y\ \delta z)} \quad \text{.....}y \text{ direction}$$

$$= -\frac{\delta}{\delta z\ (\rho V_Z\ \delta x\ \delta y\ \delta z)} \quad \text{.....}z \text{ direction}$$

The total gain in the x, y, z directions is made up of the gain in each direction.

$$= \left\{\frac{\delta}{\delta x\,(\rho V_X)}+\delta\frac{\delta}{\delta y\,(\rho V_Y)}+\delta\,\frac{\delta}{\delta z\,(\rho V_Z)}\right\}\delta x\ \delta y\ \delta z \qquad \text{... (6.9)}$$

Rate of change of mass density/second within the control volume,

$$= \left(\frac{\delta\rho}{\delta t}\right)\delta x\ \delta y\ \delta z \qquad \text{... (6.10)}$$

The total gain in the x, y, z directions = Rate of change of mass density/second within the control volume. Therefore,

$$-\left\{\frac{\delta}{\delta x\,(\rho V_X)}+\frac{\delta}{\delta y\,(\rho V_Y)}+\frac{\delta}{\delta z\,(\rho V_Z)}\right\}\delta x\ \delta y\ \delta z = \left(\frac{\delta\rho}{\delta t}\right)\delta x\ \delta y\ \delta z$$

or
$$\left\{\frac{\delta}{\delta x\,(\rho V_X)}+\frac{\delta}{\delta y\,(\rho V_Y)}+\frac{\delta}{\delta z\,(\rho V_Z)}\right\} = -\left(\frac{\delta\rho}{\delta t}\right) \qquad \text{... (6.11)}$$

Equation (6.11) applies to any fluid with no exception to any flow condition. If the flow is incompressible, that is if the fluid is a liquid, the equation gets modified to the form,

$$\left\{\frac{\delta}{\delta x\,(V_X)}+\frac{\delta}{\delta y\,(V_Y)}+\frac{\delta}{\delta z\,(V_Z)}\right\} = 0 \qquad \text{... (6.12)}$$

If the flow is two-dimensional the equation gets further simplified.

$$\left\{\frac{\delta}{\delta x(V_X)}+\frac{\delta}{\delta y(V_Y)}\right\} = 0 \qquad \text{... (6.13)}$$

For one-dimensional flow,

$$\frac{\delta}{\delta x(V_X)} = 0 \qquad \text{... (6.14)}$$

or V_x = constant

or $A_1V_1 = A_2V_2$ {Refer continuity Eq. (6.5)}

If instead of being a parallelepiped the control volume was *cylindrical* then *polar co-ordinates* (r, θ and z) would be used to derive the continuity equation with r and θ corresponding to the x and y planes of the Cartesian co-ordinate system.

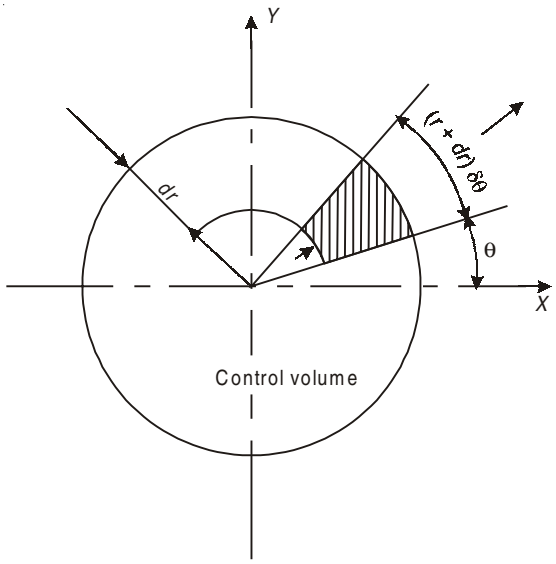


Fig. 6.4

Consider the cylindrical boundary as shown in Fig. 6.4 with center O as origin. Consider a slice of that boundary at a distance r from the center and making an angle θ with the x-axis. Let the thickness of the slice be δr

$$M_E = (\rho V_R)(r\delta\theta)\,\delta z$$

Here M_E is the mass flow rate of the fluid entering the system in unit time at radius r, ρ is mass density, V_R is velocity along the radius and ($r\delta\theta$) is the width and δz is the thickness of the slice in the z direction.

M_0 = Mass flow rate out of the system at radius ($r + \delta_R$) per unit of time

$$M_0 = \left(V_R + \frac{\delta V_R}{\delta r}\delta r\right) \times \rho\,(r + \delta r)\,\delta\theta\,\delta z$$

$$M_0 = V_R\,[\rho\,(r + \delta r)\,\delta\theta\,\delta z] + \left(\frac{\delta V_R}{\delta r}\delta r\right) \times \rho\,(r + \delta r)\,\delta\theta\,\delta z$$

$$M_0 = [V_R\,\rho r\ \ \delta\theta\,\delta z] + [V_R\,\rho\,\delta r\ \ \delta\theta\,\delta z] + \left(\frac{\delta V_R}{\delta r}\delta r\right)[\rho r\,\delta\theta\,\delta z + \rho\delta r\,\delta\theta\,\delta z]$$

The last term in the above equation can be overlooked as it is too inconsequential.

$$M_0 = [V_R\,\rho r\ \ \delta\theta\,\delta z] + [V_R\,\rho\,\delta r\ \ \delta\theta\,\delta z] + \left(\frac{\delta V_R}{\delta r}\delta r\right)[\rho r\,\delta\theta\,\delta z]$$

Now the net mass out-flow M_N in the radial direction per unit of time is $M_0 - M_E$

$$M_N = [V_R\,\rho r\ \ \delta\theta\,\delta z] + [V_R\,\rho\,\delta r\ \ \delta\theta\,\delta z] + \left(\frac{\delta V_R}{\delta r}\delta r\right)\ [\rho r\,\delta\theta\,\delta z\ ] - (\rho V_R)\,(r\delta\theta)\,\delta z$$

$$M_N = [V_R\,\rho\,\delta r\ \ \delta\theta\,\delta z] + \left(\frac{\delta V_R}{\delta r}\right)\rho r\,\delta r\delta\theta\,\delta z$$

divide throughout by r

$$M_N = \left[\frac{V_R}{r}\right]\ [\rho\,\delta r\ \ \delta\theta\,\delta z] + \left(\frac{\delta V_R}{\delta r}\right)\rho\,\delta r\delta\theta\,\delta z$$

$$M_N = \left\{\frac{V_R}{r} + \frac{\delta V_R}{\delta r}\right\}\ \rho\,\delta r\delta\theta\,\delta z$$

$$M_N = \left\{\frac{V_R}{r} + \frac{\delta V_R}{\delta r}\right\}\ \rho\,\delta V$$

Likewise the net mass out-flow in the direction of θ (M_θ) and in the direction of z(M_Z) per unit of time

$$M_\theta = \left\{\frac{\delta V_\theta}{r\delta r}\right\}\rho\,\delta V$$

$$M_Z = \left\{\frac{\delta V_Z}{\delta z}\right\}\rho\,\delta V$$

The net mass gain (M_G) across the control volume is $M_N + M_\theta + M_Z$

$$M_G = \left\{\frac{V_R}{r} + \frac{\delta V_R}{\delta r}\right\}\rho\delta V + \left\{\frac{\delta V_\theta}{r\,\delta\theta}\right\}\rho\delta V + \left\{\frac{\delta V_Z}{\delta z}\right\}\rho\,\delta V$$

M_G = 0 for incompressible fluids since density remains constant and hence there is no accumulation of mass.

$$M_G = \left\{\frac{V_R}{r} + \frac{\delta V_R}{\delta r}\right\} \rho\, \delta V + \left\{\frac{\delta V_\theta}{r\,\delta\theta}\right\} \rho\, \delta V + \left\{\frac{\delta V_Z}{\delta z}\right\} \rho\, \delta V = 0$$

Net efflux across the control volume is given by,

$$\left\{\frac{V_R}{r} + \frac{\delta V_R}{\delta r}\right\} + \left\{\frac{\delta V \theta}{r\delta\theta}\right\} + \left\{\frac{\delta V_Z}{\delta z}\right\} = 0 \qquad \text{... (6.15)}$$

or

In case of two dimensional-flow the equation gets further simplified to the form,

$$\left\{\frac{V_R}{r} + \frac{\delta V_R}{\delta r}\right\} + \left\{\frac{\delta V \theta}{r\delta\theta}\right\} = 0 \qquad \text{... (6.16)}$$

6.2.2 Velocity and Acceleration of Fluid Particles

More often than not the motion of fluid particles is three-dimensional and unsteady. Therefore straight forward equations for velocity and acceleration that we are familiar with, doesn't always hold.

Velocity in general is a function of space and time. Whereas for one dimensional steady flow $V = f(x)$, if the flow is unsteady $V = f(x, t)$. For two-dimensional steady flow $V = f(x, y)$ and if the flow is unsteady then $V = f(x, y, t)$. In case of three dimensional unsteady flow $V = f(x, y, z, t)$.

The change in velocity δv between points A and B is given by

$$\delta v = \frac{\partial v}{\partial s}(\partial s) + \frac{\partial v}{\partial t}\partial t \qquad \text{... (6.17)}$$

This implies that the change δv is the sum of the changes in velocity that occurred while traversing the distance A–B in a specific instant and the change that occurred at point B itself in time ∂t.

If V_X, V_Y and V_Z are the components of velocities in the x, y and z directions respectively then,

$$V_X = V_X(x, y, z, t),$$
$$V_Y = V_Y(x, y, z, t)$$

and
$$V_Z = V_Z(x, y, z, t)$$

Total change of velocities in each direction as the particle moves from one position to the other in the x-direction is given by,

$$\delta v_x = \frac{\partial v_X}{\partial X}(\partial X) + \frac{\partial v_Y}{\partial Y}(\partial Y) + \frac{\partial v_Z}{\partial Z}(\partial Z) \qquad \text{... (6.18)}$$

The resultant velocity is $\sqrt{v_X^{\,2} + v_Y^{\,2} + v_Z^{\,2}}$ ... (6.19)

The resultant velocity in vector form

$$V = iv_X + jv_Y + kv_Z \qquad \text{... (6.20)}$$

In Eq. (6.17) it was shown that the change in velocity between two points A and B was made up of two parts. Even the total *acceleration* of fluid particles is composed of two parts, *convective acceleration* plus *local or temporal acceleration.* Convective acceleration occurs because at any specific instant the velocities at each point during the movement of fluid particles from point $A - B$ varies. Local acceleration is due to the change in velocity occurring at point B in time ∂t. Thus $\left(\frac{V\partial v}{\partial s}\right)$ represents the convective acceleration and $\left(\frac{\partial v}{\partial t}\right)$ represents the temporal acceleration.

The total acceleration $$a = \left(\frac{V\partial v}{\partial s}\right) + \left(\frac{\partial v}{\partial t}\right) \quad \text{... (6.21)}$$

The components of acceleration a_x, a_y and a_z, in the x, y and z directions can be derived using partial differential equations and are given below.

$$a_x = v_x \frac{\partial v_x}{\partial x} + v_Y \frac{\partial v_x}{\partial y} + v_Z \frac{\partial v_x}{\partial z} + \frac{\partial v_x}{\partial t} \quad \text{...(,6.22)}$$

$$a_y = v_x \frac{\partial v_y}{\partial x} + v_Y \frac{\partial v_Y}{\partial y} + v_Z \frac{\partial v_y}{\partial z} + \frac{\partial v_y}{\partial t} \quad \text{...(6.23)}$$

$$a_z = v_x \frac{\partial v_z}{\partial X} + v_Y \frac{\partial v_z}{\partial Y} + v_Z \frac{\partial v_z}{\partial z} + \frac{\partial v_z}{\partial t} \quad \text{..(6.24)}$$

The resultant acceleration is given by $a = \sqrt{a_X^2 + a_Y^2 + a_Z^2}$ ... (6.25)

In vector form $$a = ia_x + ja_Y + ka_Z \quad \text{... (6.26)}$$

The above equations hold well as long as the velocity and acceleration are linear, that is, if the fluid particles are moving in a straight path. If they move along a curved path since the velocity keeps changing in direction there will be two components to the acceleration, one that is tangential to the radius and the other that is perpendicular to it. For circular motion therefore $a = a_t + a_N$ where a_t is *tangential acceleration* and a_N is *acceleration normal to it*.

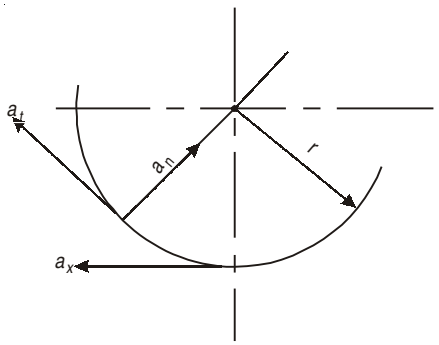


Fig. 6.5 *Circular motion*

Mathematically acceleration of a fluid particle executing a circular motion is expressed as,

$$a = \left[\left(\frac{V\partial v}{\partial s}\right)+\left(\frac{\partial v}{\partial t}\right)\right]+\left(\frac{V^2}{r}\right) \qquad \ldots (6.27)$$

Notes : Hence forward the notation for components of velocity in the x, y and z directions shall be u, v, and w respectively.

6.2.3 Types of Motion of a Fluid Particle

While discussing the types of motion executed by a fluid element and its effects, it was assumed that the flow is *two-dimensional*. Since major changes occur in the x–y plane, the other axis is neglected for the sake of simplification. Also, the fluid is assumed as *ideal fluid* so that the effects of viscosity could be neglected.

A stream of fluid may have a linear motion or a circular motion. But the fluid particles within the stream will be executing a motion on their own mass centers. The nature of motion of these fluid particles all of which may occur individually or collectively, is broadly classified as under.

(1) Linear translation or irrotational (2) Rotational

(3) Linear deformation (4) Angular deformation

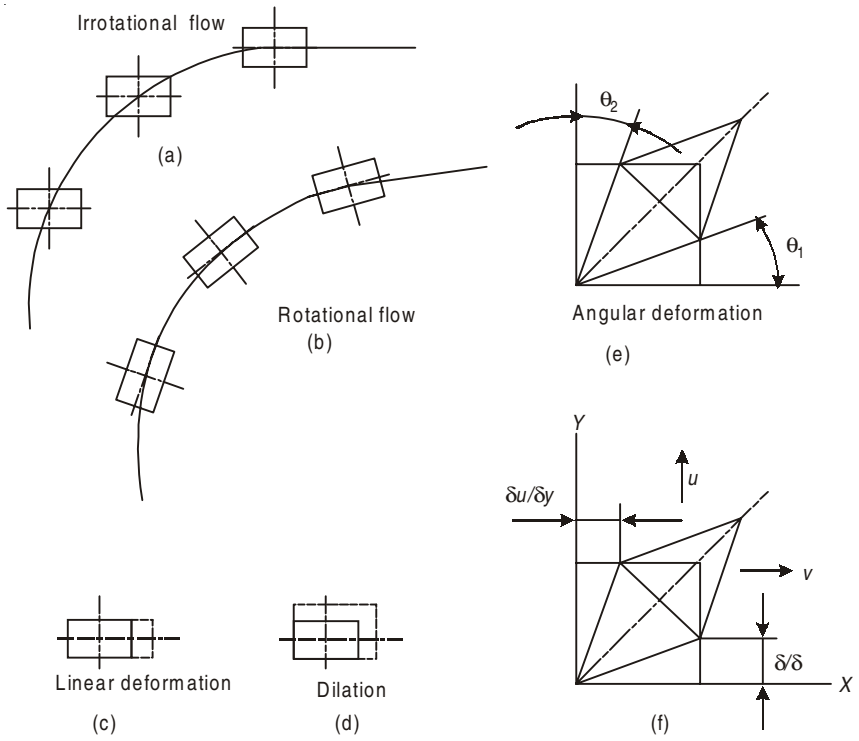


Fig. 6.6

In linear translation or *irrotational flow* the fluid particle (Fig. 6.6a) may move from one position to another but its axis remains parallel to its original position whereas in *rotational flow* the fluid particle while moving from one position to another also rotates on its own axis (Fig. 6b). Consequently its axis or mass center takes up a new position every time as compared to the previous position. This phenomenon is attributed to the turning moment associated with shear stresses. Hence it could be assumed that rotational flow is more likely where effects of viscosity are significant. It is logical therefore to also conclude that the flow is irrotational where the effect of viscous shear is insignificant.

Linear deformation could be pure elongation (Fig. 6.6c) or dilatation/volume expansion (Fig. 6.6d). In angular deformation there is a change in the angle between the two adjacent sides of a fluid particle (Fig. 6.6e). The average change in the angles defines the net angular deformation. If $\partial q_1$and ∂q_2 are the angles subtended by each side, with respect to the horizontal and the vertical respectively then,

$$\text{Angular deformation is } = \frac{1}{2}[\partial\theta_1 + \partial\theta_2] \qquad \ldots (6.28)$$

It can be shown that the shear strain rate

$$\lambda = \frac{1}{2}\left\{\left(\frac{\partial v}{\partial x}\right)+\left(\frac{\partial u}{\partial y}\right)\right\} \qquad \ldots (6.29)$$

where, $\left(\frac{\partial v}{\partial x}\right)$ and $\left(\frac{\partial u}{\partial y}\right)$ are the rates of angular rotations (Fig. 6.6f) of the two mutually perpendicular edges of the fluid element in the x–y plane. Similarly in the other two planes,

$$\lambda = \frac{1}{2}\left\{\left(\frac{\partial v}{\partial z}\right)+\left(\frac{\partial w}{\partial y}\right)\right\} \; y\text{–}z \text{ plane} \qquad \ldots (6.30)$$

$$\lambda = \frac{1}{2}\left\{\left(\frac{\partial u}{\partial z}\right)+\left(\frac{\partial w}{\partial x}\right)\right\} \; x\text{–}z \text{ plane} \qquad \ldots (6.31)$$

where *rotation* is involved the components for angular velocity (ω) for a three dimensional flow are,

$$\left.\begin{aligned}\omega_x &= \frac{1}{2}\left\{\left(\frac{\partial w}{\partial y}\right)-\left(\frac{\partial v}{\partial z}\right)\right\}\\ \omega_y &= \frac{1}{2}\left\{\left(\frac{\partial u}{\partial z}\right)-\left(\frac{\partial w}{\partial x}\right)\right\}\\ \omega_z &= \frac{1}{2}\left\{\left(\frac{\partial v}{\partial x}\right)-\left(\frac{\partial u}{\partial y}\right)\right\}\end{aligned}\right| \qquad \ldots (6.32)$$

The resultant rotation is given by $\omega = \sqrt{\omega_X^2 + \omega_Y^2 + \omega_Z^2}$ ... (6.33)

For a two-dimensional flow in the x–y plane

$$\omega_z = \frac{1}{2}\left\{\left(\frac{\partial v}{\partial x}\right) - \left(\frac{\partial u}{\partial y}\right)\right\} \quad \text{... (6.34)}$$

6.2.4 Circulation and Vorticity

The concept of circulation is based on the fact that since the fluid element is rotating, there must be a resultant peripheral velocity. The sum of the products of velocity and distance around the contour of such an element is called *circulation.* In other words it is the line integral of velocity around the element.

Mathematically circulation $\Gamma = \int_A V \cos\theta \, . \, ds$ ... (6.35)

The concept of circulation becomes important while studying flows around the wings of an aircraft which is an aero-foil section or around the blades of compressors and turbines of a gas turbine engine.

Vorticities according to stokes theorem is defined as circulation per unit of closed area or sum of vorticities within the area of the contour.

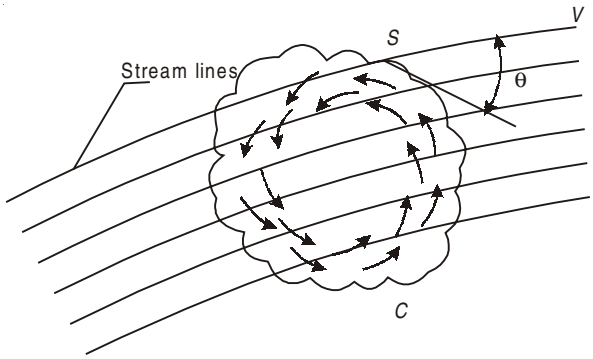


Fig. 6.7

Thus vorticity $\Omega = \frac{\Gamma}{A}$ ... (6.36)

If flow possess vorticity then it may be assumed that the flow is rotational

and $\omega = \frac{\Omega}{2}$ or $\frac{1}{2}\left\{\left(\frac{\partial v}{\partial x}\right) - \left(\frac{\partial u}{\partial y}\right)\right\}$ ... (6.37)

By analogy if the motion is purely translatory, that is, no rotation is involved then it can be assumed, provided the distortion is symmetrical that,

$$2\omega_z = 0 = \left\{\left(\frac{\partial v}{\partial x}\right) - \left(\frac{\partial u}{\partial y}\right)\right\} \quad \text{... (6.38)}$$

Since angular velocity is a vector, vorticity Ω can be expressed in vector form thus,

$$\Omega = \left\{\left(\frac{\partial w}{\partial y}\right) - \left(\frac{\partial v}{\partial z}\right)\right\} i + \left\{\left(\frac{\partial u}{\partial z}\right) - \left(\frac{\partial w}{\partial x}\right)\right\} j + \left\{\left(\frac{\partial v}{\partial x}\right) - \left(\frac{\partial u}{\partial y}\right)\right\} k$$

or $$\Omega = \Delta \times v \text{ or curl } v \quad \text{... (6.39)}$$

For a two dimensional flow in the x–y plane

$$\Omega = \left\{\left(\frac{\partial v}{\partial x}\right) - \left(\frac{\partial u}{\partial y}\right)\right\} \quad \text{... (6.40)}$$

6.2.5 Stream Function and Velocity Potential

Of the various flow lines that were earlier discussed, streamlines are the ones that are purely theoretical. Nevertheless they perform an important function; they define the flow pattern. It is therefore important to define them mathematically and assign a numerical symbol so that the flow rates within the flow pattern so described could be assessed.

The *equation for a stream line* which establishes its slope is obtained by considering the similarities between the velocity triangle and the displacement triangle for a two-dimensional flow can be written as,

$$\left(\frac{dx}{u}\right) = \left(\frac{dy}{v}\right) \quad \text{...(6.41)}$$

Let us consider the flow between two points two points A and B in a fluid region ACB and ADB as shown in Fig. 6.8. The flow rate across any stream line is a function of the psositional co-ordinates dx and dy at that point. This means that the partial derivative of a stream function in any direction gives its velocity component at right angles to it. Thus the stream function $\psi = \psi(x, y)$ and is applicable to a two dimensional steady flow stream. Stream function is un-definable for three dimensional flows. If a stream function (ψ) exists then it is indicative of the existence of flow which may or may not be rotational. If the stream function satisfies the Laplace equation then it is likely that the flow is ir-rotational. The relationship that exists between stream function and the velocity components can be defined thus,

$$u = \left(\frac{\partial \psi}{\partial y}\right) \text{ and } v = -\left(\frac{\partial \psi}{\partial x}\right) \quad \text{... (6.42)}$$

(The negative sign indicates that the flow is negative from right to left.)

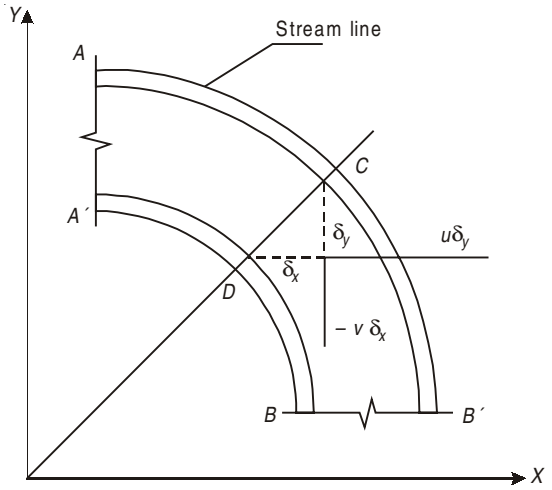


Fig. 6.8

From the continuity Eq. (6.12)

$$\left\{\frac{\delta}{\delta x}(V_x)+\frac{\delta}{\delta y}(V_Y)+\frac{\delta}{\delta z}(V_z)\right\} = 0$$

For a two dimensional steady flow, $\frac{\delta}{\delta x}(V_x) + \frac{\delta}{\delta y}(V_Y) = 0$ which is the same thing as,

$$\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right) = 0$$

The values of u and v (from Eq. 6.39) if substituted,

$$\frac{\partial}{\partial x}\left\{\left(\frac{\partial \psi}{\partial y}\right)\right\}+\frac{\partial}{\partial y}\left\{-\left(\frac{\partial \psi}{\partial x}\right)\right\} = 0$$

$$\frac{\partial^2 \psi}{\partial x\,\partial y}-\frac{\partial^2 \psi}{\partial x\,\partial y} = 0 \qquad \ldots (6.43)$$

Equation (6.40) shows that stream function satisfies the continuity equation. Continuity equation and stream function thus are complimentary, in that each proves the existence of the other. Stream function may not confirm whether the flow is rotational or ir-rotational but it certainly confirms the continuity equation.

The condition for irrotational flow is $\left(\frac{\partial v}{\partial x}\right) - \left(\frac{\partial u}{\partial y}\right) = 0$, which means,

$$\frac{\partial}{\partial x}\left\{-\left(\frac{\partial \psi}{\partial x}\right)\right\} + \frac{\partial}{\partial y}\left\{-\left(\frac{\partial \psi}{\partial y}\right)\right\} = 0 \text{ from which,}$$

$$\left(\frac{\partial^2 \psi}{\partial x^2}\right) + \left(\frac{\partial^2 \psi}{\partial y^2}\right) = 0 \text{ in the Laplace form} \qquad \text{... (6.44)}$$

In polar co-ordinates, $v_r = \frac{1}{r}\left(\frac{\partial \psi}{\partial \theta}\right)$ and $v_t = \left(\frac{\partial \psi}{\partial r}\right)$ ... (6.45)

Velocity potential (φ) like the stream function is a scalar function. *The negative derivative of velocity potential in any direction gives the fluid velocity in that direction.* Velocity potential for a three-dimensional steady flow is given by,

$$u = -\left(\frac{\partial \varphi}{\partial x}\right), v = -\left(\frac{\partial \varphi}{\partial y}\right)$$

and $$w = -\left(\frac{\partial \varphi}{\partial z}\right) \qquad \text{... (6.45)}$$

The continuity equation for a three dimensional incompressible flow is,

$$\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial z}\right) = 0$$

The values of u, v and w (from Eq. 6.41) if substituted,

$$\frac{\partial}{\partial x}\left\{-\left(\frac{\partial \varphi}{\partial x}\right)\right\} + \frac{\partial}{\partial y}\left\{-\left(\frac{\partial \varphi}{\partial y}\right)\right\} + \frac{\partial}{\partial y}\left\{-\left(\frac{\partial \varphi}{\partial z}\right)\right\} = 0$$

or $$\left(\frac{\partial^2 \varphi}{\partial x^2}\right) + \left(\frac{\partial^2 \varphi}{\partial y^2}\right) + \left(\frac{\partial^2 \varphi}{\partial z^2}\right) = 0 \qquad \text{... (6.46)}$$

This again is a Laplace equation.

In polar co-ordinates, $v_r = \left(\frac{\partial \varphi}{\partial \theta}\right)$ and $v_t = -\frac{1}{r}\left(\frac{\partial \varphi}{\partial \theta}\right)$ ... (6.47)

A relationship between stream function (ψ) and velocity potential (φ) may now be established because both satisfy the laplace equation. This will enable us to determine a stream function for each velocity potential. While velocity potential

(φ) exists only for irrotational flow, stream function (ψ) applies to both rotational and ir-rotational flows. Thus for irrotational compressible flow since both stream function and velocity potential satisfy the Laplace equation,

$$u = \left(\frac{\partial \varphi}{\partial x}\right) = -\left(\frac{\partial \psi}{\partial y}\right)$$

and

$$v = \left(\frac{\partial \varphi}{\partial y}\right) = \left(\frac{\partial \psi}{\partial x}\right) \quad \text{... (6.48)}$$

Equation (6.46) is the Cauchy–Riemann equations for irrotational flow.

The line along which the velocity potential (φ) is constant (φ = *C*) is defined as the *Equipotential line.* Equi-potential lines and stream lines at all points of intersection are always at right angles to each other. In other words the products of the slopes of these curves is numerically = 1.

Mathematically, $\left(\frac{u}{v}\right) \times \left(-\frac{v}{u}\right) = -1$

This in effect means a *flow net* could be formed from out of the streamlines and equi-potential lines by plotting the streamlines at equal distances, say $\psi = a_1$, a_2, a_3, a_4, etc., at a section where the flow is rectilinear. The next step is to superimpose on them the plot of $\varphi = b_1, b_2, b_3, b_4$, etc. The pattern obtained represents a flow net. Higher the number of stream lines, greater will be the accuracy. *Flow nets* basically help streamline objects for better aero or fluid dynamics. They also help in determining velocity and pressure distribution in a given boundary.

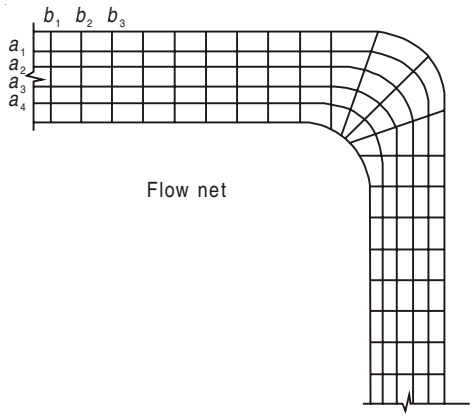


Fig. 6.9

TYPICAL PROBLEMS AND SOLUTIONS

Problem 6.1 *Water flows through a pipe 40 mm in diameter at a velocity of 6 m/sec. Determine whether the flow is laminar or turbulent. Assume* $\mu = 1.3 \times 10^{-3}$ *kg/m-sec and* $\rho = 1000$ *kg/m*^{3}.

Solution

From Eq. (6.1)

For a fluid flowing through a circular pipe $N_R = \left(\frac{VD}{\nu}\right)$

If N_R is < 2000, then the flow is described as laminar. N_R > 4000 indicates turbulent flow.

$$\nu \text{ is kinematic viscosity} = \frac{\nu}{\rho}$$

$$= \frac{(1.3\times10^{-3})}{1000}$$

$$= (1.3 \times 10^{-6}) \text{ m}^2/\text{sec.}$$

V is flow velocity = 6 m/sec

and D is pipe inside φ = 0.040 m

Therefore $$N_R = \frac{(6 \times 0.040)}{(1.3 \times 10^{-6})}$$

$$= 184615$$

Hence the flow is turbulent.

Problem 6.2 *Determine the mean velocity inside a rectangular air-duct measuring 0.4 m* × *0.6 m if the rate of flow 60 liters/sec. What will be velocity at the reduced section if the duct steps down to 0.2 m* × *0.3 m? Assume* ρ *to be constant.*

Solution

$$\text{Mean velocity } v = \frac{Q}{A}$$

where Q = 60 liters/sec

or $$\frac{60}{1000} = 0.06 \text{ m}^3/\text{sec}$$

and $$A = 0.4 \times 0.6$$

$$= 0.24 \text{ m}^2$$

Therefore, $$v = \frac{0.06}{0.24}$$

$$= 0.25 \text{ m/sec}$$

The velocity v_1 at the reduced section is obtained by applying the continuity equation,

$$Av = A_1 v_1,$$

where

$$A_1 = 0.2 \text{ m} \times 0.3 \text{ m}$$
$$= 0.06 \text{ m}^2$$

$$v_1 = \frac{Av}{A_1}$$

$$= \frac{(0.24 \times 0.25)}{0.06}$$

$$= 1 \text{ m/sec}$$

Problem 6.3 *Water at 80 °C is flowing through the convergent-divergent section of a pipe. The diameter and velocity at section 1–1 is 75 mm and 7 m/sec respectively. The diameter at section 2–2 is 110 mm. Find the velocity at section 2–2, the mass flow rate, volume flow rate and the weight flow rate.*

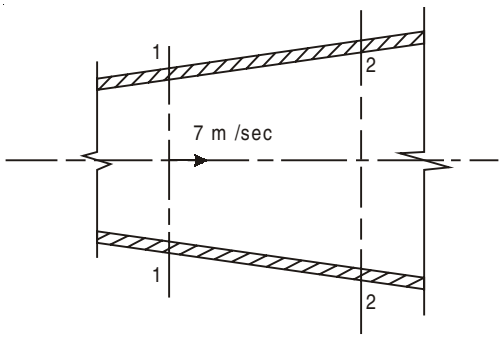


Fig. 6.10

Solution

From the continuity Eq. (6.5) we have, $A_1V_1 = A_2V_2$

$$A_1 = 0.7854\ (0.075)^2$$
$$= 4.4 \times 10^{-3} \text{ m}^2$$
$$A_2 = 0.7854\ (0.110)^2$$
$$= 9.5 \times 10^{-3} \text{ m}^2$$

$$V_2 = \frac{A_1V_1}{A_2}$$

$$= \frac{(4.4 \times 10^{-3}) \times 7}{(9.5 \times 10^{-3})}$$

$$V_2 = 3.24 \text{ m/sec}$$

Volume flow rate $Q = A_1V_1$

$= (4.4 \times 10^{-3}) \times 7$

$Q = 0.0308 \text{ m}^3/\text{sec}$

Weight flow rate $W = \gamma Q$

$= 9.53 \times 0.0308$

$W = 0.29 \text{ kN/ sec.}$

(γ of water at 80°C = 9.53 kN/m^3)

The mass flow rate $M = \rho Q$

$= 971 \times 0.0308$

$M = 29.9 \text{ kg/sec.}$

(ρ of water at 80°C = 971 kg/m^3)

Problem 6.4 *A round steel pipe is enclosed inside a square tube as shown in Fig. 6.11. Water flows through both the pipes at the same velocity. Determine the velocity if the combined volume flow rate = 40 liters / mt.*

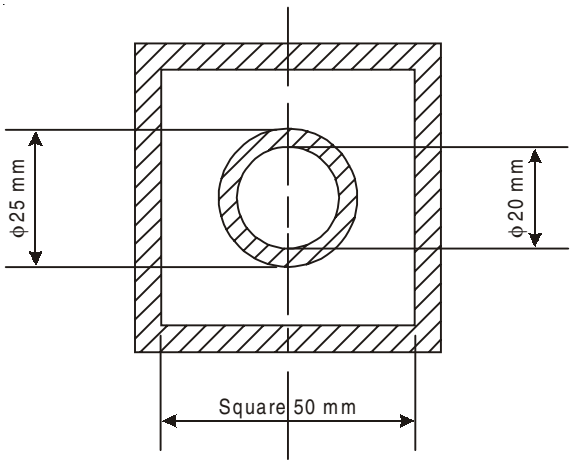


Fig. 6.11

Solution Let the net area of the square pipe be A_1 and the flow passage area of the round pipe be A_2.

A_1 = Inside area of the square pipe – outside area of the round pipe

$A_1 = (0.05 \times 0.05) - 0.7854\ (0.025)^2$

$= 2.01 \times 10^{-3} \text{ m}^2$

$A_2 = 0.7854\ (0.020)^2$

$= 3.14 \times 10^{-4} \text{ m}^2$

The combined flow rate $= Q$

$= 40 \text{ liters/mt}$

$= \dfrac{(40 \times 1000)}{60}$

$= 666.66 \text{ cm}^3/\text{sec or } 6.66 \times 10^{-4} \text{ m}^3/\text{sec}$

$$Q = A_1V_1 + A_2V_2$$

but V_1 is given $= V_2$

$$Q = V_1(A_1 + A_2)$$

$$= V_1(2.01 \times 10^{-3} + 3.14 \times 10^{-4})$$

$$6.66 \times 10^{-4} = V_1(2.01 \times 10^{-3} + 3.14 \times 10^{-4})$$

Therefore $$V_1 = \frac{(6.66 \times 10^{-4})}{(2.01 \times 10^{-3} + 3.14 \times 10^{-4})}$$

$$V_1 = 0.2865 \text{ m/sec}$$

Problem 6.5 *Water flows through a pipe of varying cross section such as the one shown in Fig. 6.12. Determine the velocity and volume flow rates at sections, 1 and 3, from the given data.*

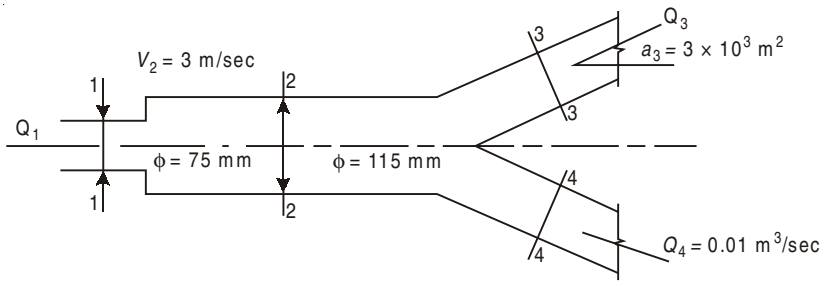


Fig. 6.12

Solution

From the continuity Eq. (6.5) we have,

$$A_1V_1 = A_2V_2$$

$$A_1 = 0.7854 \times (0.075)^2$$

$$= 4.4 \times 10^{-3} \text{ m}^2$$

$$A_2 = 0.7854 \times (0.115)^2$$

$$= 10.3 \times 10^{-3} \text{ m}^2$$

$$V_1 = \frac{(A_2V_2)}{A_1}$$

$$= \frac{\{(10.3 \times 10^{-3}) \times 3\}}{4.4 \times 10^{-3}}$$

$$= 7 \text{ m/sec}$$

$$Q_1 = A_1V_1$$

$$= (4.4 \times 10^{-3}) \times 7$$

$$= 0.0308 \text{ m}^3\text{/ sec.}$$

$$Q_1 = Q_2 = (Q_3 + Q_4)$$

$$0.0308 = (Q_3 + 0.01)$$

Therefore, $$Q_3 = 0.0208 \text{ m}^3/\text{sec}.$$

$$Q_3 = A_3 V_3$$

Therefore, $$V_3 = \frac{Q_3}{A_3}$$

$$V_3 = \frac{0.0208}{(3 \times 10^{-3})} = 6.9 \text{ m/sec}.$$

Problem 6.6 *A jet of water issues out of a φ 10 mm nozzle vertically upwards. If the velocity at the exit point of the nozzle is 20 m/sec, determine the velocity and the diameter at a distance 5 m from the nozzle.* (*Fig. 6.13*).

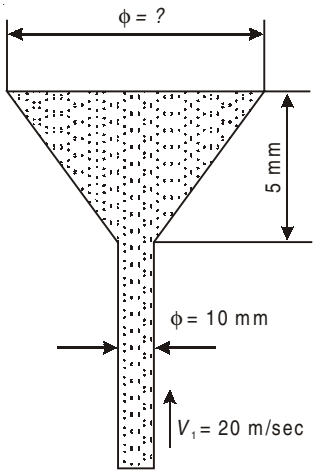


Fig. 6.13

Solution Velocity at a distance 5 m from the nozzle exit point can be determined from the fundamental equation

$$V_2^2 = u^2 + 2gh$$

Initial velocity $u = 20$ m/sec

$g = -9.81$ m/sec

and $h = 5$ m

Therefore, $$V_2^2 = (20)^2 + 2(-9.81) \times 5$$

$$= 301.9$$

or $$V_2 = 17.375 \text{ m/sec}.$$

Now from the continuity Eq. (6.5) we have,

$$A_1 V_1 = A_2 V_2$$

Here $A_1 = 0.7854\ (0.01)^2$

$= 7.854 \times 10^{-5}\ \text{m}^2$

$$A_2 = \frac{A_1 V_1}{V_2}$$

$$= \frac{(7.854 \times 10^{-5})\ (20)}{17.375}$$

$= 9.04 \times 10^{-5}\ \text{m}^2$

Therefore $d_2 = -\frac{\sqrt{(9.04 \times 10^{-5})}}{0.7854} = 0.0107\ \text{m}$

$= 10.7\ \text{mm}$

Problem 6.7 *The diameter of a diffuser varies from 200 mm to 300 mm over a length of 1.5 m. (1) Determine the total acceleration of flow at the mid section assuming constant flow @ 0.2 m³/sec, (2) Determine the total acceleration at the mid section at time t = 2.5 sec if the rate of flow increases uniformly from 0.2 m³ /sec to 0.4 m³/sec in 5 sec.*

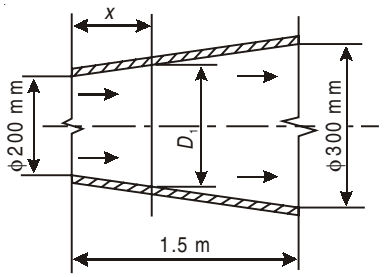


Fig. 6.14

Solution

(1) *When the flow is constant*

Diameter at a distance x from inlet

$$D_x = d + \left\{\frac{(D - d)}{L}\right\} \times (x)$$

$$D_x = (0.2) + \left\{\frac{(0.3 - 0.2)}{1.5}\right\} \times (x)$$

$= 0.2 + 0.066\ x$

Area at distance x $A_X = 0.7854\ (0.2 + 0.066\ x)^2$

Flow velocity at distance x. $V_x = \frac{Q}{A_X} = \frac{Q}{0.7854\ (0.2 + 0.066\ x)^2}$

$$\text{Rate of change of velocity} = \frac{dV_x}{dx}$$

$$= \frac{d}{dx}\left[\frac{Q}{0.7854\,(0.2 + 0.066\,x)^2}\right]$$

$$\frac{dV_x}{dx} = \frac{-2Q(0.066)}{0.7854\,(0.2 + 0.066\,x)^3}$$

$$= \frac{-0.168Q}{(0.2 + 0.066\,x)^3}$$

Total acceleration a_x = convective acceleration + local acceleration. (Ref. Eq. 6.21)

$$a_x = V_x\left(\frac{\partial v}{\partial x}\right) + \left(\frac{\partial v}{\partial t}\right)$$

$$\text{In this case } \left(\frac{\partial v}{\partial t}\right) = 0 \text{ since the flow is steady.}$$

Therefore $$a_x = V_x\left(\frac{\partial V_x}{\partial x}\right)$$

$$a_x = \left[\frac{Q}{0.7854\,(0.2 + 0.066\,x)^2}\right] \times \left\{\frac{-0.168Q}{(0.2 + 0.066\,x)^3}\right\}$$

$$a_x = \left\{\frac{-0.168\,Q^2}{0.7854\,(0.2 + 0.066\,x)^5}\right\}$$

At mid section where $x = 0.75$ m,

$$a_x = \left\{\frac{-0.168 \times (0.2)^2}{0.7854\,(0.2 + 0.066 \times 0.75)^5}\right\}$$

$$= -8.85 \text{ m/sec}^2$$

The negative sign is indicative of the decreasing velocity

(2) *When flow varies*

$$\text{At time } t = 2 \text{ sec}$$

$$Q = 0.2 + \frac{(0.4 - 0.2)}{5} \times 2$$

$$= 0.28 \text{ m}^3/\text{sec}$$

$$\text{Local acceleration } a_L = \left(\frac{\partial V}{\partial t}\right)$$

$$a_L = \frac{\partial}{\partial t}\left\{\frac{Q}{0.7854\,(0.2 + 0.066\,x)^2}\right\}$$

$$= \left\{\frac{1}{0.7854\ (0.2\ +\ 0.066\,x)^2}\right\} \times \left(\frac{\partial Q}{\partial t}\right)$$

$$a_L = \left\{\frac{1}{0.7854\ (0.2+0.066\,x)^2}\right\} \times \left\{\frac{0.28-0.2}{2.5}\right\}$$

$$= \left\{\frac{0.032}{0.7854\ (0.2+0.066\,x)^2}\right\}$$

At mid section where $x = 0.75$ m

$$a_L = \left\{\frac{0.032}{0.7854\ (0.2+0.066\times 0.75)^2}\right\}$$

$$= 0.65 \text{ m/sec}^2$$

Convective acceleration $a_c = V\left(\frac{\partial V}{\partial x}\right)$

$$a_c = \left\{\frac{-0.168Q^2}{0.7854\ (0.2+0.066\,x)^5}\right\}$$

At mid section where $x = 0.75$ m

and $Q = 0.28 \text{ m}^3/\text{sec}$

$$a_c = \left\{\frac{-\ 0.168(0.28)^2}{0.7854\ (0.2+0.066\times 0.75)^5}\right\}$$

$$= -17.12 \text{ m/sec}^2$$

Total acceleration $a_T = -17.12 + 0.65$

$$= -16.97 \text{ m/sec}^2$$

Problem 6.8 *The diameter of a diffuser varies from 300 mm to 500 mm over a length of 2 m. Determine the total acceleration at a section 1.5 m from the smaller end if the rate of flow at some point was found to be 150 liters/sec, uniformly increasing @ 30 liters/sec.*

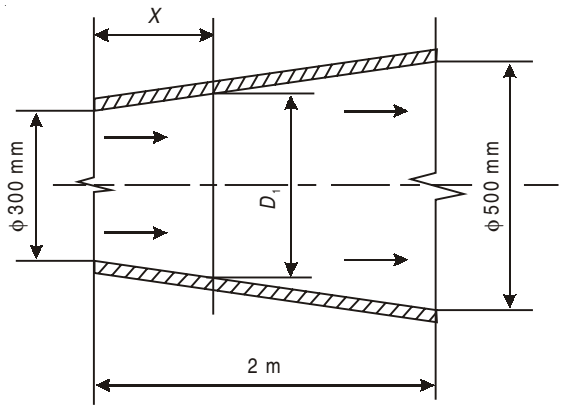


Fig. 6.15

Solution

Diameter at a distance x from inlet

$$D_x = d + \left\{\frac{(D-d)}{L}\right\} \times (x)$$

$$D_x = (0.3) + \left\{\frac{(0.5-0.3}{2}\right\} \times (x)$$

$$= (0.3 + 0.1\,x)$$

At x = 1.5 m,

D = (0.3 + 0.1 × 1.5)

= 0.45 m

Area at distance x = 1.5 m

A = 0.7854 $(0.45)^2$

= 0.159 m^2

Total acceleration a_x = convective acceleration + local acceleration. (Ref. Eq. 6.21)

$$a_x = V_x \left(\frac{\partial v}{\partial x}\right) + \left(\frac{\partial v}{\partial t}\right)$$

In this case since the flow is unsteady $\left(\frac{\partial v}{\partial t}\right)$ is given by,

$$a_L = \frac{\partial}{\partial t}\left\{\frac{Q}{A}\right\} = \left\{\frac{1}{A}\right\} \times \left(\frac{\partial Q}{\partial t}\right)$$

$$= \left\{\frac{1}{0.159}\right\} \times 0.03$$

$$= 0.1886 \text{ m/sec}^2$$

(1000 liters/sec = $1 m^3$/sec, 30 liters/sec = (30/1000) = 0.03 m^3/sec.

Convective acceleration $a_c = V_x \left(\frac{\partial V_x}{\partial x}\right)$

$$V_x = \frac{Q}{A_x} = \left\{\frac{\frac{150}{1000}}{0.7854\,(0.3 + 0.1\,x)^2}\right\}$$

$$= \left\{\frac{0.190}{(0.3+0.1\,x)^2}\right\} \quad \text{... (a)}$$

$$\left(\frac{\partial V_x}{\partial x}\right) = \left\{\frac{(-2)(0.1)(0.190)}{(0.3 + 0.1x)^3}\right\}$$

$$= \left\{\frac{0.038}{(0.3+0.1x)^3}\right\} \quad \text{... (b)}$$

From (a) and (b) convective acceleration

$$a_C = \left\{\frac{0.190}{(0.3+0.1x)^2}\right\} \times \left\{\frac{-0.038}{(0.3+0.1x)^3}\right\}$$

$$= \left\{\frac{-0.0072}{(0.3+0.1x)^5}\right\}$$

$$\text{At } x = 1.5 \text{ m}$$

$$a_C = \left\{\frac{-0.0072}{(0.3+0.1\times1.5)^5}\right\}$$

$$= -0.390 \text{ m/sec}^2$$

$$a_T = a_L + a_C$$

$$= 0.1886 + (-0.390)$$

$$= -0.2014 \text{ m/sec}^2$$

Problem 6.9 *While determining the velocity of flow over a sphere of radius 200 mm, (Fig. 6.16) the equation for upstream longitudinal velocity at a distance 'x' from the center of the sphere was* $u_1 = u_2\left(\frac{1-r^3}{x^3}\right)$. *If* u_2, *the undisturbed velocity down stream, is 70 m/sec determine the convective acceleration at x = 400 mm and at 200 mm.*

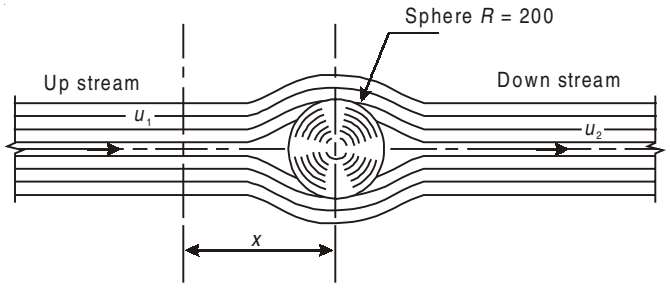


Fig. 6.16

Solution

Convective acceleration $a_c = V_x\left(\frac{\partial V_x}{\partial x}\right)$. In the present case the same thing can be written as,

$$a_c = u_1 \left(\frac{\partial u_1}{\partial x} \right) \text{ which means,}$$

$$a_c = u_2 \left(\frac{1-r^3}{x^3} \right) \frac{\partial}{\partial x} u_2 \left(\frac{1-r^3}{x^3} \right)$$

$$a_c = u_2 \left(\frac{1-r^3}{x^3} \right) u_2 \left\{ \frac{\partial}{\partial x} \left(\frac{1-r^3}{x^3} \right) \right\}$$

$$a_c = u_2 \left(\frac{1-r^3}{x^3} \right) u_2 \left\{ \left(\frac{3r^3}{x^4} \right) \right\}$$

If we now substitute the given values,

$$a_c = 60 \left(\frac{1- 0.200^3}{0.400^3} \right) \times 60 \left\{ \left(\frac{3 \times 0.200^3}{0.400^4} \right) \right\}$$

$$= 2953.125 \text{ m/sec}^2$$

At x = 200 mm the convective acceleration is zero because,

$$60 \left(\frac{1 - 0.200^3}{0.200^3} \right) = 0$$

Problem 6.10 *Show that the resultant velocity vector $V = i\,(8–2x) + j\,(7+5y) + k(6 – 3z)$ of an incompressible flow satisfies the continuity equation.*

Solution From Eq. (6.6), we know that the resultant velocity vector V is of the form $V = iu + jv + kw$. On comparing it with the given equation, $V = i(8–2x) + j(7 + 5y) + k(6–3z)$, it follows that,

$$u = (8 – 2x)$$

$$v = (7 + 5y)$$

and $$w = (6 – 3z)$$

The continuity equation for a three dimensional flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} = (–2), \quad \frac{\partial v}{\partial y} = + 5, \quad \frac{\partial w}{\partial z} = (–3)$$

Since (–2 + 5 –3) = 0 the continuity equation is satisfied.

Problem 6.11 *Check whether the velocity vector $V = i(20\,xyt + z) + j(–10\,y^2t + z + 6y) + k(– 6z + x^2\,t)$, satisfies the continuity equation and determine its velocity and acceleration vectors at point (1, 2, 1) and at t = 1.*

Solution

(i) From Eq. (6.6), we know that the resultant velocity vector V is of the form $V = iu + jv + kw$. On comparing it with the given equation, $V = i(20\,xyt + z) + j(-10y^2t + z + 6y) + k(-6z + x^2\,t)$ it follows that,

$$u = (20\,xyt + z),$$
$$v = (-10\,y^2t + z + 6y)$$

and
$$w = (-6\,z + x^2t),$$

The continuity equation for a three dimensional flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} = (+20\,yt),$$

$$\frac{\partial v}{\partial y} = (-20\,yt + 6)$$

and
$$\frac{\partial w}{\partial z} = -6$$

Since $(+20yt - 20yt + 6 - 6) = 0$ the continuity equation is satisfied.

(ii) The velocity components are

$$u = (20\,xyt + z),$$
$$v = (-10\,y^2t + z + 6\,y)$$

and
$$w = (-6\,z + x^2t).$$

At points (1, 2,1) and at $t = 1$

meaning at $x = 1$

$y = 2$

$z = 1$

and $t = 1$, these components are,

$u = (20 \times 1 \times 2 \times 1 + 1) = 41,$

$v = (-10 \times 2^2 \times 1 + 1 + 6 \times 2) = -27,$

$w = (-6 \times 1 + 1) = -5$

Hence the velocity vector at (1, 2,1) and at

$t = 1$, is

$V = 41\,i - 27\,j - 5\,k$

(iii) From Eq. (6.26) acceleration vector

$$a = ia_x + ja_Y + ka_Z,$$
$$u = (20\,xyt + z),$$

$$a_x = \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z}$$

$$a_x = 20\,xy + (20\,xyt + z)\,(20\,yt) + (-10\,y^2t + z + 6\,y)\,(20\,xt) + (-6\,z + x^2t)\,(1)$$

At (1, 2,1) and at t = 1

$a_x = (20 \times 2) + (20 \times 2 + 1)(20 \times 2) + (-10 \times 2^2 + 1 + 6 \times 2)(20) + (-6 + 1)$

= 1135 units

$v = (-10\, y^2 t + z + 6\, y)$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$a_y = (-10y^2) + (20xyt + z)(0) + (-10y^2 t + z + 6y)(-20yt + 6) + (-6z + x^2 t)(1)$

At (1, 2,1) and at t = 1

$a_y = (-10 \times 2^2) + (0) + (-10 \times 2^2 + 1 + 6 \times 2)(-20 \times 2 + 6) + (-6 + 1)$

= 1197 units

$w = (-6z + x^2 t)$.

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$a_z = (x^2) + (20xyt + z)(2xt) + (-10y^2 t + z + 6y)(0) + (-6z + x^2 t)(-6)$

At (1,2,1) and at t = 1

$a_z = (1) + (20 \times 2 + 1)(2) + (0) + (-6 + 1)(-6)$

= 113 units.

Therefore, acceleration vector at (1, 2,1) and at

t = 1, is $a = 1135i + 1197j + 113k$.

Problem 6.12 *Examine whether the velocity components $u = 4x + 5y$ and $v = 5y - 4x$ represent a possible flow.*

Solution We have to examine whether the given components comply with the continuity equation for 2-D fluid flow.

This means whether $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$\frac{\partial u}{\partial x} = 4$$

and

$$\frac{\partial v}{\partial y} = 5$$

Since $4 + 5 \neq 0$, the given components do not represent a 2-D steady flow for an in compressible fluid.

Problem 6.13 *Determine the missing velocity component from out of the given components. $u = 4x^2y$ and $v = 8y^2z$.*

Solution

The continuity equation for a three dimensional flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$8xy + 16yz + \frac{\partial w}{\partial z} = 0$$

or
$$\frac{\partial w}{\partial z} = -(8xy + 16yz)$$

$$\text{Third component } z = \int -(8xy + 16yz)\, dz$$
$$= -(8xyz + 8yz^2) + f(x)$$

Problem 6.14 *The velocity vector for a two dimensional flow field is represented by* $V = 6x^3i - 18x^2yj$. *Determine whether the flow is rotational or irrotational. If rotational what is the angular velocity and the shear strain rate?*

Solution

$$\text{Given } V = 6x^3i - 18x^2yj.$$
$$u = 6x^3$$
and
$$v = -18x^2y$$

Therefore,
$$\frac{\partial u}{\partial y} = 18x^2$$

and
$$\frac{\partial v}{\partial y} = -18x^2$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = +18x^2 - 18x^2 = 0$$

This proves that the possibility of flow exists.

The condition for irrotaionality is $\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$

$$\frac{\partial u}{\partial y} = 0$$

and
$$\frac{\partial v}{\partial x} = -36xy$$

Since $\frac{\partial u}{\partial y} \neq \frac{\partial v}{\partial x}$ the flow is rotational.

$$\text{Angular velocity } \omega = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$$

$$w = \frac{1}{2}\,[-36xy - 0\,]$$
$$= -18xy$$

$$\text{Shear strain rate } \varepsilon = \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$

$$\varepsilon = \frac{1}{2}\left[-36xy + 0\right]$$

$$= -18xy$$

Problem 6.15 *The velocity vectors for a three dimensional in-compressible flow are as follows. $V = (4 + 3y + 2z)\, i - (5 + 9x - 6z)\, j + (7 + 8x + 10y)\, k$. Do the vectors represent fluid flow? Further examine whether the flow is rotational and if so determine vorticity and rotation.*

Solution

$$u = (4 + 3y + 2z),$$

$$v = -(5 + 9x - 6z)$$

and

$$w = (7 + 8x + 10y)$$

$$\frac{\partial u}{\partial x} = (0)$$

$$\frac{\partial v}{\partial y} = 0.$$

and

$$\frac{\partial w}{\partial z} = 0$$

Since the continuity equation is satisfied the velocity components do represent fluid flow.

The flow is rotational because, $\frac{\partial u}{\partial y} = (3) \neq \frac{\partial v}{\partial x} = (-9)$

Also if $\nabla \times v = 0$ it indicates irrotaionality.

$$\nabla \times v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$\nabla \times v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (4+3y+2z) & (5+9x-6z) & (7+8x+10y) \end{vmatrix}$$

$$\nabla \times v = i\left\{\frac{\partial}{\partial y}(7+8x+10y) - \frac{\partial}{\partial z}(5+9x-6z)\right\}$$

$$+ j\left\{\frac{\partial}{\partial z}(4+3y+2z)-\frac{\partial}{\partial x}(7+8x-10y)\right\}$$

$$+ k\left\{\frac{\partial}{\partial x}(5+9x+6z)-\frac{\partial}{\partial y}(4+3y-2z)\right\}$$

Since $\nabla \times v = i\,(10-6) + j\,(2-8) + k\,(-9-3) \neq 0$ the flow is not irrotational.

$$\text{Vorticity } \Omega = (\nabla \times v) = 4i - 6j - 12k$$

and $$\text{Rotation } \omega = \frac{\Omega}{2} = \frac{1}{2}\,(4i - 6j - 12k)$$

Problem 6.16 *Obtain the stream function for the following velocity components assuming the given functions satisfy the continuity equation.* $u = 10\,xy$ *and* $v = -5xy^2$.

Solution

Equation (6.39) establishes the relation between velocity components and stream function thus,

$$u = \left(\frac{\partial \psi}{\partial y}\right) \text{and } v = -\left(\frac{\partial \psi}{\partial x}\right)$$

$$v = \left(\frac{\partial \psi}{\partial x}\right)$$

$$= -(-5xy^2) = 5xy^2$$

$$\psi = \int 5xy^2\, dx$$

$$= \left(\frac{5x^2}{2}\right) y^2 + C \qquad \text{... (a)}$$

(Here C is constant of integration which can be a function of y. Let us therefore differentiate ψ w.r.t to y)

$$\left(\frac{\partial \psi}{\partial y}\right) = \left(\frac{5x^2}{2}\right)(2y) + \frac{dC}{dy}$$

$$\left(\frac{\partial \psi}{\partial y}\right) = (5x^2y) + \frac{dC}{dy}.$$

But $$\left(\frac{\partial \psi}{\partial y}\right) = u = 10\,xy$$

Therefore by equating the two values,

$$(5\,x^2y) + \frac{dC}{dy} = 10xy$$

$$\frac{dC}{dy} = 10xy - 5x^2y$$

$$C = \int (10xy - 5x^2y)\,dy$$

$$= 5xy^2 - 5x^2\left(\frac{y^2}{2}\right) -.$$ If we now substitute this value of C in Eq. (a), we obtain the relevant stream function.

$$\psi = -\left(\frac{5x^2}{2}\right)y^2 + 5xy^2 - 5x^2\left(\left(\frac{y^2}{2}\right)\right.$$

$$= 5(xy^2 - x^2y^2)$$

Problem 6.17 *State the necessary and sufficient condition for the velocity potential to exist in a two dimensional incompressible flow. Assuming two dimensional flow determine from the velocity component $u = (4x^2 - 8y)$, (i) the unknown velocity component (ii) form of the velocity potential.*

Solution The necessary and sufficient condition for the velocity potential to exist in a two dimensional incompressible flow is that it must satisfy the continuity equation and the flow must be irrotational.

(i) $u = (4x^2 - 8y)$,

Therefore $$\frac{\partial u}{\partial x} = 8x$$

$$= -\left(\frac{\partial v}{\partial y}\right)$$

$$v = \int 8x\,dy$$

$$= -8xy + C$$

Therefore the velocity components are $u = (4x^2 - 8y)$

and $$v = -8xy$$

(ii) Velocity potential φ

$$\left(\frac{\partial \varphi}{\partial x}\right) = -u = -(4x^2 - 8y) \qquad ...(a)$$

Hence, $$\int \partial\varphi = \int (4x^2 - 8y)\,\partial x$$

$$\varphi = -\left[\frac{4x^3}{3} - 8xy\right] + C \qquad \text{... (b)}$$

$$\left(\frac{\partial \varphi}{\partial y}\right) = -v$$

$$= -(-8xy) = 8xy. \qquad \text{... (c)}$$

Hence, $$\int \partial\varphi = \int (8xy)\, \partial y$$

$$\varphi = (4xy^2) + C \qquad \text{...(d)}$$

Differentiate (d) with respect to y

$$\left(\frac{\partial \varphi}{\partial y}\right) = 8x + \frac{\partial C}{\partial y} \qquad \text{... (e)}$$

But from (c), $$\left(\frac{\partial \varphi}{\partial y}\right) = 8xy$$

If we now equate (e) and (c) $8x + \dfrac{\partial C}{\partial y} = 8xy$

or $$\frac{\partial C}{\partial y} = 8xy - 8x \text{ and upon integrating,}$$

$$C = \int ((8xy - 8x)\, \partial y$$

$$= (4xy^2 - 8xy)$$

Substitute this value of C in Eq. (d) to obtain the form of velocity potential.

$$\varphi = (4xy^2) + (4xy^2 - 8xy)$$

$$= 8\,(xy^2 - xy)$$

Problem 6.18 *Determine the velocity potential* φ *for the flow field characterized by the equation* $\psi = x^3 y - x y^3$.

Solution

(i) For a given stream function flow is possible but velocity potential cannot be defined until we determine whether the flow is irrotational. That is whether ψ satisfies the Laplace equation.

For the function $$\psi = x^3 y - x y^3.$$

$$\left(\frac{\partial \psi}{\partial x}\right) = -v$$

$$= -(3x^2 y - y^3) \text{ and } \left(\frac{\partial^2 \psi}{\partial x^2}\right) = -6xy$$

$$\left(\frac{\partial \psi}{\partial y}\right) = (x^3 - 3x y^2) \text{ and } \left(\frac{\partial^2 \psi}{\partial y^2}\right) = 6xy$$

Since $6xy - 6xy = 0$ the Laplace equation is satisfied that is,

$\left(\frac{\partial^2 \psi}{\partial x^2}\right) + \left(\frac{\partial^2 \psi}{\partial y^2}\right) = 0$ and hence the flow is irrotational and hence the velocity potential can be defined.

(ii) *Velocity potential* φ. The given stream function is $\psi = x^3 y - x y^3$

The velocity component $u = \left(\frac{\partial \psi}{\partial y}\right) = (x^3 - 3x y^2)$

The velocity component $v = -\left(\frac{\partial \psi}{\partial x}\right) = -(3x^2 y - y^3)$

$$\left(\frac{\partial \varphi}{\partial x}\right) = -\left(\frac{\partial \psi}{\partial y}\right) = u = (x^3 - 3x y^2) \quad \text{... (a)}$$

Hence, $\int \partial\varphi = \int (x^3 - 3x y^2)\partial x$

$$\varphi = \left[\frac{x^4}{4} - \left(\frac{3x^2}{2}\right)y^2\right] + C \quad \text{... (b)}$$

$$\left(\frac{\partial \varphi}{\partial y}\right) = -\left(\frac{\partial \psi}{\partial x}\right) = v = -(3x^2 y - y^3) \quad \text{... (c)}$$

Hence, $\int \partial\varphi = -\int (3x^2 y - y^3)\partial y$

$$\varphi = -\left[3x^2\left(\frac{y^2}{2}\right) - \frac{y^4}{4}\right] + C \quad \text{... (d)}$$

Differentiate $\varphi = \left[\frac{x^4}{4} - \left(\frac{3x^2}{2}\right)y^2\right] + C$ .....(b) with respect to y

$$\left(\frac{\partial \varphi}{\partial y}\right) = -3x^2 y + \frac{\partial C}{\partial y} \quad \text{... (e)}$$

But from (c), $\left(\frac{\partial \varphi}{\partial y}\right) = -(3x^2y - y^3)$ ... (f)

If we now equate (e) and (f) $-3x^2y + \frac{\partial C}{\partial y} = -(3x^2y - y^3)$

or $\frac{\partial C}{\partial y} = y^3$ and upon integrating,

$$C = \int y^3 \, \partial y$$

$$= \frac{y^4}{4}$$

Substitute the value of C in Eq. (d)

$$\varphi = -\left[3x^2\left(\frac{y^2}{2}\right) - \frac{y^4}{4}\right] + \frac{y^4}{4}$$

$$= -3x^2\left(\frac{y^2}{2}\right) + \frac{y^4}{2}$$

$$= -\left(\frac{y^2}{2}\right)(3x^2 - y^2)$$

Problem 6.19 *In a 2D flow field of an incompressible fluid the equation for velocity potential and stream function are* $\varphi = -\left(\frac{y^2}{2}\right)(3x^2 - y^2)$ *and* $\psi = x^3 y - x y^3$. *Determine the resultant velocity and its direction at points (4, 5).*

Solution The velocity components u and v, in the x and y directions have to be determined from the given equations.

(i) $$\varphi = -\left(\frac{y^2}{2}\right)(3x^2 - y^2)$$

$$u = \left(\frac{\partial \varphi}{\partial x}\right) = -\frac{6xy^2}{2}$$

$$v = \left(\frac{\partial \varphi}{\partial y}\right) = -(3x^2y - 3y)$$

at points (4, 5) $u = -\frac{(6 \times 4 \times 5^2)}{2}$

$$= -300 \text{ units}$$

$$v = -[(3 \times 4^2 \times 5) - 2 \times 5]$$
$$= -230 \text{ units}$$

The resultant velocity $V = \sqrt{u^2 + v^2}$

$$= \sqrt{(-300)^2 + (-230)^2}$$
$$= 378 \text{ units}$$

The direction of V is given by its slope.

For the curve $\varphi = \text{Constant}\left(\frac{\partial y}{\partial x}\right)$

$$= \frac{u}{v} = \frac{300}{230} = 1.30$$

Since $\tan\theta = 1.30, \theta = 52.43°.$

(ii) $\psi = x^3 y - x y^3$

$$u = \left(\frac{\partial \psi}{\partial y}\right) = x^3 - 3xy^2$$

$$v = -\left(\frac{\partial \psi}{\partial x}\right)$$
$$= -(3x^2 y - y^3)$$

At points (4, 5) $u = 4^3 - 3 \times 4 \times 5^2$

$$= -236$$
$$v = -(3 \times 4^2 \times 5 - 5^3)$$
$$= -115$$

The direction of ψ is given by its slope.

For the curve ψ = Constant

$$\left(\frac{\partial y}{\partial x}\right) = \frac{u}{v}$$
$$= \frac{-115}{-236} = 0.48$$

Since $\tan\theta = 0.48, \theta = 25.6°.$

Problem 6.20 *If* $\varphi = -\left(\frac{y^2}{2}\right)(3x^2 - y^2)$ *determine the discharge between streamlines at velocity points (1, 3) and (3, 3).*

Solution First determine the velocity components u and v for the given function.

$$u = \left(\frac{\partial \varphi}{\partial x}\right) = -\frac{6xy^2}{2}$$

$$v = \left(\frac{\partial \varphi}{\partial y}\right) = -(3x^2y - 2y)$$

Now to find the discharge between streamlines we have to obtain the stream function.

$$\left(\frac{\partial \varphi}{\partial x}\right) = \left(\frac{\partial \psi}{\partial y}\right) = u$$

Therefore $(\partial \psi) = u\ \partial y$

$$\left(\frac{\partial \varphi}{\partial y}\right) = \left(\frac{\partial \psi}{\partial x}\right) = v$$

Therefore $(\partial \psi) = -v\ \partial x$

Therefore $(\partial \psi) = -v\ \partial x + u\ \partial y$

$$= \left[-\left(3x^2y - 2y\right)dx - \left(3xy^2\right)dy\right]$$

Upon integrating $\psi = (x^3y - xy^3 + C)$

Discharge at points (1, 3) and (3, 3) between streamlines, $x^3y - xy^3$

$$\psi_{(1,3)} - \psi_{(3,3)} = (3 - 27) - (27 \times 3 - 3 \times 27)$$
$$= -24 \text{ units.}$$

Problem 6.21 *In a two dimensional flow field given the velocity components, $u = -2y$ and $v = 2x$. Sketch the streamlines and equi-potential lines. Assume the velocity components satisfy both the continuity and Laplace equations.*

Solution From the velocity components given, we have to obtain both the velocity potential φ and the stream function ψ.

(i) Velocity potential.

$$u = \left(\frac{\partial \varphi}{\partial y}\right) = -2y,$$

Therefore $(\partial \varphi) = u\ \partial y$

$$= -2y\ \partial y$$

$$v = \left(\frac{\partial \varphi}{\partial x}\right) = 2x,$$

Therefore $(\partial \varphi) = v\ \partial x$

$$= 2x\ \partial x$$

or $(\partial \varphi) = v\ \partial x + u\ \partial y$

or $(\partial \varphi) = 2x\ \partial x - 2x\ dy$

Upon integrating, $\varphi = x^2 - y^2 + C$

(ii) Stream function

$$\left(\frac{\partial \varphi}{\partial x}\right) = \left(\frac{\partial \psi}{\partial y}\right) = v = 2x$$

Therefore $(\partial \psi) = 2x\, dy$. Upon integrating,

$$\psi = 2xy + C$$

The constant of integration C can be ignored.

Therefore stream function $\psi = 2xy$

These curves $x^2 - y^2 = C$ and $2xy = C$ represent a rectangular hyperbola. The slope of φ lines ($\varphi = C$ curves) $= \left(-\frac{u}{v}\right) = -\left(-\frac{2y}{2x}\right) = \left(\frac{y}{x}\right)$ and ψ lines ($\psi = C$ curves) $= \left(\frac{v}{u}\right) = \left(-\frac{2x}{2y}\right) = \left(-\frac{x}{y}\right)$ and the product of their slopes = (–1) which shows they always intersect each other orthogonally. A sketch of a few of these curves is shown in Fig. (6.17). The stream lines represent the flow of a jet of incompressible fluid against a fixed plane.

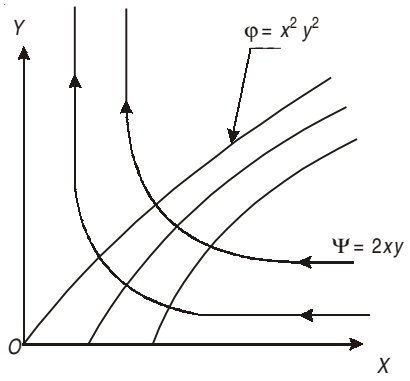


Fig. 6.17

Problem 6.22 *For the velocity vector $V = 4xi + 4yj + 8zk$ determine whether the flow is rotational or irrotational and obtain the equation for the stream line passing through points (1, 2, 3).*

Solution

The equation for the stream line is given by $\left(\frac{\partial x}{u}\right) = \left(\frac{\partial y}{v}\right) = \left(\frac{\partial z}{w}\right)$ ...(a)

$$u = 4x,$$

$$v = 4y,$$

and $$w = 8z.$$

Substituting these values in (a) we have,

$$\left(\frac{\partial x}{4x}\right) = \left(\frac{\partial y}{4y}\right) = \left(\frac{\partial z}{8z}\right)$$

First integrate equations involving x and y;

$$\frac{1}{4}\log x = \frac{1}{4}\log y + \log C$$

Therefore $$y = C_1 x$$

Hence $$C_1 = \frac{y}{x} \quad \text{... (b)}$$

Likewise we now integrate equations involving x and z;

$$\frac{1}{4}\log x = \frac{1}{8}\log z + \log C_2$$

Therefore $$z = C_2 x^2$$

Hence $$C_3 = \frac{z}{x^2} \quad \text{... (c)}$$

If we now substitute the given co-ordinates (1, 2, 3) in (b) and (c)

$$C_1 = \frac{y}{x}$$

$$= \frac{2}{1} = 2.$$

Hence $$y = 2x$$

and $$C_3 = \frac{z}{x^2} = \frac{3}{1} = 3$$

$$= 3 \times 1 = 3.$$

Hence $$z = 3x^2$$

The equation of stream lines are, $y = 2x$ and $z = 3x^2$

QUESTIONS BANK

Section A

Review Questions

1. Explain how the application of boundary layer concepts has helped scientists to analyze fluid flow phenomenon with a relatively high degree of accuracy?
2. Define the terms 'Convective acceleration, temporal acceleration and tangential acceleration.
3. Based on the velocity equation given, how would you conclude whether the flow is steady or unsteady?

4. State an exception to the principle 'Matter is neither created nor destroyed'. Explain how, this principle lead to the development of the continuity equation.
5. Distinguish clearly between steady flow and unsteady flow, uniform and non uniform flow.
6. Define stream line and streak line, velocity potential and stream function.
7. Define an Equi-potential line and state the conditions necessary to conclude a flow as irrotational.
8. Define a flow net and state the condition necessary for the flow nets to be formed.
9. Define circulation and Vorticity. What factors you would consider to explain that in case of ideal fluids the flow is always irrotational?
10. Sketch the velocity profiles for ideal fluid, real fluid, turbulent flow and uniform ir-rotational flow.

Section B

Multiple Choice Questions

Choose the right answer

1. A three dimensional compressible flow is said to be steady if

   (a) $\frac{\partial v}{\partial t} = 0$ (b) $\frac{\partial p}{\partial t} = 0$

   (c) $\frac{\partial \rho}{\partial t} = 0$ (d) All of these

   (e) None of these
2. In non-uniform flows the velocity vector

   (a) $\frac{\partial V}{\partial t} = 0$ (b) $\frac{\partial V}{\partial t} \neq 0$

   (c) $\frac{\partial V}{\partial s} = 0$ (d) None of these
3. A fluid element during its movement might have a motion which is
   (a) rotational translation
   (b) irrotational translation
   (c) pure angular or linear distortion
   (d) All of these
   (e) None of these
4. The function $v = f(x, y, z, t)$ defines
   (a) three dimensional steady flow

(b) two dimensional unsteady flow
(c) three dimensional unsteady flow
(d) None of these

5. The equation $\rho_1 A_1 V_1 = \rho_2 A_2 V_2$ represents
   (a) the continuity equation for steady compressible flow
   (b) the continuity equation for steady in-compressible flow
   (c) the continuity equation for two dimensional steady compressible flow
   (d) None of these
6. If ψ exists then it establishes
   (a) that the flow is irrotational
   (b) that the flow is rotational
   (c) that the flow is possible
   (d) that the flow satisfies the laplace equation.
7. If the flow is steady as well as irrotational then it is described as
   (a) unsteady flow (b) two dimensional flow
   (c) turbulent flow (d) potential flow
8. Stream function is applicable only to
   (a) steady and uniform flows
   (b) steady and ir-rotational flows
   (c) three dimensional flows (d) two dimensional flows.
9. If the velocity potential φ satisfies the laplace equation it means the flow is
   (a) steady, compressible and irrotational
   (b) uniform and steady
   (c) rotational
   (d) steady, incompressible and irrotational.
10. The continuity equation $\left(\frac{\partial u}{\partial x}\right)+\left(\frac{\partial v}{\partial y}\right)+\left(\frac{\partial w}{\partial z}\right)=0$ holds good
    (a) for all flows whether compressible or incompressible
    (b) only for steady in-compressible flow
    (c) steady or unsteady in compressible flow
    (d) None of these

Section C

Problems

1. Water at 70° C is flowing through a pipe. If at any arbitrary sections 1 and 2, the flow velocities are is 8 m/sec and 2 m/sec respectively, calculate the pipe inside diameter at section 2, if the pipe inside diameter at section 1 is 50 mm. (*Ans.* 100 mm)

2. A conical duct has a diameter ratio of 1 : 3 increasing uniformly over a length of 1 metre. Water at 30°C enters the duct at the smaller end which has a diameter of 1 metre @ 8 metres/sec. Determine the velocity at 0.6 metres from the entry point and the mass flow rate.

   (*Ans.* 1.65 m/sec, 6254.8 kg/sec)

3. If the discharge through the duct in example (2) was found to increase at the rate of 1.5 m^3/sec, determine the local and total accelerations at 0.6 meters from the entry point. (*Ans.* 0.39 m/sec^2 and – 4.57 m/sec^2)

4. For a steady and incompressible flow examine whether the following velocity components can exist.

   (1) $u = (4x + y^2)$ and $v = (-4y + x^2)$

   (2) $(6xt)i + (xy^2)j + (-2xyz - 6tz)k$

   (3) $u = (3x + y)$, $v = (3y + x)$ and $z = (3x + z)$

5. Given the velocity field $V = 2x^2yi - 2xy^2j + 4tk$ determine the acceleration vector at point (1,1,1) and t =1. ($a = 4i + 4j + 4k$)

6. In a two dimensional flow field, $u = 4y$ and $v = -4x$. Determine the circulation and rotation about a circle of radius a ($\Gamma = 25.12\ a^2$, $\omega = 4$ in appropriate units).

7. Examine whether the velocity vector $V = -xi + 2yj + (5 - z)k$ signifies a rotational or irrotational flow and accordingly determine its vorticity and rotation.

8. Determine the velocity components of the velocity potential function $\varphi = 9xy$ and find the discharge between streamlines at (2,4) and (3, 4).

9. State the condition that an irrotational flow must satisfy. Write down the Cauchy–Riemann equation that relates to velocity potential and stream function. What is the physical significance of the term $v \times \nabla$?

10. State the condition necessary for the potential flow to exist. In a two dimensional in-compressible flow $u = (x^2 - y)$ and $v = (1-2xy)$. Determine the stream function and the velocity potential.

Answers to Multiple Choice Questions

1. (d)	**2.** (b)	**3.** (d)	**4.** (c)	**5.** (a)
6. (c)	**7.** (d)	**8.** (d)	**9.** (d)	**10.** (c)

Chapter 7

Equation of Motion and Energy Equation

The theory and practice of Fluid Mechanics is resting basically on Newton's three fundamental laws of motion. We have already used the first law $\Sigma F = 0$ in earlier chapters while dealing with hydro-static forces, buoyancy and floatation. The present chapter will relay extensively on the second law of motion ($F = ma$) and later chapters, on the third law of motion.

We have already discussed the constraints of Lagrangian method of analysis and the need for an alternative approach while analyzing the phenomenon of fluid flow. While the Eulerian control volume concepts mostly fulfilled that need it was only after the development of *Reynolds transport theorem* that scientists could convert from Lagrangian system and apply the Newton's second law to fluid particles within a control volume which in turn lead to the development of what is now famously known as the *Euler's equation of motion.* While Newton's momentum Equation with the assistance of Reynolds transport theorem lead to the development of Euler's equation of motion, the Bernoulli's equation originated from the Euler's equation of motion.

Euler's equation of motion basically establishes a relation between pressure, velocity, elevation and density of fluid particles within a control volume. The equation is developed for an inviscid flow along a stream line. Since there is no flow across a stream line the focus is only on the velocity in the direction of its assumed flow. This makes the derivation of the equation simpler. Also, in the derivation of this equation an ideal flow is assumed thus eliminating the viscous forces. Further the flow is assumed to be steady and hence $\left(\frac{\partial v}{\partial t}\right) = 0$.

Imagine a cylindrical control volume between stations 1 and 2 of length δs. Let the velocity at the entry be v and at the exit be $(v + \delta v)$. Let the area and

pressure at the entry point be δA and p respectively and let the corresponding parameters at the exit be $(p + \delta p)$ and δA . Let ρ be the density of the fluid.

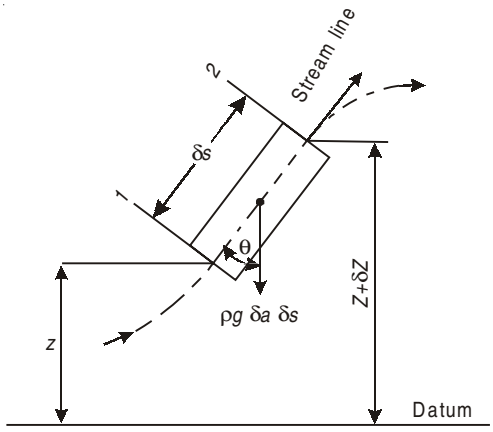


Fig. 7.1

There is a gain in momentum within the control volume = $\rho\delta A v\ (\delta v)$, where $\rho\delta A v$ is the mass flow rate in unit time and (δv) is change in velocity.

Now this gain in momentum is due to the resultant of all the forces acting during the motion of the fluid particle along the stream line. The forces are,

1. Pressure forces at station 1 in the direction of motion = $(p \times \delta A)$
2. Pressure forces at station 2 in the opposite direction = $(p + \delta p)\ \delta A$
3. Force due to the weight of the element which has a component opposing the motion $= (\rho g\ \delta A\ \delta s) \cos\theta = (\rho g\ \delta A\ \delta s)\left(\dfrac{\delta z}{\delta s}\right)$

[As δs increases the vertical component δz also increases such that $\cos\theta = \left(\dfrac{\delta z}{\delta s}\right)$]

$$\text{Net force causing flow} = (p \times \delta A) - (p + \delta p)\,\delta A - (\rho g\ \delta A\ \delta s)\left(\frac{\delta z}{\delta s}\right)$$

$$= (p \times \delta A) - p\delta A - \delta A\ \delta p - (\rho g\ \delta A\ \delta z)$$

$$\text{Net force causing flow} = -\delta A\ \delta p - (\rho g\ \delta A)\,(\delta z)$$

According to Newton's second law the net force causing the flow must be equal to the gain in momentum. Therefore,

$$-\delta A\ \delta p - (\rho g\ \delta A)\,(\delta z) = \rho\delta A v\ (\delta v),$$ Divide through by $(-\rho\ \delta A)$

$$\left(\frac{1}{\rho}\right)\delta p + g\ \delta z = -v\ (\delta v)$$ we now re-arrange to obtain the

Euler's equation for steady frictionless flow along a stream line,

$$\frac{\delta p}{\rho} + g\,\delta z + v\,\delta v = 0 \qquad \ldots (7.1)$$

Integration of Eq. (7.1) assuming the fluid as incompressible leads to the *Bernoulli's equation*. $\frac{P}{\rho} + gZ + \frac{v^2}{2} = C$. We may now re-arrange to obtain the common form in which it is stated.

$$Z + \frac{v^2}{2g} + \frac{P}{\rho} = C \qquad \ldots (7.2)$$

C is referred to as the Bernoulli's constant.

Each term in Eq. (7.2) represents a particular form of energy per unit weight possessed by the fluid in motion.

Z refers to the *potential energy* that the fluid possessed due to its elevation. The term $\frac{v^2}{2g}$ refers to the kinetic energy due to the velocity and $\frac{P}{\rho}$ is the pressure energy. Together, they represent the total energy per unit weight that the fluid element possesses. If we take weight factor into consideration Eq. (7.2) can be written as,

Total energy $$E = wZ + \frac{wv^2}{2g} + \frac{wP}{\gamma}.$$

If E represents the total energy at one point of the flow stream, as the fluid element moves to the next point, the total energy

$$E_1 = wZ_1 + \frac{wv_1^2}{2g} + \frac{wP_1}{\gamma}.$$

If in between these two points if no energy is lost or gained then according to the principle of conservation of energy $E = E_1$.

Therefore, $$wZ + \frac{wv^2}{2g} + \frac{wP}{\gamma} = wZ_1 + \frac{wv_1^2}{2g} + \frac{wP_1}{\gamma} \qquad \ldots (7.3)$$

If we eliminate w because it is common on either side,

$$Z + \frac{v^2}{2g} + \frac{P}{\gamma} = Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} \qquad \ldots (7.4)$$

This is the extended version of the Bernoulli's Eq. (7.2). This equation is good for (1). Incompressible flow (2). Inviscid flow (3). Irrotational flow (4) One dimensional flow under steady state conditions.

If we consider each term in Eq. (7.3) it represents some form of energy and therefore has the units N-m whereas each term in Eq. (7.4) represents energy/ unit weight and therefore has the units $\frac{\text{N} - \text{m}}{\text{N}}$ = m. Since m is a unit of height, each term in Eq. (7.4) can be expressed as, elevation head, velocity head and pressure head.

TYPICAL PROBLEMS AND SOLUTIONS

Problem 7.1 *Water at 4°C is flowing through a converging-diverging duct. At a certain point in the flow stream the following values were recorded. Area of the flow passage = 9.6 × 10^{-4} m^2. Pressure = 200 kPa (gauge) and flow velocity = 3.5 m/sec. At a certain other point up stream the following value was recorded. Area of the flow passage = 3.8 × 10^{-3} m^2. Determine the pressure and velocity at this point if the difference in elevation between the two points is 0.75 m.*

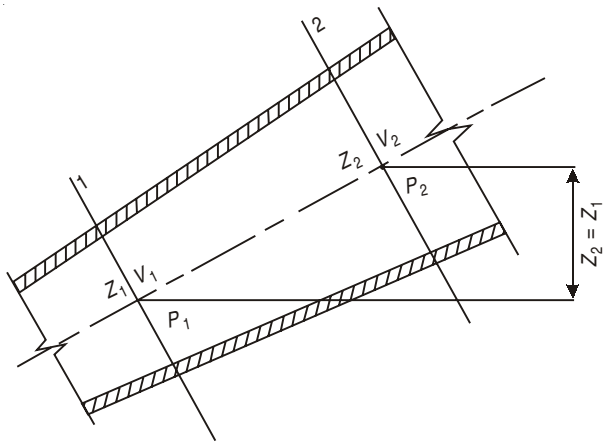


Fig. 7.2

Solution

Given:

$$A_1 = 9.6 \times 10^{-4}\ \text{m}^2$$
$$A_2 = 3.8 \times 10^{-3}\ \text{m}^2$$
$$p_1 = 200\ \text{kPa (gauge)}$$
$$p_2 = \text{to be determined}$$
$$v_1 = 3.5\ \text{m/sec}$$
$$v_2 = \text{to be determined}$$
$$Z_2 - Z_1 = 0.75\text{m or } (Z_1 - Z_2 = -0.75\ \text{m})$$
$$\gamma = 9.81\ \text{kN/m}^3$$

and $g = 9.81\ \text{m/sec}^2$

v_2 can be determined straight away from the continuity equation.

$$v_2 = v_1\left(\frac{A_1}{A_2}\right) = 3.5\left(\frac{9.6 \times 10^{-4}}{3.8 \times 10^{-3}}\right)$$
$$= 0.88\ \text{m/sec.}$$

We may now apply the Bernoulli's equation to obtain the value of p_2

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} = Z_2 + \frac{v_2^2}{2g} + \frac{P_2}{\gamma}.$$

Since it is p_2 that we need the equation could be re-arranged suitably in stages as shown.

$$\frac{P_2}{\gamma} = Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} - Z_2 - \frac{v_2^2}{2g}$$

$$\frac{(P_2 - P_1)}{\gamma} = (Z_1 - Z_2) + \frac{(v_1^2 - v_2^2)}{2g}$$

$$(P_2 - P_1) = \gamma \left\{ \frac{(Z_1 - Z_2) + (v_1^2 - v_2^2)}{2g} \right\}$$

$$P_2 = P_1 + \gamma \left\{ \frac{(Z_1 - Z_2) + (v_1^2 - v_2^2)}{2g} \right\}$$

We may now substitute the known values.

$$P_2 = 200 + 9.81 \left\{ \frac{(-0.75) + (3.5^2 - 0.88^2)}{2 \times 9.81} \right\}$$

$$= 198.38 \text{ gauge}$$

Problem 7.2 *Water at 4°C is flowing through the duct shown in Fig. (7.3). Calculate the velocity at point 1 and the mass flow rate if the pressure difference between points 2 and 1 = 15 kN/m².*

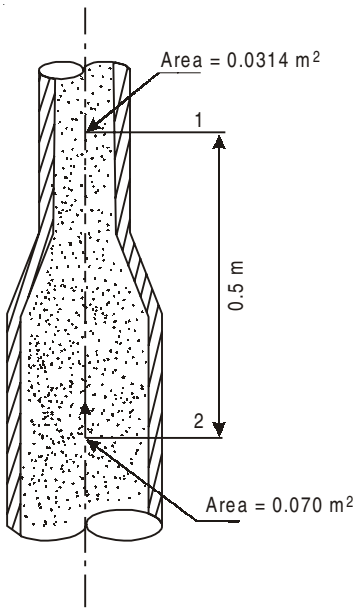


Fig. 7.3

Given: $A_1 = 0.0314\ m^2$

$A_2 = 0.070\ m^2$

$(P_2 - P_1) = 15\ kN/m^2$, $(Z_1 - Z_2)$

$= 0.5$ m, γ_{water}

$= 9.81\ kN/m^3$

and $g = 9.81\ m/sec^2$

The only value unknown in the Bernoulli's equation is the velocity. So the equation may be re-arranged suitably.

$$P_2 = P_1 + \gamma \left\{ \frac{(Z_1 - Z_2) + \left(v_1^2 - v_2^2\right)}{2g} \right\}$$

$$(P_2 - P_1) = \gamma (Z_1 - Z_2) + \gamma \frac{\left(v_1^2 - v_2^2\right)}{2g}$$

Dividing throughout by γ

$$\frac{(P_2 - P_1)}{\gamma} = (Z_1 - Z_2) + \frac{\left(v_1^2 - v_2^2\right)}{2g}$$

$$\frac{\left(v_1^2 - v_2^2\right)}{2g} = \frac{(P_2 - P_1)}{\gamma} - (Z_1 - Z_2)$$

$$(v_1^2 - v_2^2) = 2g \left\{ \frac{(P_2 - P_1)}{\gamma} - (Z_1 - Z_2) \right\}$$

We may now substitute the known values

$$(v_1^2 - v_2^2) = 2 \times 9.81 \left\{ \frac{(15)}{9.81} - (0.5) \right\}$$

$$= 20.19 \quad \text{... (a)}$$

From the continuity equation $\frac{V_2}{V_1} = \frac{A_1}{A_2}$

$$\frac{V_2}{V_1} = \frac{0.0314}{0.070}$$

$$= 0.448$$

$$\frac{V_2^2}{V_1^2} = (0.448)^2$$

$$= 0.2007$$

or $V_2^2 = (0.2007\ V_1^2)$

We may substitute this value of V_2^2 in Eq. (a)

$$v_1^2 - 0.2007\ V_1^2 = 20.19$$

or $$v_1^2\ (1 - 0.2007) = 20.19$$

or $$v_1^2\ (0.7993) = 20.19$$

or $$v_1^2 = \frac{20.19}{0.7993}$$

$$= 25.25$$

$$v_1 = 5.025 \text{ m/sec.}$$

$$\text{Mass flow rate} = \rho Q$$

$$= 1000\ (A_1 V_1)$$

$$= 1000 \times 0.0314 \times 5.025$$

$$= 157.85 \text{ kg/sec}$$

Problem 7.3 *Water is siphoned off from a tank as shown in Fig. 7.4. The I.D. of the siphon = 30 mm. Calculate (1) Volume flow rate (2) pressure at points 1, 2, 3, 4 and 5. Neglect all losses.*

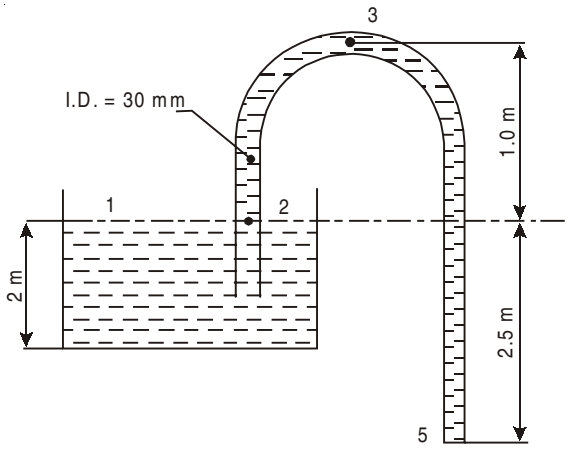


Fig. 7.4

Solution

In order to determine the volume flow rate we have to first determine the flow velocity from the Bernoulli's equation. So let us recall the Bernoulli's equation here. If we apply the equation between points 1 and 5,

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} = Z_5 + \frac{v_5^2}{2g} + \frac{P_5}{\gamma} \qquad \text{... (b)}$$

(i) Since the tank surface area is very large compared to the siphon area, the flow velocity at point 1, can be neglected. Hence $v_1 = 0$. Consequently $\frac{v_1^2}{2g} = 0$.

(ii) Since point 1 is exposed to atmosphere, pressure at this point can be taken = 0, consequently $\frac{P_1}{\gamma} = 0$.

(iii) Since point 5 is exposed to atmosphere $P_5 = 0$. Consequently $\frac{P_5}{\gamma} = 0$.

If we substitute these results in Eq. (b), it reduces to the form,

$$Z_1 = Z_5 + \frac{v_5^2}{2g}$$

or $$Z_1 - Z_5 = \frac{v_5^2}{2g}$$

or $$v_5 = -2g\,(Z_1 - Z_5)$$
$$= -2 \times 9.81\,(2.5)$$
$$= 7 \text{ m/sec}$$

Volume flow rate $$Q = Av_5$$
$$= [0.7854 \times (0.030)^2\,] \times 7$$
$$= 4.9 \times 10^{-3} \text{ m}^3\text{/sec}$$

Since at points 1 and 5 we have already established the fact that pressure = 0, we have to only determine the pressure at points 2, 3 and 4

Let us now apply Bernoulli's equation to points 1 and 2

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} = Z_2 + \frac{v_2^2}{2g} + \frac{P_2}{\gamma}$$

Since v_1 and $P_1 = 0$ equation reduces to the form,

$$Z_1 = Z_2 + \frac{v_2^2}{2g} + \frac{P_2}{\gamma}$$

If we re-arrange the equation suitably,

$$P_2 = \gamma\left[(Z_1 - Z_2) - \frac{v_2^2}{2g}\right] \qquad \text{... (c)}$$

$$(Z_1 - Z_2) = 0$$
$$v_2 = v_5$$
$$= 7 \text{ m/sec.}$$

Substitute these values in Eq. (c)

$$P_2 = 9.81 \times \left[\frac{(0) - 7^2}{(2 \times 9.81)}\right]$$

$$= -24.5 \text{ kPa (indicates vacuum pressure)}$$

P_3 can be obtained in the same manner as P_2 by applying Bernoulli's equation to points 1 and 3. If we re-write Eq. (b) to suit points 1 and 3,

$$P_3 = \gamma\left[(Z_1 - Z_3) - \frac{v_3^2}{2g}\right]$$

$$= 9.81 \times \left[\frac{(-1) - 7^2}{(2 \times 9.81)}\right]$$

$$= -25.5 \text{ kPa}$$

$(Z_1 - Z_3) = -1$ because Z_3 is at a higher level than Z_1.

Problem 7.4 *A siphon draws oil from a tank shown in Fig. 7.5 and delivers it through a nozzle of diameter 20 mm. The I.D of the siphon is 50 mm. Assuming that there are no energy losses, calculate the discharge through the jet in liters/sec and the pressure at points 2, 3, 4 and 5.*

Solution This problem is similar to the previous problem except for the fact that the discharge velocity at the nozzle is different from the velocity elsewhere in the siphon. As before let us apply the Bernoulli's equation to points 1 and 5 and obtain the discharge through the jet.

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} = Z_5 + \frac{v_5^2}{2g} + \frac{P_5}{\gamma}$$

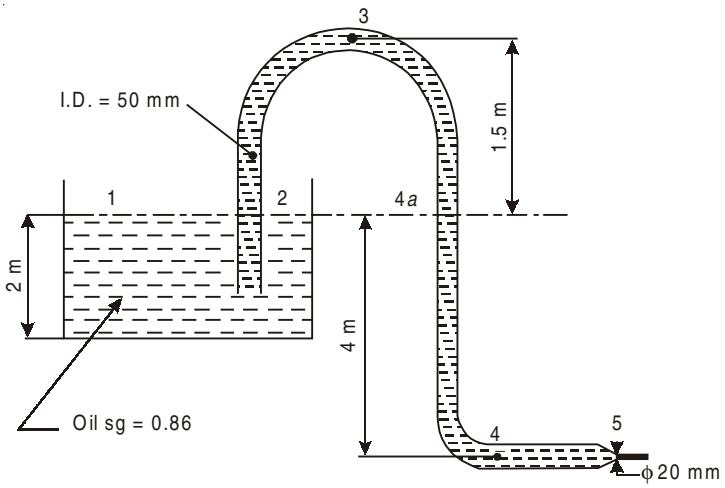


Fig. 7.5

Solution (i) Since the tank surface area is very large compared to the siphon area, the flow velocity at point 1, can be neglected. Hence $v_1 = 0$. Consequently $\frac{v_1^2}{2g} = 0$.

(ii) Since point 1 is exposed to atmosphere, pressure at this point can be taken = 0, consequently $\frac{P_1}{\gamma} = 0$.

(iii) Since point 5 is exposed to atmosphere $P_5 = 0$. Consequently $\frac{P_5}{\gamma} = 0$.

If we substitute these results in Eq. (b), it reduces to the form,

$$Z_1 = Z_5 + \frac{v_5^2}{2g}$$

or $$Z_1 - Z_5 = \frac{v_5^2}{2g}$$

or $$v_5 = -2g\,(Z_1 - Z_5)$$
$$= -2 \times 9.81\,(4)$$
$$= 8.85 \text{ m/sec}$$

Volume flow rate $$Q = A_5\, v_5$$
$$= [0.7854 \times (0.020)^2\,] \times 8.85$$
$$= 2.78 \times 10^{-3} \text{ m}^3\text{/sec}$$

Since at points 1 and 5 we have already established the fact that pressure = 0, we have to only determine the pressure at points 2, 3, 4 and 5.

Let us now apply Bernoulli's equation to points 1 and 2

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} = Z_2 + \frac{v_2^2}{2g} + \frac{P_2}{\gamma}$$

Since v_1 and $P_1 = 0$ equation reduces to the form,

$$Z_1 = Z_2 + \frac{v_2^2}{2g} + \frac{P_2}{\gamma}$$

If we re-arrange the equation suitably,

$$P_2 = \gamma\left[(Z_1 - Z_2) - \frac{v_2^2}{2g}\right] \quad \text{... (d)}$$

$$(Z_1 - Z_2) = 0$$

$$v_2 = \frac{Q}{A_2}$$

$$= \frac{(2.78 \times 10^{-3})}{0.7854\ (0.050)^2}$$

$$= 1.418 \text{ m/sec}$$

Substitute these values in Eq. (d)

$$P_2 = 9.81 \times \left[\frac{(0) - 1.418^2}{(2 \times 9.81)}\right]$$

$$= -1.005 \text{ kPa (indicates vacuum pressure)}$$

P_3 can be obtained in the same manner as P_2 by applying Bernoulli's equation to points 1 and 3. If we re-write Eq. (7.8) to suit points 1 and 3,

$$P_3 = \gamma\left[(Z_1 - Z_3) - \frac{v_3^2}{2g}\right]$$

$$= 9.81 \times \left[\frac{(-1.5) - 1.418^2}{(2 \times 9.81)}\right]$$

$$= -15.7 \text{ kPa}$$

$(Z_1 - Z_3) = -1.5$ because Z_3 is at a higher level than Z_1

$P_{4A} = P_2 = -1.005$ kPa

because the velocity and elevation points are the same.

We now apply the Bernoulli's equation to points 1 and 4 to determine the pressure at point 4.

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} = Z_4 + \frac{v_4^2}{2g} + \frac{P_4}{\gamma}$$

Since P_1 and v_1 both = 0

$$Z_1 = Z_4 + \frac{v_4^2}{2g} + \frac{P_4}{\gamma}$$

or

$$P_4 = \gamma\left[(Z_1 - Z_4) - \frac{v_4^2}{2g}\right]$$

$$= 9.81\left[\frac{(4) - 1.418^2}{(2 \times 9.81)}\right]$$

$(\because v_4 = v_2 = 1.418 \text{ m/sec})$

$$P_4 = 38.23 \text{ kPa}$$

Bernoulli's equation finds its application in all problems related to incompressible fluid flow but it is fundamentally useful in applications involving flow measurements. *Venturi meters, Orifice meter and Pitot tube* are the flow measuring devices which rely on Bernoulli's principles. These devices and many

more are discussed in detail under the chapter 'Flow Measurement and Laboratory Experiments'. Here they are discussed only briefly.

Venturi meter is a device used to measure the velocity of flow in a system and is often a permanent fixture in a flow line. It owes its existence to continuity equation which relates velocity of flow to the area of flow passage. A differential manometer is often an integral part of the Venturi meter. Venturi meters have a discharge co-efficient, (C_d) varying between 0.9 – 1 depending on their design. This accounts for flow losses. Their inside diameter ratio between the main body and the throat are controlled within certain limits.

Problem 7.5 *Figure 7.6 shows a Venturi meter. Determine the difference in levels of the gauge fluid if the flow velocity in the throat section is 15 m/sec.*

Solution We may apply the Bernoulli's equation to points 1 and 2. Since both the points are at the same level $Z_1 = Z_2$. Therefore the Bernoulli's equation is reduced to the form,

$$\frac{v_1^2}{2g} + \frac{P_1}{\gamma} = \frac{v_2^2}{2g} + \frac{P_2}{\gamma}$$

$$\frac{(P_1 - P_2)}{\gamma} = \frac{(v_2^2 - v_1^2)}{2g}$$

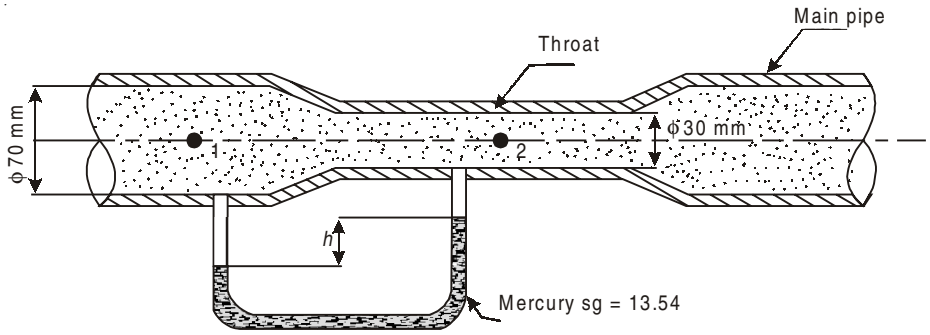


Fig. 7.6 *Venturi meter*

But $\frac{(P_1 - P_2)}{\gamma}$ is the difference in pressure-heads between points 1 and 2 say, P_H

Therefore $$P_H = \frac{(v_2^2 - v_1^2)}{2g} \quad \text{... (e)}$$

From the continuity Equation $(0.7854)\ (0.070)^2\ v_1 = (0.7854)\ (0.030)^2\ v_2$

$$v_1 = \frac{\left[(0.030)^2\ v_2\right]}{(0.070)^2}$$

$$= \frac{\left[(0.030)^2\ 15\right]}{(0.070)^2}$$

$$= 2.755 \text{ m/sec}$$

Substituting the values of v_1 and v_2 in Eq. (a),

$$P_H = \frac{(15^2 - 2.755^2)}{2 \times 9.81}$$

$$= 11.08 \text{ m}$$

In a horizontal differential U-tube manometer the difference in the gauge fluid level (h) and the difference in pressure heads (P_H) have the following relation.

$$P_H = h\left[\left(\frac{S_{gg}}{S_{gV}}\right) - 1\right] \qquad \ldots (7.5)$$

This relation holds if the specific gravity of the gauge fluid (S_{gg}) is heavier than the fluid flowing past the venturi (S_{gV}). If the gauge fluid is lighter then,

$$P_H = h\left[1 - \left(\frac{S_{gg}}{S_{gV}}\right)\right] \qquad \ldots (7.6)$$

In the present case since mercury is heavier ($Sg = 13.54$) relation (7.6) holds,

$$P_H = h\left[\left(\frac{S_{gg}}{S_{gV}}\right) - 1\right]$$

$$= h\left[\left(\frac{13.54}{1}\right) - 1\right]$$

$$= h\ (12.54),$$

Therefore,

$$h = \frac{P_H}{12.54}$$

$$= \frac{11.08}{12.54} = 0.94 \text{ m}$$

Problem 7.6 *Calculate the volume flow rate in LPM of water at 4°C through a horizontal Venturi meter with an inlet diameter = 50 mm and throat diameter = 25 mm if the gauge fluid is oil of sg = 0.8 and the difference in levels of the fluid in the differential manometer = 300 mm. Take* $C_d = 0.9$.

Solution We may apply the Bernoulli's equation. Since it is a horizontal

Venturi meter, the datum heads $Z_1 = Z_2$. Therefore the Bernoulli's equation is reduced to the form,

$$\frac{v_1^2}{2g} + \frac{P_1}{\gamma} = \frac{v_2^2}{2g} + \frac{P_2}{\gamma}$$

$$\frac{(P_1 - P_2)}{\gamma} = \frac{(v_2^2 - v_1^2)}{2g}$$

But $\frac{(P_1 - P_2)}{\gamma}$ is the difference in pressure-heads between points 1 and 2 say, P_H

Therefore $$P_H = \frac{(v_2^2 - v_1^2)}{2g} \quad \text{... (b)}$$

The gauge fluid in this case being oil has a specific gravity less than that of water. We will therefore use Eq. (7.6).

$$P_H = h\left[1 - \left(\frac{S_{gg}}{S_{gV}}\right)\right]$$

$$= 0.3\left[1 - \left(\frac{0.8}{1}\right)\right]$$

$$= 0.06$$

From the continuity equation,

$$(0.7854)\,(0.050)^2\, v_1 = (0.7854)\,(0.025)^2\, v_2$$

$$\left(\frac{v_1}{v_2}\right) = \frac{(0.025)^2}{(0.050)^2}$$

$$= 0.25$$

$$\frac{v_1^2}{v_2^2} = 0.25^2$$

$$= 0.0625$$

or $$v_1^2 = 0.0625\, v_2^2$$

Substituting the values of P_H and v_1^2 in Eq. (b) above,

$$0.06 = \frac{(v_2^2 - 0.0625\, v_2^2)}{2g}$$

$$v_2^2\ (1 - 0.0625) = (2 \times 9.81)\ 0.06$$

$$v_2 = -\frac{\sqrt{(2 \times 9.81)\ 0.06}}{0.9375}$$

$$= 1.12 \text{ m/sec}$$

$$Q = C_d\,(A_{2 \times} v_2)$$

$$= 0.9\ (0.7854)\ (0.025)^2 \times 1.12$$

$$= 4.9 \times 10^{-4} \text{ m}^3\text{/sec}$$

$$Q = (4.9 \times 10^{-4}) \times 60{,}000$$

$$= 29.6 \text{ liters/minute.}$$

(1 m^3/sec = 60,000 LPM)

Problem 7.7 *Water at 4°C flows through a horizontal Venturi meter having an inlet φ = 25 cm and throat φ = 10 cm. Pressure at the inlet = 20 × 10^4 kPa and at the throat = 0.25 m of Hg. Determine the discharge. C_d = 0.95.*

Solution Since it is horizontal Venturi meter $Z_1 = Z_2$. Therefore the Bernoulli's equation is reduced to the form,

$$\frac{(P_1 - P_2)}{\gamma} = \frac{\left(v_2^2 - v_1^2\right)}{2g} \qquad \text{... (a)}$$

$$P_2 = 0.25 \text{ m of Hg}$$

$$= 250 \text{ mm of Hg} = -33.2 \text{ kPa}$$

(Since 1 mm of Hg = 132.8 kPa, 0.250 mm of Hg = 33,200 Pa or 33.2 kPa.) Also, since we are concerned with only gauge pressure in all of these applications, 33.2 kPa indicates negative pressure since it is below atmospheric pressure which is ≈ 100 kPa).

Substituting the values of pressure both given and the one determined in Eq. (a) above,

$$\frac{(20 \times 10^4 + 33.2)}{9.81} = \frac{\left(v_2^2 - v_1^2\right)}{2g}$$

$$\frac{\left(v_2^2 - v_1^2\right)}{2g} = 20{,}390\ .7 \qquad \text{... (b)}$$

From the continuity equation, $(0.7854)\ (0.25)^2\ v_1 = (0.7854)\ (0.10)^2\ v_2$

$$\left(\frac{v_1}{v_2}\right) = \frac{(0.10)^2}{(0.25)^2} = 0.16$$

$$\frac{v_1^2}{v_2^2} = 0.16^2 = 0.0256$$

or $v_1^2 = 0.0256\ v_2^2$

Substituting the value of v_1 in (b) above,

$$\frac{(v_2^2 - 0.0256\ v_2^2)}{2g} = 20{,}390\ .7$$

$$v_2^2\ \frac{(1 - 0.0256\ )}{2g} = 20{,}390\ .7$$

$$v_2^2\ \frac{(0.9744\ )}{2g} = 20{,}390\ .7$$

$$v_2^2 = \frac{(20{,}390\ .7 \times 2 \times 9.81)}{0.9744} = 410.57$$

$$v_2 = 20.26 \text{ m/sec.}$$

$$Q = C_d\,(A_2 \times v_2)$$

$$= 0.95\ [(0.7854)\ (0.10)^2 \times 20.26]$$

$$= 0.15 \text{ m}^3\text{/sec.}$$

Problem 7.8 *Oil flows from point 1–2 in a vertical Venturi meter. Area at the entrance to the Venturi = 0.0314 m^2 and at the throat = 4.4 × 10^{-3} m^2. All other details are as given in the Fig. 7.7. Calculate the pressure difference between A and B and the discharge. Take C_d = 0.95.*

Solution We may apply the Bernoulli's equation to points 1 and 2.

$$\frac{v_1^2}{2g} + \frac{P_1}{\gamma} + Z_1 = \frac{v_2^2}{2g} + \frac{P_2}{\gamma} + Z_2$$

$$\frac{(P_1 - P_2)}{\gamma} = (v_2^2 - v_1^2\ )\ 2g + (Z_2 - Z_1) \qquad \text{... (a)}$$

$$P_H = (v_2^2 - v_1^2\ )\ 2g + (Z_2 - Z_1) \qquad \text{... (b)}$$

In a *Vertical differential U-tube manometer* the difference in the gauge fluid level (h) and the difference in pressure heads (P_H) have the following relation.

$$P_H = h\left[\left(\frac{Sgg}{SgV}\right) - 1\right]$$

Here, $S_{gg} = 13.54$,

$S_{gV} = 0.9$

and $h = 0.20$ m

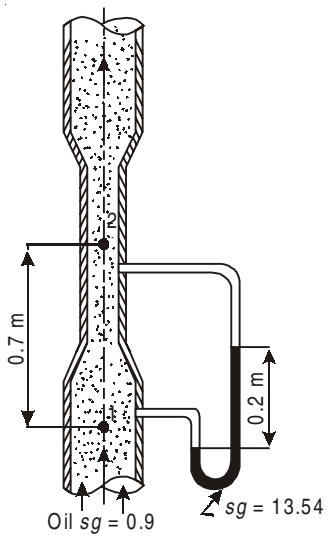


Fig. 7.7 *Vertical venturi meter*

$$P_H = 0.20\left[\left(\frac{13.54}{0.9}\right)-1\right] = 2.80.$$

Substitute this value of P_H in Eq. (b) above.

$$2.80 = (v_2^2 - v_1^2)\,2g + (0.7)$$

$$(v_2^2 - v_1^2) = 2g\,(2.80 - 0.7)$$

$$= 2 \times 9.81\,(2.10) = 41.20 \qquad \text{... (c)}$$

From the continuity equation,

$$0.0314\,(v_1) = (4.4 \times 10^{-3})\,(v_2)$$

$$\frac{(v_1)}{(v_2)} = \frac{\left(4.4 \times 10^{-3}\right)}{0.0314} = 0.140$$

or $$v_1 = 0.140 \times v_2$$

Substituting this value of in Eq. (c) above

$$v_2^2 - (0.140 \times v_2)^2 = 41.20$$

$$v_2^2\,(1 - 0.140^{\,2}) = 41.20$$

or $$v_2^2\,(0.980) = 41.20$$

or $$v_2 = -\frac{41.20}{0.980}$$

$$= 6.48 \text{ m/sec}$$

Discharge $Q = C_d (A_2 v_2)$
$= 0.95 (4.4 \times 10^{-3}) 6.48$
$= 0.027 \text{ m}^3/\text{sec}.$

$$\frac{(P_1 - P_2)}{\gamma} = P_H = 2.80$$

Therefore,

Pressure difference $(P_1 - P_2) = \gamma (2.80)$
$= 9.81(2.80)$
$= 27.46 \text{ kPa}$

Problem 7.9 *Oil of specific gravity 0.9 flows through an inclined Venturi meter from point 1 – 2 as shown in Fig. 7.8. Area at the entrance to the Venturi = 0.0314 m^2 and the area at the throat section is 0.00785 m^2. If the flow velocity at the throat section is 3 m/sec, calculate the deflection h of the manometer.*

Solution In case of inclined Venturi meters the differential manometer reading is inclusive of the differences in both the pressure head and the datum head and where the Sg. of the gauge fluid (Sgg) is higher than the Sg of the fluid passing through the Venturi (SgV), then this difference is given by,

$$\left(\frac{P_1}{\gamma} + Z_1\right) - \left(\frac{P_2}{\gamma} + Z_2\right) = P_H = h\left[\left(\frac{S_{gg}}{S_{gV}}\right) - 1\right] \quad \ldots (7.7)$$

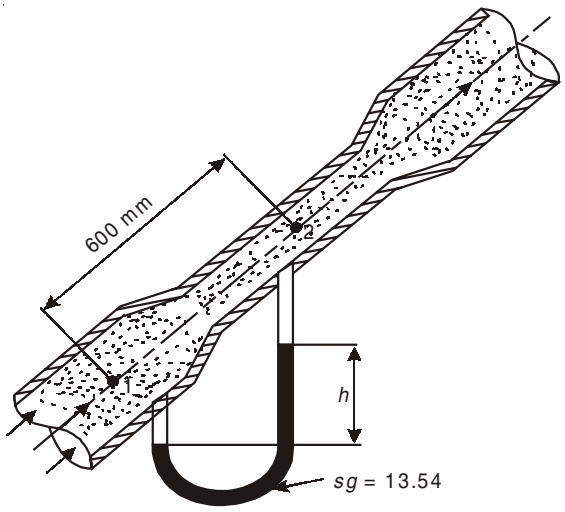


Fig. 7.8 *Inclined venturi meter*

Where the Sg of the gauge fluid is lower than the Sg of the fluid passing through the Venturi the difference is given by Eq. (7.7)

Let us apply the Bernoulli's equation.

$$\frac{v_1^2}{2g} + \frac{P_1}{\gamma} + Z_1 = \frac{v_2^2}{2g} + \frac{P_2}{\gamma} + Z_2$$

We may re-arrange the equation suitably,

$$\left(\frac{P_1}{\gamma} + Z_1\right) - \left(\frac{P_2}{\gamma} + Z_2\right) + \frac{v_1^2}{2g} - \frac{v_2^2}{2g} = 0$$

or
$$P_H + \frac{v_1^2}{2g} - \frac{v_2^2}{2g} = 0 \qquad \text{... (a)}$$

In the present case we use Eq. (7.6) since the gauge fluid is heavier to obtain the desired values.

$$P_H = h\left[\left(\frac{S_{gg}}{S_{gV}}\right) - 1\right]$$

$$= h\left[\left(\frac{13.54}{0.9}\right) - 1\right] = (14.04)\, h.$$

We may substitute this value of (P_H) in Eq. (a) above.

$$(14.04)h + \frac{v_1^2}{2g} - \frac{v_2^2}{2g} = 0 \qquad \text{... (b)}$$

$$v_2 = 3 \text{ m/sec (given)},$$
$$Q = A_2 v_2$$
$$= 0.00785 \times 3$$
$$= 0.023 \text{ m}^3\text{/sec.}$$

$$v_1 = \frac{Q}{A_1}$$

$$= \frac{0.023}{0.0314}$$
$$= 0.75 \text{ m/sec.}$$

Substituting the values of V_1 and V_2 in Eq. (b) above,

$$(14.04)\, h + \frac{0.75^2}{2g} - \frac{3^2}{2g} = 0$$

$$(14.04)\, h + 0.0286 - 0.458 = 0$$
$$(14.04)\, h = 0.4294$$
or
$$h = 0.030 \text{ m}$$
$$= 30 \text{ mm}$$

Problem 7.10 *For the configuration shown in Fig. 7.9 calculate the pressure at (2), the exit velocity of the jet of water and the time required to empty the tank T.*

Solution

The Bernoulli's equation,

$$\frac{v_1^2}{2g} + \frac{P_1}{\gamma} + Z_1 = \frac{v_2^2}{2g} + \frac{P_2}{\gamma} + Z_2$$

here, v_1 and P_1 can be neglected since at point 1 the area of surface is huge compared to the given pipe size and the flow velocity will be relatively small. Since the surface is exposed to atmosphere, gauge pressure at point 1 = 0.

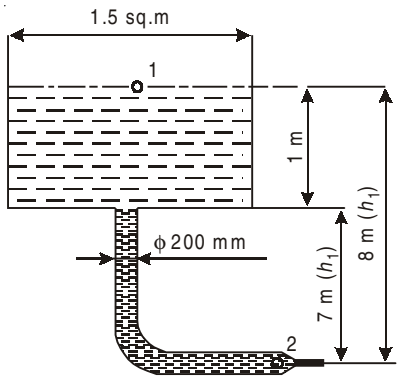


Fig. 7.9

Therefore the Bernoulli's equation reduces to the form,

$$Z_1 = \frac{v_2^2}{2g} + \frac{P_2}{\gamma} + Z_2$$

$$\left(Z_1 - Z_2 - \frac{v_2^2}{2g}\right)\gamma = P_2 \qquad \text{... (a)}$$

Exit velocity V_E can be obtained directly by using the Torricelli's equation.

$$V_E = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 8}$$
$$= 12.50 \text{ m/sec.}$$
$$Q = A_E \times V_E$$
$$= 0.7854\ (0.05)^2 \times 12.50$$
$$= 0.0245 \text{ m}^3\text{/sec.}$$

$$V_2 = \frac{Q}{A_2}$$

$$= \frac{0.0245}{0.7854(0.2)^2}$$

$$= 0.780 \text{ m/sec.}$$

Substituting the value of V_2 in Eq. (a), P_2 can obtained.

$$\left(8 - \frac{0.780^2}{2} \times 9.81\right) 9.81 = P_2 = 78.17 \text{ kPa}$$

Time required to empty the tank is the time taken for the water level to descend from level $h_1 - h_2$. Since the pressure head falls as the water level decreases, the flow velocity decreases and consequently the volume flow rate. But the volume flow rate at any point of time dt through the nozzle = $q(dt) = A_n \times v_n \times dt$. This must be equal to the volume of fluid emptied.

Therefore, $$A_n \times v_n \times dt = -A_T\, dh \qquad \text{... (7.9)}$$

where, A_T = area of the tank *and* dh = depth of water removed. Since $v_n = -2gh$ equation can be simplified to the form,

$$dt = -\left[\frac{\left(\frac{A_T}{A_n}\right)}{\sqrt{2g}}\right] h^{-\frac{1}{2}}\, dh$$

Therefore, $$\int_{t_1}^{t_2} dt = -\left[\frac{\left(\frac{A_T}{A_n}\right)}{\sqrt{2g}}\right] \int_{h_1}^{h_2} h^{-\frac{1}{2}}\, dh$$

(After integrating and simplifying)

$$t_2 - t_1 = \left[\frac{2\left(\frac{A_T}{A_n}\right)}{\sqrt{2g}}\right] \left(h_1^{\frac{1}{2}} - h_2^{\frac{1}{2}}\right) \qquad \text{... (7.10)}$$

We may now substitute the following values in Eq. (7.10).

A_T = area of the tank
= 1.5 m^2,
A_n = area of the nozzle
= (0.7854) (0.050)^2
= 1.96 $\times 10^{-3}$ m^2,
h_1 = 8 m

and $h_2 = 7$ m

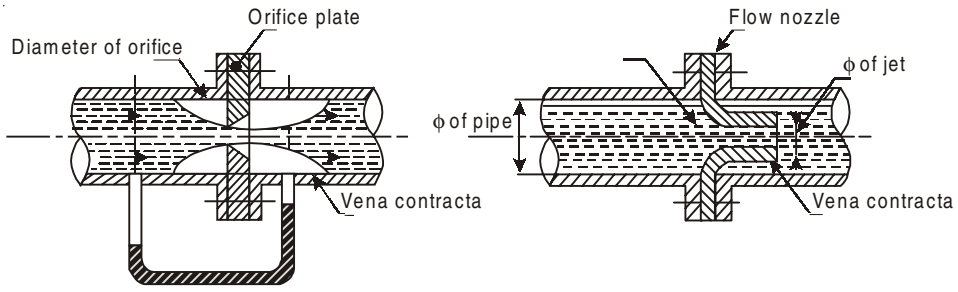
$$t_2 - t_1 = \left[\frac{2\left(\frac{1.5}{1.96\times10^{-3}}\right)}{\sqrt{2\times9.81}}\right]\left(8^{\frac{1}{2}} - 7^{\frac{1}{2}}\right)$$

$$= 126.2 \text{ seconds.}$$

Where very accurate flow measurement values are not desired a simple *orifice plate* introduced between two sections of a pipe would more or less give the same results that an elaborate arrangement such as a Venturi meter would. The orifice plate is simply a round plate with a central hole which is assembled in such a way that the hole and the ID of the two pipe sections are concentric with each other. Orifice plate is economical in terms of manufacturing cost and space occupied. Figure 7.10 shows a system with an orifice plate. An orifice plate has a sharp edged orifice and the flowing fluid is exposed to sudden contraction whereas in a Venturi meter the diameter tapers down to a smaller size gradually. The errors could be attributed to this limitation of the orifice plate.

The diameter of the jet of water issuing out of an orifice plate is substantially smaller than the diameter of the orifice. Therefore the diameter of the jet at the point called the *vena contracta* must be considered for purposes of calculating the flow velocity instead of the diameter of the orifice. The *coefficient of contraction* for an orifice plate is ≈ 0.620. Where approximation would suffice, this value if multiplied by the diameter of the orifice would give us the diameter of the jet. The coefficient of contraction can also be obtained from the equation,

$$C_c = \frac{1}{\left[\frac{2 - a_0}{a_1}\right]} \quad \text{... (7.11)}$$



(a) Flow measurement using orifice plate (b) Flow measurement using flow nozzle

Fig. 7.10

$$a_j = C_c \times a_0$$

The area of the jet can also be determined by using an equation derived from the continuity and Bernoulli's equations.

$$a_j = \frac{C_c\ (a_0 \times a_1)}{\sqrt{a_1^2 - a_0^2}} \qquad \ldots (7.12)$$

Here a_j = area of the jet at vena contracta, a_0 = area of the orifice and a_1 = area of the main pipe. *This equation applies to all sharp edged orifices.*

A Flow nozzle is an improvement over the Orifice plate. Refer Fig. 10.7(b). It has a well rounded orifice instead of a sharp edged orifice. It therefore has higher values of coefficient of contraction between 0.7 – 0.8 leading to lower values of head loss.

Problem 7.11 *An orifice plate with an orifice φ = 10 cm is placed between two sections of a pipe having an inside φ = 40 cm. Oil of sg = 0.9 flows through the pipe. A differential U-tube manometer connected to either side of the plate shows a pressure difference = 35 cm of Hg. Determine the volume flow rate through the plate.*

Solution

Area of the orifice $a_0 = 0.7854\ (0.10)^2$

$= 7.854 \times 10^{-3}\ m^2.$

Area of the pipe $a_1 = 0.7854\ (0.40)^2 = 0.125\ m^2$

$$C_c = \frac{1}{\left[2 - \left(\frac{a_0}{a_1}\right)\right]}$$

$$C_c = \frac{1}{\left[2 - \left(\frac{0.7854 \times 10^3}{0.125}\right)\right]}$$

$$C_c = 0.516$$

From Eq. (7.10) the differential head

$$P_H = h\left[\left(\frac{S_{gg}}{S_{gV}}\right) - 1\right]$$

$$P_H = 0.35\left[\left(\frac{13.54}{0.9}\right) - 1\right]$$

$= 4.9$ m of oil

Volume flow rate Q = area of the jet $\times \sqrt{2g\ P_H}$

$$Q = (C_c \times a_0) \times \sqrt{2g\ P_H}$$

Substitute the relevant values.

$$Q = [0.516\,(7.854 \times 10^{-3})\,] \times \sqrt{2 \times 9.81 \times 4.9}$$
$$Q = 0.038\ \text{m}^3/\text{sec.}$$

Problem 7.12 *An orifice plate is fitted between two pipes. The diameter of the orifice = 100 mm and that of the pipe is 200 mm diameter. The pressure gauge fitted on either side of the orifice plate reads 250 kN/m^2 and 120 kN/m^2. Determine the discharge of water through the orifice. Assume $C_c = 0.58$.*

Solution

$$\text{Area of the orifice } a_0 = 0.7854(0.1)^2$$
$$= 7.854 \times 10^{-3}\ \text{m}^2$$
$$\text{Area of the pipe } a_1 = 0.7854(0.2)^2$$
$$= 0.0316\ \text{m}^2$$
$$P_1 = 250\ \text{kN/m}^2$$
$$P_2 = 120\ \text{kN/m}^2$$
$$\text{Pressure head } h = \frac{(P_1 - P_2)}{\gamma}$$
$$= \frac{(250 - 120)}{9.81}$$
$$= 13.5\ \text{m of water}$$
$$\text{Volume flow rate } Q = C_c\,(a_0 \times a_1) \times \frac{\sqrt{2gh}}{\sqrt{a_1^2 - a_0^2}}$$

Substitute the relevant values

$$Q = \frac{0.58\,(7.854 \times 10^{-3}) \times 0.0316 \times \sqrt{2 \times 9.81 \times 13.5}}{\sqrt{[0.0316^2 - (7.854 \times 10^{-3})^2]}}$$
$$Q = 2.5\ \text{m}^3/\text{sec.}$$

A pitot tube is another simple but effective device used to measure flow velocity. In its simplest form it is no more than a L shaped hallow glass tube with rounded corner. If placed in open channels as shown in Fig. 7.11 (a), the flow velocity can be measured directly by measuring the height h to which the liquid raises. In such cases $V = C_v \sqrt{2gh}$, where C_v = coefficient of velocity. However measurement of flow velocity of a fluid flowing through a pipe is not as simple or as direct. But before we understand the reasons let us understand the principles of flow measurement while using a simple pitot tube.

The principle of flow measurement in a simple pitot tube is based on the fact that when a fluid flowing through a pipe suddenly loses its velocity at any point in the flow stream as it happens at the point where the fluid comes in contact with the small opening in the pitot tube, the consequent loss of kinetic

energy at that point results in a gain in pressure energy. This component is called the *dynamic pressure.* In addition the fluid has the *static pressure.* These two components together constitute what is termed as the *stagnation pressure.*

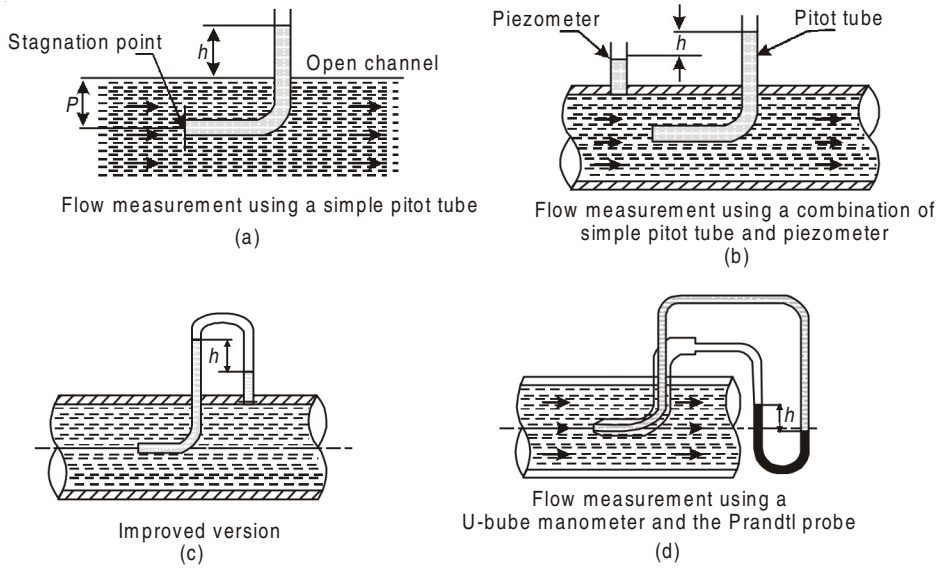
Mathematically, $\frac{P_0}{\gamma} = \frac{P}{\gamma} + \frac{v^2}{2g}$, where P_0 is the pressure at the stagnation point. In terms of the flow velocity the equation can be written as

$$v = C_v \sqrt{2g \frac{(P_0 - p)}{\gamma}}$$

$$= C_v \sqrt{2g\,\Delta h} \qquad \text{... (7.13)}$$

In an open channel, $p = \gamma d$, therefore $v = \sqrt{2g\,[(h\,gd) - gd]}$ and hence we end up with the equation $v = \sqrt{2gh}$. On the other hand if the fluid is flowing through a pipe, $(P_0 - p)$ must be determined using special techniques which can measure both static and stagnation pressure. A simple pitot tube in combination with a piezometer (Fig. 7.11 (b) is one such method. Figures 7.11 (c) and 7.11 (d) shows other improved versions.

In case of Fig. 7.11 (d), the difference in pressure head is determined by using the equation $P_H = h\left[\left(\frac{S_{gg}}{S_{gV}}\right) - 1\right]$. Here S_{gg} represents the specific gravity of the gauge fluid and S_{gv} the specific gravity of the fluid flowing through the pipe.



Flow measurement using a simple pitot tube
(a)

Flow measurement using a combination of simple pitot tube and piezometer
(b)

Improved version
(c)

Flow measurement using a U-bube manometer and the Prandtl probe
(d)

Fig. 7.11

Problem 7.13 *A pitot tube is connected to a pipe carrying water at 4°C. The diameter of the pipe is 0.4 m. Static pressure is equivalent to 12 cm of Hg. The stagnation pressure is 2×10^4 Pa. The average velocity is 0.8 times the velocity at the center where the pitot tube is placed. Determine the discharge. Assume C_v = 0.95.*

Solution Since the discharge is calculated on the basis of average velocity, we have to determine V_A first. But this requires that we determine the centerline velocity since $V_A = 0.85 \times V_c$

$$\frac{P}{\gamma} = \left(\frac{12}{100 \times 13.54}\right)$$

$$= 1.62 \text{ m (Vacuum)}$$

$$\frac{P_0}{\gamma} = \frac{(2 \times 10^4)}{9810}$$

$$= 2.03 \text{ m}$$

$$\Delta h = [2.03 - (-1.62)]$$

$$= 3.65 \text{ m.}$$

Flow velocity along the center

$$V_c = C_v \sqrt{2g \times \Delta h}$$

$$\Delta h = 0.95 \sqrt{2 \times 9.81 \times 3.65}$$

$$= 8 \text{ m/sec.}$$

$$V_A = 0.8 \times V_c$$

$$= 0.8 \times 8$$

$$= 6.4 \text{ m/sec.}$$

$$Q = 0.7854\ (0.4)^2 \times 6.4$$

$$= 0.8 \text{ m}^3\text{/sec.}$$

Problem 7.14 *A pitot-static tube with the mercury differential U-tube manometer was used to measure velocity of oil flow through a pipe. The difference in mercury level was found to be 15 cm. Determine the flow velocity. Assume C_v = 0.9 and Sg. of oil = 0.85.*

Solution

From Eq. (7.8) we have,

$$P_H = h\left[\left(\frac{Sgg}{SgV}\right) - 1\right]$$

$$h = 15 \text{ cm}$$

$$= 0.15 \text{ m}$$

$$S_{gg} = 13.54$$
$$S_{gV} = 0.85$$
$$P_H = 0.15 \left[\left(\frac{13.54}{0.85}\right) - 1\right]$$
$$= 2.24 \text{ m of oil.}$$

Flow velocity $$V = C_v \sqrt{2g P_H}\ .$$
$$= 0.9 \sqrt{2 \times 9.81 \times 2.24}$$
$$= 5.9 \text{ m/sec.}$$

The Bernoulli's equation is valid only for incompressible ideal fluids since the specific weight is assumed to remain constant throughout. Steady and irrotational flow is invariably assumed. Also the system cannot gain or loose energy between two points either due to mechanical systems in between or due to friction. Further no loss or gain of energy due to heat transfer is permitted.

The expanded version of Bernoulli's equation called the *general energy equation* overcomes some of these limitations.

$$Z_1 + \frac{V_1^2}{2g} + \frac{P_1}{\gamma} + h_A - h_C - h_L = Z_2 + \frac{V_2^2}{2g} + \frac{P_2}{\gamma} \qquad \ldots (7.14)$$

where h_A, h_C and h_L represent the values for energy added, energy consumed and energy lost, respectively.

Before we apply the general energy equation to solve practical problems we have to have an idea about how and in what form energy addition, removal and losses occur. Generally a mechanical, hydraulic or a pneumatic pump can be the device that adds energy to the system. Turbines, hydraulic motors or a hydraulic cylinders act as energy consuming devices. Energy losses occur due to resistance to flow offered by flow passages, turbulence due to sudden expansion and contraction etc. These are discussed in great detail in Chapter 11. Here we are merely interested in appreciating the various forms energy loses that occur during high velocity fluid flow through long conductors such as pipes and control systems such as valves and fittings. Their values are usually given and we merely have to account for them.

Problem 7.15 *Water is being supplied through a 75 mm ID pipe to irrigation at point 2 from the reservoir 1 @ 0.05 m^3/sec. Water is also pumped from the same reservoir at point 2 @ 0.015 m^3/sec to serve the community at point (3) through a pipe of ID 25 mm. Calculate the energy loss due bends, gate valve and other fittings between points 1 and 2 and the energy addition at point (3). Assume the energy loss between point 2 and 3 = 3.5 N-m/N. Also determine the HP required to drive the pump.*

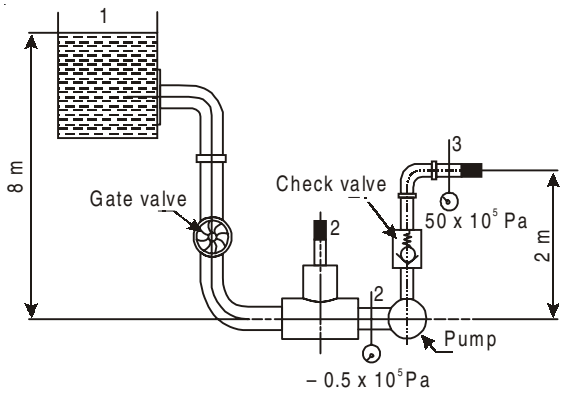


Fig. 7.12

From Eq. (7.14)

$$Z_1 + \frac{V_1^2}{2g} + \frac{P_1}{\gamma} + h_A - h_C - h_L = Z_2 + \frac{V_2^2}{2g} + \frac{P_2}{\gamma}$$

Since there is no energy addition or consumption between points 1 and 2, the above equation simplifies to,

$$Z_1 + \frac{V_1^2}{2g} + \frac{P_1}{\gamma} + h_L = Z_2 + \frac{V_2^2}{2g} + \frac{p_2}{\gamma}$$

Since it is h_L that we have to determine we re-write the equation suitably,

$$Z_1 + \frac{V_1^2}{2g} + \frac{P_1}{\gamma} - Z_2 - \frac{V_2^2}{2g} - \frac{P_2}{\gamma} = h_L$$

or

$$h_L = Z_1 - Z_2 + \frac{V_1^2}{2g} - \frac{V_2^2}{2g} - \frac{P_2}{\gamma} + \frac{P_1}{\gamma}$$

Since $p_1 = 0$, $p_2 = 0$ and v_1 can be neglected equation further simplifies

$$h_L = (Z_1 - Z_2) - \frac{V_2^2}{2g}$$

$$V_2 = \frac{Q}{A}$$

$$= \frac{0.05}{0.7854(0.075)^2}$$

$$= 11.3 \text{ m/sec}$$

$$h_L = (8) - \left[\frac{(11.3)^2}{2g}\right]$$

$$= 1.5 \text{ m or } 1.5 \text{ Nm/N}$$

The second part of the question can be solved by applying general energy equation between points 2 and 3.

$$Z_2 + \frac{V_2^2}{2g} + \frac{P_2}{\gamma} + h_A - h_C - h_L = Z_3 + \frac{V_3^2}{2g} + \frac{P_3}{\gamma}$$

Since we have to determine energy addition h_A we may rearrange the equation accordingly. At the same time we eliminate h_C since there is no energy consumption between points 2 and 3.

$$h_A = \frac{(P_3 - P_2)}{\gamma} + (Z_3 - Z_2) \frac{+(V_3^2 - V_2^2)}{2g} + h_L$$

$$h_A = \frac{\left[50 \times 10^5 - (-0.5 \times 10^5)\right]}{9.81} + (2) + \frac{(V_3^2 - V_2^2)}{2g} + 3.5$$

$$h_A = 520.28 + \frac{(V_3^2 - V_2^2)}{2g} \quad \text{... (a)}$$

$$v_3 = \frac{Q}{A_3}$$

$$= \frac{(0.015)}{0.7854\ (0.025)^2}$$

$$= 30.55 \text{ m/sec}$$

$$v_2 = \frac{Q}{A_2}$$

$$= \frac{(0.015)}{0.7854\ (0.075)^2}$$

$$= 3.40 \text{ m/sec.}$$

Substituting the values of v_3 and v_2 in Eq. (a)

$$h_A = 520.28 + \frac{(30.55^2 - 3.40^2)}{2 \times 9.81}$$

$$= 46.9 \text{ m or } 46.9 \text{ N-m/N}$$

Power required to drive the pump.

$$P = (h_A) \times \gamma \times Q$$

$$= 46.9 \times 9810 \times 0.015$$

$$= 6.901 \text{ N.m/sec or Watts.}$$

Problem 7.16 *A pump (Fig.7.13) draws water in oil emulsion from a tank whose surface is 0.5 m below the pump center line. The suction pressure is 2.5 mm of mercury (gauge) and the suction line has an ID of 35 mm. The pump delivers 50 LPM via a fluid conductor of ID 25 mm to a fluid motor located 0.75 m above the pump centerline. The total energy loss due fittings and valves = 2.5*

N-m / N. The pressure at the delivery port of the pump = 2 ×10^7 Pa and the pressure at the exit port of the fluid motor = 2 × 10^5 Pa. Determine the energy consumed by the fluid motor and its SHP if the efficiency of the motor is 0.9. The sg of the fluid may be assumed to be the same as water.

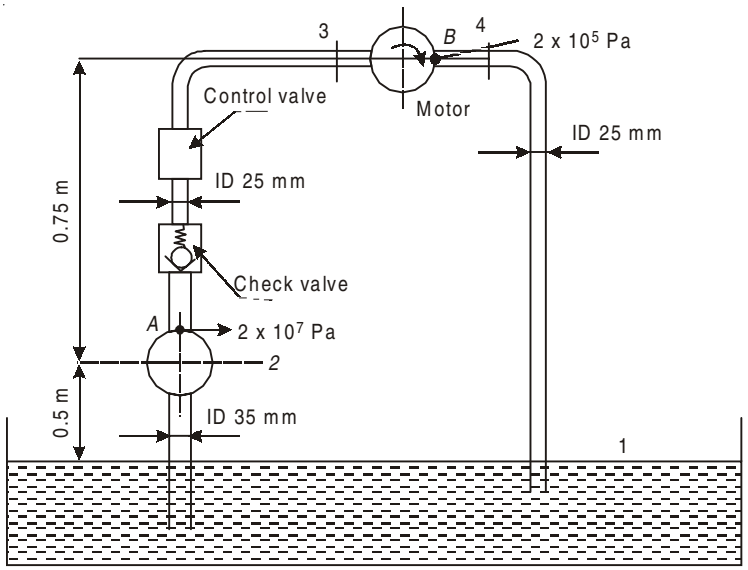


Fig. 7.13

Solution Let us apply the general energy equation to points 1 and 2,

$$Z_1 + \frac{V_1^2}{2g} + \frac{P_1}{\gamma} + h_A - h_C - h_L = Z_1 + \frac{V_2^2}{2g} + \frac{P_2}{\gamma}$$

Between point 1 and 2 there is no energy consumed or loss of energy and since we are interested in h_A, the equation reduces to,

$$h_A = (Z_2 + Z_1) + \frac{(V_2^2 - V_1^2)}{2g} + \frac{(P_2 - P_1)}{\gamma}$$

$$h_A = (0.5) + \frac{(V_2^2 - V_1^2)}{2g} + \frac{[2\times10^7 - (-332)]}{9810}$$

(1mm of Hg = 132.8 Pa)

$$h_A = 2{,}039 + \frac{(V_2^2 - V_1^2)}{2g} \qquad \text{...(b)}$$

$$\left(Q = \frac{50 \text{ LPM}}{60{,}000} = 8.33 \times 10^{-4} \text{ m}^3/\text{sec}\right)$$

From the continuity equation

$$v_1 = \frac{(A_2\ v_2\ )}{A_1}$$

$$= \frac{(8.33 \times 10^{-4})}{0.7854\ (0.035)^2}$$

$$= 0.86 \text{ m/sec}$$

$$v_2 = \frac{Q}{A_2}$$

$$= \frac{(8.33 \times 10^{-4})}{0.7854\ (0.025)^2}$$

$$= 1.7 \text{ m/sec}$$

Substitute the values in Eq. (b) above.

$$h_A = 2.039 + \frac{(1.7^2 - 0.86^2)}{2} \times 9.81$$

$$= 2039.1 \text{ m}$$

We may now apply the general energy equation to points 1 and 4 to determine the energy consumed by the motor.

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} + h_A - h_C - h_L = Z_4 + \frac{v_4^2}{2g} + \frac{P_4}{\gamma}$$

We may rearrange the equation to our convenience,

$$-h_C = (Z_4 - Z_1) + \frac{\left(v_4^2 - v_1^2\right)}{2g} + \frac{\left(P_4 - P_1\right)}{\gamma} - h_A + h_L$$

or

$$h_C = h_A - (Z_4 - Z_1) - \frac{\left(v_4^2 - v_1^2\right)}{2g} - \frac{\left(P_4 - P_1\right)}{\gamma} - h_L \quad \ldots(a)$$

$$v_4 = \frac{Q}{A_4} = \frac{(8.33 \times 10^{-4})}{0.7854\ (0.025)^2}$$

$$= 1.7 \text{ m/sec.}$$

and $v_1 = 0.86$ m/sec.

Substitute all the relevant values in Eq. (a) above

$$h_C = 2039.1 - (1.25) - \frac{(1.7^2 - 0.86^2)}{2g}$$

$$- \frac{[2 \times 10^5 - (-332)]}{9810} - 2.5$$

Energy consumed by the motor = h_C = 2014.7 N-m/N

The shaft horse power or the SHP is the net power available after the efficiency factor has been taken into account. h_C indicates the power delivered by the fluid to the motor. At this point the efficiency of the motor has not been taken into account. Hence,

$$\begin{aligned} \text{SHP} &= h_C \times \gamma \times Q \times \text{efficiency} \\ &= 2014.7 \times 9810 \times (8.33 \times 10^{-4}) \times 0.9 \\ &= 14{,}817 \text{ N-m/sec} \\ \text{SHP} &= 14.8 \text{ kW} \end{aligned}$$

The students are aware of the limitations of Bernoulli's equation and limitations of the expanded version of the Bernoulli's equation called the general energy equation.

The steady flow energy equation (7.18) overcomes all of the limitations of the Bernoulli's equation and is applicable to any fluid, real or ideal, liquid, vapour or gas, the only condition being the flow must be steady and energy transfer continuous. This is because it takes into account the loss or gain in internal energy of the fluid due to molecular activity and the consequent changes in enthalpy.

The steady flow energy equation is based on energy balance. *The net energy supplied per unit mass (E) is the sum of the energy gain within the system (δQ) and the energy leaving the system (δW)*. An elaborate derivation is not intended here, so the equation is written directly.

$$gZ_1 + \frac{1}{2} V_1^2 + H_1 + q - w = gZ_2 + \frac{1}{2} V_2^2 + H_2 \qquad \ldots (7.18)$$

Here H represents enthalpy at locations 1 and 2 of the stream tube and = $\left(\frac{p}{\rho} + e\right)$ where e represents the internal energy, q is the net inflow of heat and w is the net out flow of work per unit mass. Here, w is either negative or positive depending upon whether work was done on the fluid by a mechanical system such as a pump or work was extracted from the system by a mechanical device such as a turbine.

The equation can also be expressed in the form,

$$\frac{P_1}{\rho_1} + e_1 + gZ_1 + \alpha_1 \frac{1}{2} V_1^2 + q - w = \frac{P_2}{\rho_2} + e_2 + gZ_2 + \alpha_2 \frac{1}{2} V_2^2 \qquad \ldots (7.19)$$

Kinetic energy Correction In a stream tube the velocity across the tube is considered uniform but this is not so in case flow through a pipe. The kinetic energy of the fluid per unit mass varies in tune with the velocity. Necessary correction must therefore be applied to account for this variation. Thus true velocity $v_T = (\alpha v^2_{\text{ave}})$ where α is the velocity correction factor and v_{ave} is the average velocity. α varies between 1–1.10 for turbulent flows and for laminar flow in a pipe = 2. Generally in fluid mechanics computations α is always assumed = 1.

Kinetic energy correction factor $\alpha = \frac{1}{A}\int_A \left(\frac{v}{V}\right)^3 dA$ ... (7.20)

v = average velocity, v = velocity at any radius R normal to the elemental area dA and A = area of entire cross section of the stream tube at any section. (Fig. 7.14)

As per Prandtl's one-seventh power law the velocity distribution for *turbulent flow* in a pipe is approximately given by,

$$\frac{v}{v_{max}} = \left(\frac{y}{R_0}\right)^{\frac{1}{7}} \quad \text{... (7.21)}$$

For turbulent flow through pipes $\frac{v}{v_{max}} = 0.81$

For *a laminar flow* through a pipe the velocity distribution is given by,

$$\frac{v_R}{v_{max}} = \left[1-\left(\frac{R}{R_0}\right)^2\right] \quad \text{... (7.22)}$$

For laminar flow through pipes $\frac{v}{v_{max}} = \frac{1}{2}$ where v is the average velocity.

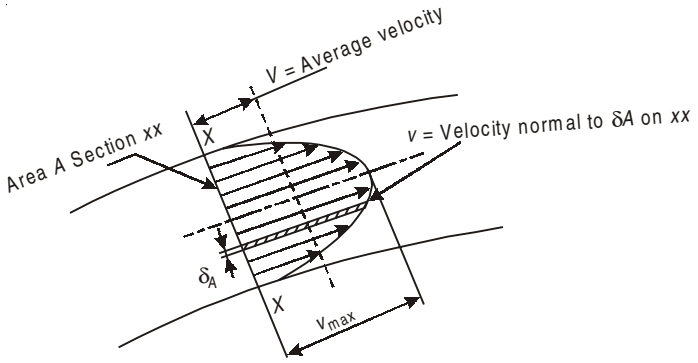


Fig. 7.14

v_R is velocity at any radius R from the center of the pipe, v_{max} is the maximum velocity at the center of the pipe and R_0 is the pipe radius.

Problem 7.17 *Mass flow rate through an air compressor is 0.7 kg/sec. The velocity, specific volume and pressure at entry is 7.5 m/sec, 0.80 m^3/kg and 1 bar respectively. The corresponding values at exit are 5 m/sec, 0.2 m^3/kg and 5 bar. The internal energy of the air at exit is greater that at entry by 80 kJ/kg. Heat removed through a cooling system 50 kJ/sec. Calculate the power required to drive the compressor and the cross sectional areas at the inlet and outlet pipes.*

Solution

Let us recall the steady flow energy Eq. (7.20)

$$\left(\frac{P_1}{\rho_1 + e_1}\right) + gZ_1 + \alpha_1 \frac{1}{2} V_1^2 + q - w = \left(\frac{P_2}{\rho_2 + e_2}\right) + gZ_2 + \alpha_2 \frac{1}{2} V_2^2$$

The equation can be rearranged suitably keeping in mind that $Z_1 = Z_2$ and $q = 0$ since there is no heat addition or removal through an external source within the control volume.

$$-w = \left(\frac{P_2}{\rho_2 + e_2}\right) - \left(\frac{P_1}{\rho_1 + e_1}\right) + \alpha_2 \frac{1}{2} V_2^2 - \alpha_1 \frac{1}{2} V_1^2$$

Taking α_1 and α_2 = 1, and regrouping the above,

$$-w = \left(\frac{P_2}{\rho_2 + e_2}\right) - \left(\frac{P_1}{\rho_1 + e_1}\right) + \frac{1}{2}\left[V_2^2 - \frac{1}{2} V_1^2\right] \quad \text{... (a)}$$

P_2 = 5 bar
= 5×10^5 N/m^2,

$\rho_2 = \left(\frac{1}{0.2}\right) = 5$ kg/m^3
= 5×9.81
= 49 N/m^3,

e_2 = 80×1000
= 80,000 J/kg
= $\frac{80{,}000}{9.81}$
= 8155 N.m / N,

e_1 = 0

P_1 = 1 bar
= 1×10^5 N/m^2,

$\rho_1 = \left(\frac{1}{0.80}\right)$
= 1.25 kg/m^3
= 1.25×9.81
= 12.26 N/m^3

V_2 = 5 m/sec

V_1 = 7.5 m/sec

Substituting the above values in Eq. (a) above,

$$-w = \left[\left(\frac{5 \times 10^5}{49}\right) + 8155\right] - \left[\left(\frac{1 \times 10^5}{12.26}\right) + 0\right] + \frac{1}{2}(7.5^2 - 5^2)$$

$$-w = (10{,}204 + 8155) - (8156.6) + (-15.625)$$

$$-w = 10186.77 \text{ Nm/N}$$

(Negative sign indicates work outflow)

Total power required = 9.81 × mass flow rate × w + power required to remove heat through the cooling system @ 50 kJ/sec = 50,000 J/sec = 50,000 Nm/sec)

$$P = 9.81 \times 0.7 \times 10186.77 + 50{,}000$$
$$= 1199552.5 \text{ Nm/sec}$$
$$= 120 \text{ kW}$$

Area at inlet and outlet of the pipe

$$\text{Mass flow rate} = 0.7 \text{ kg/sec}$$

and $$\rho \text{ at inlet} = \rho_1 = \left(\frac{1}{0.8}\right)$$

$$\rho_1 = 1.25 \text{ kg/m}^3$$

Therefore $$Q = \frac{0.7}{1.25}$$

$$Q = 0.56 \text{ m}^3/\text{sec}$$

$$\text{Inlet pipe area} = \frac{Q_i}{\text{inlet velocity}}$$

$$= \frac{0.56}{7.5}$$

$$= 0.0746 \text{ m}^2$$

$$\text{Outlet pipe area} = \frac{Q_0}{\text{outlet velocity}}$$

$$= \frac{\left(\frac{0.7}{5}\right)}{5} = 0.0128 \text{ m}^2$$

Problem 7.18 *The velocity distribution for a turbulent flow through a pipe is given by,* $\frac{v}{v_{\max}} = \left(\frac{y}{R_0}\right)^{\frac{1}{9}}$ *where y is the distance from the wall and* R_0 *is the radius of the pipe. Determine the kinetic energy correction factor.*

Solution

From Eq. (7.20)

Kinetic energy correction factor $\alpha = \frac{1}{A} \int_A \left(\frac{v}{V}\right)^3 dA$ ... (a)

The given distribution is $\frac{v}{v_{max}} = \left(\frac{y}{R_0}\right)^{\frac{1}{9}}$ ... (b)

For turbulent flow through pipes $v_A = 0.81\, v_{max}$

or $$v_{max} = \left(\frac{1}{0.81}\right)$$

$$= 1.234\, v_A$$

Substituting this value of v_{max} in (b)

$$\frac{v}{1.234 v_A} = \left(\frac{y}{R_0}\right)^{\frac{1}{9}}$$

or, $$\frac{v}{v_A} = 1.234 \left(\frac{y}{R_0}\right)^{\frac{1}{9}}$$

Substituting this value of $\frac{v}{v_A}$ in (a)

$$\alpha = \frac{1}{A} \int_A (1.234)^3 \left(\frac{y}{R_0}\right)^{\frac{3}{9}} dA$$

$$\alpha = \frac{1}{\pi R_0^2} \int_0^{R_0} \left[(1.234)^3 \left(\frac{y}{R_0}\right)^{\frac{1}{3}}\right] 2\pi r\, dr$$

$$\alpha = \frac{2 \times (1.234)^3}{R_0^2} \int_0^{R_0} \left[(1.234)^3 \left(\frac{y}{R_0}\right)^{\frac{1}{3}}\right] r\, dr$$

$$\alpha = \frac{2 \times (1.234)^3}{R_0^2} \int_0^{R_0} \left(\frac{y}{R_0}\right)^{\frac{1}{3}}\Bigg] (R_0 - y) dy$$

$(r = R_0 - y)$

$$\alpha = 1.037$$

QUESTIONS BANK

Section A

Review Questions

1. State the Euler's equation of motion and explain how it lead to the development of Bernoulli's equation.
2. Explain the principle of a Venturi meter and derive an expression for discharge through a Venturi meter.
3. Distinguish clearly between Venturi meter, orifice meter and a flow nozzle.
4. Define dynamic pressure, static pressure and stagnation pressure as applicable to a pitot tube. Distinguish between a simple pitot tube and a pitot-static tube.
5. How is the Bernoulli's equation different from the general energy equation?
6. Name the variables on which the value of head loss due to friction in pipes and head loss due to fittings and values depends.
7. Derive the Euler's equation of motion and list out all of the assumptions.

Section B

Multiple Choice Questions

Choose the right answer

1. The constituents of the total energy in a pipe flow is measured with respect to
   (a) the center line of the pipe (b) top edge of the pipe
   (c) any horizontal line (d) bottom edge of the pipe
2. Euler's equation cannot be integrated without first establishing a relation between
   (a) pressure and density (b) velocity and elevation
   (c) specific weight and volume (d) None of these
3. The units of measurement of the total energy in the Bernoulli's equation is
   (a) Pa (b) N/m^2
   (c) N-m (d) N-m/N
4. Piezometric head is
   (a) Velocity head (b) Pressure head
   (c) Velocity head and pressure head
   (d) Velocity head and elevation head
5. For proper flow measurement it is important that a pitot static tube is held

(a) concentric with the pipe diameter
(b) slightly above the center line of the pipe with its opening in the direction of flow
(c) concentric with the pipe diameter with its opening in the direction of flow
(d) concentric with the pipe diameter with its opening facing the direction of flow

6. A simple pitot tube cannot be used to measure flow velocity in a pipe because
   (a) it cannot measure total pressure
   (b) static pressure
   (c) stagnation pressure
   (d) dynamic pressure
7. The pitot static tube is so called because
   (a) it can measure static pressure
   (b) it can measure dynamic pressure
   (c) both static and stagnation pressure simultaneously
   (d) None of these
8. For flow measurement in open channel the most ideal instrument is
   (a) the venturimeter (b) pitot static tube
   (c) a flow nozzle (d) a simple pitot tube

Section C

Problems

1. Water at 4°C is flowing through a constant diameter pipe of ID 60 mm @ 3.5 m/sec and exerts a pressure of 35 N/cm^2 at entry and 25 N/cm^2 at exit. The difference in elevation between entry and exit is 1.5 m. Determine the total energy/unit weight of the fluid and loss of energy between entry and exit. (*Ans.* 63.2 N.m/N, and 8.7 m)
2. Water is flowing through a 5 m long pipe inclined at an angle of 30° to the horizontal. At the entry the pipe has cross sectional area 0.0314 m^2 and at the exit 0.125 m^2. If the velocity at the entry point is 2.5 m/sec, determine the discharge. (*Ans.* 21.6 kN/m^2)
3. An inverted U-tube manometer placed between the main line and the throat sections of an inclined Venturimeter shows a difference in level = 200 mm. The gauge fluid has a specific gravity = 0.75 and the fluid passing through the Venturimeter has a specific gravity = 0.9. The area of the main pipe = 0.0314 m^2 and that of the throat section = 7.854×10^{-3} m^2. Determine the flow rate through the Venturi if the head loss between the main and throat section is 0.2 times the kinetic head.
   (*Ans.* 7.06×10^{-3} m^3/sec)

4. A the simple pitot tube is shown in Fig. P.4. Determine the velocity at point 1 and the pressure at point 2. (*Ans.* 1.375 kPa, 0.88 m/sec)

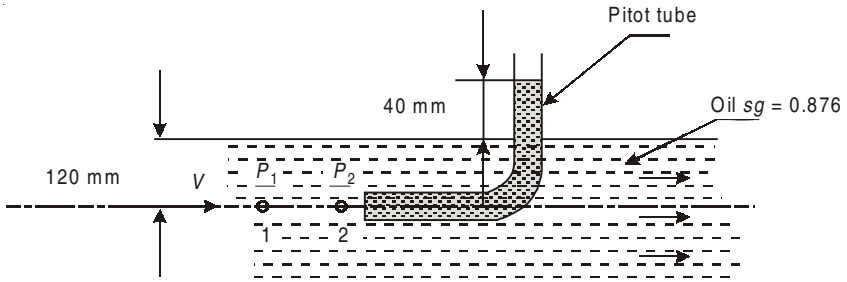


Fig. P. 4

Answers to Multiple Choice Questions

1. (c) **2.** (a) **3.** (d) **4.** (d)
5. (d) **6.** (d) **7.** (c) **8.** (d)

Chapter **8**

Impulse Momentum Equation and Its Applications

Under equations of motion we were basically concerned with the relation between pressure, velocity, elevation and density of fluid particles and the energy balance within a control volume. We were not concerned with the effect of this energy either on the fluid conductor itself or on the external objects on which it impacts.

In this chapter we are concerned with the forces due to fluids in motion and its effect on objects external to it and on objects through which it is flowing.

The magnitude or direction of a high velocity jet of fluid issuing out of a pipe cannot be changed unless an external force is applied. It takes so many fire fighters to keep the hose steady on their object of interest because an external force is required to change the direction of the high velocity jet of water issuing out of the nozzle. The Pelton wheel in a hydro-electric power station is another example that demonstrates the effect of forces due to fluid in motion and how these forces can be harnessed to our advantage.

The *Linear momentum equation* and the *Angular moment equation* are the equations that basically define and quantify the forces due to fluids in motion. *Both are based on* the Newton's second law of motion. Angular momentum equation also called *moment of momentum equation* is applied to rotating fluid mass. The linear momentum equation has two variations. *The Momentum equation* and the *Impulse momentum equation.*

The momentum equation states 'rate of change of momentum or the momentum exchange across a control volume is equal to the impressed force'.

Mathematically, $F = m(v_2 - v_1)$ ... (8.1)

Here v_2 and v_1 are velocity at the inlet and exit to the control volume and m is the mass flow rate per unit of time, $(\rho_1 a_1 v_1)$ at entry and $(\rho_2 a_2 v_2)$ at the exit.

Flow is assumed to be steady and non-uniform and the direction of F is in the direction flow. More elaborately, Eq. (8.1) could be stated thus.

In a steady flow through a pipe, for equilibrium the force components within a control volume in the x and y planes must be equal to the components of the moment in the corresponding planes.

Mathematically,

$$W + P_1A_1 + P_2A_2 + R = M_1 + M_2 \quad \text{... (8.2)}$$

W is the weight of the fluid within the control volume, P_1A_1 is the pressure and area at the entrance to the control volume, P_2A_2 is the pressure and area at the exit to the control volume, R is the reaction which acts at the center of gravity of the control volume. M_1 and M_2 together represents the momentum exchange that occurred between the inlet and outlet.

$$M_1 = -\rho Q v_1$$
$$M_2 = \rho Q v_2$$

Here v_1 and v_2 represent the velocity at the inlet and outlet.

In Fig. 8.1 the directions of the forces and the moments as the fluid flows through a pipe bend is shown. The directions of the reactions R_x and R_y is assumed initially and validated after determining their magnitude.

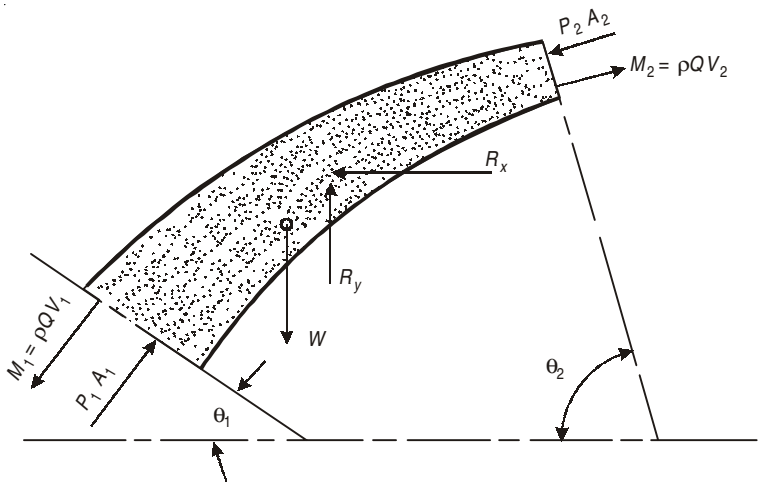


Fig. 8.1

Equation 8.2 may now be applied to the control volume shown in Fig. 8.1.

$$W + P_1A_1 + P_2A_2 + R = M_1 + M_2$$

$$P_1A_1 \cos\theta_1 - P_2A_2 \cos\theta_2 - R_x = \rho Q v_2 \cos\theta_2 - \rho Q v_1 \cos\theta_1 \quad \text{... (8.3)}$$

$$R_y - W - P_2A_2 \sin\theta_2 + P_1A_1 \sin\theta_1 = \rho Q v_2 \sin\theta_2 - \rho Q v_1 \sin\theta_1 \quad \text{... (8.4)}$$

The algebraic signs are based on the direction of flow as indicated in the figure. The forces have been resolved in to their components along the x and y axis.

The impulse momentum equation states 'rate of change of momentum across a control volume must be equal to the impressed force acting on a body for a period of time'.

$$F(\Delta t) = m\,(v_2 - v_1) \qquad \ldots (8.5)$$

Here $F(\Delta t)$ is called the *impulse.*

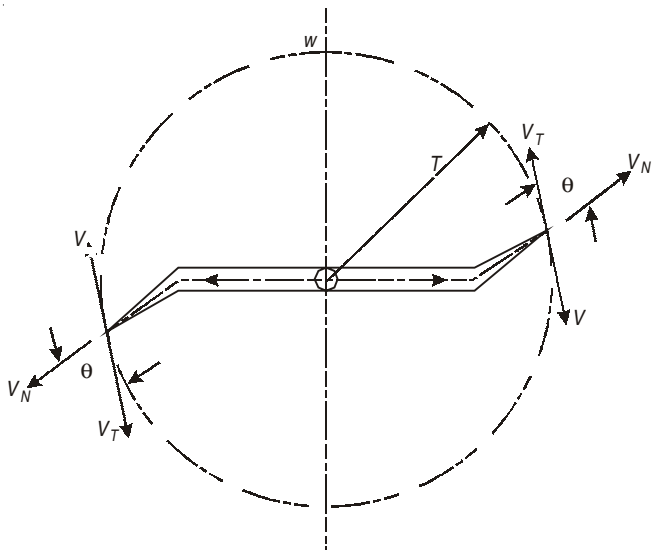
The moment of momentum equation states that the rate of change of angular momentum of a fluid as it enters and leaves the control volume is equal to the impressed torque.

Torque on the fluid by a rotating element = angular momentum of the fluid leaving the control volume – angular momentum of the fluid entering the control volume.

Thus $$T = \rho Q\,[(r\,v_T)_2 - (r\,v_T)_1] \qquad \ldots (8.6)$$

where, r is the shortest radial distance to the tangential component of the absolute velocity v.

Whether the momentum is linear or angular the fluid leaving the control volume exerts an equal and opposite reaction on the solid boundary of the control volume. In case of angular momentum the effect of such a reaction could be observed in lawn sprinklers.

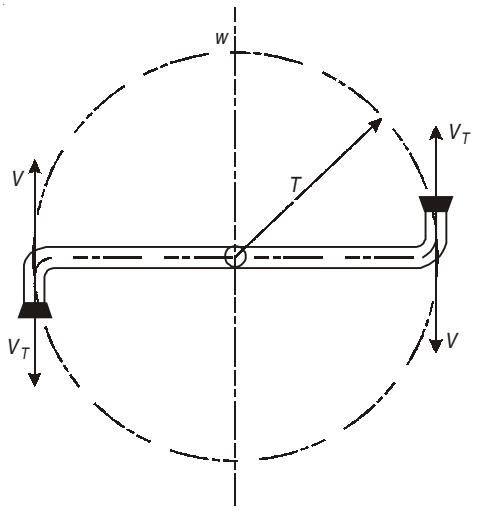


Symmetrical sprinkler

Fig. 8.2

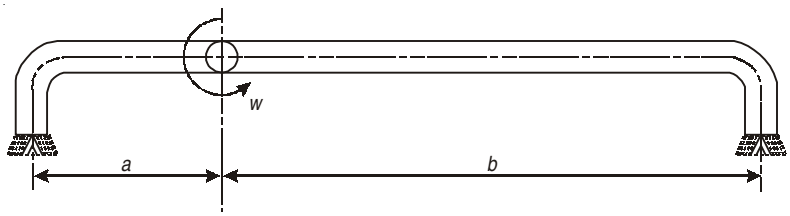
Figures 8.2, 8.3 and 8.4 show different types of lawn sprinklers. In symmetrical lawn sprinklers the outlet nozzles are symmetrical with the center of rotation and invariably discharge water in opposite directions thus creating a turning moment in the clockwise and anti-clockwise directions. The angle of discharge θ may be less than 90° as in Fig. 8.2 or may be = 90° as in Fig. 8.3.

Torque generation in asymmetrical sprinkler is due to the difference in the turning moment between the two arms.



Sysmmetrical sprinkler

Fig. 8.3



Asymmetrical sprinkler

Fig. 8.4

8.1 MOMENTUM CORRECTION FACTOR

In Chapter 7, while discussing steady flow energy equation we introduced the kinetic energy correction factor (Eq. 7.21) to take care of the velocity variations across a stream tube. For the same reasons a momentum correction factor β has to be introduced in the momentum equation since velocity variations affects the rate of change of momentum.

$$\beta = \left(\frac{1}{A}\right) \int_A \left(\frac{v}{v}\right)^2 dA \qquad \ldots (8.7)$$

Here v is the actual velocity and v is the mean velocity across a pipe of cross sectional area A.

Thus true momentum in unit time = β (Mass per unit time × mean velocity). Accordingly Eq. (8.2) stands modified thus,

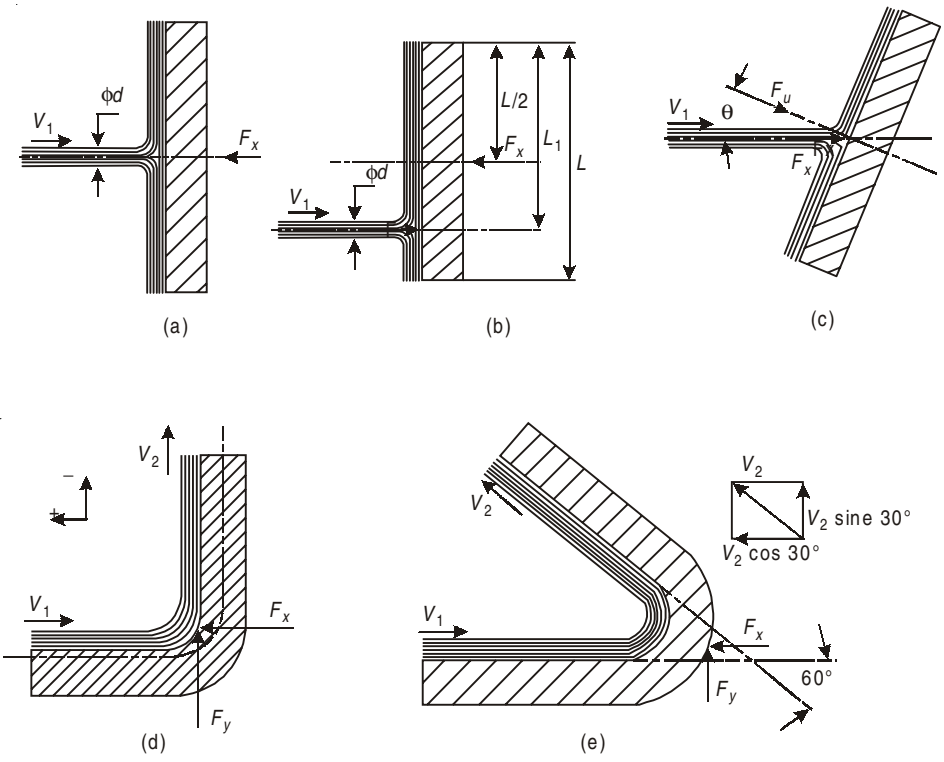
$$W + P_1A_1 + P_2A_2 + R = \beta_1 M_1 + \beta_2 M_2 \quad \text{... (8.8)}$$

The value of β depends on the shape of the pipe cross section and the shape of the velocity distribution curve. It is a non-dimensional factor whose value is always > 1. For laminar flow in a straight round pipe it has a value = $\left(\frac{4}{3}\right)$ and has a value = 1 for uniform flows.

8.1.1 Fluid Forces on Stationary Objects such as Flat Plates and Curved Vanes

When a jet of water strikes centrally on a flat stationary plate placed vertically, as in Fig. (8.5 a) the reaction force $F_x = \rho Q (v_2 - v_1)$, where v_2 is the final velocity = 0 and v_1 is the initial velocity = $- v_1$. Thus,

$$F_x = \rho Q v_1 \quad \text{... (8.9)}$$



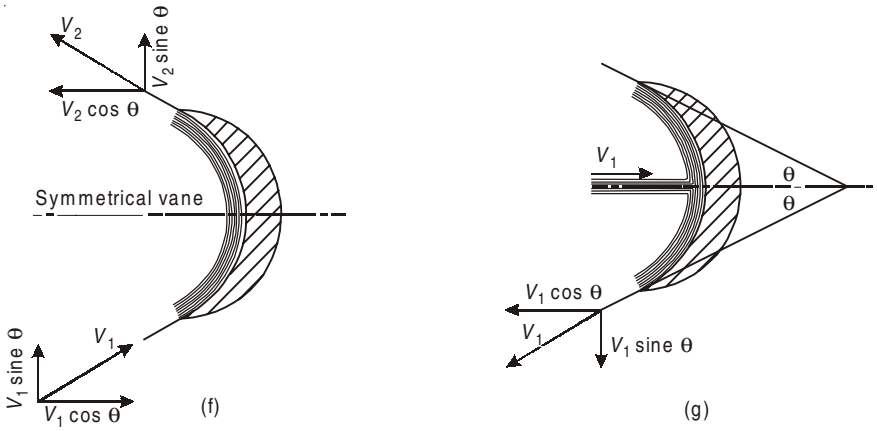


Fig. 8.5

When a jet of water strikes eccentrically on a flat stationary plate placed vertically, as in Fig. (8.5 b) the reaction force required, to keep the plate in equilibrium is,

$$F_x = (\rho\, Q\, v_1)\left(\frac{2L_1}{L}\right) \qquad \ldots (8.10)$$

When a jet of water strikes centrally on a flat stationary plate inclined at an angle θ to the direction of the jet, as in Fig. (8.5 c) the force normal to the plate,

$$F_n = \rho\, Q\, (v_1 \text{ sine } \theta) \qquad \ldots (8.11)$$

The components of this force in the x and y directions are,

$$F_x = F_n\, (\text{sine } \theta) \qquad \ldots (8.12)$$

$$F_y = F_n\, (\cos \theta) \qquad \ldots (8.13)$$

When a jet of water is deflected by a right angled stationary plate as in Fig. (8.5 d) the reaction forces on the plate are,

$$F_x = F_y = \rho\, Q\, (v_2 - v_1).$$

In case of F_x $\quad v_2 = 0$

and $\quad v_1 = -1,$

Hence $\quad F_x = \rho\, Q\, v_1 \qquad \ldots (8.14)$

In case of F_y $\quad v_1 = 0,$

Hence $\quad F_y = \rho\, Q\, v_2 \qquad \ldots (8.15)$

When a jet of water strikes a curved stationary plate as in Fig. (8.5 e) the reaction forces on the plate are,

$$F_x = \rho\, Q\, (v_1 - v_2 \cos \theta) \qquad \ldots (8.16)$$

Where, v_1 is the velocity in the x-direction and $v_2 \cos \theta$ is the component of v_2 in the x-direction.

$$F_y = \rho\, Q\, v_2 \text{ sine } \theta \qquad \ldots (8.17)$$

The component of v_1 in the direction of $y = 0$

When a high velocity liquid jet strikes a curved stationary vane tangentially as shown in Fig. (8.5 f), the reaction forces are,

$$F_x = \rho Q [v_1 \cos\theta - (-v_2 \cos\theta)]$$

Since $v_1 = v_2$

$$F_x = \rho Q [2v_1 \cos\theta] \quad \text{... (8.18)}$$

$$F_Y = \rho Q [v_1 \sin\theta - v_2 \sin\theta)]$$

$$= 0 \text{ since } v_1 = v_2$$

When a high velocity liquid jet strikes a curved stationary vane at the center as shown in Fig. (8.5 g), the reaction forces are,

$$F_x = \rho Q [v_1 - (-v_2 \cos\theta)]$$

$$F_x = \rho Q [v_1 + v_2 \cos\theta]$$

Since $v_1 = v_2$

$$F_x = \rho Q [v_1 + v_1 \cos\theta]$$

$$F_x = \rho Q v_1 [1 + \cos\theta] \quad \text{... (8.19)}$$

The above examples should be sufficient to help the student to get a fair idea of the fluid forces on stationary objects. As we solve a variety of problems a comprehensive idea would emerge.

In all of these cases since the fluid is assumed to be an ideal fluid, there are no forces parallel to the plate as no shear stresses are involved. Also since the plate is stationary the work done = 0.

8.1.2 Fluid Forces on Moving Objects such as Flat Plates and Curved Vanes

While determining the fluid forces on moving objects we consider the relative motion of the fluid with respect to the object. This is because the effective velocity with which the jet strikes the plate or the vane $V_e = (v_1 - u)$ where v_1 is the velocity of the jet and u is the velocity of the plate or the vane.

From Eq. (8.9) if a jet of fluid strikes a *stationary* flat plate centrally then the reaction force $F_x = \rho Q v_1$

If the plate is moving, $F_x = \rho Q (v_1 - u) = \rho Q (V_e)$ ... (8.20)

If a straight jet strikes a *moving plate placed at angle* θ to the direction of the jet centrally, the normal force on the plate,

$$F_N = \rho Q (v_1 - u) \sin\theta$$

$$= \rho Q (V_e) \sin\theta \quad \text{... (8.21)}$$

If we resolve this force in a direction perpendicular to the jet,

$$F_x = \rho Q (V_e) \sin\theta \times \sin\theta$$

$$= \rho Q (V_e) \sin^2\theta \quad \text{... (8.22)}$$

$$F_y = \rho Q (V_e) \sin\theta \times \cos\theta \quad \text{... (8.23)}$$

If a straight jet strikes a moving curved vane as shown in Fig. (8.6)

$$F_x = \rho Q (V_e) \cos\theta - (-\rho Q V_e)$$

$$= \rho Q V_e (1 + \cos\theta) \quad \text{... (8.24)}$$

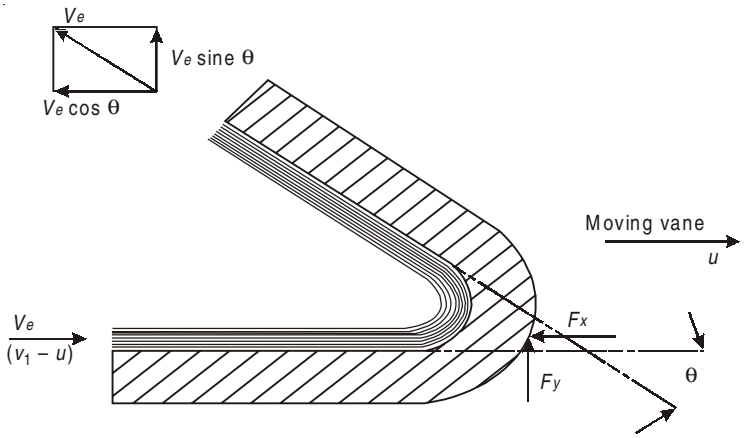


Fig. 8.6

$$F_y = \rho Q (V_e) \text{ sine } \theta \quad \text{... (8.25)}$$

In all of these cases $Q = AV_e$ ... (8.26)

Jet striking a symmetrical moving curved vane centrally (Ref. Fig. 8.6)

$$F_x = \rho Q (V_e) \cos \theta - (-\rho Q V_e)$$
$$= \rho Q V_e (1 + \cos \theta) \quad \text{... (8.27)}$$
$$F_Y = 0 \text{ due to symmetry} \quad \text{... (8.28)}$$

Force exerted when a jet of water strikes a single curved unsymmetrical vane at the inlet tip.

Note: In case of symmetrical vanes $\varphi = \theta$ (Refer Fig. 8.7) inlet angle = outlet angle

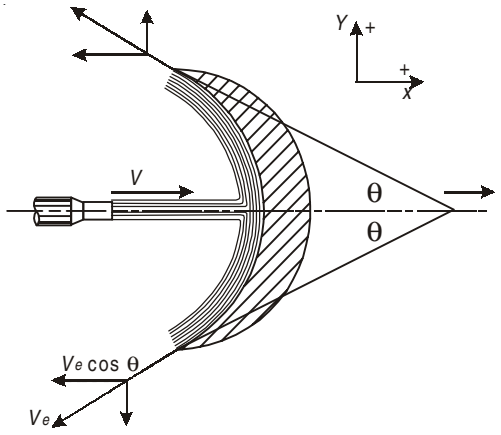


Fig. 8.7

For minimum energy loss it is necessary that the relative velocity of the jet

at the inlet and outlet must be tangential to the vane. This depends on the blade angles at the inlet and the outlet tip. Fluid would then be deflected tangentially to the vane without impact or shock. If such a condition is assumed then force in the direction of movement of the vane is equal to rate of change of momentum in that direction or mass deflected per second times the change of velocity in that direction.

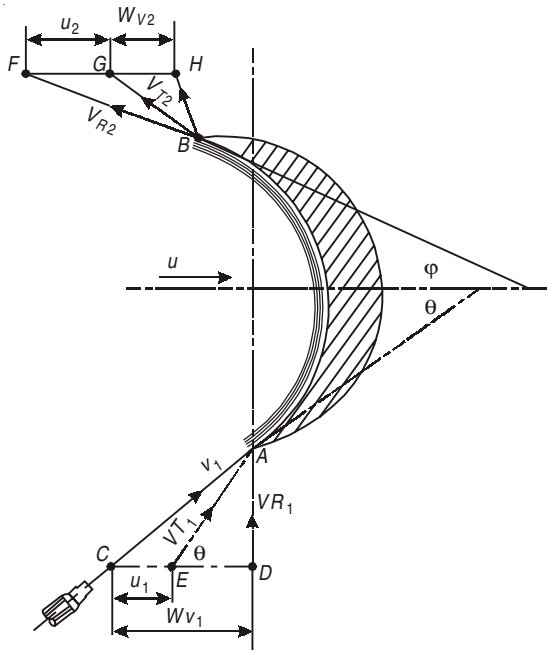


Fig. 8.8

8.1.3 Conditions at Inlet Triangle ADC

The jet strikes the vane at an angle α and with a velocity v_1. The fluid is deflected and enters the vane at an angle θ and with a relative and tangential velocity V_{T1}. *ED* represents the horizontal component of $V_{T1} = (V_{T1} \cos \theta)$ which is in the direction of the vane. V_{R1} represents the radial component of V_{T1}. $CE + ED = W_{V1}$ represent the component of velocity in the direction of motion of the vane at inlet. W_{V1} is also called the whirl velocity.

8.1.4 Conditions at Outlet Triangle BHF

The jet leaves the vane at angle β and with a velocity v_2. V_{T2} is the tangential velocity at exit and φ is the tip angle at exit. Components of v_2 at exit are W_{V2} and V_{R2} designated as whirl velocity and radial velocity at exit, respectively.

If vane surface is assumed to be smooth the exit velocity will be tangential to the vane at exit and under such conditions V_{T1} may be assumed equal V_{T2}. Consequently $u_1 = u_2 = u$ where u is the velocity of the vane.

Force in the direction of movement of the vane.

$$F_x = \rho A V_{T1} [V_{T1} \cos\theta - (- V_{T2} \cos \varphi)] \quad \text{... (a)}$$

$$V_{T1} \cos \theta = W_{V1} - u_1 \text{ and } V_{T2} \cos \varphi = W_{V2} + u_2$$

$$F_x = \rho A V_{T1} [W_{V1} - u_1 - (- W_{V2} + u_2)]$$

$$F_x = \rho A V_{T1} [W_{V1} - u_1 + u_2 + W_{V2}]$$

$$F_x = \rho A V_E [W_{V1} + W_{V2}] \quad \text{... (b)}$$

(since $u_1 = u_2$)

Equation (b) applies only where $\beta < 90°$.

If $\beta = 90°$ then

$$W_{V2} = 0$$

and hence Eq. (b) reduces to the form,

$$F_x = \rho A V_{T1} [W_{V1}] \quad \text{... (c)}$$

If $\beta > 90°$ then, $F_x = \rho A V_{T1} [W_{V1} - W_{V2}]$ ... (d)

Force Exerted when a Jet of Water Strikes Normally a Series of Flat Paddles Mounted on the Periphery of a Rotor

In case of a series of vanes/plates mounted on a rotor any one plate is always in contact with the jet resulting in the entire mass of water being utilized in doing useful work. In this particular case relative velocity before impact is $(v - u)$ and after impact it is zero.

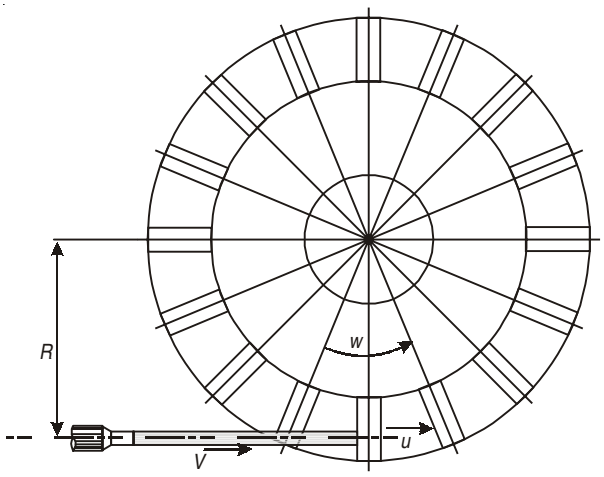


Fig. 8.9

F_x = (mass of water striking the plate)

[relative velocity before striking – velocity after striking]

$$F_x = (\rho Q v) [(V_e) - 0]$$

here $Q = (A \times v)$ since entire mass is utilized.

$$F_x = (\rho Q v)(V_e) \quad \text{... (8.29)}$$

Work done $= F_x \times u$

$$= (\rho Q v)(V_e)(u) \quad \ldots (8.30)$$

Here u is the velocity of the vane/plate.

$$\text{Efficiency} = \frac{\left(\frac{\text{Work done}}{\text{sec}}\right)}{\left(\frac{\text{Kinetic energy of the jet}}{\text{sec}}\right)}$$

$$\eta = \frac{(\rho Q v)(V_e)(u)}{\left(\frac{1}{2} mv^2\right)}$$

$$= \frac{[\rho (Av)(v)](V_e)(u)}{\left[\frac{1}{2}(\rho Av)v^2\right]}$$

$$\eta = \left[\frac{2u(V_e)}{v^2}\right] \quad \ldots (8.31)$$

η will be maximum when $\left(\frac{d\eta}{du}\right)$=0,that is when vane velocity is half the jet velocity$\left(u = \frac{v}{2}\right)$.

Force exerted when a Jet of Water Strikes Normally a Series of Curved Vanes Mounted on the Periphery of a Rotor. (Figure 8.10)

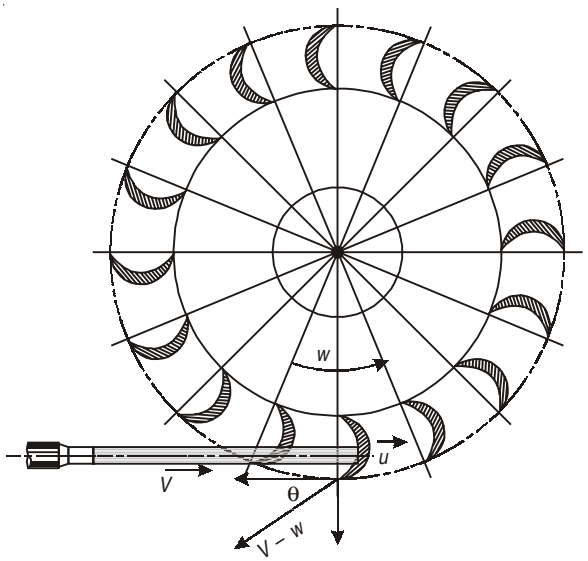


Fig. 8.10

$$F_x = \rho Q [(V_e) - (-V_e \cos \theta)$$
$$= \rho Q V_e (1 + \cos \theta) \quad \text{... (8.32)}$$
$$F_Y = 0 \text{ due to symmetry} \quad \text{... (8.33)}$$

Here again since the entire mass of water is converted in to useful work

$$Q = Av$$

Here again maximum efficiency is when $u = \frac{v}{2}$.

Force exerted when a jet of water strikes at an angle to the inlet tip of a series of curved vanes mounted on the periphery of a rotor

Vanes are curved because they can extract maximum useful energy from the fluid. Such vanes are found on the rotors of turbo-machinery such as the Pelton wheel. Since in theory, the friction is neglected by assuming the vanes are perfectly smooth, the relative velocity at entry and exit is always tangential to the vane.

Figure 8.11 shows a disc which is free to rotate on its center O and on whose periphery, a series of curved vanes are mounted. The inner radius of the disc is R_i and the outer radius is R_0. If ω is the angular velocity then, $u_1 = \omega R_0$ and $u_2 = \omega R_i$. The jet flow could be from A–B or from B–A depending on whether the jet enters through the inner or outer periphery.

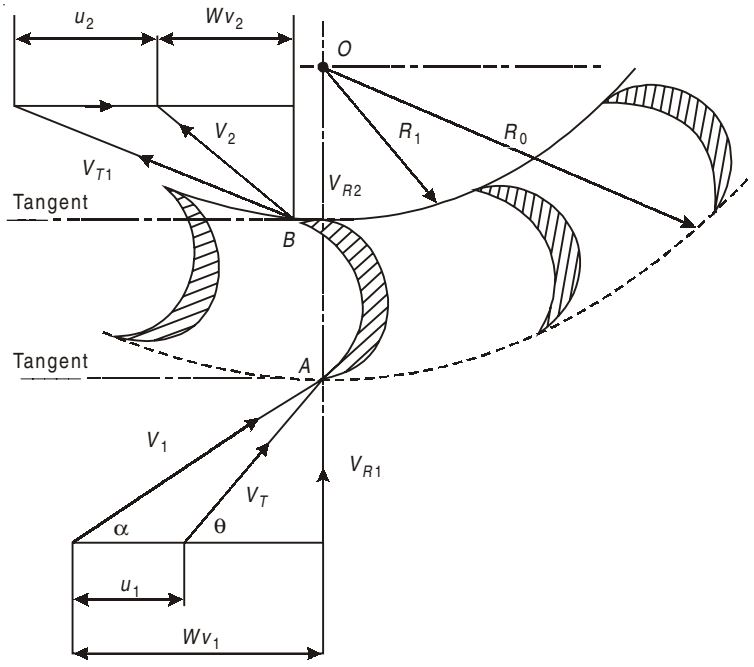


Fig. 8.11

From the moment of momentum equation,

Torque exerted on the wheel by the jet = rate of change of angular momentum.

$$T = \left(\frac{\text{Angular momentum}}{\text{Sec at the inlet}} - \frac{\text{Angular momentum}}{\text{Sec at the outlet}} \right)$$

$$T = (\text{momentum at inlet} \times \text{radius at inlet} - \text{momentum at outlet} \times \text{radius at outlet})$$

$$T = [\rho A v_1 \times W_{V1} \times R_0 - (-\rho A v_1 \times W_{V2} \times R_i)]$$

$$T = \rho A v_1 [W_{V1} \times R_0 + W_{V2} \times R_i] \quad \ldots(8.34)$$

$$\text{Work done } W = \rho A v_1 [W_{V1} \times R_0 + W_{V2} \times R_i] \omega \quad \ldots(8.35)$$

$$F_x = \rho A v_1 [W_{V1} + W_{V2}] \quad \ldots (8.36)$$

8.2 JET PROPULSION

We have thus far discussed the force exerted by a jet of fluid issuing out of a nozzle on objects external to it and how this force varies when the jet strikes a moving object as compared to a stationary object. It is important to appreciate at this point that if the nozzle itself was not held firmly in position during this process it would have made little impact on the objects on which jet was striking because the reaction force would have displaced the nozzle from its original position. This is exactly the phenomenon that causes the lawn sprinklers about which we have already mentioned (Ref. Fig. 8.2 to 8.4) to rotate about their centers. The nozzle through which the jet issues out is free to move about its axis and hence the nozzle is simply obeying the Newton's third law of motion, "action and reaction are equal and opposite" *Jet propulsion could therefore be defined as the propulsion or the displacement of an object in a direction opposite to the direction of a jet of fluid issuing out of a nozzle contained within.*

This is the governing principle for the jet aircraft propulsion system, the rocket propulsion system or even the simple mechanized boat propulsion system.

In the aircraft propulsion system the atmospheric air is drawn in and compressed in a multi stage axial flow compressor. This high pressure air is converted in to a stream of hot gas in a combustion chamber by admitting a small amount of fuel and then expanded in a turbine before it is discharged at a very high velocity through the tail pipe back in to the atmosphere to create the forward thrust required to propel the aircraft.

In a rocket propulsion system as the rocket is required to cruise in space where there is no oxygen, the rocket carries its own oxygen supplies in addition to fuel to keep the combustion process going so that forward thrust is always maintained.

In a mechanized speed-boat on the other hand Ref. Fig. (8.12) a hydraulic pump (P) driven by a prime mover imparts energy to water admitted through an opening either in the bow (b) of the vessel or through the sides(s) and this

water is let out of a jet pipe or a nozzle (N) mounted at the stern which in turn creates the forward thrust (F_x) required to propel the boat. The propelling force

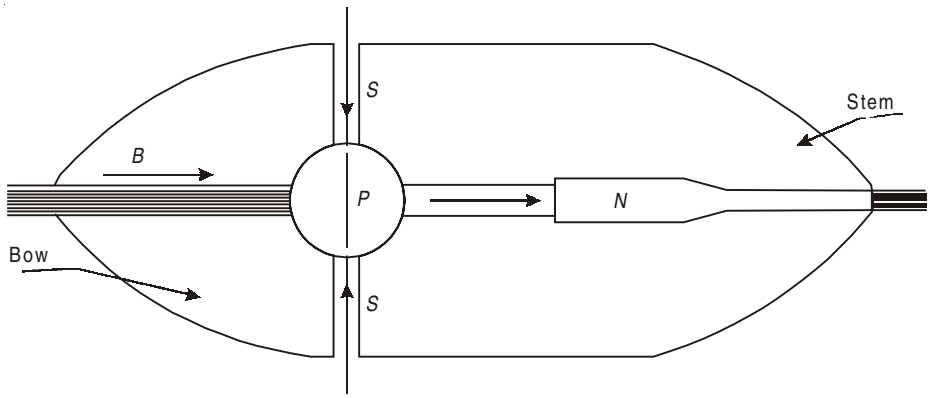


Fig. 8.12

and the work done remains the same in either case but the hydraulic efficiency is lower if the water is admitted through the sides as the pump is not assisted at all at the intake point and hence must exert that much more to provide all of the energy required.

(i) *Let us first consider straight intake* (B) *at the bow.*

Mass of water discharged through the jet/sec = ρQ

$$= (\rho A v_R)$$

v_R is the relative velocity $= [v - (-u)]$

$$= (v + u)$$

where v = velocity of the issuing jet and u is the velocity of the boat. u is negative because its direction is opposite to v. A = area of the jet

$$\text{Propelling force } F_x = \frac{\text{Mass of water discharged}}{\text{sec}} \times \text{Change in velocity}$$

$$F_x = (\rho A v_R) \times (v_R - u)$$

$$F_x = (\rho A v_R) \times (v + u - u)$$

$$F_x = (\rho A v_R) \times (v) \qquad \text{... (8.37)}$$

$$\frac{\text{Work done}}{\text{sec}} = (F_x \times u)$$

$$\frac{\text{Work done}}{\text{sec}} = (\rho A v_R) \times (v) \times (u)$$

$$\frac{\text{Work done}}{\text{sec}} = (\rho A v_R) \times (v_R - u) \times (u) \qquad \text{... (8.38)}$$

Hydraulic efficiency η = ($F_x \times u$) / kinetic energy of the jet.

Kinetic energy of the jet = $\frac{1}{2} mv^2$

$= \frac{1}{2} (\rho A v_R) (v_R - u)^2$

Hydraulic efficiency $\eta = \frac{(\rho A v_R) \times (v) \times (u)}{\frac{1}{2}(\rho A v_R) (v_R - u)^2}$

Hydraulic efficiency $\eta = \frac{(\rho A v_R) \times (v_R - u) \times (u)}{\frac{1}{2}(\rho A v_R) (v_R - u)^2}$

Upon simplification $\eta = \frac{2u}{(v_R + u)}$ ... (8.39)

(ii) *When the in take is through the sides S*

Kinetic energy of the jet per unit of time in this case = $\left[\frac{1}{2}(\rho A v_R) (v^2_R)\right]$

since the velocity of the boat is in no way assisting the intake. Therefore, hydraulic efficiency $\eta = \frac{(\rho A v_R) \times (v_R - u) \times (u)}{\frac{1}{2}(\rho A v_R) (v^2_R)}$ which upon simplifying reduces to the form, $\eta = \frac{2 (v_R - u) \times (u)}{(v^2_R)}$ ... (8.40)

8.3 PROPULSION THEORY OF PROPELLERS

Water jet propulsion system is confined to small boats where speed is the main criteria and not the hauling capacity. Also the hydraulic efficiency is low. Large ships and aircrafts make use of rotating propellers to change the momentum of the fluid in which it is submerged and develop the forward thrust. A propeller may be stationary through a moving fluid as in an exhaust air ducting system Fig. (8.23) or the propeller might be moving through a stationary fluid as in case of a ship or an aircraft. In either case the momentum exchange remains the same.

The imaginary boundary between stations *AA–BB* is called the slip-stream boundary. The fluid remains undisturbed beyond this boundary. As the fluid approaches the propeller due to reduced pressure upstream it is accelerated and suffers pressure intensification between points 1 and 2.

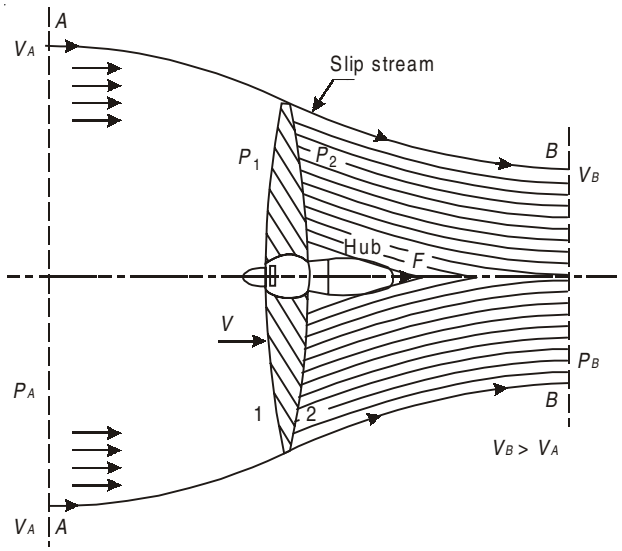


Fig. 8.13

Flow, as the slip-stream boundary now has reduced cross section.

The force exerted by the propeller $F = (P_2 - P_1) A$ ... (8.41)

Here A = Swept area of the propeller blades.

From the momentum equation $F = \rho Q (v_B - v_A)$ ... (8.42)

By equating the two,

or $$\rho Q (v_B - v_A) = (P_2 - P_1) A$$

or $$\rho A V (v_B - v_A) = (P_2 - P_1) A$$

$$\rho V (v_B - v_A) = (P_2 - P_1) \quad \text{... (8.43)}$$

By applying the energy equation across points (AA– 1) and (2 – BB)

$$P_A + \frac{1}{2} \rho v^2_A = P_1 + \frac{1}{2} \rho v^2$$

and $$P_2 + \frac{1}{2} \rho v^2 = P_B + \frac{1}{2} \rho v^2_B \quad \text{... (a)}$$

Equation (a) may be re-arranged in terms of P_1 and P_2.

$$P_1 = P_A + \frac{1}{2} \rho v^2_A - \frac{1}{2} \rho v^2$$

$$P_2 = P_B + \frac{1}{2} \rho v^2_B - \frac{1}{2} \rho v^2$$

$$(P_2 - P_1) = \left(P_B + \frac{1}{2} \rho v^2_B - \frac{1}{2} \rho v^2 \right) - \left(P_A + \frac{1}{2} \rho v^2_A - \frac{1}{2} \rho v^2 \right)$$

It may be noted that the datum head is the same throughout, the fluid velocity before and after the blades (v) does not change and that the velocity and pressure at station AA is the same as that at station BB. Hence,

$$(P_2 - P_1) = P_B + \frac{1}{2}\rho v^2{}_B - \frac{1}{2}\rho v^2 - P_A - \frac{1}{2}\rho v^2{}_A + \frac{1}{2}\rho v^2$$

$$(P_2 - P_1) = \frac{1}{2}\rho\,(v^2{}_B - v^2{}_A) \qquad \ldots (8.44)$$

From Eq. (8.43)

$$\rho V\,(v_B - v_A) = (P_2 - P_1) \text{ and hence,}$$

$$\rho V\,(v_B - v_A) = \frac{1}{2}\rho\,(v^2{}_B - v^2{}_A)$$

$$V = \frac{1}{2}\,(v_B + v_A) \qquad \ldots (8.45)$$

This goes to prove that the velocity through the propeller is the average of velocities up-stream and down-stream from it.

$$\text{Propulsive power} = F \times v_A$$
$$= \rho Q\,(v_B - v_A)\,v_A$$
$$\text{Power input} = \text{propulsive power + power lost}$$
$$\text{Power input} = \rho Q\,(v_B - v_A)\,v_A + \text{kinetic energy remaining}$$

$$\text{Power input} = \rho Q\,(v_B - v_A)\,v_A + \frac{1}{2}\,[\rho Q\,(v_B - v_A)^2]$$

$$\text{Theoretical efficiency } \eta = \frac{\text{(Power Input)}}{\text{(Power out put + Power lost)}}$$

$$\eta = \frac{\rho Q\,(v_B - v_A)\,v_A}{\rho Q\,(v_B - v_A)\,v_A} + \frac{1}{2}\,[\rho Q\,(v_B - v_A)^2]$$

(We eliminate ρQ)

$$\eta = \frac{(v_B - v_A)\,v_A}{(v_B - v_A)\,v_A} + \frac{1}{2}\,(v_B - v_A)^2$$

Dividing throughout by 2

$$\eta = \frac{2(v_B - v_A)\,v_A}{2(v_B - v_A)\,v_A} + (v_B - v_A)^2$$

$$\eta = \frac{2(v_B - v_A)\,v_A}{2(v_B - v_A)\,v_A} + (v_B - v_A)\,(v_B - v_A)$$

$$\eta = \frac{2(v_B - v_A)\,v_A}{(v_B - v_A)}\,[2\,v_A + (v_B - v_A)]$$

$$\eta = \frac{2v_A}{[v_A + v_B]} = \frac{2v_A}{2v}$$

$$\eta = \frac{v_A}{v} \qquad \ldots (8.46)$$

Propellers used in compressible fluid media have an efficiency rating of up to 85 percent if the velocity $v_A \leq 100$ m/sec. Beyond this a steep decline in efficiency is observed. Propellers used in in-compressible media such as in ships suffer from low efficiency because propeller diameters are small due to design constraints.

TYPICAL PROBLEMS AND SOLUTIONS

Problem 8.1 *A jet of water 60 mm in diameter is discharged in the horizontal direction with a velocity of 50 m/sec from a nozzle mounted on a trolley. Determine the force required to hold the trolley stationery. Neglect friction.*

Solution The control volume is the trolley. The net efflux of moment in the direction of the jet = $\rho Q\, v_{exit}$

$= (1000) \times [0.7854\ (0.06)^2] \times 50$

$= 14.13$ kN.

The force required to hold the boat stationery is 14.13 kN in the direction of the jet.

Problem 8.2 *A nozzle and a pipe are held together by bolts (Fig.8.13). At the entrance to the nozzle the pressure is 7×10^5 Pa and the diameter is 100 mm. At the exit to the nozzle the diameter is 30 mm. The viscosity of the fluid passing through is 0.9. Determine the force exerted by the nozzle on the bolts.*

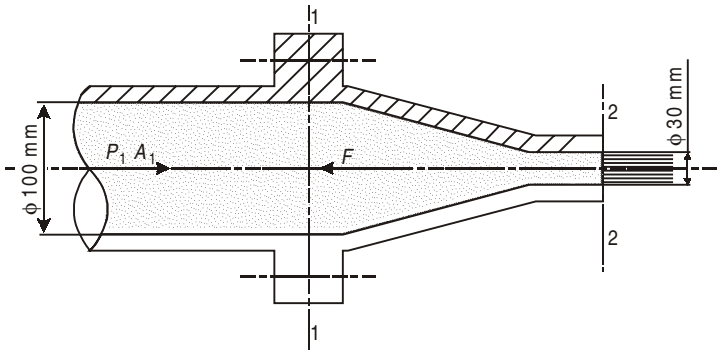


Fig. 8.14

Solution We have to apply the energy equation between points 1 and 2

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} = Z_2 + \frac{v_2^2}{2g} + \frac{P_2}{\gamma}$$

Here $Z_1 = Z_2$ and $P_2 = 0$

$$\frac{v_1^2}{2g} + \frac{P_1}{\gamma} = \frac{v_2^2}{2g} \quad \text{... (a)}$$

From the continuity equation $0.7854\,(d_1)^2\,v_1 = 0.7854\,(d_2)^2\,v_2$

$$v_2 = \left(\frac{d_1}{d_2}\right)^2 v_1$$

$$v_2 = \left(\frac{100}{30}\right)^2 v_1 = 11.10\,v_1$$

$$P_1 = \frac{\left(7 \times 10^5\right)}{0.9 \times 9810}$$

$$= 79.3 \text{ m}$$

Substituting these values in Eq. (a)

$$\frac{v_1^2}{2g} + 79.3 = \frac{\left(11.10 v_1\right)^2}{2g}$$

$$\frac{v_1^2}{2g}\,(1 - 123.4) + 79.3 = 0$$

$$(-122.4)\,\frac{v_1^2}{2g} = -79.3$$

$$v_1^2 = \frac{[(-79.3) \times 2g]}{(-122.4)}$$

$$= 12.71$$

$$v_1 = 3.56 \text{ m/sec}$$

and $v_2 = 39.5$ m/sec

$$Q = 0.7854\,(0.1)^2 \times 3.56$$

$$= 0.028 \text{ m}^3/\text{sec}$$

$$P_1 \times A_1 - F = \gamma \times Q\,(v_2 - v_1)$$

$$(7 \times 10^5) \times 0.7854\,(0.1)^2 - F = 0.9 \times 9810 \times 0.028\,(39.5 - 3.56)$$

$$F = 8.8 \text{ kN}$$

Problem 8.3 *Water at 15°C flows through a right angle elbow placed in the horizontal plane @ 0.03 cubic meters/sec. Pressure at the inlet is 500 kPa and diameter at the inlet is 300 mm. Determine the horizontal and vertical components of the reaction force at the elbow.*

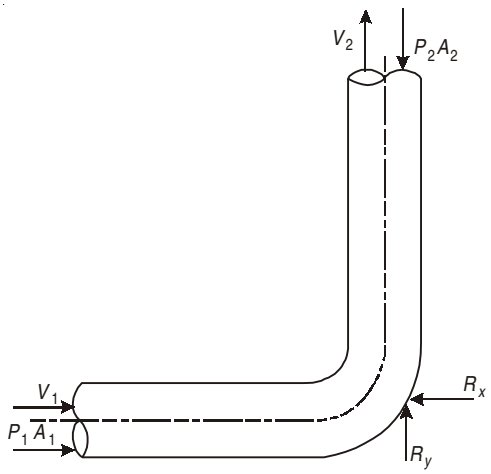


Fig. 8.15

Solution

$\rightarrow F_x$ = net reaction force in the opposite direction

$\rightarrow F_x = (R_x - P_1A_1) \leftarrow$

$\rho Q(v_2 - v_1) = R_x - P_1A_1$ ($v_2 = 0$, and $v_1 = -v_1$) hence,

$R_x - P_1A_1 = \rho Q\, v_1$

or $R_x = \rho Q\, v_1 + P_1A_1$ ... (a)

Similarly in the ↑ direction $R_y = \rho Q\, v_2 + P_2A_2$

But $P_2 = P_1$

and $v_2 = v_1$

Since the diameter of the pipe is constant.

Therefore $R_y = \rho Q\, v_1 + P_1A_1$ ... (b)

From (a) and (b)

$R_x = R_y$

$= \rho Q\, v_1 + P_1A_1$ ... (c)

For water at 15°C $\rho = 1000$ kg/m^3

$Q = 0.03$ m^3/sec

$P_1 = (500 \times 1000)$ Pa

$A_1 = 0.7854(0.3)^2$

$= 0.007$ m^2

Substituting these values in (c),

$$v_1 = \frac{Q}{A_1}$$

$$= \frac{0.03}{0.007}$$

$$= 4.28 \text{ m/sec}$$

$$R_x = R_y$$

$$= (1000)(0.03)\,(4.28) + (500000)\,(0.007)$$

$$= 128.4 + 3500$$

$$= 3628.4 \text{ N}$$

Problem 8.4 *A reducing bend shown in Fig. 8.16 has an inlet φ = 60 cm and outlet φ = 30 cm. The discharge through the bend is 0.05 m^3/sec and the pressure at the inlet and outlet is 10 kN/m^2 and 5.5 kN/m^2 respectively. The weight of water inside the bend is 3 N. Determine the reaction forces that keep the bend in equilibrium.*

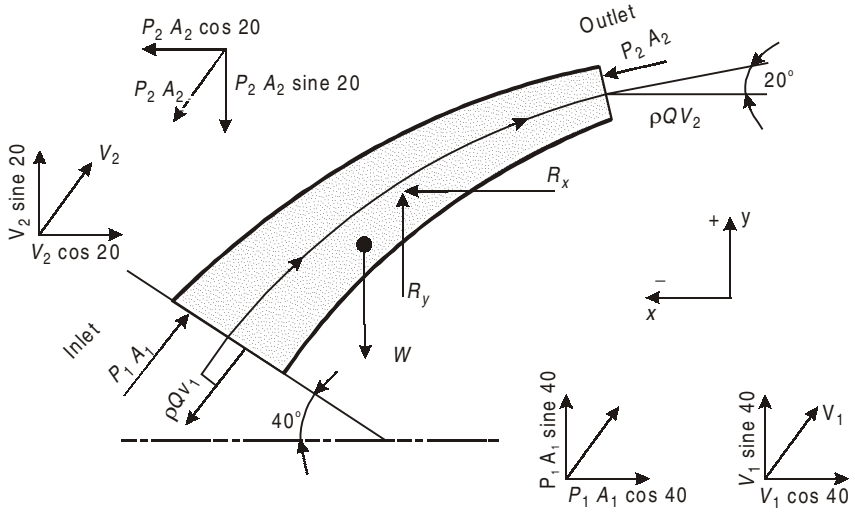


Fig. 8.16

Solution

Let us recall the momentum Eq. (8.3) and (8.4)

$$P_1A_1 \cos \theta_1 - P_2A_2 \cos \theta_2 - R_x = \rho\, Q\, v_2 \cos \theta_2 - \rho\, Q\, v_1 \cos \theta_1 \quad \text{... (a)}$$

$$R_y - W - P_2A_2 \text{ sine } \theta_2 + P_1A_1 \text{ sine } \theta_1 = \rho\, Q\, v_2 \text{ sine } \theta_2 - \rho\, Q\, v_1 \text{ sine } \theta_1 \quad \text{... (b)}$$

$$P_1 = 10 \text{ kN/m}^2$$

$$= (10 \times 1000)$$

$$= 10^4 \text{ N}$$

$$P_2 = 5.5 \text{ kN/m}^2$$
$$= (5.5 \times 1000)$$
$$= 5.5 \times 10^3 \text{ N}$$
$$A_1 = 0.7854\ (0.6)^2$$
$$= 0.28 \text{ m}^2$$
$$A_2 = 0.7854\ (0.3)^2$$
$$= 0.070 \text{ m}^2$$
$$\cos \theta_1 = \cos 40 = 0.766$$
$$\cos \theta_2 = \cos 20 = 0.939$$
$$v_1 = \frac{Q}{A_1}$$
$$= \frac{0.05}{0.28} = 0.178 \text{ m/sec}$$
$$v_2 = \frac{Q}{A_2}$$
$$= \frac{0.05}{0.07} = 0.71 \text{ m/sec}$$
$$\text{sine } \theta_1 = \text{sine } 40 = 0.642$$
$$\text{sine } \theta_2 = \text{sine } 20 = 0.342$$
$$Q = 0.05 \text{ m}^3/\text{sec}$$

Substitute the relevant values in Eqs. (a) and (b) above and solve for R_x and R_y

$$-R_x = P_1A_1\cos\theta_1 + P_2A_2\cos\theta_2 + \rho Q v_2 \cos\theta_2 - \rho Q v_1 \cos\theta_1$$
$$-R_x = (10^4 \times 0.28)\ 0.766 + (5.5 \times 10^3)(0.070)0.939 + (9810)\ (0.050)(0.71)\ (0.939) - (9810)\ (0.05)\ (0.178)\ (0.766)$$
$$-R_x = 2144.8 + 361.5 + 327 - 119$$
$$= 2714.3 \text{ N}$$
$$R_y = W + P_2A_2 \text{ sine } \theta_2 - P_1A_1 \text{sine } \theta_1 + \rho Q v_2 \text{ sine } \theta_2 - \rho Q v_1 \text{ sine } \theta_1$$
$$R_y = 3 + (5.5 \times 10^3)(0.070)\ (0.342) - (10^4 \times 0.28)\ 0.642 + (9810)(0.050)(0.71)\ (0.342) - (9810)\ (0.050)\ (0.178)\ (0.342)$$
$$= -1573.7 \text{ N}$$

Since both R_x and R_y are negative it can be concluded that the initially assumed directions are wrong. The right values for the reactions are,

$$R_x = 2714.3 \text{ N}$$

and $$R_y = 1573.7 \text{ N}$$

Problem 8.5 *Oil with a specific volume of 1.176 × 10^{-3} m^3/kg flows through a reducing bend shown in Fig. (8.17. The diameter at the inlet is 350 mm and at the outlet it is 175 mm. Pressure at the inlet is 500 kPa. If the mass flow rate is 400 kg/sec, determine the magnitude and direction of the resultant force on the bend. The pipe is in the horizontal plane.*

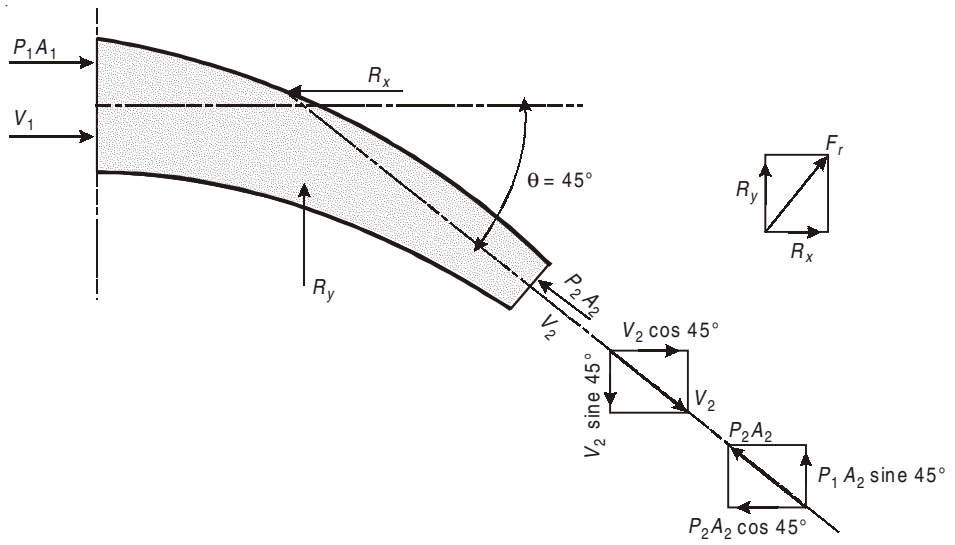


Fig. 8.17

Solution

Let us recall the momentum Eqs. (8.3) and (8.4)

$$P_1A_1\cos\theta_1 - P_2A_2\cos\theta_2 - R_x = \rho Q v_2\cos\theta_2 - \rho Q v_1\cos\theta_1 \quad \text{... (a)}$$

$$R_y - W - P_2A_2\sin\theta_2 + P_1A_1\sin\theta_1 = \rho Q v_2\sin\theta_2 - \rho Q v_1\sin\theta_1 \quad \text{... (b)}$$

The above equations require to be altered to suit the present problem.

$$P_1A_1 - P_2A_2\cos\theta - R_x = \rho Q v_2\cos\theta - \rho A_1 v_1 \quad \text{... (c)}$$

$$R_y - P_2A_2\sin\theta = \rho Q v_2\sin\theta \quad \text{... (d)}$$

We may arrange (c) and (d) in terms of R_x and R_y for convenience

$$-R_x = -P_1A_1 + P_2A_2\cos\theta + \rho Q v_2\cos\theta - \rho A_1 v_1 \quad \text{... (e)}$$

$$R_y = P_2A_2\sin\theta + \rho Q v_2\sin\theta \quad \text{... (f)}$$

Since P_2, v_1 and v_2 are not known we apply the Bernoulli's equation to obtain the value of P_2 and determine v_1 and v_2 from the mass-flow rate.

Mass flow rate = ρQ.

$$\rho = \left(\frac{1}{1.176 \times 10^{-3}}\right)$$

$$\approx 850 \text{ kg/m}^3$$

Therefore

$$Q = \frac{400}{850}$$

$$= 0.470 \text{ m}^3/\text{sec}$$

We now apply the Bernoulli's equation to obtain P_2.

$$\frac{v_1^2}{2g} + \frac{P_1}{\gamma} + Z_1 = \frac{v_2^2}{2g} + \frac{P_2}{\gamma} + Z_2$$

Since $Z_1 = Z_2$ the equation reduces to the form

$$P_2 = P_1 + \gamma \frac{\left(v_1^2 - v_2^2\right)}{2g}$$

$(\gamma = 850 \times 9.81 = 8338.5 \text{ N/m}^3)$

$$v_1 = \frac{Q}{A_1}$$

$$= \frac{0.470}{0.7854\ (0.350)^2} = 4.9 \text{ m/sec}$$

$$v_2 = \frac{Q}{A_2}$$

$$= \frac{0.470}{0.7854\ (0.175)^2} = 19.5 \text{ m/sec}$$

$$P_2 = (500 \times 1000) + \left[\frac{(4.9^2 - 19.5^2)}{2 \times 9.81}\right] \times 8338.5$$

$$= 348.6 \text{ kN/m}^2$$

$$A_1 = 0.7854\ (0.350)^2$$

$$= 0.096 \text{ m}^2$$

$$\cos\theta = \cos 45$$

$$= 0.707$$

$$A_2 = 0.7854\ (0.175)^2$$

$$= 0.024 \text{ m}^2$$

$$\text{sine } \theta = \text{sine } 45$$

$$= 0.707$$

Substitute the relevant values in Eqs (e) and (f) above and solve for R_x and R_y

$$-R_x = -P_1A_1 + P_2A_2 \cos 45 + \rho Q v_2 \cos 45 - \rho Q v_1$$

$$-R_x = -(500 \times 1000)0.096 + (348.6 \times 1000)0.024$$
$$(0.707) + (8338.5)(0.470)(19.5 \times 0.707)$$
$$-(8338.5)(0.470)(4.9)$$

$$= 32258\text{N}$$

$$-R_x = 32.25 \text{ kN}$$

(since R_x is negative the direction assumed was wrong)

$$R_y = P_2 A_2 \sin 45 + \rho Q v_2 \sin 45$$

$$R_y = (348.6 \times 1000)\ 0.024\ (0.707) + (8338.5)(0.470)(19.5) \times 0.707$$

$$= 58945.6\text{N}$$

$$R_y = 58.94 \text{ kN}$$

The resultant force $F_R = -R_x^2 + R_y^2$

$$F_R = \sqrt{-32.25^2 + 58.94^2}$$

$$= 67.158 \text{ kN}$$

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) = 61.3°$$

Problem 8.6 *A jet of oil impinges horizontally and centrally on a flat vertical plate. The diameter of the jet is 60 mm and the jet velocity is 30 m / sec. Determine, the reaction force that can hold the plate in equilibrium. Sg. of oil = 0.83.*

Solution

$$F_x = \rho Q v_1$$

$$= (0.83 \times 1000)\ [0.7854\ (0.060)^2 \times 30]\ 30$$

$$= 2.1 \text{ kN}$$

Problem 8.7 *All other factors remaining the same, if the jet of oil in problem (8.6) strikes eccentrically as shown in Fig. (8.5 b) and if the eccentricity e = 200 mm, determine the reaction force at the center of the plate.*

Solution Refer to Fig. (8.5 b)

Let $L = 1$ m

$e = 0.2$ m

This means $L_1 = 0.7$ m

$$\left(L_1 = \frac{L}{2} + 0.2 = \frac{1}{2} + 0.2 = 0.7 \text{ m}\right)$$

$$F_x = (\rho Q v_1)\left(\frac{2L_1}{L}\right)$$

$$F_x = (0.83 \times 1000)\ [0.7854\ (0.060)^2 \times 30]30$$

$$(2 \times 0.7) = 2.94 \text{ kN}$$

Problem 8.8 *A jet of water of diameter 50 mm strikes a U bend placed as shown in Fig (8.18) with a velocity = 4 m/sec. Determine the reaction force required, to keep the bend in equilibrium.*

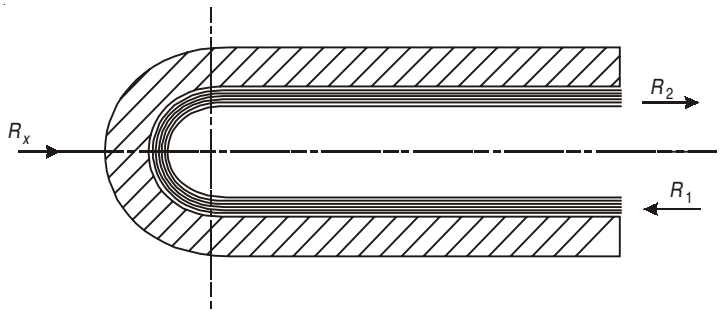


Fig. 8.18

Solution

$$F_x = \rho Q [v_1 - (-v_2)]$$
$$= (1000) [0.7854 (0.050)^2 \times 4] [v_1 + v_2)]$$
$$F_x = (1000) [0.7854 (0.050)^2 \times 4] [2v_1)]$$

Since $v_1 = v_2$.

$$F_x = (1000) [0.7854 (0.050)^2 \times 4] [2 \times 4)]$$
$$= 62.8 \text{ N}$$

Problem 8.9 *A jet of water strikes a bend shown in Fig. (8.19) with a velocity = 10 m/sec and exits with a velocity = 8 m/sec. If the mass flow rate is 0.8 kg/sec, determine the reaction forces required to keep the bend in equilibrium and the magnitude and direction of the total resultant force.*

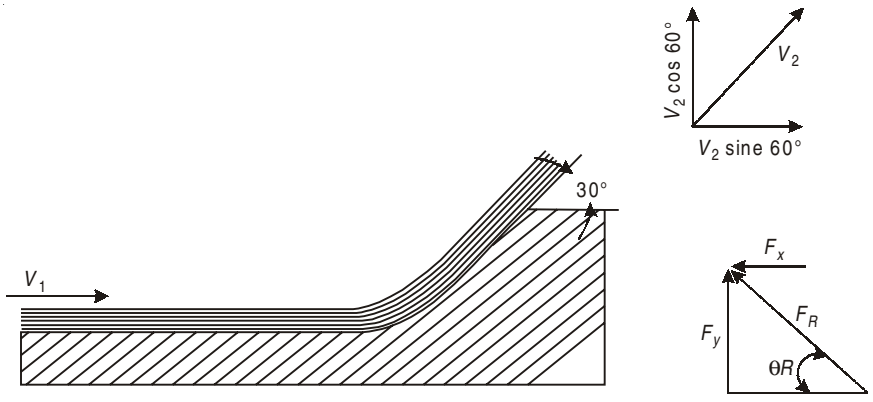


Fig. 8.19

Solution

$$F_x = \rho Q (v_1 - v_2 \text{ sine } \theta)$$

Where, v_1 is the velocity in the x-direction and v_2 sine θ is the component of v_2 in the x-direction.

$$F_y = \rho Q v_2 \cos \theta$$

The component of v_1 in the direction of $y = 0$

and $\theta = (90 - 30) = 60°$

$F_x = 0.8\,(10 - 8 \text{ sine } 60)$

$= 2.45$ kg-m/sec$^2 = 2.45$ N

$F_y = \rho\, Q\, v_2 \cos\theta$

$= 0.8\,(8 \cos 60)$

$= 3.20$ kg-m/sec$^2 = 3.20$ N

The resultant force $F_R = \sqrt{-F_x^2 + F_y^2}$

$= \sqrt{-2.45^2 + 3.20^2} = 4$ N

$\theta_R = \tan^{-1}\left(\dfrac{3.20}{2.45}\right) = 52.56°$

Problem 8.10 *A plate weighing 70 N is suspended vertically by a hinge at the top edge. The center of gravity of the plate is at a distance of 0.2 m from the hinge. A jet of water of diameter 30 mm impinges the plate horizontally with a velocity of 6 m/sec at a distance of 0.4 m from the hinge. Determine the reaction force at the point of center of gravity. Determine the velocity of the jet required to deflect the plate by 30° if the reaction force remains constant.*

Solution

(a) The reaction force $F_x = (\rho\, Q\, v_1)\left(\dfrac{2L_1}{L}\right)$

Refer to Fig. (8.15)

$F_x = (1000)\,[0.7854\,(0.030)^2\,(6)] \times 6\left(\dfrac{2 \times 0.4}{0.2}\right)$

$= 102$ N

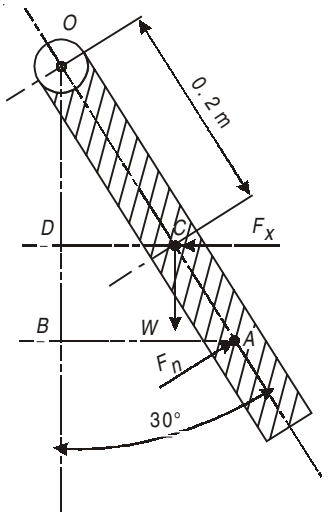


Fig. 8.20

(b) When the plate is deflected through 30° the forces acting are as shown in Fig. (8.20).

$$\text{Normal force } F_N = \rho Q (v \text{ sine } \theta) \text{ from Eq. (8.24)}$$

$$F_N = \rho(a \times v)(v \text{ sine } \theta)$$

$$= \rho(av^2)(\text{ sine } 90 - 30)$$

$$F_N = \rho(av^2)(\text{ sine } 60)$$

$$= (1000)(0.7854 \times 0.030^2)\,(v)^2 \times 0.8660$$

$$= 0.612\, v^2.$$

Taking moments about the hinge O

$$F_N \times OA = F_x \times OD + W \times DC \qquad \ldots (2)$$

$$OA = \frac{OB}{\cos 30}$$

$$= \frac{0.4}{0.8660} = 0.46 \text{ m}$$

$$OD = OC \cos 30$$

$$= 0.2\,(0.8660) = 0.173 \text{ m}$$

$$DC = OC \text{ sine } 30$$

$$= 0.2\,(0.5) = 0.1 \text{ m}$$

Substituting the above values in Eq. (2)

$$0.612\, v^2 \times 0.46 = 102 \times 0.173 + (70 \times 0.1)$$

$$v = 21 \text{ m/sec.}$$

Problem 8.11 *A jet of oil of sg. 0.85 strikes the underside of a flat plate at the center. The plate weighs 750 N and the direction of the jet is perpendicular to the plate. If the diameter of the jet is 50 mm, determine the velocity of the jet that will lift the plate.*

Solution

$$\text{The reaction force } F_x = (\rho Q v_1)$$

$$750 = (0.85)(1000)\,[0.7854\,(0.050)^2\,(v)] \times v$$

$$750 = (1.669)\, v^2$$

$$v = 21.20 \text{ m/sec}$$

Problem 8.12 *Determine the reaction forces and the work done due to the water jet impinging tangentially on the single moving vane shown in Fig. (8.21). The jet initial velocity is 70 m/sec and the velocity of the vane is 35 mm/sec. The jet diameter is 70 mm and $\theta = 60°$.*

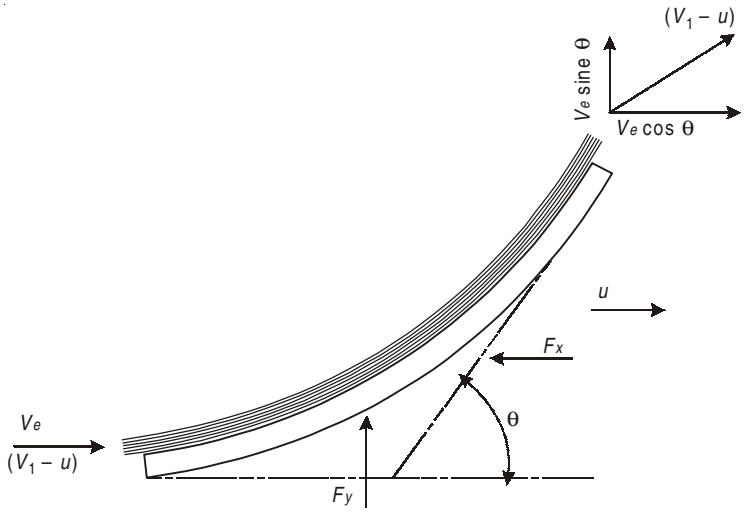


Fig. 8.21

Solution

From Eq. (8.14)

$$F_x = \rho Q (V_e) \cos\theta - (-\rho Q V_e)$$
$$= \rho Q V_e (1 + \cos\theta)$$

For this particular case since $V_e \cos\theta$ is in the same direction as V_e

$$F_x = \rho Q (V_e) \cos\theta - (\rho Q V_e)$$
$$= \rho Q V_e (1 - \cos\theta)$$
$$F_x = \rho Q V_e (1 - \cos\theta) \quad \text{... (a)}$$
$$\cos\theta = \cos 60$$
$$= 0.5,$$
$$V_e = (v_1 - u)$$
$$= (70 - 35)$$
$$= 35 \text{ m/sec}$$
$$Q = (A \times V_e)$$
$$= 0.7854 (0.070)^2 (35)$$
$$= 0.134 \text{ m}^3/\text{sec}$$

Substituting these values in Eq. (a)

$$F_x = (1000) (0.134) (35) (1 - 0.5)$$
$$= 2357 \text{ N}$$

From Eq. (8.15)

$$F_y = \rho Q (V_e) \text{ sine } \theta$$
$$= (1000) (0.134) (35) \text{ sine } 60$$
$$= 4061 \text{ N}$$

Work done $= F_x \times u$
$= (2357)(35)$
$= 82495$ Nm/sec
$= 82.5$ kW

Problem 8.13 *A jet of water 100 mm in diameter enters a moving curved vane in the direction of its motion with a velocity = 25 m/sec and exits the vane at angle of 30° to its direction of motion. If the vane is moving @ 15 m/sec, determine the force components, the magnitude and direction of the resultant force and the power developed. Refer Fig.* (*8.22*)

Solution

From Eq. (8.14)

$F_x = \rho Q (V_e) \cos\theta - (- \rho Q V_e)$
$= \rho Q V_e (1 + \cos\theta)$ ... (a)
$F_y = \rho Q (V_e) \sin\theta$ ... (b)
$V_e = (v - u)$
$= (25 - 15) = 10$ m/sec
$Q = A \times V_e$
$= 0.7854\ (0.1)^2\ (10)$
$= 0.0785$ m^3/sec.
sine θ = sine 30 = 0.5, cos θ
= cos 30 = 0.8660

[Substituting these values in (a) and (b)],

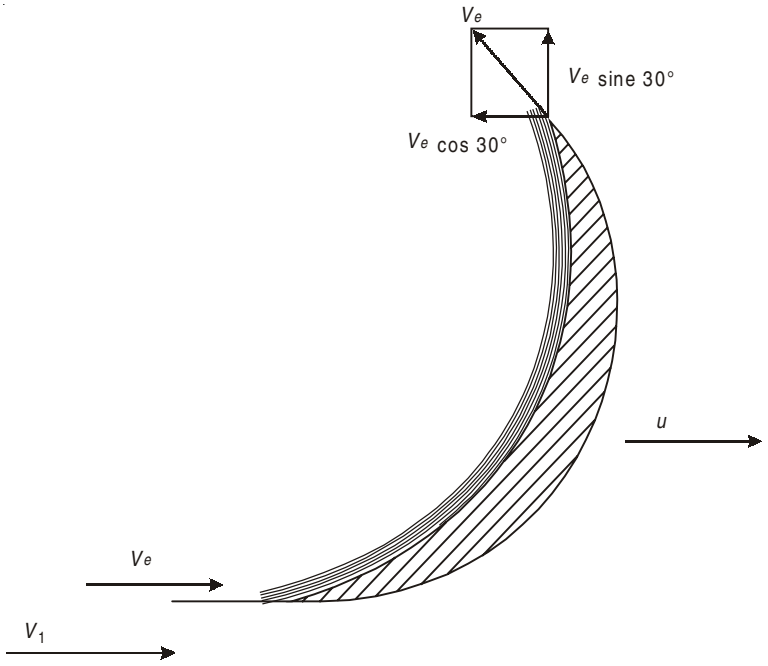


Fig. 8.22

$$F_x = (1000)\,)(0.0785)\,(10)\,(1 + 0.8660)$$
$$= 1464.8 \text{ N}$$
$$F_y = (1000)\,(0.0785)\,(10 \times 0.5) = 392.7 \text{ N}$$

The Resultant $$F_R = \sqrt{-F_x^2 + F_y^2}$$

$$= \sqrt{-1464.8^2 + 392.7^2} = 1516.5 \text{ N}$$

$$\theta_R = \tan^{-1}\left(\frac{392.7}{1464.8}\right) = 15°$$

Work done $= F_x \times u$

$$W = 1464.8 \times 15 = 21972 \text{ Nm/sec}$$

or $$W = 21.972 \text{ kW}$$

Problem 8.14 *A jet of water of diameter 75 mm strikes a series of vanes at an angle of 30° and with a velocity of 40 m/sec and leaves the vanes at right angles to the direction of motion of the vanes. The relative velocity of fluid at outlet is 0.8 times the relative velocity at inlet. The velocity of the vanes is 15 m/sec. Determine, (1) the inlet and outlet angles that will enable the fluid to enter and leave the vane without shock (2) the force exerted on the vanes in the direction of motion assuming the vanes are mounted on the periphery of a rotor.*

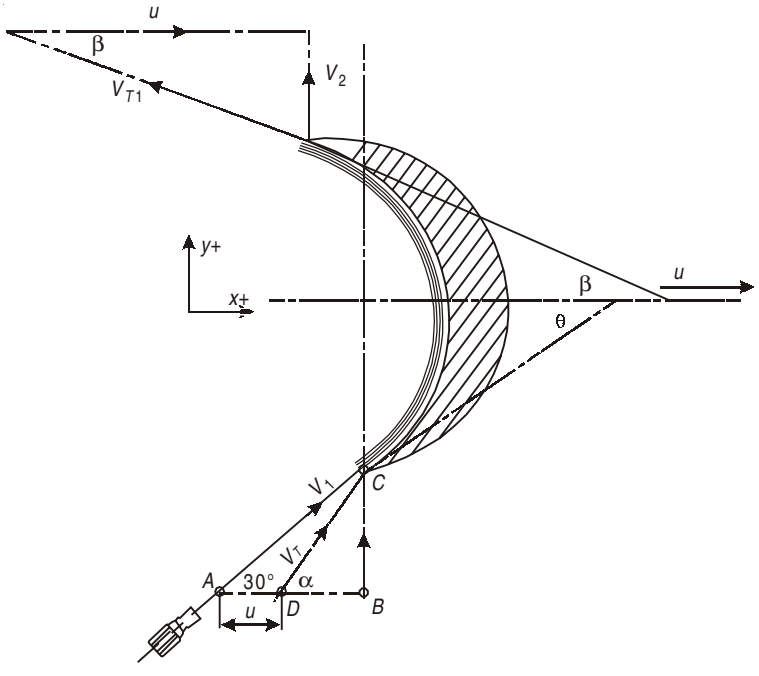


Fig. 8.23

Solution

Inlet and outlet angles α and β respectively can be obtained by referring to the respective triangles as depicted in Fig. (8.23). If the fluid has to enter and leave the vane without shock the relative velocity vectors V_T and V_{T1} must be tangential to the vane.

$$\tan \alpha = \frac{CB}{DB}$$

$$= \frac{v_1 \text{sine } 30}{(v_1 \cos\ 30°) - u}$$

$$\tan \alpha = \frac{(40 \times 0.5)}{(40 \times 0.8660) - 15}$$

$$= 1.018$$

$$\alpha = 45.5°$$

$$\cos \beta = \frac{u}{V_{T1}} = \frac{15}{V_{T1}} \quad \text{... (a)}$$

But $$V_{T1} = 0.8\,(V_T)$$

$$= 0.8 \left(\frac{CB}{\text{sine } \alpha} \right)$$

$$= 0.8 \left(\frac{v_1 \text{sine } 30°}{\text{sine } 45.5°} \right)$$

$$V_{T1} = 0.8 \left(\frac{40 \times \text{sine } 30°}{\text{sine } 45.5°} \right) = 22.44$$

Substituting this value in (a)

$$\cos \beta = \frac{15}{22.44} = 0.668$$

$$\beta = 48°$$

Force on the vane in the direction of motion,

F_x = ρ Av_1(component of v_1 in the x-direction – component of v_2 in the x-direction)

F_x = ρ Av_1(v_1 cos 30 – v_2 cos 90°)

= ρ Av_1(v_1 cos 30 – 0)

F_x = ρ Av_1(v_1 cos 30)

= (1000)(0.7854 × 0.075^2) (40) (40 × 0.8660)

F_x = 6121.4 N

Problem 8.15 *A jet of water of diameter 50 mm strikes a series of vanes mounted on the periphery of a rotor which is free to rotate. The absolute velocity of the jet at the inlet is 50 m/sec and is deflected through 120°. Assuming the vanes are symmetrical and the vane velocity is 0.4 times the jet velocity, determine,*

(1) *The inlet angle α of the jet that will enable the water to enter without shock,*

(2) *The absolute velocity of the jet at exit in magnitude and direction,*

(3) *The torque developed if the inner and outer radius of the rotor is 1.5 m and 2 m respectively, the RPM of the rotor and the power developed.*

Solution

Since the vanes are symmetrical $\theta = \varphi$

$$= \frac{(180° - 120°)}{2} = 30°$$

(i) To determine the inlet angle α, we apply the sine rule to triangle *ADB*

$$\frac{AD}{\text{sine}\,(180 - \theta)} = \frac{AB}{\text{sine}(\theta - \alpha)}$$

$$\frac{v_1}{\text{sine}\,\theta} = \frac{u_1}{\text{sine}(30 - \alpha)}$$

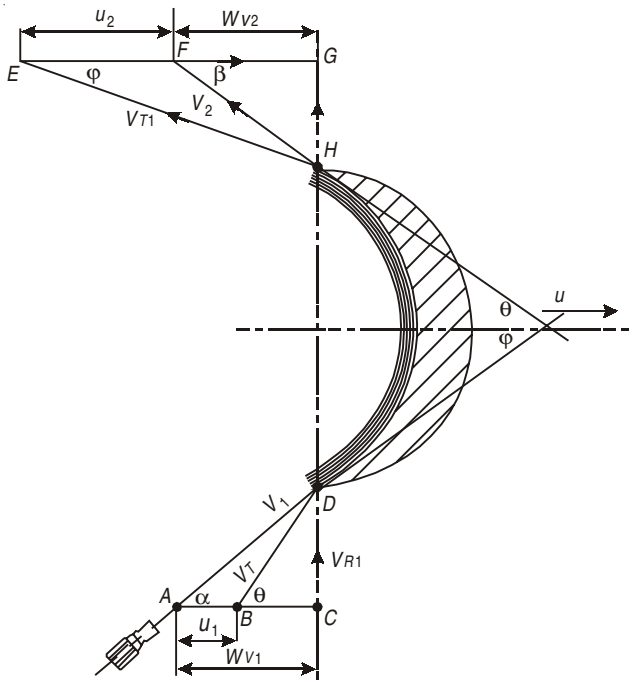


Fig. 8.24

$$\frac{50}{\text{sine } 30} = \frac{0.4(50)}{\text{sine } (30-\alpha)}$$

$$\text{sine } (30-\alpha) = \frac{[0.4(50) \text{ sine } 30]}{50}$$

$$\text{sine } (30-\alpha) = 0.2$$

or $$30-\alpha = 11.54$$

$$\alpha = 18.46°$$

Consider triangle *ABD*

$$\frac{AD}{\text{sine } (180-\theta)} = \frac{BD}{\text{sine } \alpha}$$

$$\frac{50}{\text{sine } (180-\theta)} = \frac{v_T}{\text{sine } \alpha}$$

$$\frac{50}{\text{sine } \theta} = \frac{v_T}{\text{sine } 18.46°}$$

$$\frac{50}{\text{sine } 30} = \frac{v_T}{\text{sine } 18.46°}$$

Therefore, $$v_T = 31.66 \text{ m/sec}$$

$$v_T = v_{T1}$$

$$= 31.66 \text{ m/sec}$$

(assuming that there is no flow loss due to friction over the vanes)

(ii) Direction of absolute velocity v_2 at exit

Consider the triangle *EFGH*,

$$\tan \beta = \frac{HG}{FG}$$

$$= \frac{v_{R2}}{W_{V2}} \quad \text{... (a)}$$

$$v_{T1} \cos \varphi = u_2 + W_{V2}$$

$$31.66 \cos 30 = 0.4(50) + W_{V2}$$

$$W_{V2} = 7.42 \text{ m/sec}$$

$$v_{R2} = v_{T1} \text{ sine } 30$$

$$= 31.66 \times 0.5$$

$$= 15.83 \text{ m/sec}$$

Therefore from (a) $$\tan \beta = \frac{15.83}{7.42}$$

$$= 2.133$$

or $\beta = 64.88°$

Direction of absolute velocity v_2 at exit

$= 180 - 64.88$

$= 115.12°$

Magnitude of $v_2 = \sqrt{-v^2_{R2} + W^2_{V2}}$

$= \sqrt{-15.83^2 + 7.42^2}$

$v_2 = 17.48$ m/sec

(iii) From Eq. (8.29)

$T = \rho A v_1 [W_{V1} \times R_0 + W_{V2} \times R_i]$

(Where R_0 and R_i are outer and inner radius of the rotor)

$T = (1000)\,(0.7854 \times 0.050^2)\,(50)$

$[(v_1 \cos 18.46) \times 2 + 7.42 \times 1.5]$

$T = (1000)\,(0.7854 \times 0.050^2)\,(50)$

$[(50 \cos 18.46) \times 2 + 7.42 \times 1.5]$

$T = 10{,}405$ kg-m^2/sec^2

$= 10{,}405$ Nm

Angular velocity $\omega = \dfrac{u_1}{R_0}$

$= \dfrac{0.4(50)}{2} = 10$ radians/sec

RPM of the rotor $= N$

$= \dfrac{(\omega \times 60)}{2\pi}$

$= \dfrac{(10 \times 60)}{2\pi} = 95.5$

$F_x = \rho A v_1 [W_{V1} + W_{V2}]$

$F_x = (1000)\,(0.7854 \times 0.050^2)\,(50)$

$[(v_1 \cos 18.46) + 7.42]$

$F_x = (1000)\,(0.7854 \times 0.050^2)\,(50)$

$[(50 \cos 18.46) + 7.42]$

$F_x = 5384.6$ N

Power $P = (F_x \times u)$

$= [5384.6 \times (0.4 \times 50)] = 107.7$ kW

Note : The general expression for power or work done per second is $\rho A v_1 [W_{V1}(u_1) \pm W_{V2}(u_2)]$. In the present case since $\beta < 90°$ positive sign was chosen. Also in this case $u_1 = u_2 = u$.

Problem 8.16 *A water jet propulsion system capable of overcoming a total resistance of 4000 N while cruising at 40 km/h is required to be installed on a boat. Hydraulic efficiency is 63%. Determine the discharge through the pump and the input power. Assume a straight intake system through the bow.*

Solution

From Eq. (8.39), for a straight intake system through the bow,

$$\text{Hydraulic efficiency } \eta = \frac{2u}{(v_R + u)}$$

Here, u is the velocity of the boat $= \dfrac{(40 \times 1000)}{3600}$

$= 11.11$ m/sec

$$(v_R + u) = \frac{2u}{\eta}$$

$$= \frac{(2 \times 11.11)}{0.63}$$

$= 35.27$

$v_R = 35.27 - u$

$= 35.27 - 11.11$

$v_R = 24.16$ m/sec.

(velocity of the jet relative to the ship)

Absolute velocity of the ship $v = (v_R - u)$

$= 24.16 - 11.11$

$= 13.05$ m/sec

$F_x = (\rho A v_R) \times (v_R - u)$

$4000 = (1000 \times A \times 24.16) \times 13.05$

Area of the jet $A = 0.01268$ m^2

Pump discharge $Q = A \times v_R$

$= 0.01268 \times 24.16$

$= 0.306$ m^3/ sec

$$\text{In put power required to drive the pump} = \frac{(F_x \times u)}{\eta}$$

$$= \frac{(4000 \times 11.11)}{0.63}$$

$= 70{,}539$ J/sec

$P = 70.54$ kW

Problem 8.17 *In a jet propulsion system in which water is drawn through the sides, the boat velocity is 40 km/hour and the discharge jet has a cross sectional area = 0.01268 m^2. Absolute velocity of the boat is 13.05 m/sec. Determine, the propulsive force, work done and the hydraulic efficiency.*

Solution

Velocity of the boat $= u$

$$= \frac{(40 \times 1000)}{3600} = 11.11 \text{ m/sec}$$

The propelling force $F_x = (\rho A v_R) \times (v_R - u)$

Here $v_R = (v + u)$

$= (13.05 + 11.11)$

$= 24.16$ m/sec and $(v_R - u)$

$= 13.05$ m/sec

Hence $F_x = (1000 \times 0.01268 \times 24.16) \times 13.05$

$= 3997.85$ N

Work done $= 3997.85 \times 11.11$

$= 44.416$ J/sec.

From Eq. (8.40) for side inlet

$$\text{Hydraulic efficiency } \eta = \frac{2(v_R - u) \times (u)}{(v_R^2)}$$

$$\eta = \frac{2(13.05) \times (11.11)}{(24.16^2)} = 0.496$$

$= 49.6\%$

Problem 8.18 *A symmetrical water sprinkler Fig. (8.25) discharges totally 3.3×10^{-4} m^3/sec through two of its nozzles each having an area = 1.75×10^{-4} m^2. The nozzles are 300 mm apart from the center in a horizontal plane and are inclined at 30° to the horizontal. Determine the power consumed. Neglect friction and air resistance.*

Solution

Flow velocity at $L = V_N$

$$= \frac{Q}{2A}$$

$$= \frac{(3.3 \times 10^{-4})}{2\,(1.75 \times 10^{-4})} = 0.94 \text{ m/sec.}$$

(**Note:** The total flow is equally divided between two jets)

Tangential velocity $V_T = V_N \cos 30$

$= 0.94 \times 0.8660$

$= 0.813$ m/sec

$Q_L = Q_R$

$$= \frac{(3.3 \times 10^{-4})}{2}$$

$= 1.65 \times 10^{-4}$ m^3/sec

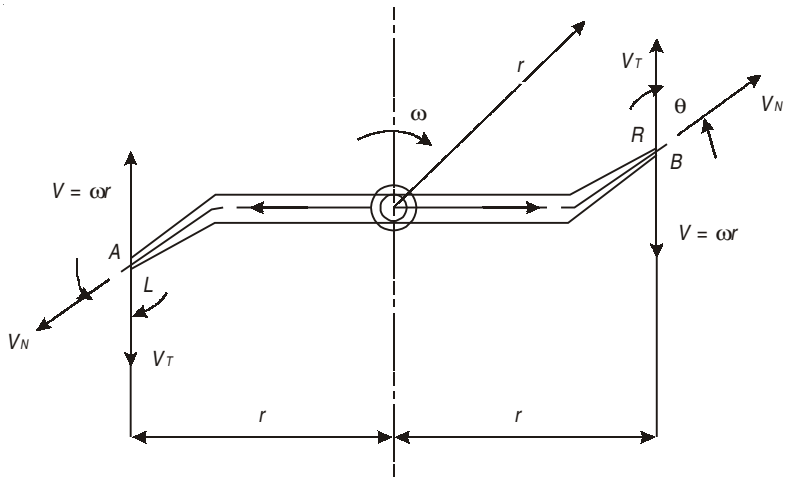


Fig. 8.25

Since it is a symmetrical sprinkler with equal nozzle areas the torque exerted by water jet at L is the same as that at R and hence,

$$\begin{aligned}
\text{Torque at } L &= (\rho Q \times V_T \times R) \\
&= \text{Torque at } R \\
&= (1000 \times 1.65 \times 10^{-4} \times 0.813 \times 0.30) \\
\text{Torque at } L &= \text{Torque at } R \\
&= 0.040 \text{ Nm} \\
\text{Total torque } T &= [\text{Torque at } L + \text{Torque at } R] \\
&= [0.040 + 0.040] \\
&= 0.080 \text{ Nm}
\end{aligned}$$

If the nozzle is free to rotate about its center then total torque $T = 0$

That is, $[(\rho Q \times V_T \times R)] = 0$

or $[(\rho Q \times (V_N \cos 30 - \omega R) \times R)] = 0$

$[1000 \times (3.3 \times 10^{-4}) \times (0.813 - 0.3\ \omega) \times 0.3] = 0$

Therefore, $\omega = 2.7$ rad/sec

$$\begin{aligned}
\text{Power consumed} &= P \\
&= T\omega \\
&= 0.080 \times 2.7 \\
&= 0.216 \text{ W}
\end{aligned}$$

Problem 8.19 *An asymmetrical sprinkler, Fig. (8.26) discharges 5×10^{-4} m^3/sec through each nozzle. Neglecting friction determine the RPM. Each nozzle has an area 2×10^{-4} m^2.*

Solution

$$V_T = \frac{Q}{A}$$

$$= \frac{(5 \times 10^{-4})}{(2 \times 10^{-4})}$$

$$= 2.5 \text{ m/sec}$$

Torque at L $= (\rho Q \times V_T \times R)$

$= (1000 \times 5 \times 10^{-4} \times 2.5 \times 0.20)$

$= 0.25$ Nm

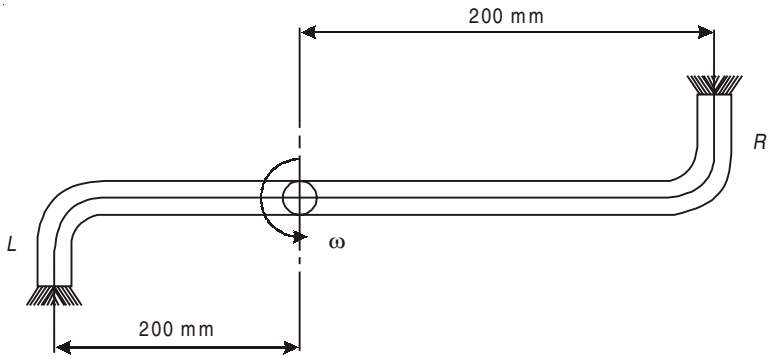


Fig. 8.26

Torque at R $= (\rho Q \times V_T \times R)$

$= (1000 \times 5 \times 10^{-4} \times 2.5 \times 0.30)$

$= 0.375$ Nm

Total torque $T = 0.25 + 0.375 = 0.625$ Nm

If the nozzle is free to rotate about its center then total torque $T = 0$

That is, $[(\rho Q \times V_T \times R)] = 0$

$[(\rho Q \times V_{T1} \times R_L) + [(\rho Q \times V_{T2} \times R_R)] = 0$ ... (a)

But, absolute velocity $V_{T1} = V_T - \omega R_L$

$= (2.5) - \omega(0.2)$

and absolute velocity $V_{T2} = V_T - \omega R_R$

$= (2.5) - \omega(0.3)$ } Substitute in (a)

$\rho Q\ [(2.5) - \omega(0.2)] \times R_L + \rho Q\ [(2.5) - \omega(0.3)] \times R_R = 0$

$\rho Q\ [(2.5) - \omega(0.2)]\ 0.2 + \rho Q\ [(2.5) - \omega(0.3)]\ 0.3 = 0$

$1000 \times 5 \times 10^{-4}\ [(2.5) - \omega(0.2)]\ 0.2 + 1000 \times 5 \times 10^{-4}\ [(2.5) - \omega(0.3)]\ 0.3 = 0$

$0.1\ [(2.5) - \omega(0.2)] + 0.15\ [(2.5) - \omega(0.3)] = 0$

$\omega = 9.6$ rad/sec

or $$N = \frac{(\omega \times 60)}{2\pi}$$

$$N = \frac{(9.6 \times 60)}{2\pi} = 92 \text{ RPM}$$

Problem 8.20 *The water sprinkler shown in Fig. 8.27 has a nozzle area = 2 × 10^{-4} m^2 and discharges totally 6 × 10^{-4} m^3/sec. Determine the RPM and the torque required to hold the arm stationary.*

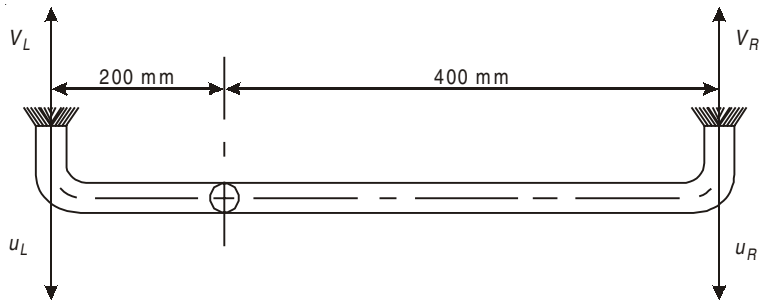


Fig. 8.27

Solution

$$\text{Flow velocity at } L = V_L = \frac{Q}{2A}$$

$$= \frac{(6 \times 10^{-4})}{2(2 \times 10^{-4})} = 1.5 \text{ m/sec.}$$

$$\text{Flow velocity at } R = V_R = \frac{Q}{2A}$$

$$= \frac{(6 \times 10^{-4})}{2(2 \times 10^{-4})} = 1.5 \text{ m/sec.}$$

(**Note:** The total flow is assumed to be equally divided between two jets)

$$Q_L = Q_R = Q_n$$

$$= \frac{Q}{2} = (3 \times 10^{-4}) \text{ m}^3 \text{/sec.}$$

Tangential velocities, $U_L = \omega R_L = \omega(0.200)$

and $U_R = \omega R_R = \omega(0.400)$

Absolute velocities; $v_{AL} = V_L + \omega R_L = 1.5 + \omega(0.200)$

and $v_{AR} = V_R + \omega R_R = 1.5\ \omega(0.400)$

Because moment of momentum of the fluid entering the sprinkler is zero, the resultant torque = 0, and there being no external torque,

$$\rho Q\,(v_{AR} \times R_R - v_{AL} \times R_L) = 0$$

(Note that the absolute velocities are in the same direction)

$$(v_{AR} \times R_R - v_{AL} \times R_L) = 0 \text{ (because } \rho Q \neq 0)$$

$$[1.5 - \omega(0.400)\,(0.4)] - [1.5 - \omega(0.200)\,(0.2)] = 0$$

Upon simplifying $\omega = 1.5$ rad/sec

and
$$N = \frac{(60 \times \omega)}{2\pi}$$

$$= \frac{(60 \times 1.5)}{2\pi} = 14.3 \text{ RPM}$$

Torque on the arm $T = \rho Q_n\,(v_{AR} \times R_R - v_{AL} \times R_L)$

$$T = \rho Q_n\,[(V_R + \omega R_R) \times R_R - (V_L + \omega R_L) \times R_L)]$$

If the arm is stationary then $\omega = 0$

$$T = \rho Q_n\,[V_R \times R_R - V_L \times R_L]$$

$$T = (1000) \times (3 \times 10^{-4})\,[1.5 \times 0.4 - 1.5 \times 0.2]$$

$$= 0.09 \text{ Nm}$$

Problem 8.21 *An aircraft travelling through still air 4000 m above sea level develops a thrust of 30 kN while displacing 300 m^3/sec through each of its two propellers. The area swept over by each propeller is 3 m^2. Theoretical efficiency = 85%. Determine the speed of the aircraft, the pressure difference across the propellers and the power required.*

Solution

From Eq. (8.42)

Thrust $F = \rho Q\,(v_B - v_A)$

$$(v_B - v_A) = \frac{F}{\rho Q}$$

$$= \frac{30{,}000}{(0.8194)\,2(300)} = 61$$

(ρ at 4000 m above seal level = 0.8194)

$$v_A = \eta \times v$$

$$= (0.85)\left(\frac{300}{3}\right) = 85 \text{ m/sec.}$$

$$v_B = 61 + 85$$

$$= 146 \text{ m/sec}$$

The speed of the aircraft $= v_A$

$$= 85 \text{ m/sec}$$

$$= \frac{(85 \times 3600)}{1000} = 306 \text{ km/hour}$$

From Eq. (8.43) pressure difference

$$(P_2 - P_1) = \rho V (v_B - v_A)$$

$$(P_2 - P_1) = (0.8194)\left(\frac{300}{3}\right)(61)$$

$$= 5 \text{ kPa}$$

$$\text{Power developed } P = (F \times v_A)/\eta$$

$$= \frac{(30000 \times 85)}{0.85} = 3000 \text{ kW}$$

QUESTIONS BANK

Section A

Review Questions

1. Distinguish clearly between jet impact and jet propulsion.
2. Discuss the significance of the equation $F = m\ (v_2 - v_1)$.
3. Name any one thing you are familiar with where rate of change of angular momentum is zero.
4. Discuss the merits and demerits of water-jet propulsion, air-jet propulsion and propulsion by propellers.
5. Explain the need for momentum correction factor and state the condition for the rotation to start in a lawn sprinkler.
6. How are fluid forces different when impacting a single stationary vane as compared to a single moving vane and a series of moving vanes?
7. In a boat installed with jet propulsion system explain the advantages of using a straight jet intake at the bow as compared to side intake system.
8. What is the condition for maximum efficiency of a propeller? Derive an equation to show that the velocity at the propeller is the average of velocities upstream and down stream from it.

Section B

Multiple Choice Questions

Choose the right answer

1. The magnitude of the resultant force required to hold a 90° elbow of 300 mm inside diameter subjected to a pressure of 1.5 Mpa in equilibrium is
   (a) 220 kN (b) 110 kN
   (c) 150 kN (d) 130 kN
   (e) None of these

2. For a jet striking a moving curved vane centrally F_x and F_y are
   (a) $\rho QV \sin^2\theta$ and 0 (b) $2r\ QV \cos\theta$ and 0
   (c) $\rho\ QV_E\ (1 + \cos\theta)$ and 0 (d) None of these
3. In jet propulsion of boats when the inlet velocities are at right angles to the direction of motion, the condition for maximum efficiency is
   (a) $u = 0.5\ v$ (b) $u = 0.75\ v$
   (c) $u = v$ (d) None of these
4. The condition for maximum efficiency for a series of curved vanes is that the vane speed must be
   (a) 0.5 times the jet speed (b) 0.75 times the jet speed
   (c) equal to the jet speed (d) None of these
5. Water flows through a U bend @ 0.5 m^3/sec, at a pressure of 100 kPa and a velocity of 5 m/sec. It exits through a nozzle at the other end at 15 m/sec. The reaction force on the bend is
   (a) 12 kN (b) 5 kN
   (c) 15 kN (d) None of these
6. Maximum efficiency of a jet of water striking tangentially a hemispherical vane moving in the direction of the jet is limited to
   (a) 60 % (b) 70 %
   (c) 80 % (d) None of these
7. Identify the impulse momentum equation from among these
   (a) $F = \rho Q(\Delta v)$ (b) $F = Q\left(\frac{\Delta v}{\Delta t}\right)$
   (c) $F(Dt) = m(\Delta v)$ (d) None of these
8. In the jet propulsion system of a boat the velocity of the vessel is 18 m/sec, the velocity of the jet is 28 m/sec and the inlet orifice faces the direction of motion. The hydraulic efficiency is
   (a) 68% (b) 72%
   (c) 76% (d) None of these

Section C

Problems

1. A jet of oil of *sg* 0.85 strikes a flat plate centrally. The diameter of the jet is 70 mm and the jet velocity is 25 m/sec. Determine the reaction force required to hold the plate stationary. (*Ans.* $F_x = 1308.47$ N)
2. Oil of *sg* 0.85 flows through a 90° bend placed in a horizontal plane. At the entrance to the bend the diameter is 300 mm and expands uniformly to φ 600 mm at the exit. If at the entrance the pressure is 150 kPa and the velocity is 30 m/sec, determine the reaction forces that keep the bend in equilibrium. The pressure at the exit is atmospheric.
   (*Ans.* $R_x = 8.7$ kN, $R_y = 1.26$ kN)

3. The total resistance of a ship while cruising at u m/sec is $6\,u^3 + 1000\,u^2$ N. The ship is propelled by a jet with straight intake at the bow with $\eta = 0.85$ while the pump has an $\eta = 0.70$. The speed of the ship = 2 m/sec. Determine the mass of water to be pumped /sec and the power required to drive the pump. (*Ans.* 5783 kg/sec and 14.45 kW)
4. An asymmetrical lawn sprinkler has straight nozzles facing one opposite the other. The left nozzle is at a distance of 200 mm and the right nozzle is at a distance of 400 mm from the center of rotation. The area of each nozzle is 7×10^{-6} m^2 and the discharge through each nozzle is = 0.5 liters/sec. Neglecting friction determine the angular velocity.
(*Ans.* 85.70 rad/sec)
5. A ship has a speed of 50 km/hour and discharges 3 m^3/sec through a propeller of diameter 0.4 m. Determine the thrust, theoretical efficiency and the power input. (*Ans.* 59,940 N, 58%, 1434 kW)

Answers to Multiple Choice Questions

1. (c) **2.** (d) **3.** (c) **4.** (a)
5. (c) **6.** (a) **7.** (c) **8.** (d)

Chapter **9**

Flow through Orifices and Mouth Pieces

The purpose of this chapter is to enable students to understand how flow through *orifices* (small or large openings) and *mouth-pieces* (short-pipe lengths) affect flow velocity and consequently the discharge.

An orifice is an integral part of any tank or a vessel while a mouth piece is an extension to that orifice in the form of a small pipe whose $\frac{L}{D}$ ratio is ≤ 3. The cross section of an orifice can be circular, triangular, rectangular or square and the edges could be sharp or bell-mouthed depending upon its configuration. A nozzle fitted to a fire extinguisher could be construed as an orifice in the broad sense of the word. Mouth pieces are either welded or screwed to a tank and they can either be internal or external to a tank. Like orifices even mouth pieces have different configurations and we will deal with each of these as and when we come to them.

Orifices are classified as either small or large in relation to the area of the tank in which they are placed. If the flow passage area of an orifice is small as compared to the area of the tank in which it is placed then flow velocity throughout discharge could be considered as uniform. This will enable us to neglect the velocity of the head of water causing the flow.

An orifice that is discharging to the atmosphere is termed *free discharge orifice* whereas an orifice discharging to another tank is called a *submerged orifice.*

Orifices cause obstruction to the free flow of a liquid and hence the actual discharge is always less than the theoretical discharge as given out by the Torricelli's equation $v_2 = \sqrt{2\,gH}$ where H is the head causing the flow. In this equation which is derived from the Bernoulli's equation v_1 the velocity of the

head of liquid causing the flow which is neglected as too small to cause any significant difference. (Fig. 9.1)

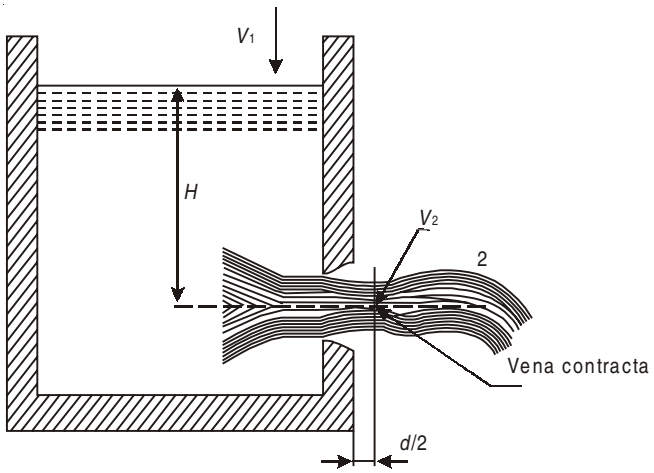


Fig. 9.1

The flow through an orifice is initially curved and as the flow gets past the orifice the stream lines become straight and parallel. The point at which stream lines become parallel is located at a distance = $\frac{d}{2}$ from the edge of the tank and this point is called *Vena contracta*. The cross sectional area at this point is less than the cross sectional area of the orifice. It is approximately = 0.62 times the area of the orifice.

For various reasons the true discharge is never the same as theoretical discharge and in order to account for the loss, various coefficients obtained through experimental methods have been introduced. These coefficients are defined below.

1. Coefficient of contraction $(C_c) = \dfrac{\text{Area of the jet at Vena Contracta}}{\text{Area of the Orifice}}$

The average value of C_c is taken to be = 0.64 and its range varies between 0.61 to 0.69

2. Coefficient of velocity $(C_v) = \dfrac{\text{Velocity of the jet at Vena Contracta } (v)}{\text{Theoretical Velocity } \left(\sqrt{2gH}\right)}$

The average value of C_v is taken to be = 0.98 and its range could vary between 0.95 to 0.99 depending on the available head of liquid and the shape of the orifice.

3. Coefficient of discharge $(C_d) = \dfrac{\text{Actual Dischage}(Q)}{\text{Theoretical Discharge}(Q_r)}$

The value of C_d could be determined if the values of C_c and C_v are known since

$$C_d = C_c \times C_v \qquad ...(9.1)$$

The average value of C_d is taken to be = 0.62 and its range could vary between 0.61 to 0.65 depending on the available head of liquid and the shape of the orifice.

Experimental methods adopted for determination of the values of C_d, C_c and C_v are described below.

9.1 DETERMINATION OF THE VALUE OF C_D

In this method the head of water in the main tank is maintained at a pre-determined level by external supply which can be varied. The measuring tank

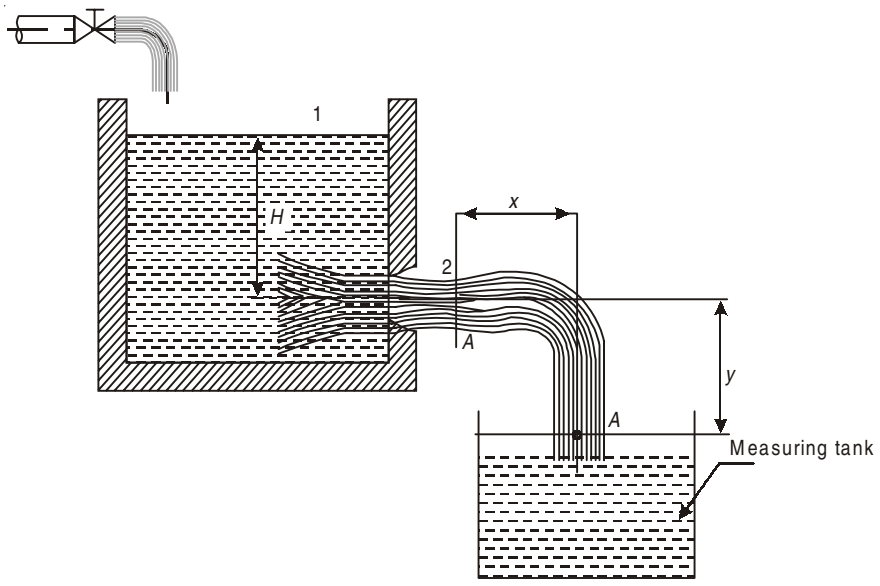


Fig. 9.2

is meant to collect the supply from the orifice in the main tank. The time period (t) that elapsed for the liquid to attain a known level in the measuring tank is noted.

By definition $$C_d = \frac{Q_{\text{Actual}}}{Q_{\text{Theo}}}$$

$$C_d = \frac{\left(\begin{array}{c}\text{Area of the measuring tank} \\ \times \text{ Rise in water level}\end{array}\right)/t}{\left(\text{Area of the orifice} \times \sqrt{2gH}\right)} \qquad ...(9.2)$$

9.2 DETERMINATION OF THE VALUE OF C_V

Referring to Fig. (9.2), consider a point A on the jet at Vena contracta. The horizontal and vertical distance travelled by the point in time t is given by

$$x = v \times t \qquad ...(a)$$

(v is the velocity of the jet at Vena contracta)

$$y = \frac{1}{2}(gt^2)$$

But from (a) $$t = \frac{x}{v}$$

$$y = \frac{1}{2}(g)\left(\frac{x}{v}\right)^2$$

$$v = \sqrt{\left(\frac{gx^2}{2y}\right)} \qquad ... (b)$$

Since by definition $$C_v = \left(\frac{v}{\sqrt{2gh}}\right)$$

$$C_v = \frac{\left(\sqrt{\frac{gx^2}{2y}}\right)}{\sqrt{2gh}} = \sqrt{\frac{x^2}{4yH}} \qquad ...(9.3)$$

9.3 DETERMINATION OF THE VALUE OF C_C

The value of coefficient contraction C_c can be either obtained directly from Eq. (9.1) as

$$C_c = \frac{C_d}{C_v} \qquad ...(9.4)$$

The value of C_c can also be obtained by practically measuring the diameter of the jet at Vena contracta (d_v) using a 4 point micrometer contraction gauge where the points are independently adjusted and made to touch the periphery of the jet. Since the orifice diameter is known,

$$C_c = \left(\frac{a_v}{a}\right) = \left(\frac{d_v}{d}\right)^2 \qquad ...(9.5)$$

Head lost due to flow through an orifice (h_L) as determined by applying the Bernoulli's equation to points 1 and 2 (Ref Fig. 9.2) is given by

$$h_L = H\,[1 - (C_v)^2] \qquad ...(9.6)$$

9.4 FLOW THROUGH MOUTH PIECES

The purpose of a Mouth piece is to improve the volume flow rate through the

orifice. In case of a straight cylindrical Mouth piece this is achieved by causing the jet to expand immediately after Vena contracta. This improves the value of C_c. If the jet expands fully to the size of the Mouth piece then the mouth piece is said to be running full and the value of C_c could be considered = 1. If the jet expansion is only partial then the Mouth piece is said to be running free. Mouth pieces although are an advantage they can contribute to frictional loss if the pipe internal surfaces are not smooth and a consequently values of C_v would be lower.

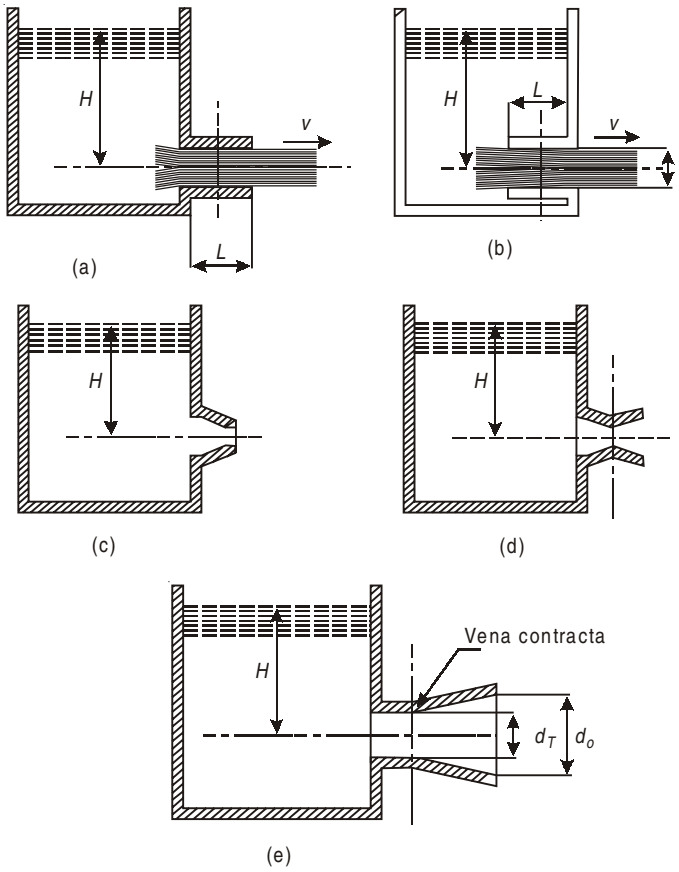


Fig. 9.3

A straight cylindrical tube whose length L = (1.5 – 3) D where D is its internal diameter could be considered a mouth piece. For the mouth piece to run full L must be at least = 2.5 D. Straight cylindrical Mouth pieces could be placed external to a tank (Fig. 9.3 a) or placed internally as shown in Fig. (9.3 b). An internal Mouth piece whose $\frac{L}{D}$ ratio = 1 is called the Borda's mouth piece. It is a free running mouth piece because the jet doesn't make full contact with the inner periphery. For this mouth piece C_c= 0.52 and C_d = 0.51 and C_v =

0.97. If the $\frac{L}{D}$ ratio is > 2.5 then an internal mouth piece is said to be running full. For this mouth piece $C_c = 1$ and $C_d = C_v = 0.707$.

Instead of being straight, cylindrical Mouth pieces could be *convergent* (9.3 c) or *divergent* (9.3 e) or both convergent and divergent (9.3 d). In such cases the improvement in volume flow rate is achieved by reducing the losses due to sudden expansion or contraction.

In case of a *convergent Mouth piece* loss due sudden enlargement could be avoided since there is a gradual change in cross section which finally converges to the diameter of the jet at Vena contracta. Both C_v and C_d improves to values as high as 0.98 and the value of $C_c = 1$.

A *divergent Mouth piece* is similar to a convergent mouth piece since it is also designed to reduce the loss due to sudden expansion but the improvement in the values of flow coefficients is not the same as in a convergent mouth piece because the jet doesn't make full contact with the surface of the Mouth piece. For this mouth piece $C_c = 1$ and C_d and C_v depends on the divergent angle and the length.

A *convergent-divergent mouth piece* improves the values of flow coefficients to almost unity as the flow velocity at Vena contracta increases because the point of convergence is at Vena contracta and there after the jet diverges. The throat diameter (d_T) of a convergent-divergent mouth piece is the diameter at Vena contracta. There is a limitation on the maximum outlet diameter (d_0) to avoid separation of flow due to excessive vacuum pressure. This is ensured by limiting the angle of divergence to $\leq 3°$. The discharge is governed by the throat diameter.

For this mouth piece $C_c = 1$

and $C_d = C_v$

$= 0.97$

9.5 FLOW THROUGH LARGE ORIFICES

All along we have been considering the effects of flow under a constant head either by assuming the flow is kept constant by external supply or by neglecting the velocity of the head of water causing the flow as too insignificant to cause appreciable errors because the area of the orifice was considered as too small in relation to the area of the tank.

If the area of the orifice is considerably large (if $H < 5$ times the diameter or height of the orifice) then the head causing the flow cannot be considered as constant. Under such conditions the discharge equation ($Q = C_d \times A \times \sqrt{2gH}$) does not give accurate results. The discharge must be considered as one occurring under a falling head because the original level of the liquid (H_1) falls till it reaches the lower level of the orifice (H_2). Also the flow velocity across the entire cross section of the orifice is not constant as in the case of a small orifice. Therefore theoretically the right discharge equation is $Q = C_d \times A \times \sqrt{2g(H_1 - H_2)}$. In case of small orifices we set $(H_1 - H_2) = H$ by neglecting H_2.

It becomes obvious therefore that in case of large orifices we have to take in to account the time t required for the fluid to fall from level $H_1$to level H_2. This time depends basically on the geometrical shape of the tank.

(1) Let us first consider a rectangular or round tank placed vertically and determine the time it takes for the liquid level to fall from the initial level H_1 to level H_2.

Referring to Fig. (9.4 a) let A be the area of the tank and a be the area of the orifice. It is assumed A remains constant throughout.

Consider an element of fluid of height ∂h

Discharge through the orifice in time ∂t

$$= a \times v \times \partial t \qquad \text{....(a)}$$

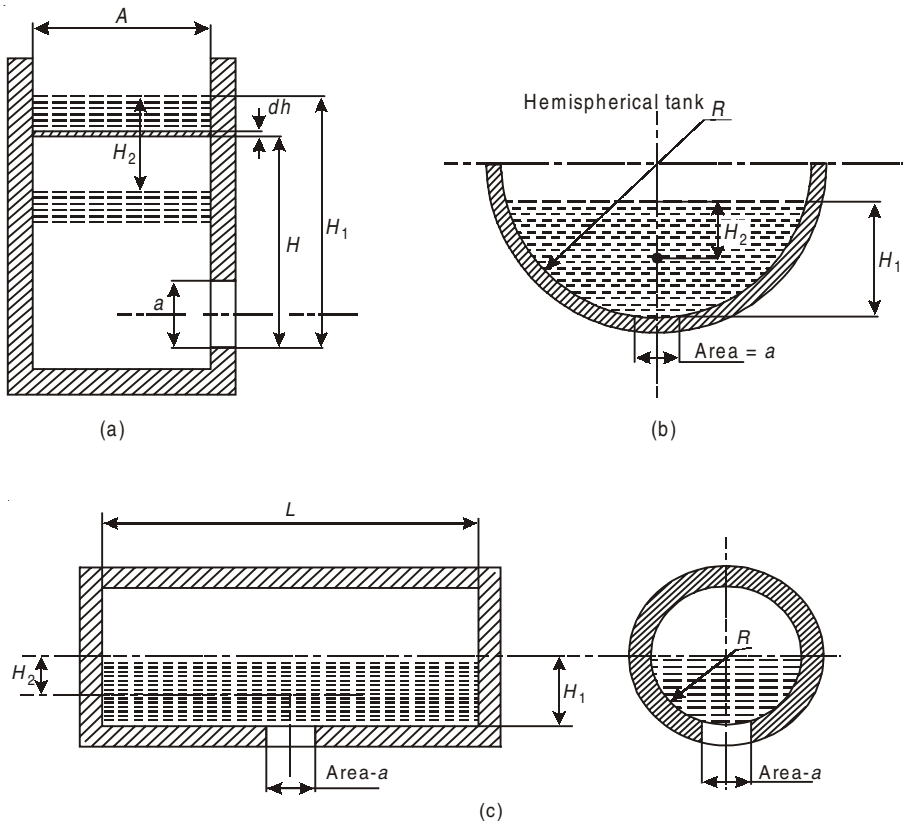


Fig. 9.5

Fall in the level of the tank during the same period

$$= -A \times \partial h \qquad \text{...(b)}$$

(here negative sign indicates the fall in level)

Equating (a) and (b) we have,

$$[-A \times \partial h\,] = [a \times v \times \partial t]$$

$$[-A \times \partial h\,] = C_d \left[a \times \sqrt{2gH} \times \partial t\right]$$

$$\partial t = \frac{[-A \times \partial h]}{C_d\ [a \times \sqrt{2gH}\,]}$$

$$t = \int \partial t = \frac{-A}{C_d\ [a \times \sqrt{2g}\,]} \int_{H_1}^{H_2} \left(\frac{\partial h}{\sqrt{H}}\right)$$

$$t = \frac{-2A\left(\sqrt{H_1} - \sqrt{H_2}\right)}{C_d\ [a \times \sqrt{2g}\,]} \qquad \ldots.(9.7)$$

$(\because H_2 = 0)$

Time required to empty the tank $T = \dfrac{2A\left(\sqrt{H_1}\right)}{C_d\ [a \times \sqrt{2g}\,]}$ $(\because H_2 = 0)$

$\ldots(9.8)$

Time (T) required for emptying tanks of various other shapes and configurations too can be derived in the same manner. Equations derived for a few of the popular configurations are given below.

(2) *Hemispherical tank* Fig. (9.4 b)

Time (t) required for the liquid level to fall from H_1 to H_2.

$$t = \frac{2\pi}{C_d\ [a \times \sqrt{2g}\,]}$$

$$\left[\left(\frac{2}{3}\right) R \left(H_1^{\frac{3}{2}} - H_2^{\frac{3}{2}}\right) - \frac{1}{5}\left(H_1^{\frac{5}{2}} - H_2^{\frac{5}{2}}\right)\right] \ldots(9.9)$$

Time (T) required for emptying the tank. ($H_2 = 0$)

$$T = \frac{2\pi}{C_d\ [a \times \sqrt{2g}\,]}$$

$$\left[\left(\frac{2}{3}\right) R \left(H_1^{\frac{3}{2}}\right) - \frac{1}{5}\left(H_1^{\frac{5}{2}}\right)\right] \qquad \ldots(9.10)$$

If the tank was initial full then ($H_1 = R$ and $H_2 = 0$)

$$T = \frac{2\pi}{C_d\ [a \times \sqrt{2g}\,]} \left[\frac{2}{3}\left(R^{\frac{5}{2}}\right) - \frac{1}{5}\left(R^{\frac{5}{2}}\right)\right]$$

$\ldots(9.11)$

(3) *Round tank placed horizontally* Fig. (9.4 c)

Time (t) required for the liquid level to fall from H_1 to H_2.

$$t = \frac{4L}{3C_d\ [a \times \sqrt{2g}]} \left[(2R - H_2)^{\frac{3}{2}} - (2R - H_1)^{\frac{3}{2}} \right] \quad \ldots(9.10)$$

Time (T) required for emptying the tank. ($H_2 = 0$)

$$T = \frac{4L}{3C_d\ [a \times \sqrt{2g}]} \left[\left(2R^{\frac{3}{2}} \right) - (2R - H_1)^{\frac{3}{2}} \right] \quad \ldots(9.12)$$

9.5.1 Discharge Through Fully Submerged/Partially Submerged Orifices

An orifice is said to be *fully submerged* if it is wholly covered by the same liquid on either side. An orifice is said to be only *partially submerged* if it is not fully covered by the liquid on any one side. In case of a partially submerged orifice the portion not covered by the liquid behaves as a free orifice and the covered portion behaves as a submerged orifice.

(1) **Fully-submerged orifice**

Consider a rectangular orifice of height h and width w.

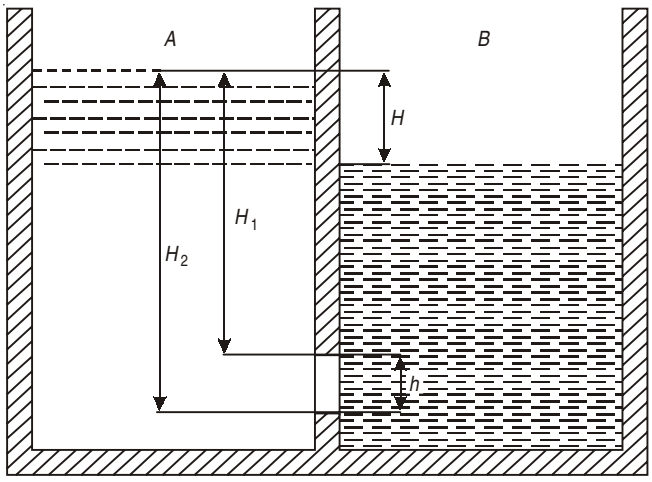


Fig. 9.6

If we consider the discharge as one occurring under a constant head or that the orifice area as too small compared to the area of the tank A then v_1 can be neglected.

Hence discharge through the orifice

and $$V_2 = \sqrt{2gH}$$

$$Q = C_d \times W\ (H_2 - H_1)\ \sqrt{2gH} \quad \ldots(9.13)$$

TYPICAL PROBLEMS AND SOLUTIONS

Problem 9.1 *A tank has a round sharp edge orifice on one of its side walls. Orifice diameter is 50 mm and is discharging water at the rate of 20 liters per sec under a constant head of 15 m. Diameter of the jet as measured at Vena contracta was found to be 40 mm. Determine C_v, C_d and C_v.*

Solution Area of the orifice = $0.7854\ (0.05)^2 = 1.963 \times 10^{-3}$ m^2

$$\text{Actual discharge } Q_A = \frac{20}{1000} = 0.020 \text{ m}^3/\text{sec}$$

$$\text{Theoretical velocity} = \sqrt{2gH} = \sqrt{2 \times 9.81 \times 15} = 17.155 \text{ m/sec}$$

$$\text{Theoretical discharge } Q_T = 1.963 \times 10^{-3} \times 17.155 = 0.033 \text{ m}^3/\text{sec}$$

$$C_d = \frac{Q_A}{Q_T} = \frac{0.020}{0.033} = 0.606$$

$$C_c = \left(\frac{d_v}{d}\right)^2 = \left(\frac{40}{50}\right)^2 = 0.64 \qquad \text{(from Eq. 9.5)}$$

$$C_v = \frac{C_d}{C_c} = \frac{0.606}{0.64} = 0.946$$

Problem 9.2 *Water is being discharged @ 0.018 m^3/sec under a constant head of 5 m through a vertical orifice of diameter 60 mm. The horizontal and vertical co-ordinates of a point on the jet trajectory as measured from Vena contracta are 2.2 m and 0.25 m respectively. Determine C_c, C_d and C_v.*

Solution Theoretical velocity = $\sqrt{2gH} = \sqrt{2 \times 9.81 \times 5} = 9.90$ m/sec

Theoretical discharge Q_T = Area of the orifice × V_T

Theoretical discharge $Q_T = 0.7854\ (0.06)^2 \times 9.90 = 0.028$ m^3/sec

$$C_d = \frac{Q_A}{Q_T} = \frac{0.018}{0.028} = 0.64 \qquad \text{...(9.13)}$$

From Eq. (9.13)

$$C_v = \sqrt{\frac{x^2}{4yH}}$$

$x = 2.2$ m and $y = 0.25$ m

$$C_v = \sqrt{\frac{(2.2)^2}{4 \times 0.25 \times 5}} = 0.968$$

From Eq. (9.4)

$$C_c = \frac{C_d}{C_v} = \frac{0.64}{0.968} = 0.66$$

Problem 9.3 *A vertical orifice of diameter 80 mm is discharging to a collecting tank under a head of 15 m. The cross sectional area of the collecting tank 0.78 m^2 and the raise in water level is 2 m in 30 sec. The diameter of jet at Vena contracta as measured through a micrometer was 55 mm. Determine C_c, C_d and C_v.*

Solution Theoretical velocity $= \sqrt{2gH} = \sqrt{2 \times 9.81 \times 15}$

$= 17.155$ m/sec

Theoretical discharge Q_T = Area of the orifice $\times V_T$

Theoretical discharge $Q_T = 0.7854\ (0.08)^2 \times 17.155$

$= 0.086\ m^3/sec$

$$\text{Actual discharge} = \frac{\text{(Area of the collecting tank} \times \text{Rise in water level)}}{t}$$

$$Q_A = \frac{0.78 \times 2}{30} = 0.052\ m^3/sec$$

$$C_d = \frac{Q_A}{Q_T} = \frac{0.052}{0.086} = 0.604$$

$$C_c = \left(\frac{d_c}{d}\right)^2 = \left(\frac{65}{80}\right)^2 = 0.66 \qquad \text{(from Eq. 9.5)}$$

$$C_v = \frac{C_d}{C_c} = \frac{0.604}{0.66} = 0.915$$

Problem. 9.4 *For a tank which has two identical orifices placed one below the other as shown in Fig. (9.7) assuming that the two jets intersect at the same point prove that (i) the jets trajectory follow the relationship $y_1 = 2y_2$ if $H_2 = 2H_1$. (ii) the jets intersect on the horizontal line through the base if $y_2 = H_1$.*

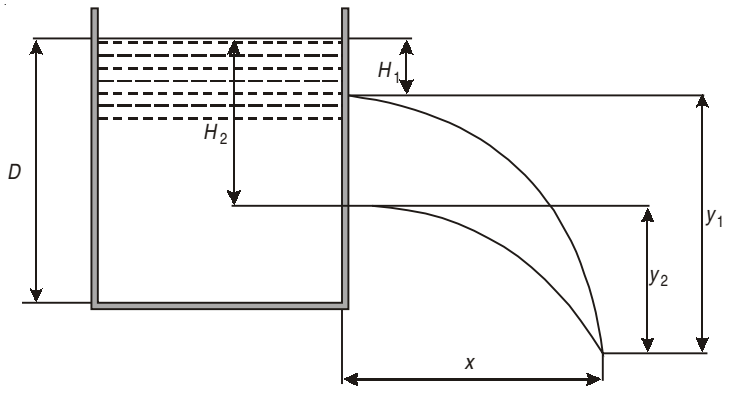


Fig.9.7

Solution Since the two orifices are identical (same in shape and size) and their coefficients of velocity are given to be equal and since the two jets intersect at the same point,

(i) $$\sqrt{\frac{x^2}{4y_1H_1}} = \sqrt{\frac{x^2}{4y_2H_2}}$$ (from Eq. 9.3)

$$y_1 H_1 = y_2 H_2$$

$$y_1 H_1 = y_2 2H_1$$

(Because it is given that $H_2 = 2H_1$)

Hence $$y_1 = 2y_2$$

(ii) In this case it is required to prove that $y_2 = H_1$

It is known that for any jet $x = v \times t$ and $y = \frac{1}{2} gt^2$ ...(Ref. Fig. 9.2)

$$y = \frac{1}{2} (g) \left(\frac{x}{v}\right)^2$$

$$y = \frac{1}{2} (g) \left(\frac{x}{\sqrt{2gH}}\right)^2$$

$$(H_2 - H_1) + y_2 = \frac{1}{2} (g) \left(\frac{x}{\sqrt{2gH_1}}\right)^2$$

$$(H_2 - H_1) + H_1 = \frac{1}{2} (g) \left(\frac{x}{\sqrt{2gH_1}}\right)^2$$

(because it is given that $y_2 = H_1$)

$$H_2 = \frac{1}{2} (g) \left(\frac{x}{\sqrt{2gH_1}}\right)^2$$

or $$x^2 = 4 H_1 H_2 \quad \text{...(a)}$$

Likewise,

$$y_2 = \frac{1}{2} (g) \left(\frac{x}{\sqrt{2gH_1}}\right)^2$$

or $$x^2 = 4 y_2 H_2 \quad \text{...(b)}$$

Equating (a) and (b) we have $y_2 = H_1$.

Problem 9.5 *The actual discharge through an orifice of diameter 15 mm was found to be 7×10^{-4} m^3/sec under a head of 2 metres of water. Determine the discharge-coefficient and the power loss at the orifice. Assume $C_v = 0.98$.*

Solution Discharge coefficient $C_d = \dfrac{Q}{\left(A \times \sqrt{2gH}\right)}$

where A is the area of the orifice

$$C_d = \frac{7 \times 10^{-4}}{\left[0.7854 \times (0.015)^2 \times \sqrt{2 \times 9.81 \times 2}\right]}$$

$$= 0.63$$

Power loss is given by $P_L = \gamma Q h_L$

where $h_L = H[1 - (C_v)^2]$ Refer Eq. (9.6)

Therefore power loss $P_L = Q \times H\ [1 - (C_v)^2]$

$$P_L = 9810 \times (7 \times 10^{-4}) \times 2\ [1 - (0.98)^2]$$

$$= 0.544 \text{ W}$$

Problem 9.6 *A closed tank contains oil of sg = 0.85 up to a height of 3 m and the rest of the space contains pressurized Nitrogen gas. Determine the gas pressure required to ensure a discharge of 0.01 m^3/sec through a horizontal orifice of diameter 30 mm. $C_d = 0.63$.*

Solution Pressure head due to Nitrogen gas

$$= \frac{P}{\gamma_0}$$

$$= \frac{P}{0.85 \times 9.81} = \frac{P}{8.34}$$

$$= 0.12\ P \text{ metres.}$$

Here P is the gas pressure in kPa

$$Q = C_d \times A \times v$$

(v = velocity of the jet at Vena contracta)

$$0.01 = 0.63 \times 0.7854\ (0.030)^2 \times \sqrt{2gH}$$

$$0.01 = 0.63 \times 0.7854\ (0.030)^2 \times \sqrt{2 \times 9.81 \times (3 + 0.12P)}$$

$$22.472 = \sqrt{2 \times 9.81 \times (3 + 0.12P)}$$

$$(22.472)^2 = 2 \times 9.81 \times (3 + 0.12P)$$

or $$P = 189.5 \text{ kPa}$$

Problem 9.7 *An external cylindrical mouth piece runs full under a constant head of 3 m and has an inside diameter = 40 mm. The flow velocity at the exit to the mouth piece is 7 m/sec. If C_v = 0.8 determine the discharge and pressure at Vena contracta.*

Solution For an external cylindrical mouth piece that runs full $C_c = 1$

Hence
$$\begin{aligned} C_d &= C_v \times C_c \\ &= 1 \times 0.8 = 0.8 \\ Q &= C_d \times A \times v \\ &= 0.8 \times 0.7854\ (0.040)^2 \times 7 \\ &= 7.037 \times 10^{-3}\ \text{m}^3/\text{sec} \end{aligned}$$

Pressure at vena contracta can be determined by applying the Bernoulli's equation. Let the free surface of water be point (0) and Vena contracta be point (1)

$$\frac{P_0}{\gamma} + \frac{v_0^2}{2g} + H_0 = \frac{P_1}{\gamma} + \frac{v_1^2}{2g} + H_1$$

The flow velocity v_0 can be neglected as too small as the area of the tank is too large compared to the flow passage area. Similarly P_0 can be neglected since the pressure is atmospheric. H_1 too can be neglected since the datum head is zero at Vena contracta.

Therefore,
$$H_0 = \frac{P_1}{\gamma} + \frac{v_1^2}{2g}$$

or
$$\frac{P_1}{\gamma} = \left(H_0 - \frac{v_1^2}{2g}\right)$$

$$\frac{P_1}{\gamma} = \left(3 - \frac{7^2}{2 \times 9.81}\right)$$

$$= 0.50\ \text{m}$$

Pressure at vena contracta $P_1 = (\gamma)\ (0.50)$

$$= 9.81 \times 0.50$$

$$= 4.90\ \text{kPa}$$

Problem 9.8 *Throat diameter of a convergent-divergent mouth piece is 50 mm and the head of water over the mouth piece is 3 metres. If the maximum vacuum pressure at the throat is limited to 6 m of water determine the outlet diameter.*

Solution

Let a_o be the area at the outlet and a_T be the area at the throat. Then from the continuity equation,

$$a_o \times v_o = a_T \times v_T.$$

$$\frac{a_O}{v_T} = \frac{v_T}{v_o}$$

$$\frac{a_O}{v_T} = \frac{\sqrt{2g\,(H_A + H - H_T)}}{\sqrt{2g\,H}}$$

(H_A= atmospheric head)

Upon simplification, $$\frac{a_o}{a_T} = \sqrt{1 + \frac{(H_A - H_T)}{H}} \qquad \ldots(9.14)$$

Here $H = 3$ m

$(H_A - H_T) = 6$ m

$a_o = 0.7854\,(d_o)^2$

and $a_T = 0.7854\,(d_T)^2$

Substituting the values in Eq. (9.14)

$$\frac{d_0^2}{d_T^2} = \sqrt{1 + \frac{6}{3}}$$

$$= 1.732$$

$$\frac{d_0}{d_T} = 1.316$$

Therefore outlet diameter $d_o = d_T \times 1.316$

$= 50 \times 1.316 = 65.8$ mm

Problem 9.9 *Find the discharge and the size of the jet of an internal mouth piece of diameter 75 mm over a head of 3 m if (i) its* $\frac{L}{D}$ *ratio = 1 (ii) its* $\frac{L}{D}$ *ratio = 2.5.*

Solution

(i) $\frac{L}{D}$ ratio = 1. This is a Borda's Mouth piece and is therefore free running. The value of C_D in this case = 0.5.

$$Q = C_D \times A \times \sqrt{2gH}$$

$$= 0.5 \times 0.7854\,(0.075)^2 \times \sqrt{2 \times 9.81 \times 3}$$

$$= 0.017 \text{ m}^3/\text{sec}$$

Size of the jet = Diameter of the jet at Vena contracta (In this case $C_C = 0.52$)

$$D_J = \sqrt{C_C \times (\text{Diameter of the mouth piece})^2}$$

$$= \sqrt{0.52\,(75)^2}$$

$$D_J = 54 \text{ mm}$$

(ii) $\frac{L}{D}$ ratio = 2.5. This is an internal Mouth piece running full. The value of C_D in this case = 0.707.

$$Q = C_D \times \text{A} \times \sqrt{2gH}$$

Hence the discharge $Q = 0.707 \times 0.7854\,(0.075)^2 \times \sqrt{2 \times 9.81 \times 3}$

$= 0.024 \text{ m}^3/\text{sec}.$

Size of the jet

D_J = (Diameter of the mouth piece)

= 75 mm since the mouth piece runs full

Problem 9.10 *Area of a tank and its orifice area are 2 m^2 and 2 × 10 $^{-3}$ m^2 respectively. Determine the time required for the liquid level to fall from H_1 to H_2. Neglect C_d and Refer Fig. (9.8).*

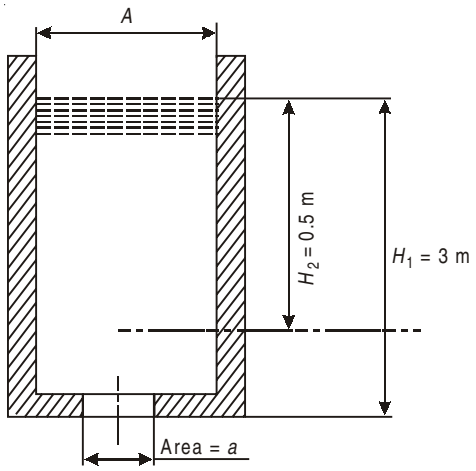


Fig. 9.8

From Eq. (9.7)

$$t = \frac{-2A\left(\sqrt{H_1} - \sqrt{H_2}\right)}{C_d\,[a \times \sqrt{2g}]}$$

$$= \frac{-2(2)\left(\sqrt{3} - \sqrt{0.5}\right)}{\left[2 \times 10^{-3} \times \left(\sqrt{2 \times 9.81}\right)\right]}$$

= 427.8 sec = 7 mt 8 sec

Problem 9.11 *It takes 5 minutes for a sphere of certain radius completely filled with liquid to empty through an orifice of diameter 20 mm. Determine inside diameter of the sphere. Neglect* C_d.

Solution

From Eq. (9.11) for a hemisphere of radius R which is initially full the time T required to empty the tank fully is given by

$$T = \frac{2\pi}{C_d\ [a \times \sqrt{2g}]} \left[\frac{2}{3}\left(R^{\frac{5}{2}}\right) - \frac{1}{5}\left(R^{\frac{5}{2}}\right)\right]$$

T = 5 minutes

= 300 sec for the sphere to empty.

For hemisphere T = 150 sec.

Area of the orifice, a = $0.7854\ (0.020)^2$

= $3.14 \times 10^{-4}\ m^2$

$$150 = \frac{2\pi}{\left[3.14 \times 10^{-4} \times \left(\sqrt{2 \times 9.81}\right)\right]} \left[\frac{2}{3}\left(R^{\frac{5}{2}}\right) - \frac{1}{5}\left(R^{\frac{5}{2}}\right)\right]$$

$$150 = 45175.26 \left[\frac{2}{3}\left(R^{\frac{5}{2}}\right) - \frac{1}{5}\left(R^{\frac{5}{2}}\right)\right]$$

$$\left[\frac{2}{3}\left(R^{\frac{5}{2}}\right) - \frac{1}{5}\left(R^{\frac{5}{2}}\right)\right] = \frac{150}{45175.26}$$

$$\left(R^{\frac{5}{2}}\right)\left[\frac{2}{3} - \frac{1}{5}\right] = (3.32 \times 10^{-3})$$

$$\left(R^{\frac{5}{2}}\right) = \frac{(3.32 \times 10^{-3})}{0.466}$$

= 0.0071

R = 0.1382 m

= 138.2 mm

or Diameter of the sphere = 276.4 mm

Problem 9.12 *A concrete water tank (Refer Fig. 9.9) is of the shape of a funnel. Determine the time it takes for the water level to drop from 5 m to 0.5 m. Neglect* C_d.

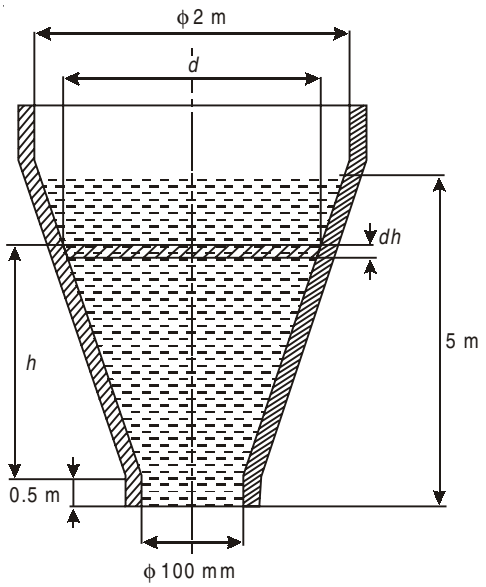


Fig. 9.9

Solution

Based on similarity and uniformity, $= \frac{(2-0.1)}{4.5} = \frac{(d-0.1)}{h}$, where d is the diameter at a distance h

$$\frac{(1.9)}{4.5} = \frac{(d-0.1)}{h}$$

$$0.42 = \frac{(d-0.1)}{h}$$

$$0.42 = (d - 0.1)$$

$$d = (0.42h + 0.1)$$

$$\text{Area at distance } h = 0.7854\,(0.422h + 0.1)^2$$

$$= 0.134\,h^2 + 0.06h + 0.00785$$

Referring to the Fig. (9.9)

$$Q \times dt = -A \times dh \qquad \ldots(a)$$

$$[C_d \times A\sqrt{2gh}]\,dt = -[0.138h^2 + 0.06h + 0.00785]\,dh$$

here A is the area of the orifice $= 0.7854\,(0.1)^2$

$$= 7.854 \times 10^{-3}\ \text{m}^2$$

Substituting in (a)

$$[C_d \times (7.854 \times 10^{-3})\sqrt{2gh}\,]\,dt = -[0.138h^2 + 0.06h + 0.00785]\,dh$$

$$dt = \frac{-[0.138h^2 + 0.06h + 0.00785]\, dh}{[C_d \times (7.854 \times 10^{-3})\sqrt{2gh}]}$$

$$\int dt = \frac{1}{[C_d \times (7.854 \times 10^{-3})\sqrt{2g}]} \int_0^{4.5} \frac{-[0.138h^2 + 0.06h + 0.00785]\, dh}{\sqrt{h}}$$

If we neglect C_d then,

$$T = 29.74 \left\{-[(0.046h^3) + (0.03h^2) + 0.00785h)](2h^{0.5})\right\}_0^{4.5}$$

$$T = 28.74 \{- [(17.78) + (2.23) + (1.49)]\}$$

$$T = 618 \text{ sec}$$

Problem 9.13 *A water tank is of the shape of a truncated cone of vertex angle 60°. The depth of water in the tank is 3 m and diameter of the base is 1 m. Determine the time it takes for the water level to drop from 3 m to 1 m through the orifice of diameter 100 mm provided at the base.* $C_d = 0.85$.

Solution

Referring to Fig. (9.10) by geometry we determine, $D = 4.464$ m

Based on similarity and uniformity, $\frac{(4.464 - 1)}{3} = \frac{(d - 1)}{h}$, where d is the diameter at a distance h

$$\frac{(4.464 - 1)}{3} = \frac{(d - 1)}{h}$$

$$1.15\, h = (d - 0.1)$$

$$d = (1.15\, h + 0.1)$$

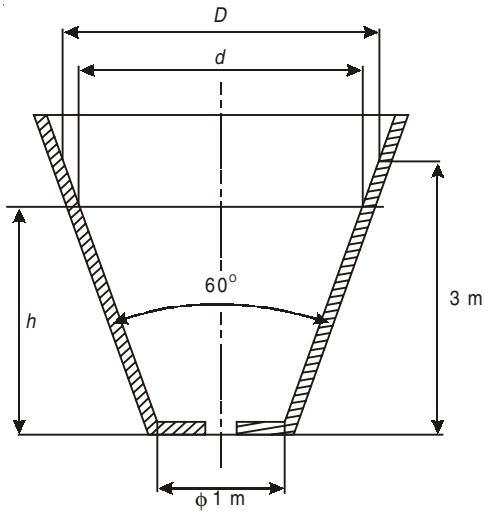


Fig. 9.10

$$\text{Area} = A$$

$$= 0.7854\, (1.15\, h + 1)^2$$

$$= 1.04\,h^2 + 1.8h + 0.7854$$

$$Q \times dt = -A \times dh \quad \text{... (a)}$$

$$[C_d \times a\sqrt{2gh}]\,dt = -(1.04h^2 + 1.8h + 0.7854)\,dh$$

Here A is the area of the orifice

$$= 0.7854\,(0.1)^2 = \mathbf{7.854 \times 10^{-3}\ m^2}$$

Substituting in (a)

$$[C_d \times (7.854 \times 10^{-3})\sqrt{2gh}\,]\,dt = -(1.04h^2 + 1.8h + 0.7854)\,dh$$

$$dt = \frac{-[1.04h^2 + 1.8h + 0.7854]\,dh}{[C_d \times (7.854 \times 10^{-3})\sqrt{2gh}\,]}$$

$$\int dt = \frac{1}{[0.85 \times (7.854 \times 10^{-3})\sqrt{2g}\,]} \int_1^3 \frac{-[1.04h^2 + 1.8h + 0.7854]\,dh}{\sqrt{h}}$$

$$T = 33.8 \int_1^3 \left\{-[1.04h^2 + 1.8h + 0.7854]\,h^{-\frac{1}{2}}\right\} dh$$

$$T = 33.8 \left\{-\left[1.346h^2 + 0.9h^2 + 0.7854h\right] 2h^{0.5}\right\}_1^3$$

$$T = 33.8\,\{-[0.34(3)^3 + 0.9(3)^2 + 0.7854(3)]\,(2)\,(3)^{0.5}]\}$$

$$T = 2289 \text{ sec} = 38 \text{ mts}$$

Problem 9.14 *Determine the discharge through a square orifice of area 2.25 m^2 placed on one side of a rectangular water tank whose $l \times b = 20\ m^2$. The water level is 5 m above the top edge of the orifice. Also determine the time it takes for the water level to fall to the level of the bottom edge. $C_d = 0.6$.*

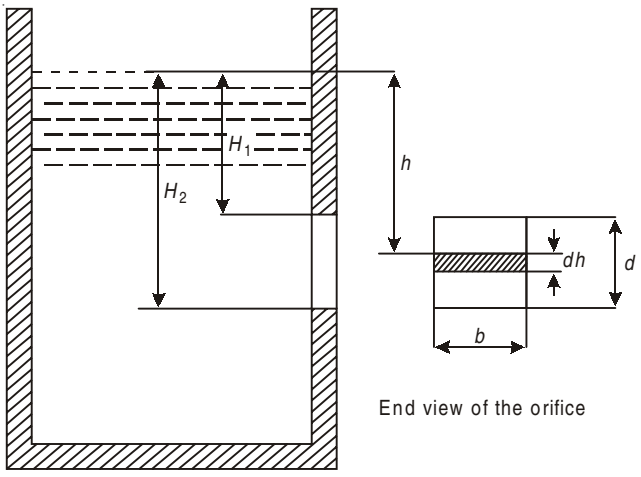


Fig. 9.11

Solution

Height of the orifice = $\sqrt{2.25}$ = 1.5 m. Since the available head is less than 5 times the height of the orifice, this could be construed as a large orifice and dealt with accordingly. The discharge (Q) through the orifice could be obtained by integrating the discharge through an elemental strip of the orifice between limits H_1 to H_2. Fig. (9.11)

$$Q = \int_{H_1}^{H_2} C_d \times b \times \sqrt{2gh}\, dh$$

$$Q = C_d \times b \times \sqrt{2g} \int_{H_1}^{H_2} \left(\sqrt{h}\right) dh$$

$$Q = \frac{2}{3} (C_d \times b \times \sqrt{2g}\,) \left(H_2^{1.5} - H_1^{1.5}\right) \quad ...(9.12)$$

$$Q = \frac{2}{3} (0.6 \times 1.5 \times \sqrt{2} \times 9.81) (6.5^{1.5} - 5^{1.5})$$

$$Q = 14.28 \text{ m}^3/\text{sec}$$

Time (T) taken for the water level to fall from H_1 to level H_2.

$$Q \times T = (\text{Area of the tank}) \times H_2$$
$$14.28 \times T = (20) \times (5 + 1.5)$$
$$T = 9.10 \text{ sec.}$$

Problem 9.15 *Find the discharge through a submerged orifice 200 mm in height and 300 mm wide if the difference in water levels between the two sides of the orifice = 1 m. $C_d = 0.6$.*

Solution

From Eq. (9.12)

$$Q = C_d \times W (H_2 - H_1) \sqrt{2gH}$$

$$Q = 0.6 \times 0.3 (0.2) \sqrt{2 \times 9.81 \times 1}$$

$$= 0.159 \text{ m}^3/\text{sec.}$$

(2) Partially-submerged orifice

A partially submerged orifice (Fig. 9.12) behaves both as a free discharge orifice and fully submerged orifice. Hence discharge Q is the sum of the discharges obtained from their respective governing equations.

Hence $Q = (Q_S + Q_F)$

(The subscripts S and F denote submerged and free orifice respectively)

$$Q = [C_d \times W (H_2 - H_1) \sqrt{2gH}\,] + \left[\frac{2}{3} (C_d \times b \times \sqrt{2g}\,) (H_2^{1.5} - H_1^{1.5})\right] \quad ...(9.13)$$

(From Eq. 9.13) (From Eq. 9.12)

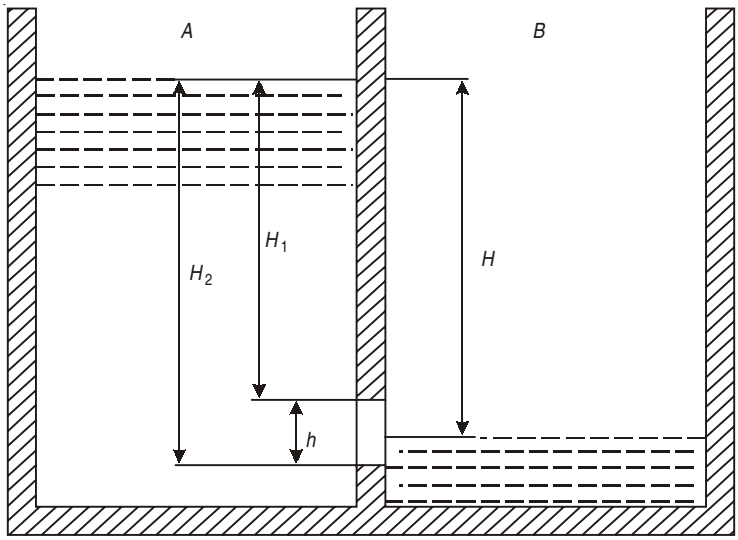


Fig. 9.12

Problem 9.16 *A rectangular tank is divided into two separate tanks by an intermediate plate. The intermediate plate has an orifice of height 0.5 m and width 0.2 m. The orifice is covered by water to half its height on one side while on the other side it is fully submerged. If the difference in water level is 2 m, determine the discharge through the orifice.* $C_d = 0.6$.

Solution From Eq. (9.15) the total discharge for a partially submerged orifice is given by,

$$Q = [C_d \times W (H_2 - H_1) \sqrt{2gH}\,] + \left[\frac{2}{3} (C_d \times b \times \sqrt{2g}) (H_2^{1.5} - H_1^{1.5})\right]$$

$$H = 2 \text{ m}$$

$$H_2 = H + \text{half the height of the orifice}$$

$$= 2 + \left(\frac{0.5}{2}\right) = 2.25 \text{ m}$$

$$H_1 = H - \text{half the height of the orifice}$$

$$= 2 - \left(\frac{0.5}{2}\right) = 1.75 \text{ m}$$

$$(H_2 - H_1) = 2.25 - 1.75 = 0.5 \text{ m}$$

$$C_d = 0.6$$

$$W = b = 0.2 \text{ m}$$

Substituting the values in the above equation,

$$Q = [0.6 \times 0.2\,(0.5)\,\sqrt{2g}\,\times 2] + \left[\frac{2}{3}\,(0.6 \times 0.2 \times \sqrt{2g}\,)\,(2.25^{1.5} - 1.75^{1.5})\right]$$

$$Q = 0.750\ \text{m}^3/\text{sec}$$

QUESTIONS BANK

Section A

Review Questions

1. Distinguish clearly between an Orifice and a mouth piece and state their uses.
2. State the need for hydraulic coefficients.
3. How do you distinguish between a mouth piece running free and a mouth piece running full?
4. With the help of sketches explain the salient features of
   (i) Borda's mouth piece
   (ii) Convergent mouth piece
   (iii) Convergent-divergent mouth piece.
5. Derive an expression to determine the ratio of areas at the outlet and at Vena contracta in case of a convergent-divergent mouth piece.
6. Explain how the coefficient of velocity is determined experimentally.
7. Explain why a large orifice is mathematically treated differently from a small orifice.
8. Define a fully submerged orifice and give instances where a need for such an orifice arises.

Section B

Multiple Choice Questions

Choose the right answer

1. An orifice is said to be discharging free if its top edge is
   (a) below the surface of water
   (b) above the surface of water
   (c) if the orifice is discharging in to another tank
   (d) if the orifice is discharging to atmosphere
2. For a sharp edged small orifice the value of C_V lies within
   (a) 0.613 – 0.69 (b) 0.62 – 0.65
   (c) 0.95 – 99 (d) 0.8 – 0.9
3. One of the ways of determining the practical value of C_C is
   (a) by measuring the $x-y$ co-ordinates of the jet
   (b) determining the actual velocity at Vena contracta
   (c) by measuring the diameter of the jet at Vena contracta
   (d) None of these

4. If D is the depth of water in the tank then for the jet to travel a maximum distance the orifice must be placed at a distance

   (a) $\frac{2}{3D}$ (b) $\frac{1}{3D}$

   (c) $\frac{1}{4D}$ (d) $\frac{1}{2D}$

5. Mouthpieces cannot be used for flow under high heads because
   (a) the value of C_V increases
   (b) the value of C_V decreases
   (c) the absolute pressure at Vena contracta increases
   (d) vaporization occurs

6. For an internal mouth piece whose $\frac{L}{D}$ ratio is 2.5, the value of C_C lies between
   (a) 0.75 – 8 (b) 0.52 – 0.55
   (c) 0.99 – 1 (d) None of these

7. For a Borda's mouth piece the value of C_C $H \approx$
   (a) 1 (b) 0.95
   (c) 0.92 (d) 0.52

8. Discharge through a large orifice is a case of
   (a) unsteady flow (b) uniform flow
   (c) all of these (d) None of these

9. For a mouth piece whose $\frac{L}{d}$ ratio is 2.5 the diameter of the jet is equal to
   (a) 0.5 d (b) 1.5 d
   (c) 0.75 d (d) None of these

10. For a mouth piece discharging under a head of 3 m the pressure at Vena contracta is
    (a) 29.43 kPa (b) 35 kPa
    (c) 27.45 kPa (d) None of these

11. Identify the values of C_C and C_V of an internal mouth piece running full from out of the given values.
    (a) 1 and 0.707 (b) 0.5 and 0.707
    (c) 0.65 and 0.95 (d) None of these

Section C

Problems

1. The inside diameter of the hose of a fire extinguisher is 125 mm and the nozzle attached to it has a diameter 25 mm. The discharge through the

nozzle is 1.5 LPS. The pressure before the nozzle is 10 kPa. Assuming that the datum heads are equal, determine the coefficient of discharge. (*Ans.* C_d = 0. 68)

2. A tank has two identical vertical orifices placed directly one below the other and their difference in head is 1m. The center of upper orifice is 4 m below the free surface of water. Determine the point of intersection of the two jets if their C_V = 0.98. (*Ans.* 8.765 m)
3. An internal mouth piece of diameter 125 mm is discharging over a head of 6 m. Determine the discharge when the mouth piece is running full and when the mouth piece is running free. (*Ans.* 0. 066 m^3/sec and 0. 094 m^3/sec)
4. Two tanks *A* and *B* are interconnected through a fully submerged orifice of diameter 100 mm. The area of tank *A* is half the area of tank *B* and the head over the orifice as measured from tank *A* is 5 m and the difference in head between the two tanks is 2.5 m. Determine the time required for the water level in the two tanks to equalize. C_d = 0.65. (*Ans.* 466 sec)
5. A reservoir discharges through a rectangular sluice gate 2 m in height and 1 m wide. If the top edge of the gate is 1m below the level of water in the reservoir determine (i) the discharge through the gate assuming the bottom edge of the gate is above the downstream level of the reservoir. (ii) What would be the percentage error if the gate was considered as a small orifice. Neglect C_d. (*Ans.* 12. 4 m^3 /sec and 1. 035 %)
6. A cylindrical tank of diameter 3 m is totally 8 m in height and contains water up to the top edge. The lower half of the tank is a hemisphere of radius 1.5 m. The orifice at the base of the hemisphere has an area = 1.76 × 10^{-2} m^2. If C_D = 0.6, find the time required to fully empty the tank. (*Ans.* 667. 9 sec)

Answers to Multiple Choice Questions

1. (d) **2.** (c) **3.** (c) **4.** (d) **5.** (d) **6.** (c)
7. (a) **8.** (a) **9.** (d) **10.** (a) **11.** (a)

Chapter

10

Flow Over Notches and Weirs

10.1 WEIRS

A Notch is an opening at the side of a tank or a channel similar to orifices but unlike orifices they have no top edge and are not submerged but extend just above the free surface level of water. A Notch is formed out of a steel plate and is basically used to measure the discharge across small tanks, channels or reservoirs. Notches could be either rectangular or triangular or trapezoidal. A variation to a trapezoidal notch is the stepped notch.

A weir on the other hand is a concrete-masonry construction and could be construed as a large notch. Weirs are used to measure the flow across a river or a spill way or to divert excess water during floods.

There are a variety of shapes and forms to a weir and are classified accordingly. They can be rectangular, triangular or trapezoidal. A trapezoidal weir with a side slope of 1 **:** 4 is called *Cippoletti Weir*. A triangular weir is rarely found since it is not preferred for large flows and a variation to this is the compound weir or a stepped weir.

The top edge of a weir could be above or extend below the free surface of water in which case the weir is said to be submerged.

The Crest or Sill is that part of a weir's surface over which the water flows and depending upon its size a weir is termed broad crested ($b > 0.47H$) or narrow crested. A weir is classified as *Ogee weir* and a *sharp crested weir* depending upon whether the crest is rounded towards the end or is left sharp.

Mathematical analysis for determining the flow across a notch is no different from that of a weir. Hence equation governing the discharge from a Notch applies as well to a Weir of the same geometrical form.

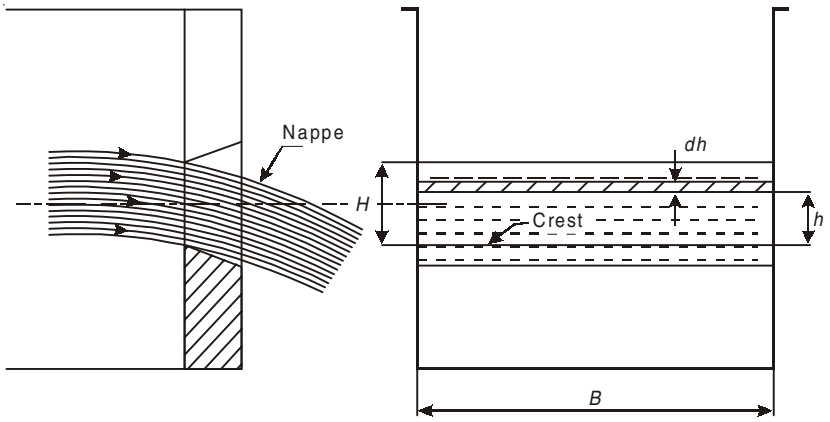
Consider a *rectangular sharp crested Notch* (Fig. 10.1 a).

Consider a small strip of thickness dh at a distance h from the free surface called 'nappe'.

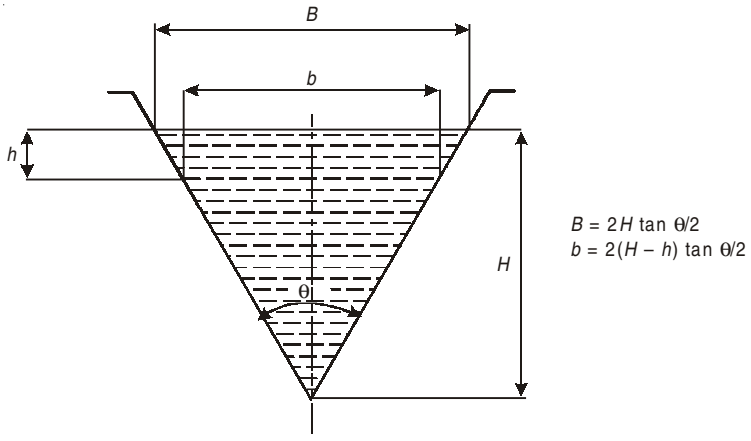
Discharge dQ through the strip = $(B)(dh)\sqrt{2gh}$

$$\text{Total discharge } Q = \sqrt{2g}\int_0^H B\sqrt{h}\,dh$$

$$= \frac{2}{3}B\sqrt{2g}\,(H^{1.5}) \qquad \ldots(10.1)$$



Rectangular sharp crested notch/ weir
(a)



V-Notch
(b)

Fig. 10.1

For a V-Notch (Fig. 10.1b) at any height h

$$b = 2(H-h)\tan\frac{\theta}{2}$$

Therefore Eq. (10.1) stands modified to that extent.

$$\text{Total discharge } Q = \sqrt{2g}\int_0^H 2(H-h)\tan\frac{\theta}{2}\sqrt{h}\,dh$$

$$\text{Total discharge } Q = 2\sqrt{2g}\left(\tan\frac{\theta}{2}\right)\int_0 (H-h)\,h^{\frac{1}{2}}\,dh$$

$$\text{Total discharge } Q = \left(\frac{8}{15}\right)\sqrt{2g}\left(\tan\frac{\theta}{2}\right)H^{2.5} \qquad \ldots(10.2)$$

For a Trapezoidal Notch (Fig. 10.2)

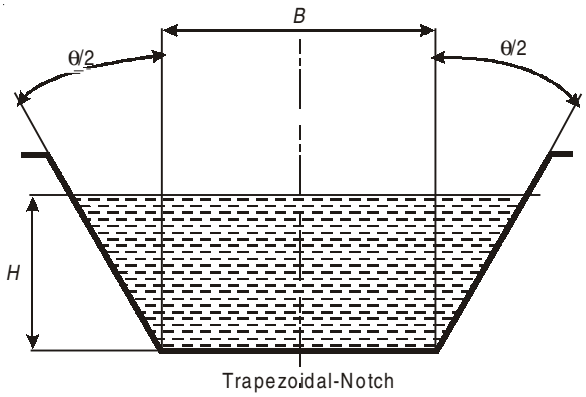


Fig. 10.2

A trapezoid is formed out of one rectangle and two triangles on either side. The two triangles together constitute a V-notch.

Therefore the total discharge is a combination of discharges from the rectangular notch and from the V-notch

$$Q = \left(\frac{2}{3}B\sqrt{2g}H^{1.5}\right)+\left(\frac{8}{15}\right)\sqrt{2g}\left(\tan\frac{\theta}{2}\right)H^{2.5} \qquad \ldots(10.3)$$

For a stepped rectangular notch (Fig. 10.3)

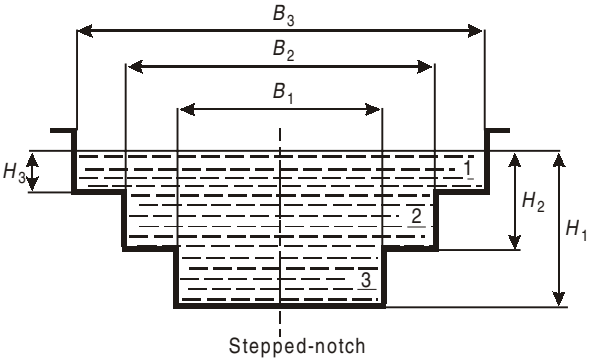


Fig. 10.3

The total discharge Q is the sum of the discharges from each step considered as a rectangular notch.

$$Q = \left(\frac{2}{3}B\sqrt{2g}H_1^{1.5}\right) + \left(\frac{2}{3}B\sqrt{2g}H_2^{1.5}\right) + \left(\frac{2}{3}B\sqrt{2g}H_3^{1.5}\right) \quad \ldots(10.4)$$

In the above equations we have neglected the kinetic energy of the flowing fluid by assuming that the approach velocity is small. But where the width of the reservoir or channel narrows down the approach velocity at the weir could be significant and hence the kinetic energy. The head causing the flow under such circumstances is $H + \alpha\left(\frac{v^2}{2g}\right)$ where α is the kinetic energy correction factor and v is mean velocity of flow obtained by dividing the discharge through the channel by the cross sectional area of the channel.

Additionally there could be end contractions due to the high velocity approach which reduces the effective breadth of the weir and consequently the discharge. The value of these end contractions is defined in terms of the head causing the flow.

For weirs the end contractions are generally = $0.1H$. Thus the effective breadth $B_e = B - 2(0.1H)$. This is called the Francis's formula.

In case of a *suppressed weir* where the breadth of the weir (B) = the breadth of the channel (B_c) there are no end contractions and correction for end contractions need not be applied. Equations as given below have been developed for determining the discharge.

Where the approach velocity is negligible $Q = \frac{2}{3}B\sqrt{2g}\ [H^{1.5}]$ ...(10.5)

If the channel narrows down towards the end of the weir the approach velocity could be significant.

$$\text{In such cases } Q = \frac{2}{3}B\sqrt{2g}\ [H + H_a]^{1.5} - H_a^{1.5} \quad \ldots(10.6)$$

Here H_a = the velocity head and $B = B_c$
= Width of the channel

The *Ogee weir* (Fig. 10.4) also called the *spill-way* is an improvement over the suppressed weir and generally over any sharp crested rectangular weir because the profile of the spill-way immediately following the sharp crest is constructed in such a way as to conform to the profile of the lower nippe of the sheet of water flowing over the weir. A represents the profile of the lower nippe of the sheet of water flowing over any sharp crested rectangular weir. B represents the flow over a spill-way. *Discharge over Ogee weir is the same that of a rectangular weir.*

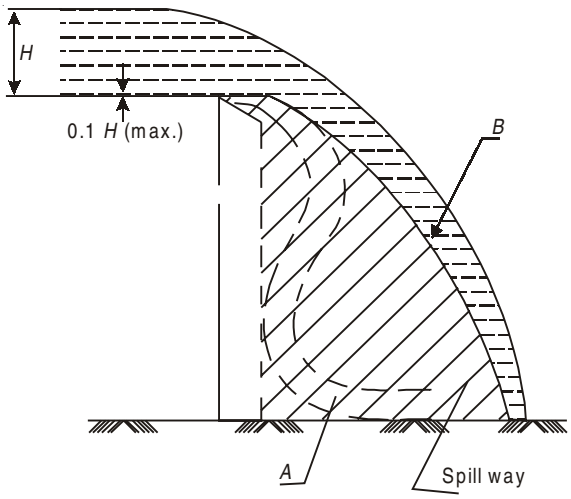


Fig. 10.4 *Ogee weir*

In case of a *Broad crested weir* (Fig. 10.5) the sill of the weir $L > 0.47H$.

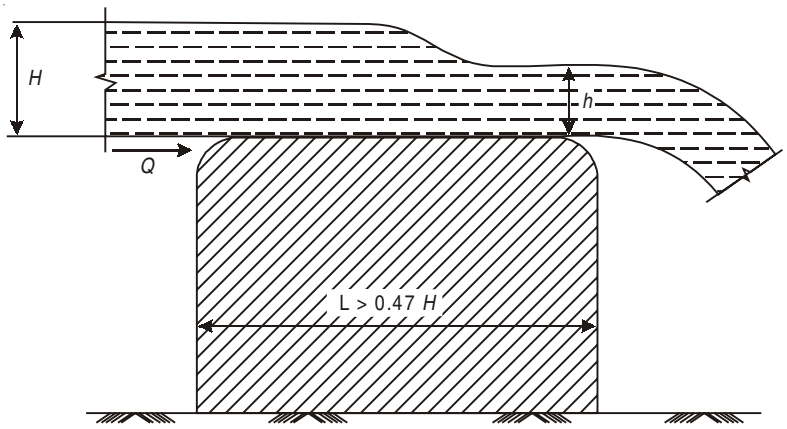


Fig. 10.5

$$Q = L \times h \left[\sqrt{2g}\ (H - h)\right]$$

$$Q = L \times \left[\sqrt{2g}\ (Hh^2 - h^3)\right]$$

For maximum discharge $h = \left(\frac{2}{3}H\right)$ a factor obtained by differentiating $(Hh^2 - h^3)$ with respect to h and setting it = 0

Therefore Q_{max} in its simplified form = 1.705 $LH^{1.5}$ ...(10.5)

The coefficient of discharge depends on the Reynolds number and the smoothness of the sill and ranges between (0.85 – 1.1)

V-Notch has distinct advantages over a weir in that it is better suited to measure small discharges, measurement of discharge is easy as only the height has to be measured and one V-Notch can measure a wide range of flows. It becomes desirable therefore to combine a rectangular notch or a weir with a V-notch to provide for different conditions that might prevail.

This resulted in the construction of the *Compound weir* (Fig. 10.6) which is a combination of stepped rectangular weir and the V-Notch.

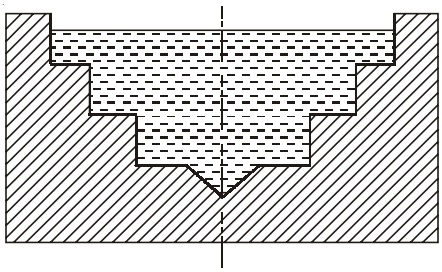


Fig. 10.6

If the water level down stream of the crest of a weir is above the crest then the weir is said to be submerged. (Fig.10.7)

The equation for total discharge over a *Submerged weir* is based on the Du-Buat's principle which treats a part of the discharge as that occurring over weir and the rest as occurring over a submerged orifice.

Q_1 = discharge over a weir

$= \frac{2}{3} B\sqrt{2g}\ (H_2 - H_1)^{1.5}$

Q_2 = discharge over a submerged orifice

$= LH_2\ \sqrt{2g}\ (H_2 - H_1)^{1.5}$

Total discharge $Q = Q_1 + Q_2$

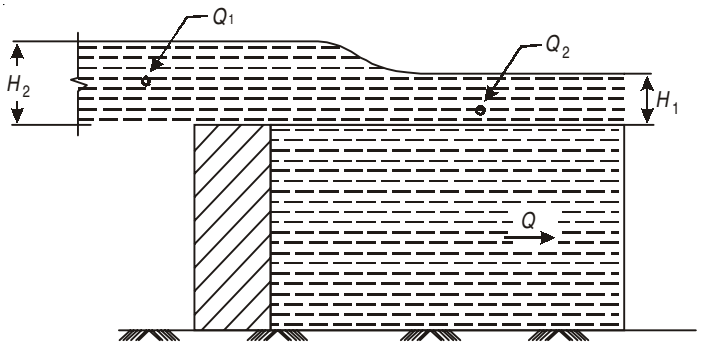


Fig. 10.7

$$Q = \frac{2}{3}B\sqrt{2g}\,(H_2 - H_1)^{1.5} + LH_2\sqrt{2g}\,(H_2 - H_1)^{1.5} \quad \ldots(10.6)$$

The head over Notch or a weir should be measured accurately. It can be shown that an error of 1% in case of a rectangular notch or a weir could lead to an error of 1.5% in the discharge values obtained. In case of a V-Notch an error of 1% could result in an error of 2.5% in the discharge values obtained.

TYPICAL PROBLEMS AND SOLUTIONS

Problem 10.1 *Find the discharge over a right angled V-Notch given that the head over the notch = 0.3 m. Take $C_d = 0.65$.*

Solution

From Eq. (10.2)

$$\text{Total discharge } Q = \left(\frac{8}{15}\right)\left(\sqrt{2g}\right)\left(\frac{\tan\theta}{2}\right)H^{2.5}$$

For a right angled notch $\theta = 90^\circ$ and hence $\tan\frac{\theta}{2} = \tan 45^\circ = 1$

$$Q = \left(\frac{8}{15}\right)\left(\sqrt{2g}\right)H^{2.5}$$

Since $C_d = 0.65$

$$Q = 0.65\left(\frac{8}{15}\right)\sqrt{2g}(0.3)^{2.5}$$

$$Q = 1.535\,(0.3)^{2.5} = 0.075 \text{ m}^3/\text{sec}$$

Problem 10.2 *The included angle formed by a trapezoidal notch = 60°. The base of the notch measures 2 m. Head over the notch = 0.5 m. $C_d = 0.6$. Determine the discharge over the notch.*

Solution

From Eq. (10.3)

$$Q = \left(\frac{2}{3}B\sqrt{2g}H^{1.5}\right) + \left(\frac{8}{15}\right)\sqrt{2g}\left(\tan\frac{\theta}{2}\right)H^{2.5}$$

$$Q = \left(\frac{2}{3}\times 2\sqrt{2g}\times 0.5^{1.5}\right) + \left(\frac{8}{15}\right)\sqrt{2g}\,(\tan 30)(0.5)^{2.5}$$

$$Q = C_d\,[(5.90)\times(0.5)^{1.5} + 1.364\,(0.5)^{2.5}] = C_d\,(2.327)$$

$$Q = (0.6)\,(2.327) = 1.396 \text{ m}^3/\text{sec}$$

Problem 10.3 *If the discharge over a right angled V-Notch is 0.075 m^3/sec, determine the percentage error in discharge if the error that occurred while measuring the head was = 3 mm. C_d = 0.65.*

Solution For a similar problem (Refer Problem 10.1) the head over the notch was given to be = 0.3 m.

It has already been stated that in case of a V-Notch an error of 1% while measuring the head could result in an error of 2.5% in the discharge values obtained.

This means Error $$E = 2.5 \frac{dh}{H} = \frac{2.5\,(3 \times 10^{-3})}{0.3}$$

$$= 0.025 = 2.5\ \%$$

Problem 10.4 *A channel is 3 m wide × 0.5 m deep. A rectangular notch of width 1 m is constructed across the channel. If the water level over the crest of the notch is 0.3 m determine the discharge taking into consideration the approach velocity. C_d = 0.65.*

Solution

From Eq. (10.6) in cases where velocity of approach is considered

$$Q = \frac{2}{3} B \sqrt{2g}\ [H + H_a]^{1.5} - H_a^{1.5} \qquad ...(a)$$

Here H_a = the velocity head and B = Width of the notch

Let us neglect the approach velocity and determine the discharge over the notch taking into account only the end contractions and the C_d

$$Q = \frac{2}{3} B \sqrt{2g}\ H^{1.5} \qquad \text{(from Eq. 10.5)}$$

$$Q^* = \frac{2}{3} C_d (B - 0.2H) \sqrt{2g}\ H^{1.5}$$

$$Q^* = \frac{2}{3} (0.65)\,(1 - 0.2 \times 0.3)\ \sqrt{2g}\ (0.3)^{1.5}$$

$$Q^* = 0.296\ \text{m}^3/\text{sec}$$

$$v_a = \frac{Q^*}{A_c} = \frac{0.296}{3 \times 0.5} = 0.197\ \text{m/sec}$$

$$H_a = \frac{V_a^2}{2g} = \frac{(0.197)^2}{2\text{g}} = 1.98 \times 10^{-3}\ \text{m}$$

Substituting the value of H_a in Eq. (a) above

$$Q = \frac{2}{3} B \sqrt{2g}\ [H + H_a]^{1.5} - H_a^{1.5}$$

$$Q = \frac{2}{3} C_d (B - 0.2H)\ \sqrt{2g}\ [H + H_a]^{1.5} - H_a^{1.5}$$

$$Q = \frac{2}{3}(0.65)(1-0.2\times0.3)\sqrt{2g}\ [0.3 + 1.98 \times 10^{-3}]^{1.5} - (1.98\times10^{-3})^{1.5}$$

$$Q = (1.804)(0.166) - (8.81\times10^{-5}) = 0.299\ \text{m}^3/\text{sec}$$

Problem 10.5 *The upstream and downstream heights of water over the crest level of a submerged weir are 0.5 m and 0.25 m respectively. If the channel is 3 m wide, determine the discharge.* $C_d = 0.6$.

Solution

From Eq. (10.6) the discharge over a submerged weir is given by,

$$Q = \frac{2}{3}B\sqrt{2g}\ (H_2 - H_1)^{1.5} + LH_2\sqrt{2g}\ (H_2 - H_1)^{1.5}$$

$$Q = \frac{2}{3}(3)\sqrt{2g}\ (0.5-0.25)^{1.5} + (3\times0.5)\sqrt{2g}\ (0.5-0.25)^{1.5}$$

$$Q = (1.1075) + (0.830) = 1.938\times C_d = 1.938\times0.6 = 1.163\ \text{m}^3/\text{sec}$$

Problem 10.6 *A rectangular sharp edged notch has an effective width of 0.75 m and the head over the sill is 200 mm. The sill is at a height of 0.250 m from the bottom of the channel which is 1.5 m wide. Calculate the discharge taking into account the velocity of approach. Take kinetic energy correction factor á = 1.1 Neglect* C_d.

Solution

From Eq. (10.6) in cases where velocity of approach is considered

$$Q = \frac{2}{3}B\sqrt{2g}\ [H + H_a]^{1.5} - H_a^{1.5} \qquad \text{...(a)}$$

Here H_a = the velocity head and B = Width of the notch

Let us neglect the approach velocity and determine the discharge over the notch.

$$Q = \frac{2}{3}B\sqrt{2g}\ H^{1.5} \qquad \text{(from Eq. 10.5)}$$

Since effective width is given we need not consider end contractions.

$$Q^* = \frac{2}{3}(0.75)\sqrt{2g}\ (0.200)^{1.5}$$

$$Q^* = 0.198\ \text{m}^3/\text{sec}$$

$$v_a = \frac{Q^*}{A_c} = \frac{0.198}{1.5}\times(0.250+0.20) = 0.293\ \text{m/sec}$$

$$H_a = \alpha\frac{V_a^2}{2g} = \frac{1.1(0.293)^2}{2g} = 4.824\times10^{-3}\ \text{m}$$

Substituting the value of H_a in Eq. (a) above

$$Q = \frac{2}{3} B \sqrt{2g} \; [H + H_a]^{1.5} - H_a^{1.5}$$

$$Q = \frac{2}{3} (0.75) \sqrt{2g} \; [0.200 + 4.824 \times 10^{-3}]^{1.5} - (4.824 \times 10^{-3})^{1.5}$$

$$Q = (2.214)(0.0.092) - (3.35 \times 10^{-4}) = 0.202 \text{ m}^3/\text{sec}$$

Problem 10.7 *Determine the maximum discharge over a broad crested weir of length 330 mm. The head over the weir on the upstream side is 250 mm. Assume a suitable value for* C_d .

Solution In case of broad crested weir C_d depends on the Reynolds number and the smoothness of the flow surface area and ranges from 0.85 –1.1. We will assume a value = 0.85.

From Eq. (10.5) the maximum discharge over a broad crested weir is given by

$$Q_{\max} = 1.705 \, LH^{1.5}$$
$$Q_{\max} = (C_d) 1.705$$
$$LH^{1.5} = (0.85)(1.705)\,(0.33)(0.250)^{1.5}$$
$$Q_{\max} = 0.059 \text{ m}^3/\text{sec}$$

Problem 10.8 *Find the discharge over a Cippoletti weir of breadth 1.5 m if the head over crest is 0.50 m.* $C_d = 0.62$.

Solution The discharge equation obtained for a trapezoidal weir applies to a Cippoletti weir as well.

From Eq. (10.3)

$$Q = \frac{2}{3} B \sqrt{2g} \, H^{1.5}) + \left(\frac{8}{15}\right) \sqrt{2g} \left(\frac{\tan\theta}{2}\right) H^{2.5} \quad \ldots(a)$$

Since the angle of the slope θ is not specified we have to determine the angle based on the fact that for a Cippoletti weir $H = 4h$.

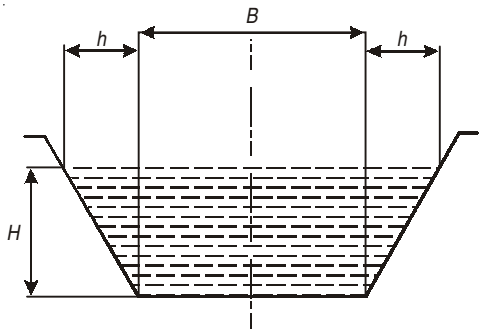


Fig. 10.8

Head over the weir $H = 0.5$ m. Therefore $h = \dfrac{0.5}{4} = 0.125$ m

$$\tan\theta = \frac{0.125}{0.5} = 0.25$$

or $$\theta = 14°$$

We may now substitute the relevant values in Eq. (a).

(Note: In case of a triangular notch θ is the included angle = 28° and hence $\dfrac{\theta}{2} = 14°$)

$$Q = \frac{2}{3}(1.5)\sqrt{2g}\,(0.5)^{1.5} + \left(\frac{8}{15}\right)\sqrt{2g}\;(\tan 14°)(0.5)^{2.5}$$

$Q = 1.670$ m^3/sec. But since C_d is given to be equal to 0.62,

$Q = (1.670) \times 0.62 = 1.035$ m^3/sec.

Problem 10.9 *A stepped notch has three steps. The width of the 1st, 2nd and 3rd step is 1.5 m, 1m and 0.75 m respectively. The head over the 1st, 2nd and 3rd step is 0.5 m, 1 m, and 1.5 m respectively. Determine the discharge over the weir if their combined $C_d = 0.65$.*

Solution The total discharge Q is the sum of the discharges from each step considered as a rectangular notch.(Eq. 10.4)

$$Q = Q_1 + Q_2 + Q_3$$

$$Q = \left(\frac{2}{3}B\sqrt{2g}H_1^{1.5}\right) + \left(\frac{2}{3}B\sqrt{2g}H_2^{1.5}\right) + \left(\frac{2}{3}B\sqrt{2g}H_3^{1.5}\right)$$

$$Q = \left[\frac{2}{3}(1.5)\sqrt{2g}\,(0.5)^{1.5} + \frac{2}{3}\times 1\sqrt{2g}\,(1)^{1.5} + \frac{2}{3}(0.75)\sqrt{2g}\,(1.5)^{1.5}\right]$$

$$Q = C_d\,(8.59) = 0.65 \times 8.59 = 5.58 \text{ m}^3\text{/sec.}$$

Time required for emptying a Tank or a Reservoir through a Notch

Let us first consider a *rectangular notch* and determine the time it takes for the liquid level to fall from the initial level H_1 to level H_2.

Let A be the area of the reservoir/tank. It is assumed A remains constant throughout. Let h be the head over the notch.

$$\text{Discharge through the notch} = C_d\left(\frac{2}{3}B\left(\sqrt{2g}\right)\times h^{1.5}\right) \qquad \text{...(a)}$$

Let at any instant of time the reservoir level decrease by a height $[-dh]$

$$\text{Then } -(A)\times\left(\frac{dh}{dt}\right) = Q = C_d\left(\frac{2}{3}B\left(\sqrt{2g}\right)\times h^{1.5}\right)$$

$$-(A \times dh) = Q = C_d \left[\frac{2}{3}B\left(\sqrt{2g}\right)\times h^{1.5}\right]dt$$

$$dt = -\frac{(A \times dh)}{C_d}\left[\frac{2}{3}B\left(\sqrt{2g}\right)\times h^{1.5}\right]$$

$$\int_0^T dt = \int_{H_1}^{H_2}\left[-\frac{(A)}{C_d}\left[\frac{2}{3}B\left(\sqrt{2g}\right)\right]\times h^{-1.5}\right]$$

$$T = \left[-\frac{(A)}{C_d}\left[\frac{2}{3}B\left(\sqrt{2g}\right)\right]\right]\left\{\frac{\left[h^{-1.5+1}\right]}{[-1.5+1]}\right\}_{H_1}^{H_2}$$

$$T = \left[-\frac{(A)}{C_d}\times\frac{2}{3}B\left(\sqrt{2g}\right)\right]\left[(-2)\frac{1}{\sqrt{H_2}}-\frac{1}{\sqrt{H_1}}\right]$$

$$T = \left[\frac{3(A)}{C_d}\times B\left(\sqrt{2g}\right)\right]\left[\frac{1}{\sqrt{H_2}}-\frac{1}{\sqrt{H_1}}\right] \quad ...(10.1)$$

The same way the time required for emptying a tank/reservoir through a V- notch can be derived.

In case of a *V-Notch* time required for the level to fall from $H_1 - H_2$

$$T = \left[\frac{5(A)}{4C_d}\left(\tan\frac{\theta}{2}\right)\left(\sqrt{2g}\right)\right]\left[\frac{1}{\sqrt{H}_2^{1.5}}-\frac{1}{\sqrt{H}_1^{1.5}}\right]$$

...(10.2)

Problem 10.10 *Determine the time required to lower the water level in a reservoir measuring 70 m × 70 m from 4 m to 3 m through a right angled V-notch.* $C_d = 0.6$.

Solution

From Eq. (10.2)

$$T = \left[\frac{5(A)}{4C_d}\tan\frac{\theta}{2}\left(\sqrt{2g}\right)\right]\left[\frac{1}{\sqrt{H}_2^{1.5}}-\frac{1}{\sqrt{H}_1^{1.5}}\right]$$

$$T = \left[\frac{5(4900)}{4(0.6)}\tan\frac{90}{2}\left(\sqrt{2g}\right)\right]\left[\frac{1}{\left(\sqrt{3}\right)^{1.5}}-\frac{1}{\left(\sqrt{4}\right)^{1.5}}\right]$$

= 194.5 sec

QUESTIONS BANK

Section A

Review Questions

1. Differentiate between a Notch and a Weir.
2. What do you understand by the term *velocity of approach*? How does it affect the initial head over the Notch or a Weir?
3. How is a Cippoletti weir different from a trapezoidal weir?
4. Define an Ogee weir, a sharp crested weir and a broad crested weir. When is a weir said to be submerged? How do you find the discharge in case a weir is submerged?
5. How is suppressed weir different from a normal weir and what does Francis's formula relate to ?
6. State the distinct advantages of a V-Notch over a rectangular notch and explain the need for a compound weir.
7. Why is it that the head over a V-Notch or must be measured more accurately than the head over a rectangular notch.
8. Derive an expression for the time required for emptying a reservoir using a V-Notch.
9. Derive an expression for maximum discharge over a broad crested weir.
10. Name the weir that uses Du-Buat's formula for determining the discharge.

Section B

Multiple Choice Questions

Choose the right answer

1. The difference between an orifice and a notch is
   (a) orifice cannot be used to in rivers and channels.
   (b) an orifice is an absolute necessity in a closed tank where as a Notch is need based.
   (c) a notch is without its top edge.
   (d) an orifice can be plugged and unplugged but it is not possible to plug a notch.
   (e) All of these.
2. Which among there is not used in practice
   (a) rectangular weir
   (b) triangular notch
   (c) triangular weir
   (d) stepped weir
   (e) broad crested weir

3. Another name for a spillway is
   (a) submerged weir
   (b) Cippoletti weir
   (c) suppressed weir
   (d) Ogee weir
4. In a suppressed weir the crest or sill length is
   (a) 0.5 times the width of the channel
   (b) equal to the width of the channel
   (c) 0.75 times the width of the channel
   (d) None of these
5. End contractions are not considered in case of
   (a) suppressed weirs
   (b) V-notch
   (c) rectangular weir
   (d) submerged weir
   (e) broad crested weir
6. The effect of approach velocity is not considered in case of
   (a) rectangular weir
   (b) V-notch
   (c) Cippoletti weir
   (d) submerged weir
   (e) suppressed weir
7. Head over a rectangular notch was 0.150 m. The error committed while measuring the head was 2 mm. The percentage error in discharge is
   (a) 1.5 % (b) 2.5%
   (c) 2% (d) 3%
8. Notches are fabricated out of
   (a) reinforce concrete
   (b) metallic paste
   (c) steel plates
   (d) any metal

Section C
Problems

1. The depth of a channel is 0.5 m and the flow velocity is 2 m/sec. It is proposed to use a sharp crested suppressed weir. Determine the head over the weir. Neglect C_d. (*Ans.* 0.485 m)
2. Using the Francis's formula for end contractions determine the discharge over a rectangular weir 2 m wide. The height of water above the crest is 0.5 m. $C_d = 0.65$. (*Ans.* 1.29 m^3/sec)

3. Determine the maximum height of the Ogee weir if the crest is at a height of 3 m, the breadth of the weir is 5 m and the discharge over the weir 2.240 m^3 /sec and C_d = 0.6. (*Ans.* 3.046 m)
4. A rectangular weir of breadth 1m and a right angled V-notch of base 0.5 m together form a compound weir. The head over the compound weir from the apex of the V-notch is 0.5 m. Determine the discharge if their combined C_d = 0.6. (*Ans.* 0.277 m^3/sec)
5. Determine the percentage error that occurs while using a right angled V-notch if the error in measurement of head over the notch is 2 mm. The discharge is 0.3 m^3/sec and C_d = 0.6. (*Ans.* 0.93%)

Answers to Multiple Choice Questions

1. (e) **2.** (c) **3.** (d) **4.** (b)
5. (a) **6.** (e) **7.** (c) **8.** (c)

Chapter

11

Flow Through Pipes

This chapter deals with *steady and uniform flow of incompressible fluids* through pipes. In dealing with the flow of fluids through pipes we are mainly concerned that the three basic parameters of flow, the volume flow rate, the pressure and the velocity remain constant within reasonable limits, from point of entry to the point of exit. We use the phrase 'within reasonable limits' because engineers have over the years studied the phenomenon of fluid flow through pipes and have come to accept the fact that while energy losses in pipe flows cannot be entirely avoided they can be controlled to a large extent by following proper design procedures.

Fluids flowing through a pipe with a certain velocity encounters resistance to flow. If the flow passages are adequately sized based on the recommended fluid velocity, then this resistance is limited to *pipe wall friction*, *turbulence due to sudden enlargement or contraction* and *sharp bends*. This resistance is a component of the total energy loss as the fluid flows from point to point. Head lost due to friction forms a major portion of this component while the others are termed *minor losses*.

Resistance to flow due to friction in the pipes cannot be avoided unless the walls are made entirely smooth which is neither possible nor feasible especially when you consider industrial fluid flow systems such as the pipe line carrying crude oil from the dock yard to the refinery hundreds of kilometres away or the municipal water supply system with its complicated piping net work. There have been several equations developed by different scientists over the years which quantify the loss due to pipe wall friction and prominent among these is the *Darcy–Weisbach* Eq. (11.1).

In the general energy equation (Refer Chapter 7) there is a term (h_L) which denotes head loss due to energy losses. Of these, energy loses loss due to pipe

wall friction is one component. This loss is proportional to the square of the velocity of the fluid and to the $\frac{L}{d}$ ratio of the pipe. That is, $h_f \left(\frac{L}{d}\right) \frac{v^2}{2g}$, where L is the length of the pipe, d is the inside diameter of the pipe and v average flow velocity. If we now introduce the constant of proportionality f called the *friction factor* the *Darcy–Weisbach* equation would emerge.

$$h_f = f\left(\frac{L}{d}\right) \frac{v^2}{2g} \qquad \ldots(11.1)$$

The above equation applies to both laminar (R_e< 2000) and turbulent (R_e> 4000) flows through long straight round pipes.

Here h_f is the head loss due to friction with units N.m/N or simply m. D is the inside diameter of the pipe in meters, L is the length of the pipe in metres and v is mean flow velocity in m/sec. f is the *friction factor,* a dimensionless quantity introduced to correct the possible errors in assessing the losses.

If instead of *friction factor* (f) Darcy's original *coefficient friction* (f_1) is used in the Eq. (11.1) then f must be taken as = $4f_1$ and resulting equation would be,

$$h_f = 4f_1\left(\frac{L}{d}\right) \frac{v^2}{2g} \quad \text{called the Darcy's equation} \qquad \ldots (11.2)$$

$$f_1 = \frac{16}{R_e}$$

(for R_e = Reynolds's number < 2000) ... (11.2 a)

$$f_1 = \frac{0.079}{R_e^{0.25}}$$

(for R_e = Reynolds's number 4000 – 10^6) ...(11.2 b)

The friction factor is not a constant but a variable whose value depends on the diameter of the pipe, length of the pipe, flow velocity, density of the fluid, viscosity and on the pipe roughness factors. It becomes important therefore that the value of f must be carefully chosen to avoid significant errors in head loss.

It has already been emphasized that *Darcy–Weisbach* equation holds good for both laminar and turbulent flows. But where only laminar flow is involved then the immeasurable quantity f can be overlooked since in case of laminar flows the energy loss is independent of pipe surface roughness but depends on a measurable quantity such as viscosity. Based on these considerations a separate equation for laminar flow called *Hagen–Poiseuille* Eq. (11.3) emerged. This equation has been verified many times over experimentally and is found to give correct results for flows with R_e< 2000.

$$h_f = \frac{32\,\mu L v}{\gamma D^2} \qquad \ldots(11.3)$$

If we now equate (11.1) and (11.3) since both apply equally well to laminar flows and solve for friction factor f we can obtain the value of f straight away instead of referring to charts and graphs as is the case when using Darcy's equation for turbulent flows.

$$f = \frac{64\,\mu g}{v\gamma D}$$

$$f = \frac{64\,\mu}{v\rho D} = \frac{64}{R_e} \qquad \ldots(11.4)$$

where R_e is the Reynolds's number

It is now established that either Darcy–Weisbach equation or Hagen–Poiseuille's equation can be used alternately to determine the energy loss due to friction in laminar flows $R_e < 2000$ and the friction factor f can be determined by simply knowing the Reynolds's number. Since laminar flow is restricted to low flow velocities it is only of academic interest because in practical engineering applications turbulent flow conditions are more likely to occur.

Determination of friction factor for turbulent flows cannot be obtained by mathematical means as turbulent flow is unpredictable, independent of Reynolds's number and the values can vary moment to moment. In case of turbulent flows ($R_e > 4000$) f depends on a factor called the *relative roughness* which is a ratio of the average pipe wall roughness (ε) and the pipe diameter D. Surface roughness value ε is not an absolute value but compares to the smoothness of glass and hence is a measure of relative roughness. Pipe wall roughness depends on the material of which the pipe is made and the manufacturing methods employed.

For instance coated seamless steel or copper tubes which are drawn to their final size have a lower value of ε (0.0015 mm) as compared to electrically resistance welded (ERW) pipes (0.046 mm). Cast Iron pipes have a value of ε = 0.25. Large concrete pipe have a value between 0.3 mm – 3 mm.

Stanton diagram is a plot of friction factor and R_e on a log-log scale and can be used to obtain the values of f or obtain the same value for flows $R_e < 100{,}000$ directly through Blasius empirical formula

$$f = \frac{(0.316)}{R_e^{0.25}} \qquad \ldots(11.5)$$

11.1 CHEZY'S EQUATION FOR LOSS OF HEAD DUE TO FRICTION

$$V = C\sqrt{mi}$$

where, v is the mean velocity, C is the Chezy's constant = $\sqrt{\frac{w}{f}}$, m is the hydraulic

radius which is a ratio of area of flow to the wetted perimeter $\left(\frac{A}{P}\right)$ and $i = \frac{h_f}{L}$, represents loss of head per unit length. *However since this equation adopts itself better to flow over open channels it is not used in this chapter.*

In solving pipe problems associated with turbulent flow where pipe friction is the only factor contributing to energy loss Darcy–Weisbach equation in combination with continuity equation and Moody's diagram can be used to determine the unknown quantity.

11.2 MOODY'S DIAGRAM (APPENDIX-2)

Moody diagram is a plot of friction factor f and Reynolds's number R_e on a logarithmic scale and is by far the most popular means adopted to determine the value of f. Moody diagram is divided in to three zones. (1) Laminar zone where $f = \frac{64}{R}$, (2) critical transition zone ($2000 < R_e < 4000$) for which no data is available, (3) Complete turbulent zone ($R_e > 4000$) in which curves for different values of $\left(\frac{\varepsilon}{D}\right)$ are plotted. Some references use $\left(\frac{D}{\varepsilon}\right)$ or $\left(\frac{R}{\varepsilon}\right)$ where R refers to pipe radius. In Moody's diagram shown very high values of $\left(\frac{\varepsilon}{D}\right)$ indicate rough pipes and very low values indicate a smooth pipe such as glass.

Darcy–Weisbach equation holds good for *non-circular sections* if the factor called *hydraulic radius* (R) *which is a ratio of the area of the section to its wetted perimeter* is introduced in place of pipe diameter (d).

For a square section of sides b

Hydraulic radius $$R = \frac{\text{Area}}{\text{Wetted perimeter}}$$

$$= \frac{b^2}{4b} = \left(\frac{b}{4}\right)$$

$$b = 4R = d$$

Equation (10.2) stands modified thus

$$h_f = f\left(\frac{L}{4R}\right)\frac{v^2}{2g} \qquad \ldots(11.6)$$

called the Darcy's equation

$$\text{Likewise } R_e = \frac{v\,\rho\,(4R)}{\mu}$$

$$= \frac{v\,(4R)}{v} \qquad \ldots(11.7)$$

The relative roughness $\frac{\varepsilon}{D}$ is modified to $\frac{\varepsilon}{4R}$ $\qquad \ldots(11.8)$

11.3 SEPARATION LOSSES ALSO CALLED MINOR LOSSES

11.3.1 Loss due to Valves and Fittings

We have all along been talking of flow through simple pipe lengths. We have not considered the entire piping system consisting of control valves, fittings such as elbows, T-joints, loses at points of entry and exit, expansion joints that connect two pipes of different diameters and the consequent loss of head that the fluid suffers as it flows through these expansions and contractions. All of these loses are termed as 'minor losses'.

As can be observed from the Darcy–Weisbach equation the head loss due to friction is directly proportional to the length of the pipe. The loss of head due to valves and fittings taken together constitute a very small portion of the loss of head due to friction where the length of the pipe line is considerably long ($L >$ 100 × pipe *ID*). This is the reason why they are termed *Minor losses*. If the length of the pipe is short then head loss due to valves and fittings will form the major portion of the energy loss and therefore needs to be accounted for. Minor losses can be neglected if they constitute less than 5% of head loss due to friction.

Fluid flowing through a complicated piping system is similar to a man running an obstacle race. It is never a smooth or straight forward flow. The fluid passing through a contraction is similar to the passengers in a bus trying to exit through the door all at the same time.

The losses in valves and fittings, are proportional to the square of the velocity of the fluid that is, h_L ; $\frac{v^2}{2g}$.

$$h_L = K\left(\frac{v^2}{2g}\right) \qquad \ldots(11.9)$$

K an empirical factor is the coefficient of resistance which depends on the particular valve or fitting, v is flow velocity in meters/sec. The value of K is generally expressed in terms of the equivalent length ratio of the pipe and the friction factor. Equivalent length ratio is constant for a particular valve or a fitting.

Equivalent length (L_E) is the length of a pipe of the same inside diameter as the pipe through which the fluid is passing immediately preceding the valve or the fitting and whose coefficient of resistance is same as the valve or the fitting.

Values of equivalent length ratio $\left(\frac{L_E}{D}\right)$ for various valves and fittings have been determined experimentally and the value of K is obtained by simply multiplying this factor with the friction factor (f) obtained through Reynolds's number or through Moody's diagram. Thus $K = \left(\frac{L_E}{D}\right) f$ (Table 11.1).

Minorlosses can be expressed in terms of the equivalent L_E of a pipe that has the same discharge and head loss by equating the Darcy–Weisbach equation to the head loss due to valves and fittings.

Thus, $$f\left(\frac{L}{d}\right)\frac{v^2}{2g} = k\frac{v^2}{2g}$$

$$L_E = \frac{kD}{f} \qquad \ldots(11.11)$$

Table 11.1 *Values of resistance coefficient K for different valves and fittings*

1.	90° flanged elbow	30 f
2.	90° one side threaded elbow	50 f
3.	Return bend	50 f
4.	T- connector	20 f
5.	Ball check valve	150 f
6.	Butterfly valve	45 f
7.	Foot valve	75 f
8.	Globe valve –fully open	340 f
9.	Gate valve – fully open	8 f
10.	Gate valve – ½ open	160 f

11.3.2 Energy Losses due to Sudden Expansion

In a complex piping system pipe diameters may vary from one section to another. If the pipe cross sectional area suddenly increases or decreases at any section while the volume flow rate remains constant then the only other parameter that gets affected is the flow velocity. Therefore even in this case the energy loss is proportional to the velocity head that is, $\frac{V_1^2}{2g}$ or equal to $k\left(\frac{V_1^2}{2g}\right)$.

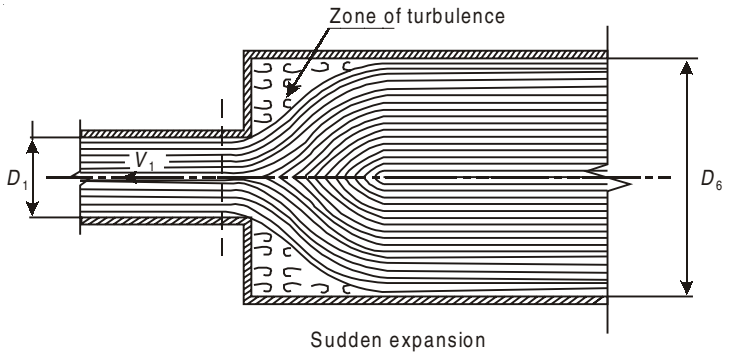


Fig. 11.1

The coefficient of Resistance k in case of sudden expansion is determined from the equation

$$k = \left[1 - \left(\frac{A_1}{A_2}\right)\right]^2 \qquad \ldots(11.11)$$

or

$$h_E = \left[1 - \left(\frac{A_1}{A_2}\right)\right]^2 \left(\frac{V_1^2}{2g}\right) \qquad \ldots(11.11\text{ a})$$

This equation gives erroneous results for flows beyond 1.5 m/sec. In such cases one has to resort to experimental data available in the form of charts and graphs.

11.3.3 Energy Loss due to Sudden Contraction

The coefficient k in case of sudden contraction which occurs when the flowing fluid has to pass through a much smaller diameter compared to the diameter

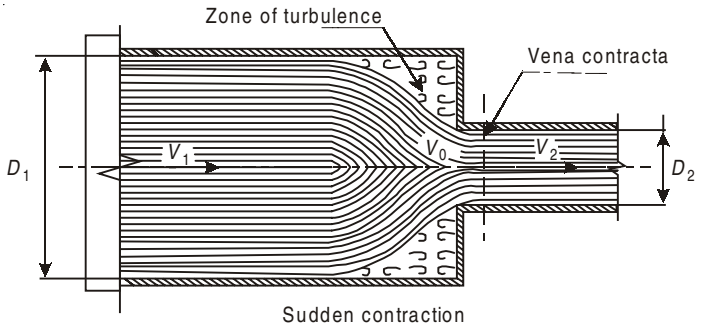


Fig. 11.2

immediately preceding the point of entry. The value of k in this case can be computed from the value of the coefficient of contraction at the point of vena contracta that occurs as the fluid enters the smaller section.

Thus $$k = \left[\frac{1}{C_c - 1}\right]^2$$

Head loss due to sudden contraction $h_L = \left[\frac{1}{C_c - 1}\right]^2 \left[\frac{v_2^2}{2g}\right]$ ...(11.12)

In this case v_2 is the fluid velocity at a point beyond vena contracta or,

Head loss due to sudden contraction $h_L = \frac{(v_0 - v_2)^2}{2g}$ ...(11.13)

In this case v_0 is the fluid velocity at vena contracta.

11.3.4 Energy Loss in Case of Gradual Expansion

Gradual expansion is an improvement over sudden expansion as the transition from smaller diameter to a large diameter is less abrupt and this factor appreciably reduces energy loss. However determination of energy loss in this

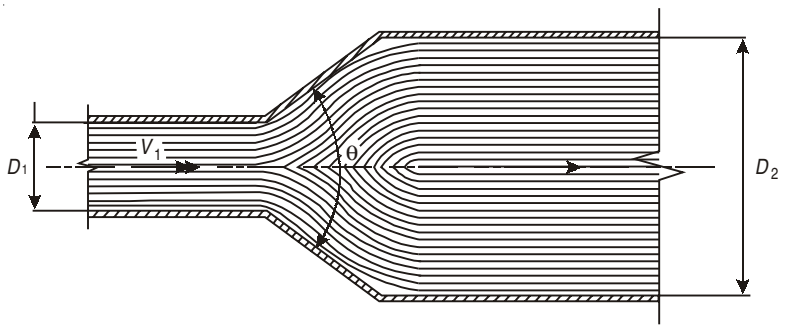


Fig. 11.3

case is complicated because the loss is a function of the diameter ratios, cone angle θ, pipe-friction loss over the length of expansion and the pressure increase in the direction of flow. We are therefore constrained to depend on experimentally determined values for the loss coefficient k.

Head loss due to gradual expansion $h_L = k\frac{(v_1 - v_2)^2}{2g}$ ...(11.14)

11.3.5 Energy Loss due to Gradual Contraction

Gradual contraction reduces significantly the energy loss the same way gradual expansion does. Whereas gradual expansion causes pressure increase downstream

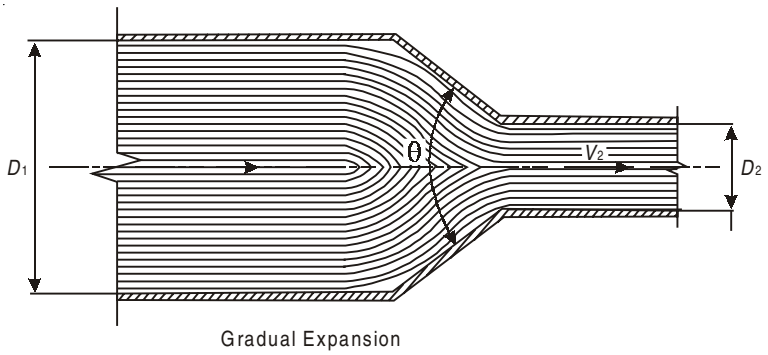


Fig. 11.4

and for this reason it is often termed as a diffuser, gradual contraction causes velocity increase down stream and hence could be termed as a nozzle. Even in this case we have to relay on experimental data for values of resistant coefficient k.

Head loss due to gradual expansion $h_L = k \dfrac{(v_2)^2}{2g}$ ... (11.15)

11.3.6 Energy Loss at the Point of Exit

If the fluid flowing through a pipe enters a reservoir or a tank the kinetic energy of the flowing fluid is totally dissipated because the flow velocity inside the tank is almost reduced to zero due to the huge difference in the area of the pipe as

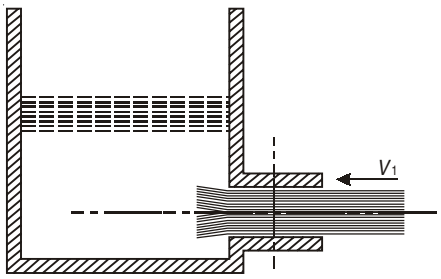


Fig. 11.5

compared to the area of the tank or the reservoir. This is called the exit loss and the coefficient of resistance in this case is always taken to be = 1 regardless of the form of the mouth piece at the point of exit.

Thus the loss at exit $h_L = \dfrac{v_1^2}{2g}$ ...(11.16)

11.3.7 Energy Loss at the Point of Entry

The *energy loss at the point of entry* to a pipe from a reservoir or a tank is also proportional to the square of the velocity head and the value of k depends on the form of the mouth piece. For a sharp edged mouth piece $k = 0.5$. If the edges are

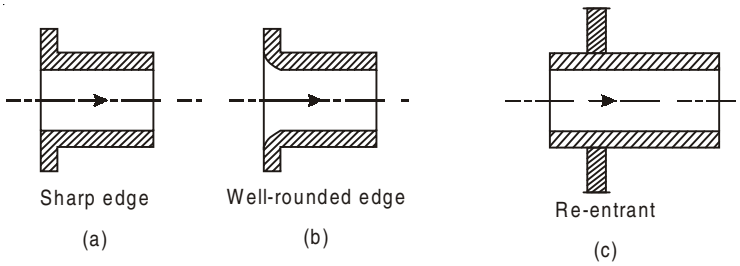


Fig. 11.6

Well-rounded the value of k lies between 0.01 – 0.05 and in case of a re-entrant mouth piece the value of k lies between 0.8 – 1.

11.4 PARALLEL PIPE LINE SYSTEMS

If two or more pipes run parallel to each other and if all of them draw fluid from a single source then such a system is termed as a parallel flow piping system. (Fig. 11.7)

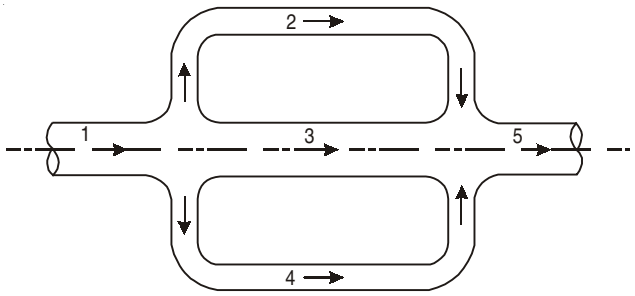


Fig. 11.7

$$Q_1 = Q_5 = Q_2 + Q_3 + Q_4 \quad \ldots(11.17)$$

Head lost between points 1 and 5 = Head lost at 3 = Head lost at 4 = Head lost at 2.

11.5 HYDRAULIC GRADIENT AND ENERGY GRADIENT LINES

Hydraulic gradient lines (HGL) is a plot of Piezometric head $\left(\frac{p}{\gamma} + z\right)$ available

at various locations as the fluid flows through a closed conduit whereas Energy gradient lines (EGL) is a plot of total energy available at each section. Thus HGL is a line connecting the values of $\left(\frac{p}{\gamma+z}\right)$ at desired locations whereas EGL is a line connecting the values $\left(\frac{p}{\gamma}+z+\frac{v^2}{2g}+\text{all losses}\right)$ at the same locations. Since here we are considering the total head EGL is also designated as total

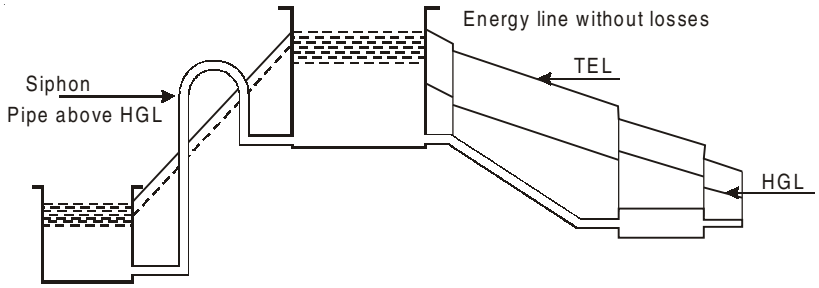


Fig. 11.8

energy line or simply as TEL. If a section of a pipe line lies above the HGL it is indicative of the presence of partial vacuum at that section and the negative pressure induces *siphon effect* at this section. This negative pressure should not fall below 3 m of water to prevent the flowing fluid from cavitation and vaporization effects.

Hydraulic gradient lines (HGL) and energy gradient lines (EGL) help engineers develop a better comprehension of the total flow loss by hi-Lighting flow losses as they occur at each section. HGL always lies below the EGL and the vertical difference between the two at any point is equal to the velocity head at that point. If the diameter of the fluid transmission pipe is uniform throughout then the slopes of HGL and EGL are equal which means the two lines run parallel to each other.

In case of a diffuser or a suddenly expanding pipe at the point of transition there is drop in the total energy line (EGL) whereas there is a raise in the hydraulic gradient line (HGL). This is in order because although there is a pressure increase the total energy of the fluid suffers a loss as the velocity head is affected.

11.7 THE SIPHON

The siphon is that portion of a pipe line which is above the hydraulic gradient line (HGL). Refer Fig. (11.8). It sometimes becomes necessary to lay the pipe line connecting two reservoirs over a hillock or an elevation which is at a higher level than the hydraulic gradient line. In all such instances it is the induced

vacuum pressure that causes the flow at that point. It should be ensured that this negative pressure is well above 3 m of water absolute to prevent fluid separation due to entrapment of dissolved air and other gases at the peak. The upper limit for any section of the siphon is 7 m of water from the HGL. It is also desired that the inlet leg of the siphon measured along the siphon should be as short as possible in order to limit head loss due to friction and consequent loss in positive pressure.

11.8 POWER TRASMITTED THROUGH PIPES

Power transmitted through a pipe is the net power available for use at the point of delivery which is the difference between the power available at the source and the power lost due to friction and power lost due to minor losses.

Thus
$$P_T = \gamma Q H - \gamma Q h_L$$
$$= \gamma Q [H - h_L] \quad \ldots(11.18)$$

Its units are Nm/sec or J/sec or Watts if γ is expressed N/m^3, Q in m^3/sec and H in metress.

As can be seen from Eq. (11.18) power transmitted is proportional only to the flow velocity as all other factors such as pipe diameter and the available head are constants.

The efficiency of transmission $\eta = \dfrac{[H - h_L]}{H}$ ...(11.19)

The efficiency of transmission is maximum when $h_L \leq \dfrac{H}{3}$

The above statement can be mathematically proved thus.

From (11.18),

$$P_T = \gamma Q [H - h_L]$$

$$P_T = \gamma (Av) \left[H - f\left(\frac{L}{d}\right)\frac{v^2}{2g} \right]$$

$$P_T = \gamma (A) \left[HV - f\left(\frac{L}{d}\right)\frac{v^3}{2g} \right]$$

P_T is maximum $\dfrac{d(P)}{dv} = 0$ That is,

$$\frac{d}{dv}\left\{\gamma(A)\left[HV - f\left(\frac{L}{d}\right)\frac{v^3}{2g}\right]\right\} = 0$$

$$\gamma (A) \left[H - f\left(\frac{L}{d}\right)\frac{3v^2}{2g} \right] = 0$$

$$\left[H - f\left(\frac{L}{d}\right)\frac{3v^2}{2g}\right] = 0, \qquad \text{... since } \gamma(A) \neq 0$$

$$H = 3f\left(\frac{L}{d}\right)\frac{v^2}{2g}$$

or $$H = 3\,h_L$$

or $$h_L = \frac{H}{3}$$

11.9 FLOW THROUGH A NOZZLE

A nozzle is a device that converts pressure energy in a pipe to kinetic energy. This converted kinetic energy and the kinetic energy inherent in the flowing fluid together form a high velocity jet that exits the nozzle. A nozzle is fitted to the delivery end of a pipe wherever there is need for a high velocity jet of fluid such as in fire extinguishers or in penstock pipes which deliver high velocity jet of water to the Pelton wheel. In pipes equipped with a nozzle the hydraulic gradient line falls sharply to the level of the nozzle as the diameter decreases. In its simplest form it is no more than a tapering collar whose flow passage area sharply decreases over a small length.

The exit velocity (v_2) of the jet at the end of the nozzle could be obtained from the continuity equation.

$Q_1 = Q_2$ where Q_1 and Q_2 constitute discharge through the pipe and the discharge through the nozzle respectively.

$$A_1 v_1 = A_2 v_2.$$

or $$v_1 = \frac{(A_2\, v_2)}{A_1} \qquad \ldots(a)$$

Head at the outlet to the nozzle

$$H = h_L + \frac{v_2^2}{2g} \qquad \ldots(b)$$

$$H = f\left(\frac{L}{D}\right)\frac{\left(\frac{A_2\, v_2}{A_1}\right)^2}{2g} + \frac{v_2^{\,2}}{2g} \quad \ldots\text{from (a) above}$$

$\ldots(c)$

Solving for v_2

$$v_2 = \sqrt{\frac{gH}{\left(\frac{L}{D_1}\right)\left(\frac{A_2}{A_1}\right)^2}} \qquad \ldots(11.20)$$

$$\text{Transmission efficiency} = \frac{\text{Power available at the nozzle}}{\text{Power at the inlet to pipe}}$$

$$= \frac{\text{K.E. of the jet}}{\gamma\, QH}$$

$$\text{Transmission efficiency} = \eta = \frac{\left[\left(\frac{1}{2}\right) m v_2^2\right]}{\gamma\, QH} \qquad \ldots(11.21)$$

Power transmitted will be maximum when head lost due to friction is no more than $\frac{1}{3}$ of the available head. This condition will enable us to determine the diameter of the jet that will transmit maximum power

From (b) above $$H = h_L + \frac{v_2^2}{2g}$$

$$3\, h_L = h_L + \frac{v_2^2}{2g}$$

$$2\, h_L = \frac{v_2^2}{2g}$$

$$2\left[f\left(\frac{L}{D_1}\right)\frac{\left(\frac{A_2 v_2}{A_1}\right)^2}{2g}\right] = \frac{v_2^2}{2g} \qquad \ldots\text{from (a) above}$$

$$\frac{A_1}{A_2} = \sqrt{\frac{(2\, f\, L)}{D_1}} \qquad \ldots(11.22)$$

Here A_1 and A_2 represent area of the pipe and the area of the nozzle respectively, f is the friction factor in the Darcy–Weisbach equation, L is the length of the pipe and D_1 is the inside diameter of the pipe. v_1 and v_2 represent the flow velocity in pipe and the flow velocity through the nozzle respectively. 'm' is the mass flow rate = ρQ.

11.10 WATER HAMMER

Water hammer is a phenomenon that occurs in long pipe lines such as penstock pipes when the energy contained in the flowing fluid is brought to rest by

instantaneous closure of the valve. The momentum of the fluid is not destroyed but gets converted as elastic energy which results in the formation of sinusoidal pressure waves that travel back and forth through the pipe. The magnitude of these pressure waves basically depends on the length of the pipe, flow velocity and the speed of valve closure. The positive and negative pressure waves cause elastic deformation of the pipe walls which has the effect similar to hammering. How soon these waves dampen out depends on the pipe wall friction and the elasticity of the pipe material and the Bulk modulus of the fluid. The pressure raise due sudden closure of valve must be taken into consideration while deciding on the pipe wall thickness to prevent pipe burst. In penstock pipes used in hydro-electric power generation since penstock control valves are designed to automatically close in case excess load, *surge tank erected between the fore bay and the penstock and connected to the supply line absorbs the pressure energy effectively thereby reducing the rate of deceleration of water.*

The closure of the valve is said to be *sudden or instantaneous* if the time period (t) required for the pressure waves to travel back and forth once, along the entire pipe length is less than $\left[\frac{2L}{C}\right]$ where L is length of the pipe and C is the velocity of the pressure wave. If t is greater than $\left[\frac{2L}{C}\right]$, it indicates *gradual closure*. By gradual closure we mean fluid velocity undergoes uniform retardation as the initial velocity $v \to 0$.

An equation for the pressure raise due to *gradual closure* can be derived from the Newton's law of motion $F = ma$, if we assume that the liquid is incompressible.

$$F = m \times (\text{retardation}) = m \times \frac{v}{t}$$

$$F = \frac{\rho AL \times v}{t} \text{ kg-m/sec}^2$$

This must be equal to force due to pressure wave = $\Delta p\, A$. Equating the two forces and solving for Δp, we obtain an equation for pressure raise when the valve is closed gradually.

$$\Delta p = \frac{\rho L v}{t} \qquad \ldots(11.23)$$

Pressure head

$$H = \frac{\Delta p}{\gamma} = \frac{\rho L v}{\gamma t}$$

$$= \frac{\rho L v}{\gamma g t} = \frac{L v}{g t} \qquad \ldots(11.24)$$

11.10.1 Instantaneous Closure—Rigid Pipes

An equation for pressure raise due to instantaneous closure of a valve in a *rigid* pipe line can be obtained by equating loss in kinetic energy to gain in strain energy. This based on the law that energy can neither be created nor destroyed but can be converted from one form to the other. Hence if the pipe is entirely rigid the kinetic energy lost now exists as strain energy $\left(\frac{\sigma^2}{2E}\right)$ × volume

Thus kinetic energy = strain energy stored

$$\frac{1}{2} mv^2 = \left[\frac{(\Delta p)^2}{2K}\right] \times A \times L$$

$$\frac{1}{2} (\rho AL)v^2 = \left[\frac{(\Delta p)^2}{2K}\right] \times A \times L$$

$$\frac{1}{2} (\rho)v^2 = \frac{(\Delta p)^2}{2K}$$

$$(\Delta p)^2 = K(\rho)v^2$$

$$\Delta p = v\sqrt{K(\rho)}$$

$$\Delta p = v\sqrt{\frac{K(\rho)^2}{\rho}}$$

$$\Delta p = v(\rho)c \qquad \ldots(11.25)$$

Here $$C = \sqrt{\frac{K}{\rho}}$$

represents the velocity of sound in water ...(11.26)

11.10.2 Gradual Closure—Elastic Pipes

If the material has a definite yield point it indicates that the pipe has elastic properties and hence not totally rigid. In such a case all of the strain energy is not stored in water because the material of the pipe due to its elastic property absorbs some of it by yielding within the elastic limit. Hence the velocity of the pressure wave C in this case is $< \sqrt{\frac{K}{\rho}}$.

For elastic materials such as steel $C = \dfrac{\sqrt{\frac{K}{\rho}}}{\sqrt{1 + \frac{KD}{Et}}}$ ...(11.27)

Pressure raise is given by $\Delta p = \dfrac{\sqrt{\rho\, v^2}}{\sqrt{\dfrac{1}{\dfrac{K+D}{Et}}}}$ ...(11.28)

While solving the problems related to water hammer the following values have been assumed.

Bulk modulus k of water = 2×10^9 GN/m^2.

Young's modulus e for steel = 200×10^9 GN/m^2.

TYPICAL PROBLEMS AND SOLUTIONS

Problem 11.1 *Fuel oil* $\rho = 852$ *kg/m*^{3} *and* $\mu = 0.003$ *Pa.s flows through a 200 mm diameter castiron pipe 30 m long and with an average velocity = 5 m/sec. Determine the energy loss.*

Solution

In all of these problems unless specified we have to first determine R_e to identify whether the flow is laminar or turbulent and then proceed accordingly.

$$R_e = \frac{v\,\rho D}{\mu}$$

$$= \frac{(5)\,(852)\,(0.2)}{0.003} = 284{,}000.$$

Since $R_e > 4000$, the flow is turbulent.

Cast Iron pipes have a value of $\varepsilon = 0.25$. Relative roughness $\dfrac{\varepsilon}{D} = \dfrac{0.25}{200} = 1.25 \times 10^{-3}$. If we now refer to Moody's chart (Appendix-2) by interpolation we obtain the value of $f = 0.022$. We may now use Darcy–Weisbach equation to determine the energy loss as all other parameters are given.

$$h_f = f\left(\frac{L}{d}\right)\frac{v^2}{2g}$$

$$= \frac{0.022\left(\dfrac{30 \times 1000}{200}\right)(5)^2}{2g} = 4.20 \text{ N.m/N}$$

Problem 11.2 *Heavy fuel oil* $\rho = 906$ *kg/m*^{3} *and* $\mu = 0.107$ *Pa.s flows through a 100 mm diameter pipe 50 m long with an average velocity = 2 m/sec. Determine the energy loss.*

Solution

$$R_e = \frac{v\rho D}{\mu}$$

$$= \frac{2 \times 906 \times 0.1}{0.107}$$

$$= 1693.45.$$

Since $R_e < 2000$, the flow is laminar. We can determine the friction factor straight from Eq. (11.4)

$$f = \frac{64}{R_e}$$

$$= \frac{64}{1693.45}$$

$$= 0.037$$

$$h_f = f\left(\frac{L}{d}\right)\frac{v^2}{2g}$$

$$= \frac{0.037\left(\frac{50 \times 1000}{100}\right)(2)^2}{2g}$$

$$= 3.77 \text{ N.m/N}$$

Problem 11.3 *Determine the energy loss if water at 70° C is flowing through a commercial quality steel pipe of 30 mm inside diameter and 10 m long at 8 m/sec. Take* $\nu = 4.1 \times 10^{-7}$ *m*^{2}*/sec and* $\rho = 978$ *kg/m*^{3}.

Solution

$$R_e = \frac{vD}{v}$$

$$= \frac{(8 \times 0.03)}{4.1 \times 10^{-7}}$$

$$= 585365.85$$

Since the flow is turbulent relative roughness must be determined.

Commercial quality steel pipes have a value of $\varepsilon = 0.046$ mm. (Refer Moody's diagram). Relative roughness $\frac{\varepsilon}{D} = \frac{0.046}{30} = 1.533 \times 10^{-3}$. If we now refer to Moody's chart (Appendix-2) by interpolation we obtain the value of $f = 0.024$. We may now use Darcy–Weisbach equation to determine the energy loss as all other parameters are given.

$$h_f = f\left(\frac{L}{d}\right)\frac{v^2}{2g}$$

$$= (0.024)\left(\frac{30 \times 1000}{30}\right)\frac{(8)^2}{2g}$$

$$= 78.28 \text{ N.m/N}$$

Problem 11.4 *Determine the head loss if linseed oil* $\mu = 3.3 \times 10^{-2}$ *N-sec/*m^2 *and* $\rho = 930$ *kg/*m^3 *is flowing through a 150 mm diameter cold drawn seamless steel tube 4m long. The volume flow rate is 150 liters/sec.*

Solution

$$R_e = \frac{v\rho D}{\mu}$$

$$= \frac{(Av\rho D)}{\mu\left(\frac{\pi}{4}D^2\right)}$$

$$R_e = \frac{(4Q\rho)}{\mu(\pi D)}$$

$$R_e = \frac{(4 \times 0.150 \times 930)}{(3.3 \times 10^{-2})(\pi \times 0.150)}$$

$$= 35{,}882$$

Since $R_e < 100{,}000$ we may use the Blasius empirical formula as given in Eq. (11.5.)

$$f = \frac{(0.316)}{R_e^{0.25}}$$

$$= \frac{(0.316)}{35882^{0.25}} = 0.023$$

$$v = \frac{Q}{A}$$

$$= \frac{(0.150)}{0.7854 \times (0.150)^2}$$

$$= 8.48 \text{ m/sec}$$

$$h_f = f\left(\frac{L}{d}\right)\frac{v^2}{2g}$$

$$= (0.023)\left(\frac{4 \times 1000}{150}\right)\frac{(8.48)^2}{2g} = 2.248 \text{ m}$$

Problem 11.5 *Using Darcy's original equation solve Problem 11.4.*

Solution

Let us recall Eqs (11.2), (11.2 a) and (11.2 b)

$$h_f = 4f_1\left(\frac{L}{d}\right)\frac{v^2}{2g}$$

$$f_1 = \frac{16}{R_e}$$

(for R_e = Reynolds's number < 2000)

$$f_1 = \frac{0.079}{R_e^{0.25}}$$

(for R_e = Reynolds's number $4000 - 10^6$)

Since in this case $R_e > 2000$. We use (11.2.b) to determine f_1

$$f_1 = \frac{0.079}{R_e^{0.25}}$$

$$= \frac{0.079}{35882^{0.25}} = 5.74 \times 10^{-3}.$$

$$v = 8.48 \text{ m/sec}$$

$$h_f = 4(5.74 \times 10^{-3})\left(\frac{4 \times 1000}{150}\right)\frac{(8.48)^2}{2g}$$

$$= 2.243 \text{ m}$$

Problem 11.6 *A pump is delivering kerosene from point A to B at normal temperature and at a pressure 6 times the atmospheric pressure at mean sea level. The pump is located 15 m below the point of delivery and the liquid is being conveyed through a plastic hose ID = 60 mm and 300 m long. If Q = 50 LPM, determine the pressure at point A. Take friction factor for plastic hose as = 0.020.*

Solution

Kerosene at normal temperature has the following properties.

$$\rho = 823 \text{ kg/m}^3$$

and $$\mu = 1.64 \times 10^{-3} \text{ N-sec/m}^2$$

Pressure at point $$B = P_B$$

$$= 101 \text{ kPa} \times 6$$

$$= 606 \text{ kPa}$$

Since pressure at A is required to be determined from the pressure at B, we have to use the general energy equation. (Refer Chapter 7)

$$Z_A + \frac{v_A^2}{2g} + \frac{P_A}{\gamma} + h_A - h_C - h_L = Z_B + \frac{v_B^2}{2g} + \frac{P_B}{\gamma}$$

where, h_A, h_C and h_L represent the values for energy added, energy consumed and energy lost, respectively. Since no energy is added or lost between points A and B and since the flow velocity remains the same between the two points, the above equation reduces to the form,

$$Z_A + \frac{P_A}{\gamma} - h_L = Z_B + \frac{P_B}{\gamma}$$

$$\frac{P_A}{\gamma} - \frac{P_B}{\gamma} = (Z_B - Z_A) + h_L$$

$$(P_A - P_B) = \gamma\,[(Z_B - Z_A) + h_L]$$

$$P_A = P_B + \gamma\,[(Z_B - Z_A) + h_L] \qquad \ldots(a)$$

To determine the value of P_A we have to only determine h_L since all other quantities are known or can be determined. Since friction factor f has been specified we need not determine R_e but apply the Darcy–Weisbach equation straight away after determining the flow velocity.

$$Q = \frac{50}{60{,}000}$$

$$= 8.33 \times 10^{-4}\ \text{m}^3/\text{sec}$$

$$v = \frac{Q}{A}$$

$$= \frac{(8.33 \times 10^{-4})}{0.7854 \times (0.060)^2}$$

$$= 0.295\ \text{m/sec}$$

$$h_f = f\left(\frac{L}{d}\right)\frac{v^2}{2g}$$

$$= (0.020)\left(\frac{300 \times 1000}{60}\right)\frac{(0.295)^2}{2g}$$

$$= 0.044\ \text{m}$$

We may now substitute the relevant values in Eq. (a) above

$$P_A = 606 + \left[\frac{(823 \times 9.81)}{1000}\right](15 + 0.044)$$

$$= 727\ \text{kPa}$$

Problem 11.7 *Discharge through a pipe of length 150 m is 50 LPS. The head lost due friction is 3 m and the friction factor is 0.022. Determine the pipe inside diameter.*

Solution

$$h_f = f\left(\frac{L}{d}\right)\frac{v^2}{2g}$$

$$h_f = f\left(\frac{L}{d}\right)\left[\frac{\left(\frac{Q}{A}\right)^2}{2g}\right]$$

$$h_f = f\left(\frac{L}{d}\right)\left[\frac{\left(\frac{Q}{0.7854 \times d^2}\right)^2}{2g}\right]$$

$$h_f = f\left(\frac{L}{d}\right)\left[\frac{\left(\frac{Q^2}{0.7854^2 \times d^4}\right)}{2g}\right]$$

$$d^5 = \left[\frac{f}{h_f}\right](0.0826\, LQ^2) \qquad \ldots(11.29)$$

$$d^5 = \frac{0.022}{3\,(0.0826 \times 150 \times 0.05^2)}$$

$$= 2.27 \times 10^{-4}$$

$$d = 0.18675 \text{ m} = 186.75 \text{ mm}$$

Problem 11.8 *Oil of specific gravity 0.895 and $v = 6.7 \times 10^{-5}$ m^2/sec is flowing through 0.15 m diameter pipe @ 20 LPS. Determine the power required to maintain the flow over a length 500 m.*

Solution

$$R_e = \frac{(4Q\ \rho)}{\mu\ (\pi\ D)}$$

$$= \frac{(4Q)}{v\ (\pi\ D)}$$

$$= \frac{\left[4 \times \frac{20}{1000}\right]}{6.7 \times 10^{-5} \times \pi\ (0.15)}$$

$$R_e = 2533.8$$

Since R_e < 100,000 we may use the Blasius empirical formula as given in (11.5)

$$f = \frac{(0.316)}{R_e^{0.25}}$$

$$= \frac{(0.316)}{2533^{0.25}} = 0.044$$

$$v = \frac{Q}{A}$$

$$= \frac{(0.020)}{0.7854 \times (0.150)^2} = 1.13 \text{ m/sec}$$

$$h_f = f\left(\frac{L}{d}\right)\frac{v^2}{2g}$$

$$= 0.044 \left(\frac{500}{0.150}\right) \frac{(1.13)^2}{2g}$$

$$= 9.545 \text{ N.m/N}$$

Power required for maintaining flow rate

$$P = h_f \times \gamma \times Q$$
$$= (9.545) \times (0.895 \times 9.81 \times 1000) \times 0.020$$
$$= 1676 \text{ N.m/sec}$$
$$P = 1676 \text{ W} = 1.676 \text{ kW}$$

Problem 11.9 *Water supply to a domestic house hold is from an overhead tank 45 m above the tap. If the pipe ID is 20 mm and the friction factor is 0.04 determine the maximum volume flow rate through the tap assuming the pipe is vertical. Neglect separation losses.*

Solution Let us choose two convenient points 1 and 2, 1 being the free surface level of water in the overhead tank and 2 being the discharge point and apply the general energy equation.

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} + h_A - h_C - h_L = Z_2 + \frac{v_2^2}{2g} + \frac{P_2}{\gamma} \qquad \ldots(a)$$

Neglecting both the velocity head and the pressure head at point (1) and the pressure head at point (2) and keeping in mind that there was no addition or consumption of energy between these two points, let us re-write Eq. (a).

$$h_L = (Z_1 - Z_2) - \frac{v_2^2}{2g} \qquad \ldots(b)$$

$$h_f = \left(\quad \right) \frac{v_2^2}{2} \qquad L_f$$

Eliminating h_f and h_L from (b) and (c) by equating the two quantities,

$$(Z_1 - Z_2) + \frac{v_2^2}{2g} = f\left(\frac{L}{d}\right)\frac{v_2^2}{2g}$$

$$(Z_1 - Z_2) = f\left(\frac{L}{d}\right)\frac{v_2^2}{2g} + \frac{v_2^2}{2g}$$

$$(Z_1 - Z_2) = \frac{v_2^2}{2g}\left[f\left(\frac{L}{d}\right) + 1\right]$$

$$45 = \frac{v_2^2}{2g}\left[0.04\left(\frac{45}{0.020}\right) + 1\right]$$

$$45 = \frac{v_2^2}{2g}[91]$$

$$v_2 = 3.115 \text{ m/sec}$$

$$Q_{\max} = A \times v_2$$
$$= 0.7854(0.020)^2 \times 3.115$$
$$= 9.786 \times 10^{-4} \text{ m}^3 \text{/sec} = 58.7 \text{ LPM}$$

(**Note :** 1 m^3/sec = 60,000 LPM)

Problem 11.10 *A circular duct uniformly tapers down to 200 mm from 400 mm over a length of 15 m. If the discharge through the duct is 0.2 m^3/sec, assuming that the pipe is running full and the friction factor is 0.030 for the entire section determine the energy loss.*

Solution

$$h_f = f\left(\frac{L}{d}\right)\frac{v^2}{2g} \qquad \text{.....from (11.1)} \qquad \text{...(a)}$$

$$h_f = f\left(\frac{L}{d}\right)\left[\frac{Q^2}{2g \times a^2}\right]$$

$$h_f = f\left(\frac{L}{d}\right)\frac{16Q^2}{2g \times (\pi^2 d^4)}$$

$$h_f = f\left(\frac{L}{d^5}\right)\frac{16Q^2}{2g \times \pi^2}$$

$$h_f = 0.08263\, f(L)\frac{Q^2}{d^5} \qquad \text{...(b)}$$

Consider a section at a distance X from the end where φ = 400 mm.

Diameter d at section $\quad X = [0.40] - \left[\dfrac{(0.40 - 0.20)}{15}\right] X$

Diameter d at section $\quad X = 0.40 - 0.0133\,X$

Referring to Eq. (b) above,

$$h_{f\text{–}x} = 0.08263\, f\,(dx) \left(\frac{Q^2}{d^5}\right)$$

$$h_{f\text{–}x} = 0.08263(0.030) \left[\frac{(2)^2}{d^5}\right] dx$$

$$h_{f\text{–}x} = (9.9 \times 10^{-5}) \frac{1}{(0.40 - 0.0133\,X)^5}\, dx$$

$$h_{f\text{–}x} = (9.9 \times 10^{-5})\,(0.40 - 0.0133\,X)^{-5}\, dx$$

$$h_f = (9.9 \times 10^{-5}) \int_0^{15} (0.40 - 0.0133\,X)^{-5}\, dx$$

$$h_f = 1.10 \text{ m}$$

Problem 11.11 *Determine the pressure drop across a fully open gate valve fixed to a 150 mm ID commercial quality steel pipe through which oil @ 30 LPS is being discharged. Take sg of oil = 0.85 and* $\upsilon = 6.7 \times 10^{-5}$ *m²/sec.*

Solution

Applying Bernoulli's equation to points upstream and downstream of the gate valve,

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} + h_A - h_C - h_L = Z_2 + \frac{v_2^2}{2g} + \frac{P_2}{\gamma} \qquad \ldots(a)$$

If we neglect Z_1, v_1, h_A, h_C, Z_2 and v_2 as they are irrelevant to the present case, the equation stands modified thus,

$$\frac{P_1}{\gamma} - h_L = \frac{P_2}{\gamma}$$

or $$\frac{P_1}{\gamma} - \frac{P_2}{\gamma} = h_L.$$

or $$(P_1 - P_2) = \gamma\, h_L.$$

Here $(P_1 - P_2)$ represents the required pressure drop and hence we have to determine only the head loss due to the gate valve.

$$h_L = k\left(\frac{v^2}{2g}\right) \quad \ldots(b)$$

From Eq. (11.10), $L_E = \frac{kD}{f}$

or $$k = \frac{fL_E}{D} \quad \ldots(c)$$

$$v = \frac{Q}{A}$$

$$= \frac{\left(\frac{30}{1000}\right)}{0.7854\,(0.150)^2} = 1.7 \text{ m/sec}$$

$$R_e = \frac{vD}{v}$$

$$= \frac{(1.7 \times 0.150)}{6.7 \times 10^{-5}} = 3806$$

Since R_e < 100,000 we may use the Blasius empirical formula as given in Eq. (11.5)

$$f = \frac{(0.316)}{R_e^{0.25}}$$

$$= \frac{(0.316)}{3806^{0.25}} = 0.040$$

Substituting the value f = 0.040 and the value of $\frac{L_E}{D}$ = 8 from Table 11.1 in Eq. (c)

We obtain the value of $k = 0.040 \times 8 = 0.32$

Substituting the value of k in Eq. (b) above we obtain the value of head loss.

$$h_L = 0.32\left(\frac{1.7^2}{2g}\right) = 0.047 \text{ m}$$

But $$(P_1 - P_2) = \gamma h_L$$
$$= (9.81 \times 0.85) \times 0.047$$
$$= 0.39 \text{ kN/m}^2$$

The low value of the pressure drop is indicative of the high efficiency of the gate valve as compared to a globe valve. Theoretically the loss across a gate valve is no more than 2.6% of the loss across a globe valve.

Problem 11.12 *Water is being conveyed through a 0.44 m diameter and 500 m long commercial quality steel pipe. If the head loss due to friction is 6 m and the value of coefficient of resistance k = 12, determine the discharge.*

Solution

For a commercial quality steel pipe ε = 0.046 mm (Reference Moody's diagram)

Therefore $$\frac{\varepsilon}{D} = \frac{0.046}{440}$$

$$= 1.045 \times 10^{-3}$$

By interpolation in the Moody's diagram the corresponding value $f = 0.02$

From Eq. (11.10), $$L_E = \frac{kD}{f}$$

$$= \frac{(12 \times 0.44)}{0.02}$$

$$= 264 \text{ m}$$

$$h_f = f\left(\frac{L}{d}\right)\frac{v^2}{2g} \qquad \text{...from (11.1)}$$

$$6 = 0.02\left[\frac{(L + L_E)}{d}\right]\frac{v^2}{2g}$$

$$6 = 0.02\left[\frac{(500 + 264)}{0.44}\right]\frac{v^2}{2g}$$

$$v = 1.84 \text{ m/sec}$$

$$Q = 0.7854\ (0.44)^2 \times 1.84$$

$$= 0.279 \text{ m}^3\text{/sec}$$

$$= 280 \text{ LPS}$$

Problem 11.13 *Oil of density 850 kg/m^3 is pumped @ 60 LPM through a commercial quality steel pipe of ID = 25 mm and total length 75 m. Determine the total head loss and the consequent pressure drop for the system shown in Fig. 11.10.*

Solution

The total head loss comprises of,

1. Head lost due to pipe wall friction
2. Head lost due to Globe valve, check valve and fittings.

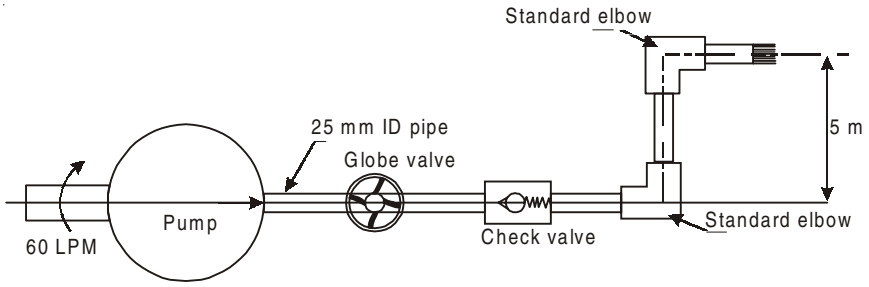


Fig. 11.10

(i) Head loss due to friction.

$$R_e = \frac{v\,\rho\,D}{\mu}$$

$$= \frac{(Av\,\rho\,D)}{\mu\left(\frac{\pi}{4}\right)}D^2.$$

$$R_e = \frac{(4Q\rho)}{\mu(\pi D)}$$

(Q = 60/60,000 = 0.001 m^3/sec)

$$R_e = \frac{(4 \times 0.001 \times 850)}{(2.99 \times 10^{-3}\ )\,(\pi \times 0.025)}$$

$$= 14.478$$

(For oil of density 850 kg/m^3 the dynamic viscosity $\mu = 2.99 \times 10^{-3}$ Pa-s)

$$f_1 = \frac{0.079}{R_e^{0.25}} \qquad \text{(for } R_e = 4000 - 10^6\text{)}$$

Since in this case $R_e > 2000$. We use (11.2 b) to determine f_1

$$f_1 = \frac{0.079}{R_e^{0.25}}$$

$$= \frac{0.079}{14478^{0.25}}$$

$$= 7.20 \times 10^{-3}$$

$$h_f = 4f_1\left(\frac{L}{d}\right)\frac{v^2}{2g}$$

$$[v = \frac{Q}{A} = \frac{0.001}{0.7854\,(0.025)^2} = 2 \text{ m/sec}]$$

$$h_f = 4 \times 7.20 \times 10^{-3} \left(\frac{75}{0.025}\right)\frac{(2)^2}{2g}$$

$$= 17.6 \text{ m}$$

(ii) Head loss – globe valve + Head loss – Check valve + Head loss – Elbow

$$k\,\frac{v^2}{2g} \quad + k\,\frac{v^2}{2g} \quad + k\,\frac{v^2}{2g}$$

$$(340\,f)\frac{v^2}{2g} \quad + (150\,f)\frac{v^2}{2g} \quad + 2(30\,f)\frac{v^2}{2g}$$

(From Table 11.1)

$$(340 \times 7.20 \times 10^{-3})\,\frac{v^2}{2g} + (150 \times 7.20 \times 10^{-3})\,\frac{v^2}{2g} + 2\,(30 \times 7.20 \times 10^{-3})\,\frac{v^2}{2g}$$

$$2.44\left(\frac{v^2}{2g}\right) + 1.08\left(\frac{v^2}{2g}\right) + 0.432\left(\frac{v^2}{2g}\right)$$

$$3.95\left(\frac{v^2}{2g}\right) = \frac{3.95\,(2)^2}{2g}$$

$$= 0.805 \text{ m}$$

$$\text{Total Head loss} = 17.6 + 0.805$$

$$= 18.40 \text{ m}$$

Now to determine the pressure drop we apply general energy equation to points 1 and 2

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} + h_A - h_C - h_L = Z_2 + \frac{v_2^2}{2g} + \frac{P_2}{\gamma} \qquad \ldots(a)$$

Since $(v_1 = v_2)$ and since h_A, h_C, are irrelevant to the present case, the equation stands modified thus,

$$Z_1 + \frac{P_1}{\gamma} - h_L = Z_2 + \frac{P_2}{\gamma}$$

or $$(Z_1 - Z_2) + \frac{(P_1 - P_2)}{\gamma} = h_L.$$

or
$$(P_1 - P_2) = (\gamma)[h_L - (Z_1 - Z_2)]$$
$$= (850 \times 9.81)\,[18.4 - (5)]$$
$$= 111736 \text{ N/m}^2$$

Problem 11.14 *Two reservoirs A and B 150 m apart are connected by a straight steel pipe with an inside diameter = 50 mm. The difference in water surface levels between the two reservoirs is 20 m. If the f = 0.025, determine the discharge.*

Solution

The available head 20 m = loss of head due to friction in pipe + the loss at entrance to the pipe + the loss at exit to the pipe.

$$20 = \frac{f\,L\,v^2}{2gd} + 0.5\left[\frac{v^2}{2g} + \frac{v^2}{2g}\right]$$

.....(Ref. Eq. 11.16)

$$20 = \frac{(0.025)(150)\,v^2}{2g(0.050)} + 0.5\left[\frac{v^2}{2g} + \frac{v^2}{2g}\right]$$

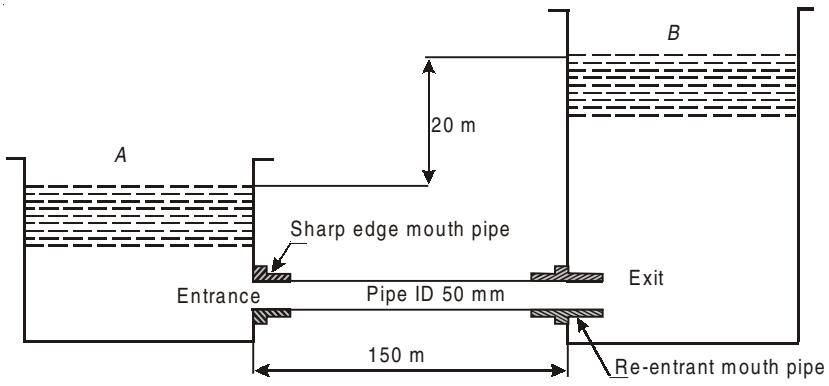


Fig. 11.11

$$20 = \frac{v^2}{2g}\left[\frac{(0.025)(150)}{(0.050)}\right] + 0.5 + 1$$

$$20 = \frac{v^2}{2g}[76.5]$$

$$v = 2.26 \text{ m/sec}$$
$$Q = 0.7854\,(0.050)^2 \times 2.26$$
$$= 4.43 \times 10^{-3} \text{ m/sec}$$

Problem 11.15 *A section of a pipe through which water is flowing @ 0.05 m^3/sec suddenly expands from 150 mm diameter to 250 mm diameter. Determine the head loss due to sudden expansion and the pressure difference between the two sections.*

Solution

Loss due to sudden expansion $h_L = k\left(\frac{v_1^2}{2g}\right)$

The coefficient of Resistance k in case of sudden expansion is determined from the Eq. (11.11)

$$k = \left[1-\left(\frac{A_1}{A_2}\right)\right]^2$$

$$A_1 = 0.7854\,(0.150)^2 = 0.0176\ \text{m}^2.$$

$$A_2 = 0.7854\,(0.250)^2 = 0.0490\ \text{m}^2.$$

$$v_1 = \frac{Q}{A_1}$$

$$= \frac{0.05}{0.017} = 2.94\ \text{m/sec}$$

$$v_2 = \frac{Q}{A_2}$$

$$= \frac{0.05}{0.049} = 1.02\ \text{m/sec}$$

$$k = \left[1-\left(\frac{0.0176}{0.049}\right)\right]^2 = 0.410$$

Therefore,

$$h_L = k\left(\frac{v_1^2}{2g}\right)$$

$$= 0.410\ \frac{(2.94)^2}{2g} = 0.183\ \text{m}$$

The pressure difference between the two sections is obtained through the general energy equation.

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} + h_A - h_C - h_L = Z_2 + \frac{v_2^2}{2g} + \frac{P_2}{\gamma} \qquad \ldots(a)$$

Since the datum heads are equal and since there is no addition or consumption of energy between the two sections, the equation reduces to the form,

$$\frac{v_1^2}{2g} + \frac{P_1}{\gamma} - h_L = \frac{v_2^2}{2g} + \frac{P_2}{\gamma}$$

$$\frac{(P_2 - P_1)}{\gamma} = \frac{\left(v_1^{\,2} - v_2^{\,2}\right)}{2g} - h_L$$

$$\frac{(P_2 - P_1)}{\gamma} = \frac{\left(2.94^2 - 1.02^2\right)}{2g} - 0.183$$

$$\frac{(P_2 - P_1)}{\gamma} = \frac{\left(2.94^2 - 1.02^2\right)}{2g} - 0.183$$

$$= 0.204 \text{ m}$$

$$(P_2 - P_1) = \gamma\,(0.204)$$

$$= 9.81(0.204) = 2 \text{ kPa}$$

Problem 11.16 *Water flowing through a 200 mm diameter pipe experiences sudden contraction to 125 mm diameter. If the discharge is 0.030 m^3/sec and the difference in the pressure head as measured by a water-mercury horizontal differential U-tube manometer is 40 mm, determine the head loss due to contraction.*

Solution

In a horizontal differential U-tube manometer the difference in the gauge fluid level (h) and the difference in pressure heads (P_H) have the following relation. Refer Eq. (5.7)

$$P_H = h\left[\left(\frac{Sgg}{Sgv}\right) - 1\right]$$

$$\frac{(P_1 - P_2)}{\gamma} = h\left[\left(\frac{Sgg}{Sgv}\right) - 1\right]$$

Here *Sgg* and *Sgv* refer to the specific gravity of gauge fluid which in this case is mercury and the specific gravity of the pipe fluid which in this case is water.

$$\frac{(P_1 - P_2)}{\gamma} = 0.040\left[\left(\frac{13.6}{1}\right) - 1\right]$$

$$= 0.504 \text{ m}$$

Applying general energy equation to points 1 and 2 which represent points before and after transition respectively,

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} + h_A - h_C - h_L = Z_2 + \frac{v_2^2}{2g} + \frac{P_2}{\gamma} \quad \ldots(a)$$

$$\frac{(P_1 - P_2)}{\gamma} = \frac{\left(v_2^2 - v_1^2\right)}{2g} + h_L. \quad \ldots(b)$$

$$V_2 = \frac{Q}{A_2}$$

$$= \frac{0.030}{0.7854 \times (0.125)^2}$$

$$= 2.44 \text{ m/sec}$$

$$V_1 = \frac{Q}{A_1}$$

$$= \frac{0.030}{0.7854 \times (0.200)^2}$$

$$= 0.955 \text{ m/sec}$$

Substituting these values in Eq. (b) above,

$$0.504 = \frac{(2.44^2 - 0.955^2)}{2g} + h_L.$$

$$h_L = 0.247 \text{ m}$$

(h_L = represents the loss due to contraction)

Problem 11.17 *For the arrangement shown in Fig. 11.12, determine the volume flow rate through the system. The system fluid is hydraulic oil (sg = 0.880). Resistance coefficient K for the equal elbow = 0.856 and for the reducing elbow K = 0.80.*

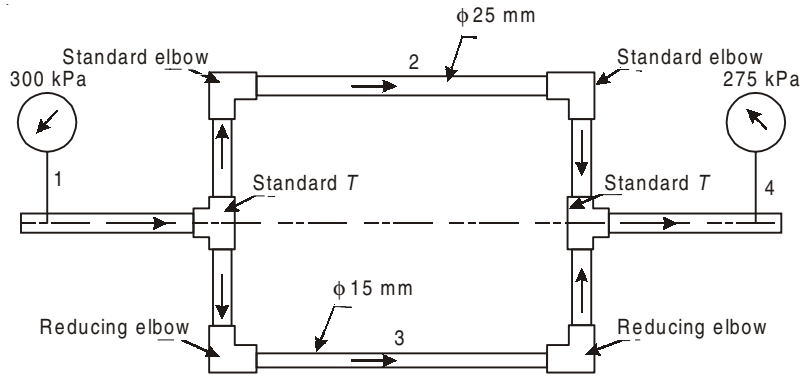


Fig. 11.12

Since the pressure drop between points 1 and 4 is given we can apply the general energy equation to obtain the head loss.

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} + h_A - h_C - h_L = Z_4 + \frac{v_4^2}{2g} + \frac{P_4}{\gamma} \quad \ldots(a)$$

Since $(v_1 = v_4)$, $(Z_1 = Z_4)$ and since h_A and h_C, are irrelevant to the present case, the equation stands modified thus,

$$\frac{P_1}{\gamma} - h_L = \frac{P_4}{\gamma}$$

or $$\frac{(P_1 - P_4)}{\gamma} = h_L$$

or $$\frac{(300 - 275)}{0.880(9.81)} = h_L$$

$$h_L = 2.89 \text{ m}$$

Head lost at stage 2 $= h_2$

$$= 2 K_2 \frac{(v_2)^2}{2g}$$

$$= 2 \times 0.856 \times \frac{(v_2)^2}{2g}$$

$$= 0.0872\ v_2^2$$

Head lost at stage 3 $= h_3$

$$= 2 K_3 \frac{(v_3)^2}{2g}$$

$$= 2 \times 0.80 \times \frac{(v_3)^2}{2g}$$

$$= 0.0815\ v_3^2$$

$$h_L = h_2$$

$$= 0.0872\ v_2^2 = 2.89$$

Hence, $v_2 = 5.75$ m/sec

$$h_L = h_3 = 0.0815\ v_3^2 = 2.89$$

Hence, $v_3 = 5.95$ m/sec

$$Q_2 = A_2 \times v_2$$

$= 0.7854\ (0.0.025)^2 \times 5.75$

$= 0.00282\ m^3/sec$

$Q_3 = A_3 \times v_3$

$= 0.7854\ (0.015)^2 \times 5.95$

$= 0.00105\ m^3/sec$

Flow through the system $Q_4 = Q_2 + Q_3$

$= 0.00282 + 0.00105$

$= 0.00387\ m^3/sec$

Problem 11.18 *Pipes 1 and 2 each of equal length are connected in parallel as shown in Fig. (11.13). If the Darcy–Weisbach friction factor is the same for the two pipes, determine Q_1 and Q_2.*

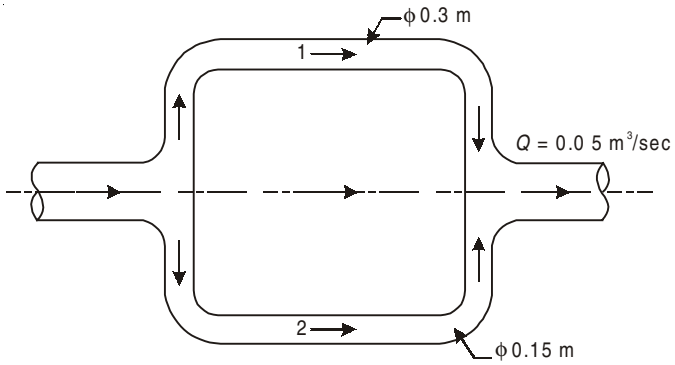


Fig. 11.13

Since head loss $h_L = h_{L-1} = h_{L-2}$,

$$h_L = f\left(\frac{L}{d}\right)\frac{v^2}{2g}$$

$$f_1\left(\frac{L_1}{d_1}\right)\frac{v_1^2}{2g} = f_2\left(\frac{L_2}{d_2}\right)\frac{v_2^2}{2g}$$

Since $L_1 = L_2$ and $f_1 = f_2$,

$$\frac{v_1^2}{d_1} = \frac{v_2^2}{d_2}$$

$$\frac{v_1^2}{v_2^2} = \frac{d_2}{d_1}$$

$$= \frac{0.30}{0.15} = 2$$

$$v_1^2 = 2v_2^2 \quad \text{(a)}$$

Since $Q = Q_1 + Q_2$,

$$0.05 = 0.7854\,(d_1^2)\,(v_1) + 0.7854\,(d_2^2)\,(v_2)$$
$$0.05 = 0.7854\,(0.3^2)\,(v_1) + 0.7854\,(0.15^2)\,(v_2)$$
$$0.05 = 0.070\,(v_1) + 0.0176\,(v_2)$$
$$\frac{[0.05 - 0.070\,(v_1)]}{0.0176} = v_2 \quad \text{...(b)}$$

Substituting this value of v_2 in (a)

$$v_1^2 = 2\left\{\frac{[0.05 - 0.070\,(v_1)]}{0.0176}\right\}^2$$

$$v_1 = \sqrt{2}\left\{\frac{[0.05 - 0.070\,(v_1)]}{0.0176}\right\}$$

$$v_1 = 0.606 \text{ m/sec}$$
$$Q_1 = 0.7854 \times (0.3)^2 \times 0.606$$
$$= 0.042 \text{ m}^3/\text{sec}$$
$$Q_2 = Q - Q_1$$
$$= 0.05 - 0.042$$
$$= 0.008 \text{ m}^3/\text{sec}$$

Problem 11.19 *Two reservoirs with a difference in water surface level = 12 m are connected in parallel by two 75 m long steel pipes of ID 150 mm and 100 mm. If the Darcy–Weisbach friction factor for the two pipes = 0.020 determine the total discharge.*

Solution

The available head of 12 m must be lost in overcoming pipe wall friction. Since head loss is same for both pipes,

Hence
$$12 = f\left(\frac{L_1}{d_1}\right)\frac{v_1^2}{2g}$$

$$= 0.020\left(\frac{75}{0.150}\right)\frac{v_1^2}{2g}$$

$$= 0.51\,v_1^2$$

$$0.51\,v_1^2 = 12$$
$$v_1 = 4.85 \text{ m/sec}$$
$$Q_1 = 0.7854\,(0.150)^2 \times 4.85$$
$$= 0.085 \text{ m}^3/\text{sec}$$

Similarly,

$$12 = f\left(\frac{L_2}{d_2}\right)\frac{v_2^2}{2g}$$

$$= 0.020\left(\frac{75}{0.100}\right)\frac{v_2^2}{2g}$$

$$= 0.76\ v_2^2$$

$$0.76\ v_2^2 = 12$$

$$v_2 = 3.97 \text{ m/sec}$$

$$Q_2 = 0.7854\ (0.100)^2 \times 3.97$$

$$= 0.031 \text{ m}^3\text{/sec}$$

Total discharge $Q = 0.085 + 0.031$

$$= 0.116 \text{ m}^3\text{/sec}$$

Problem 11.20 *Two water reservoirs A and B are connected through a steel pipe. At the mid section C a branch from the main pipe is connected to a third reservoir D. The pressure at C is 90 kPa. The diameters of pipe AC and CB are 0.2 m and 0.25 m respectively. Assuming that the friction factor for all pipes is 0.025 determine the height H of reservoir D. All other details are as given in Fig. (11.14).*

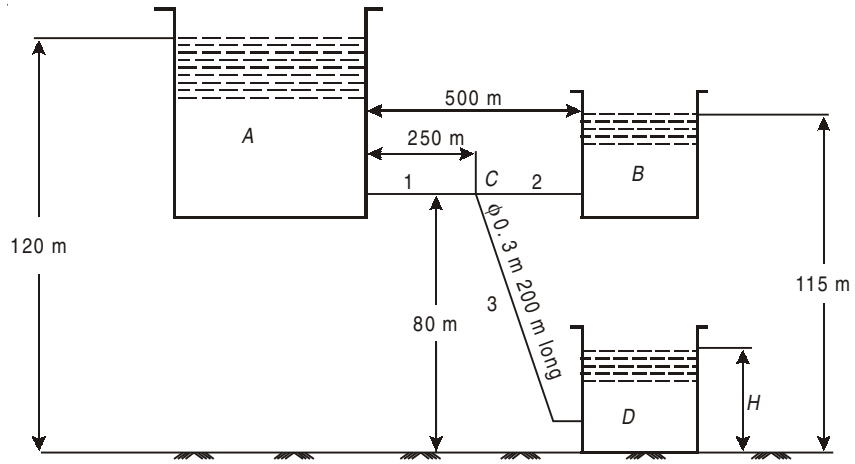


Fig. 11.14

Solution

The piezometric head (P_h) at $C = 80 + \frac{P}{\gamma}$

$$= 80 + \frac{90}{9.81}$$

$$= 89.17 \text{ m of water.}$$

Head loss in pipe 1 (AC) = 120 – 89.17 = 30.83 m

$$30.83 = f\left(\frac{L_1}{d_1}\right)\frac{v_1^2}{2g}$$

$$30.83 = 0.025\left(\frac{250}{0.20}\right)\frac{v_1^2}{2g}$$

$$v_1 = 4.4 \text{ m /sec}$$

Head loss in pipe 2 (CB) = 115 – 89.17

$$= 25.83 \text{ m}$$

$$25.83 = f\left(\frac{L_2}{d_2}\right)\frac{v_2^2}{2g}$$

$$30.83 = 0.025\left(\frac{250}{0.25}\right)\frac{v_2^2}{2g}$$

$$v_2 = 4.9 \text{ m/sec}$$

$$Q_3 = Q_1 + Q_2$$

$$= 0.7854\,(d_1)^2 \times v_1 + 0.7854\,(d_2)^2 \times v_2.$$

$$Q_3 = 0.7854\,(0.20)^2 \times 4.4 + 0.7854\,(0.250)^2 \times 4.9$$

$$Q_3 = 0.378 \text{ m}^3\text{/sec}$$

$$v_3 = \frac{Q_3}{A_3} = \frac{0.378}{0.7854\,(0.30)^2}$$

$$= 5.34 \text{ m/sec}$$

Head loss in pipe 3 (CD) $h_3 = f\left(\frac{L_3}{d_3}\right)\frac{v_3^2}{2g}$

$$h_3 = 0.025\left(\frac{200}{0.30}\right)\frac{(5.34)^2}{2g}$$

$$h_3 = 24.22 \text{ m}$$

Therefore, $H = P_h - h_3$

$$= 89.17 - 24.22 = 64.95 \text{ m}$$

Problem 11.21 *A reservoir discharges freely to atmosphere through a φ 0.2 m × 500 long pipe and a φ 0.3 m ×12 m long expanded pipe. Considering all*

losses determine volume flow rate through the pipe and draw the HGL and the EGL. Friction factor f = 0.020 for the entire pipe. All other dimensions are as per Fig. (11.15).

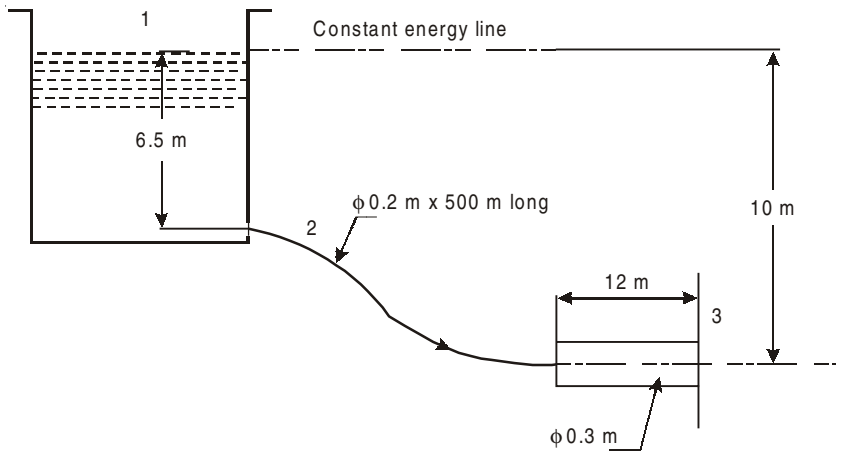


Fig. 11.15

Solution

Applying the energy equation to points 1 and 3

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} + h_A - h_C - h_L = Z_3 + \frac{v_3^2}{2g} + \frac{P_3}{\gamma} \quad \text{... (a)}$$

$$(10) + 0 + 0 + 0 - 0 - h_L = 0 + \frac{v_3^2}{2g} + 0$$

$$10 - \frac{v_3^2}{2g} - h_L = 0$$

$$\frac{v_3^2}{2g} + h_L = 10 \quad \text{...(b)}$$

h_L = Head loss at entrance + frictional loss in pipe (2) + loss due to sudden expansion + frictional loss in pipe 3

$$h_L = (h_E) + (h_{f-2}) + (h_{SE}) + (h_{f-3}) \quad \text{...(c)}$$

$$h_L = 0.5\left(\frac{v_2^2}{2g}\right) + f\left(\frac{L_2}{d_2}\right)\frac{v_2^2}{2g} + \frac{(v_2 - v_3)^2}{2g} + f\left(\frac{L_3}{d_3}\right)\frac{v_2^2}{2g} \quad \text{...(d)}$$

From continuity equation $Q_2 = Q_3$

$$A_2 v_2 = A_3 v_3$$

Hence $$v_2 = \frac{(A_3\ v_3)}{A_2}$$

$$= \left(\frac{D_3}{D_2}\right)^2 \times v_3$$

$$= \left(\frac{0.3}{0.2}\right)^2 \times v_3 = 2.25\ v_3\ .$$

$$v_2 = 2.25\ v_3 \qquad \text{... (e)}$$

From (c) and (d)

$$h_E = 0.5\left(\frac{v_2^2}{2g}\right)$$

$$= 0.5\ \frac{(2.25\ v_3)^2}{2g} = 0.129\ (v_3)^2 \qquad \text{... (f)}$$

$$h_{f-2} = f\left(\frac{L_2}{d_2}\right)\frac{v_2^2}{2g}$$

$$= 0.02\left(\frac{500}{0.2}\right)\frac{(2.25\ v_3)^2}{2g}$$

$$= 12.90\ (v_3)^2 \qquad \text{...(g)}$$

$$h_{SE} = \frac{(v_2 - v_3)^2}{2g}$$

$$= \frac{(2.25 v_3 - v_3)^2}{2g}$$

$$= \frac{(1.25 v_3)^2}{2g} = 0.08\ (v_3)^2 \qquad \text{...(h)}$$

$$h_{f-3} = f\left(\frac{L_3}{d_3}\right)\frac{v_2^2}{2g}$$

$$= 0.02\left(\frac{12}{0.3}\right)\frac{(2.25\ v_3)^2}{2g}$$

$$= 0.206\ (v_3)^2 \qquad \text{...(i)}$$

Substituting these values in (c)

$$h_L = 0.129\,(v_3)^2 + 12.90\,(v_3)^2 + 0.08\,(v_3)^2 + 0.206\,(v_3)^2$$

$$h_L = 13.315(v_3)^2 \quad \ldots(j)$$

Substituting this value of (h_L) in Eq. (b)

$$\frac{v_3^2}{2g} + 13.315\,v_3^2 = 10$$

$$0.050\,v_3^2 + 13.315\,v_3^2 = 10$$

$$v_3 = 0.86 \text{ m/sec}$$

$$Q = A_3\,v_3$$

$$= 0.7854(0.3)^2 \times 0.86$$

$$= 0.060 \text{ m}^3/\text{sec}$$

HGL and EGL (Refer Fig. 11.16)

From (f) $h_E = 0.129\,(v_3)^2$

$= 0.129\,(0.86)^2$

$= 0.095$ m

From (g) $h_{f-2} = 12.90\,(v_3)^2$

$= 12.90\,(0.86)^2$

$= 9.5$ m

From (h) $h_{SE} = 0.08\,(v_3)^2$

$= 0.08\,(0.86)^2$

$= 0.059$ m

From (i) $h_{f-3} = 0.206\,(v_3)^2$

$= 0.206\,(0.86)^2$

$= 0.152$ m

Problem 11.22 *Water flowing through 125 mm diameter pipe experiences sudden expansion to 200 mm diameter. At the transition point hydraulic gradient raises by 15 mm. Determine the discharge.*

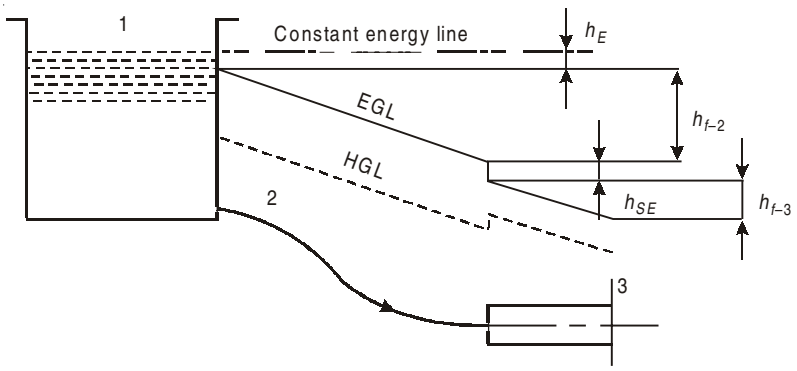


Fig. 11.16

Solution

Hydraulic gradient between any two points 1 and 2 is the difference in piezometric head and is determined by,

$$\left[Z_2 + \left(\frac{P_2}{\gamma}\right)\right] - \left[Z_1 + \frac{P_1}{\gamma}\right]$$

$$\left[Z_2 + \left(\frac{P_2}{\gamma}\right)\right] - \left[Z_1 + \frac{P_1}{\gamma}\right] = \frac{15}{1000}$$

$$= 0.015 \text{ m} \quad \ldots(a)$$

Since $Q_1 = Q_2$

$$A_1 v_1 = A_2 v_2$$

osr $$v_1 = \left(\frac{D_2}{D_1}\right)^2 \times v_2$$

$$= \left(\frac{0.200}{0.125}\right) v_2$$

$$= 1.6\, v_2$$

$$v_1 = 1.6\, v_2 \quad \ldots(b)$$

By considering the change in momentum before and after sudden expansion and by the continuity equation it can be shown that

$$h_E = \frac{(V_1 - V_2)^2}{2g} \quad \ldots(c)$$

$$h_E = \frac{(1.6V_1 - V_2)^2}{2g}$$

$$= \frac{(1.6V_2)^2}{2g}$$

$$= (0.36)\, \frac{(V_2)^2}{2g}$$

Applying energy equation to points 1 and 2

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} + h_A - h_C - h_L = Z_2 + \frac{v_2^2}{2g} + \frac{P_2}{\gamma}$$

$$\frac{(1.6v_2)^2}{2g} - \frac{(0.36)\, V_2^2}{2g} - \frac{V_2^2}{2g} = \left(Z_2 + \frac{P_2}{\gamma}\right) - \left(Z_1 + \frac{P_1}{\gamma}\right)$$

$$1.2 \left(\frac{v_2^2}{2g}\right) = \left(Z_2 + \frac{P_2}{\gamma}\right) - \left(Z_1 + \frac{P_1}{\gamma}\right)$$

$$1.2\left(\frac{v_2^2}{2g}\right) = 0.015 \quad \text{... from (a) above}$$

$$V_2 = 0.495 \text{ m/sec}$$

$$Q = Q_1 = Q_2$$

$$= 0.7854\ (0.200)^2 \times 0.495$$

$$= 0.0155 \text{ m}^3\text{/sec}$$

Problem 11.23 *A 200 mm diameter pipe line that connects two reservoirs passes over a hillock as shown in Fig. (11.17). The difference in surface water levels between the two reservoirs is 10 m and the total length of the pipe is 1000 m and the length of the pipe line from the first reservoir to the vertex of the siphon is 400 mm. Taking all losses into consideration and assuming that the vertex of the siphon lies 6 m above the hydraulic gradient line determine the discharge and the height of the vertex from the water surface level of the first reservoir. f = 0.030.*

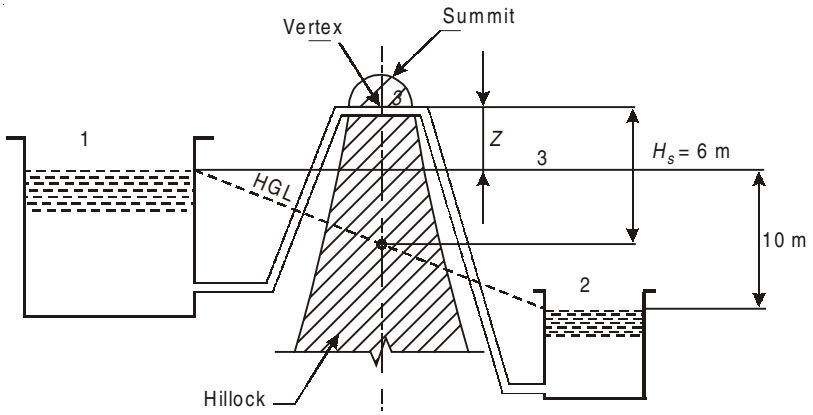


Fig. 11.17

Solution

Applying the energy equation to points 1 and 2

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} + h_A - h_C - h_L = Z_2 + \frac{v_2^2}{2g} + \frac{P_2}{\gamma}$$

Values irrelevant for this problem are scored off and the equation reduces to the form,

$$(Z_1 - Z_2) - h_L = \frac{v_2^2}{2g}$$

Since the pipe is of uniform diameter velocity is constant throughout the length and $v_1 = v_2 = v$

$$(10) - [\text{loss at entrance + pipe friction loss + exit loss}] = \frac{v^2}{2g}$$

$$(10) - \left[0.5\frac{v^2}{2g} + f\left(\frac{L}{d}\right)\frac{v^2}{2g} + \frac{v^2}{2g}\right] = \frac{v^2}{2g}$$

$$(10) - \frac{v^2}{2g}\left[0.5 + 0.030\left(\frac{1000}{0.2}\right) + 1\right] = \frac{v^2}{2g}$$

$$(10) - \frac{v^2}{2g}\ [151.5] = \frac{v^2}{2g}$$

$$152.5\ \frac{v^2}{2g} = 10$$

or $v = 1.14$ m/sec

Hence $Q = 0.7854\ (0.200)^2 \times 1.14$

$= 0.035\ m^3/sec$

Again applying the energy equation to points 1 and 3 we can obtain the height of the vertex (z_3) from the free water surface in the 1st reservoir.

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} + h_A - h_C - h_L = Z_2 + \frac{v_2^2}{2g} + \frac{P_2}{\gamma}$$

Here again values irrelevant for this problem are scored off and the equation reduces to the form,

$$\frac{P_1}{\gamma} - h_L = Z_2 + \frac{v_1^2}{2g} + \frac{P_2}{\gamma}$$

$$\frac{P_1}{\gamma} - h_L = Z_2 + \frac{v_1^2}{2g} + \left[\left(\frac{P_1}{\gamma}\right) - h_S\right]$$

$-h_S$ represents the siphon head given to be equal to 6 m of water and since it is negative pressure, negative sign is prefixed.

$$\frac{P_1}{\gamma} - h_L = Z_2 + \frac{v_2^2}{2g} + \left(\frac{P_1}{\gamma}\right) - 6$$

$$-h_L = Z_2 + \frac{v_2^2}{2g} - 6$$

$$-Z_2 = \frac{v_2^2}{2g} - 6 + h_L.$$

$-Z_2 = \frac{v_2^2}{2g} - 6$ + [loss at entrance + pipe friction loss between reservoir 1 and the vertex]

$$-Z_2 = \frac{v^2}{2g} - 6 + \left[0.5\frac{v^2}{2g} + f\left(\frac{L}{d}\right)\frac{v^2}{2g}\right]$$

(Since $v_1 = v_2 = v$)

$$-Z_2 = \frac{v^2}{2g} - 6 + \left[0.5\frac{v^2}{2g} + 0.03\left(\frac{400}{0.2}\right)\frac{v^2}{2g}\right]$$

$$-Z_2 = \frac{v^2}{2g} - 6 + 60.5\left[\frac{v^2}{2g}\right]$$

$$-Z_2 = 61.5\left[\frac{v^2}{2g}\right] - 6$$

$$= 61.5\left[\frac{1.14^2}{2g}\right] - 6$$

$$= 4.07 - 6$$

$$= -1.92 \text{ m}$$

$$Z_2 = 1.92 \text{ m}$$

Problem 11.24 *Water at 10° C enters a pipe section at a pressure of 5 × 10^6 Pa. The pressure drop measured at the point of exit is 10^6 Pa. If the power transmitted is equivalent to 100 kW, determine the efficiency of power transmission and the discharge.*

Solution

From Eq. (11.19)

$$\text{The efficiency of transmission } \eta = \frac{[H - h_L]}{H}$$

$$H = \frac{P}{\gamma} = \frac{(5 \times 10^6)}{9810} = 509.7 \text{ m}$$

$$h_L = \frac{P_D}{\gamma}$$

$$= \frac{10^6}{9810} = 102 \text{ m}$$

$$\text{Therefore} \qquad \eta = \frac{[509.7 - 102]}{509.7}$$

$$= 0.79 \text{ or } 79\%$$

Discharge could be obtained from Eq. (11.18)

$$P_T = \gamma Q\,[H - h_L]$$

$$100 = (9.81)\,Q\,[509.7 - 102]$$

$$Q = 0.025 \text{ m}^3/\text{sec}$$

Problem 11.25 *A hydraulic pump whose delivery line is 3 m below the supply tank delivers water @ 100 LPM through a nozzle incorporated smooth*

plastic hose to an irrigation facility located 15 m above. The nozzle at the end of the hose requires 150 kPa to operate effectively. If the ID of the hose is 30 mm, determine the power delivered by the pump and the pressure at the outlet to the pump. Neglect losses in the pump suction line and pressure lines.

Solution

Applying energy equation to points 1 and 2 (Ref. Fig. 11.18)

$$Z_1 + \frac{v_1^2}{2g} + \frac{P_1}{\gamma} + h_A - h_C - h_L = Z_2 + \frac{v_2^2}{2g} + \frac{P_2}{\gamma}$$

After neglecting the irrelevant factors the equation reduces to the form,

$$h_A = \frac{v_2^2}{2g} + \frac{P_2}{\gamma} + (Z_2 - Z_1)$$

$$v_2 = \frac{Q}{A_2}$$

$$= \frac{\left(\frac{100}{60{,}000}\right)}{0.7854\,(0.030)^2} = 2.35 \text{ m/sec}$$

$$h_A = \frac{(2.35)^2}{2g} + \frac{P_2}{\gamma} + (-5)$$

or

$$h_A = \frac{P_2}{\gamma} - 4.71 \qquad \ldots(a)$$

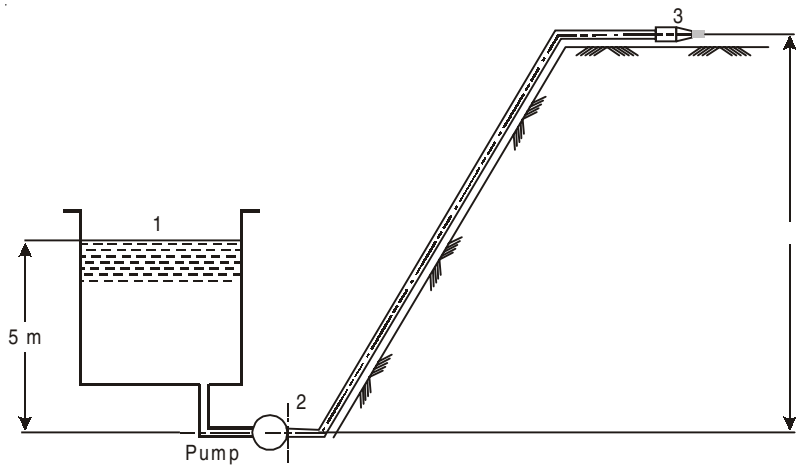


Fig. 11.18

Applying energy equation to points 2 and 3

$$Z_2 + \frac{v_2^2}{2g} + \frac{P_2}{\gamma} + h_A - h_C - h_L = Z_3 + \frac{v_3^2}{2g} + \frac{P_3}{\gamma}$$

After neglecting the irrelevant factors the equation reduces to the form,

$$Z_2 + \frac{P_2}{\gamma} - h_L = Z_3 + \frac{P_3}{\gamma}$$

$$\frac{P_2}{\gamma} = (Z_3 - Z_2) + \frac{P_3}{\gamma}$$

$$P_2 = \gamma (Z_3 - Z_2) + P_3$$

$$P_2 = 9.81\,(15) + 150$$

$$= 297 \text{ kPa}$$

From (a) above,

$$h_A = \frac{P_2}{\gamma} - 4.71$$

$$h_A = \frac{297}{9.81} - 4.71$$

$$= 25.56 \text{ m}$$

Power delivered by the pump $= \gamma Q h_A$

$$= 9.81\left(\frac{100}{60{,}000}\right)(25.56)$$

$$= 0.42 \text{ kW}$$

Problem 11.26 *A nozzle of φ = 90 mm is fitted to the end of a pipe line 100 m long. The vertical distance from the center of the nozzle tip to the free water surface in the reservoir is 12 m. If the diameter of the pipe is 200 mm and f = 0.020 determine the discharge and the transmission efficiency. Neglect minor losses.*

Solution

From Eq. (11.20)

$$v_2 = \sqrt{\frac{gH}{1 + f\left(\frac{L}{D_1}\right)\left(\frac{A_2}{A_1}\right)^2}}$$

$$v_2 = \sqrt{\frac{g\ (12)}{1 + 0.020\left(\frac{100}{0.20}\right)\left(\frac{90^2}{200^2}\right)^2}}$$

$$v_2 = 9.13 \text{ m/sec}$$

$$Q = 0.7854\,(0.09)^2 \times 9.13$$

$$Q = 0.058 \text{ m}^3/\text{sec}$$

From Eq. (11.21)

$$\text{Transmission efficiency} = \quad \eta = \frac{\left[\left(\frac{1}{2}\right) m v_2^2\right]}{\gamma Q H}$$

$$m = \text{mass flow rate} = \rho Q$$

$$\text{Hence} \qquad \eta = \frac{\left[\left(\frac{1}{2}\right) \rho Q v_2^2\right]}{\gamma Q H}$$

$$\eta = \frac{\left[\left(\frac{1}{2}\right) \rho v_2^2\right]}{\gamma H}$$

$$\eta = \frac{\left[\left(\frac{1}{2}\right) 1000 \times (9.13)^2\right]}{(9810)\,(12)}$$

$$= 0.35 = 35\%$$

Problem 11.27 *A pipe line connecting a supply reservoir has an ID = 400 mm and is 500 m long. The center of the pipe is 20 m below the free water level of the reservoir. Determine the size of the nozzle to obtain maximum efficiency. f = 0.020.*

Solution

From Eq. (11.22) for maximum efficiency

$$\frac{A_1}{A_2} = \frac{(2 f L)}{D_1}$$

$$\frac{A_1}{A_2} = \frac{(2 \times 0.020 \times 500)}{0.40}$$

$$\frac{A_1}{A_2} = 7.07$$

$$A_2 = \frac{A_1}{7.07}$$

$$= \frac{0.7854\,(0.400)^2}{7.07} = 0.0177 \text{ m}^2$$

$$\text{Hence diameter of the nozzle } D_2 = \sqrt{\frac{-0.0177}{0.7854}} = 0.150 \text{ m} = 150 \text{ mm}$$

Problem 11.28 *The discharge through a rigid penstock pipe of ID 800 mm × 2500 m long is 2 m^3/sec. Pipe wall thickness = 15 mm. If the valve controlling the discharge at the delivery end is closed in 3 seconds determine the surge pressure due to water hammer. What would be the raise in pressure if the valve is closed in 5 sec?*

Solution

From Eq. (11.25) the pressure raise as applicable to RIGID pipes is given by,

$$\Delta p = v(\rho)c$$

where v is the flow velocity $= \dfrac{2}{0.7854\,(0.8)^2}$

$$= 3.97 \text{ m/sec}$$

From (11.26) velocity of pressure wave $C = \sqrt{\dfrac{K}{\rho}}$

K = Bulk Modulus of water assumed
$= 2 \text{ GN/m}^2$
$= 2 \times 10^9 \text{ N/m}^2$
ρ = Mass density of water
$= 1000 \text{ kg /m}^3$

$$C = \sqrt{\frac{2 \times 10^9}{1000}}$$

$$= 1414 \text{ m/sec}$$

Time period $t = \left[\dfrac{2L}{C}\right]$

$$= \left[\frac{2 \times 2500}{1414}\right] = 3.53 \text{ sec}$$

Since it is given that the valve is closed in 3 sec which is less than 3.53 sec, it can be considered as rapid closure.

Pressure raise $\Delta p = v(\rho)c$
$= 3.97 \times 1000 \times 1142$
$= 4533.74 \text{ kN/m}^2$

If the valve is closed in 5 seconds then it can be considered as gradual closure since the time of closure is > 3.53 sec.

From Eq. (11.23) the pressure raise is,

$$p = \frac{\rho L\, v}{t}$$

$$= \frac{(1000 \times 2500 \times 3.97)}{5}$$

$$= 1985 \text{ kN/m}^2.$$

Problem 11.29 *A valve controlling a steel penstock pipe 2500 m long closes in 3 seconds. The Pipe ID = φ 500 mm and the flow velocity = 3 m/sec. If the thickness of the pipe is 20 mm, determine the raise in pressure.*

Solution

We have to first determine whether the closure of the valve was sudden or gradual.

The condition for gradual closure is $t > \left[\frac{2L}{C}\right] = \frac{(2 \times 2500)}{C}$ ...(a)

The velocity of pressure wave C for elastic pipe is given by Eq. (11.27)

$$C = \frac{\sqrt{\frac{K}{\rho}}}{\sqrt{1+\frac{KD}{Et}}} = \frac{\sqrt{\frac{(2 \times 10^9)}{1000}}}{\sqrt{1+\frac{2\,(500)}{200\,(20)}}}$$

$$= \frac{1414.2}{1.118} = 1265$$

Substituting the value of C in Eq. (a) above,

$$t > \left[\frac{2L}{C}\right] = \frac{(2 \times 2500)}{1265}$$

$$= 3.95 \text{ sec.}$$

Since the valve closes in 3 sec the closure is instantaneous.

The raise in pressure for sudden closure of elastic pipes is given by equation (11.29),

$$\Delta p = \frac{\sqrt{\rho\, v^2}}{\sqrt{\left[\frac{1}{\left(K+\frac{D}{Et}\right)}\right]}} = \frac{\sqrt{1000 \times 3^2}}{\sqrt{\left[\frac{1}{(2\times10^9)+\frac{500}{(200\times10^9)\,20}}\right]}}$$

$$= 4373 \text{ kN/m}^2$$

Problem 11.30 *The valve of a steel penstock was so manipulated as to cause a reduction in flow from 4 m^3/sec to 1.5 m^3/sec within a time period of 5 sec. If the length of the penstock is 3000 m, OD = 700 mm and ID = 660 mm determine the raise in pressure.*

Solution

Time period of Pulsation $t = \left[\frac{2L}{C}\right]$ ...(a)

The velocity of pressure wave C for elastic pipe is given by Eq. (11.27)

$$C = \frac{\sqrt{\frac{K}{\rho}}}{\sqrt{1+\frac{KD}{Et}}} = \frac{\sqrt{\frac{(2\times 10^9)}{1000}}}{\sqrt{1+\frac{2\,(660)}{200\,(20)}}}$$

$$= \frac{1414.2}{1.153} = 1226.5$$

Substituting the value of C in Eq. (a) above,

$$t = \left[\frac{2\times 3000}{1226.5}\right] = 4.89 \text{ sec.}$$

Since the valve closes in 5 sec it is gradual closure.

From Eq. (11.23) the pressure raise for gradual closure

$$\Delta p = \frac{\rho L v}{t}$$

$$v = \frac{Q}{A}$$

$$= \frac{(4-1.5)}{0.7854(0.660)^2} = 7.3 \text{ m /sec}$$

$$\Delta p = \frac{1000\times 3000\times 7.3}{5}$$

$$= 4380 \text{ kN /m}^2$$

QUESTIONS BANK

Section A

Review Questions

1. Differentiate between pipes connected in series and pipes connected in parallel.
2. Why are minor losses so called? When would these losses come to be regarded as major losses ? Name all the minor losses you are familiar with.

3. Name all the factors that contribute to loss of head in a closed conduit in the order of their importance and explain the methods employed to reduce these losses.
4. Define an equivalent pipe. How do you determine the size of an equivalent pipe?
5. Write down the Darcy–Weisbach equation for pressure lost and head lost due to friction in circular pipes and state the limiting condition for applying it to laminar flows.
6. Minor losses are $= K\left(\frac{v^2}{2g}\right)$ where the value of K depends on the kind of fittings or valves used. In the absence of this value of K what is the alternative method employed to determine the minor losses?
7. Distinguish between Energy gradient and hydraulic gradient lines and what does the vertical intercept at any point between these two lines represent?
8. Define a siphon and what is the theoretical limitation to the siphon head?
9. Derive an expression for power transmitted through a pipe and state the condition for maximum efficiency of transmission.
10. Differentiate between a nozzle and a diffuser and state their uses.
11. State the cause for water hammer in long pipe lines pipes. State the effect of water hammer and the methods adopted to overcome the effects.
12. Intensity of pressure wave produced in case of gradual closure of valve in a rigid pipe is given by $\left(\frac{\rho L v}{t}\right)$. This implies that the intensity of pressure should rise to infinity in case of instantaneous closure since $t = 0$. Explain why this is not so in actual practice.

Section B

Multiple Choice Questions

Choose the right answer

1. If R_E is < 2000 then right equation for determination of head lost is the
   (a) Chezy's formula
   (b) Hagen–Poiseuille equation
   (c) Darcy–Weisbach equation
   (d) Any of these
2. If R_E is > 4000 and < 100000 then friction factor can be determined from
   (a) $f = \frac{16}{R_E}$ (b) $f = \frac{(0.316)}{R_E^{0.25}}$

(c) $f = \frac{64}{R_E}$ (d) Any of these

3. Friction factor depends on
   (a) flow velocity
   (b) pipe diameter
   (c) length of the conduit
   (d) relative roughness of the pipe
   (e) All of these
4. Aging of a closed conduit is indicated by
   (a) cracks in the pipe wall
   (b) corrosion of pipe wall
   (c) increase in relative roughness of pipe walls
   (d) Any of these
5. Two pipes are considered equivalent if their
   (a) diameters and lengths are equal
   (b) flow through them are equal
   (c) friction factor and diameters are equal
   (d) All of these
6. Loss of head due to sudden expansion is determined from the flow velocity
   (a) upstream
   (b) downstream
   (c) average of these two velocities
   (d) None of these
7. Loss of head at exit is given by

   (a) $0.5 \frac{v^2}{2g}$ (b) $\frac{v^2}{2g}$

   (c) $\frac{v}{2g}$ (d) None of these
8. Which among these has a higher value of resistant coefficient
   (a) ball check valve (b) fully open globe valve
   (c) fully open gate valve (d) a standard elbow
9. The closure of a valve is considered rapid if

   (a) $t > \frac{2L}{C}$ (b) $t < \frac{2L}{C}$

   (c) $t = \frac{2L}{C}$ (d) None of these

10. Which among these are considered un-related factors while determining the pressure rise due to rapid closure of a valve in elastic pipes.
    (a) Viscosity of the fluid
    (b) Relative roughness of the pipe
    (c) Density of the fluid
    (d) Bulk Modulus of the fluid
    (e) Pipe wall thickness
    (f) All of these

Section C
Problems

1. Glycerin is flowing through a pipe of diameter 150 mm × 500 m long @ 2 m/sec. Determine the head lost due to friction. $\rho = 1258$ kg/m^3 and $\mu = 0.95$ Pa.s. (*Ans.* 21.7 m)
2. The flow rate through a pipe of ID = 70 mm × 300 m long pipe is 0.0025 m^3/sec. If the difference in elevation between the inlet and the outlet is 25 m, the outlet being at a higher level than the inlet, determine the pressure at the inlet if the outlet pressure is 750 kPa. $f = 0.020$, $\gamma = 8.45$ kN/m^3. (*Ans.* 977 kPa)
3. Water is being supplied from a reservoir to a village located on the other side of a hillock. The supply pipe line at its highest point is 2 m above the free water surface in the reservoir and the discharge point is 5 m below. The ID of the pipe is 125 mm and its total length is 20 m. The length of the pipe up to the highest point is 8 m. If the Darcy's friction factor for the pipe is 0.05 and the entrance to the pipe is sharp determine the velocity at the exit and the pressure at the pressure at the highest point. (*Ans.* $v = 2.93$ m /sec and $P_H = -25.8$ kN/m^2 vacuum)
4. Discharge through a 100 mm diameter pipe is 8.33×10^{-3} m^3/sec. The diameter suddenly expands to 175 mm and the pressure just before the transition is 30 kPa. Assuming the pipe to be horizontal, determine the pressure immediately after transition. (*Ans.* 30. 25 kPa)
5. Two reservoirs with a difference in head = 20 m are connected by two pipes of equal length but of different diameters in series. The total length of the pipe line is 1000 m and the diameters are 125 mm and 200 mm and the transition is abrupt. Darcy's coefficient f for the smaller pipe = 0.020 and for the bigger $f = 0.030$. Determine the discharge. Consider entrance and exit losses too. (*Ans.* $Q = 0.026$ m^3/sec)
6. Derive an expression for the limiting nozzle diameter for maximum power transmission. A few parallel pipes each 80 mm in diameter have

to transmit 300 kW of power through a 4000 m length of pipe lines. A pump at the outlet maintains a continuous pressure of 7000 kPa at 90% efficiency. Determine the number of pipes required. $f = 0.025$.

(*Ans.* 4 pipes)

7. A pump delivers water @ 50 LPS to an overhead tank through φ 160 mm × 500 m long pipe line. The exit to the tank is located at a height of 25 m from the pump delivery line. If the Darcy–Weisbach friction factor = 0.02 determine the power required to drive the pump. Assume 90% efficiency for the pump. (*Ans.* 10.72 kW)

Answers to Multiple Choice Questions

1. (b and c)	**2.** (b)	**3.** (e)	**4.** (c)	**5.** (c)
6. (a)	**7.** (b)	**8.** (b)	**9.** (b)	**10.** (a, b)

Chapter

12

Flow Through Open Channels

In closed conduit flow the fluid is completely enclosed by the boundary whereas in open channel flow the top surface of the fluid is invariably exposed to atmosphere. A circular pipe running half full could be considered as an open channel since the portion above the top surface is exposed to atmosphere. Storm water drains, underground sewage lines, flow of rivers, canals or man made channels to divert water from a river to a desired location for irrigation purposes are all examples of open channel flow.

The force causing motion in open channel flow is gravity caused due to variation in the datum along the flow path whereas in closed conduit flow the force causing motion is the pressure head. In closed conduit flow the cross section of the conduit is generally circular and rarely if ever square. Open channels could have any cross section. Surface roughness of the wetted perimeter is important to limit the head lost due to friction but in open channel flow the values vary over a wide limit. Unlike in closed conduit flow, in open channel flow the hydraulic gradient line coincides with top open surface of the liquid. The piezometric head is $(z + d)$, where d is the depth of flow. Piezometric head in a closed conduit at any cross section is $\left(\frac{z+p}{\gamma}\right)$. The velocity profile in closed conduit flow is symmetrical about the pipe axis, the maximum velocity occurring at the centre of the pipe. In open channel flow since the maximum velocity occurs a little below the top surface the velocity profile at any cross section is asymmetrical. Unlike in closed conduit flow the cross section is not controlled by fixed boundaries since the width or depth at any cross section can vary depending on the terrain.

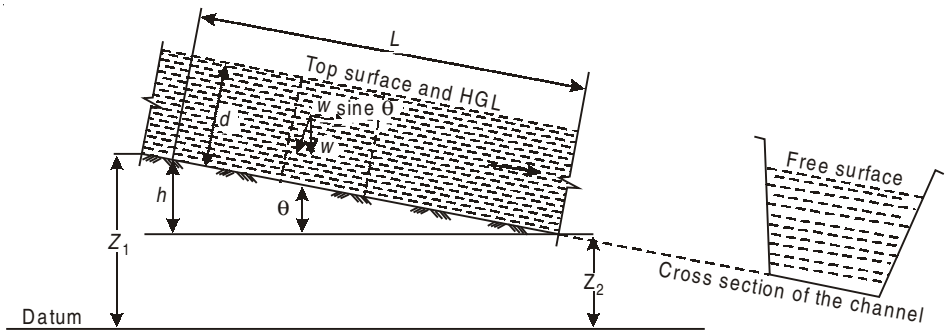


Fig. 12.1

Note: The top surface coincides with the Hydraulic gradient line.

The slope of the channel that is, the angle θ that the channel bottom surface makes with the datum (Fig 12.1) and a component of the weight of the fluid along the flow path together impart the fluid its flow velocity.

Reynolds number for open channel flow is based on the hydraulic radius R, which is a ratio of the area of cross section of the flow stream to the wetted perimeter $R = \frac{A}{WP}$. Thus Reynolds number for open channel flow is given by

$$R_E = \frac{vR}{v} \qquad \ldots(12.1)$$

v is the mean flow velocity, R is the hydraulic radius and v is the kinematic viscosity.

In open channel flow the upper limit for laminar flow is $R_E < 500$ and for turbulent flow the lower limit is $R_E > 2000$. In the transition zone (500 to 2000) the behaviour of the fluid is so unpredictable that no meaningful characterization is possible.

In addition to Reynolds number another characteristic equation used in open channel flow is the *Froude number* (F_R) which relates inertial forces to gravity. Froude number can be used to characterize flows as *sub-critical*, *critical* or *super-critical*.

$$F_R = \frac{v}{\sqrt{gD_H}} \qquad \ldots(12.2)$$

D_H is the hydraulic depth which is a ratio of the area of the channel to the breadth of the free surface of the fluid in the channel. Thus $D_H = \frac{A}{b}$.

When $F_R = 1$, that is when $v = \sqrt{gD}$ the flow is described as critical, when $F_R < 1$, the flow is considered sub-critical or tranquil and when $F_R > 1$, the flow is described as super-critical or torrential.

Flow through an open channel could be laminar or turbulent, steady or unsteady, uniform or non uniform, critical or super critical and could change from one form to the other section to section.

Uniform steady flow is possible only if the cross section of the channel and the depth of the channel remains constant within the section of interest. The section of such a channel is termed Prismatic. *In other words uniform and steady flow is possible only in prismatic channels.*

If the depth of the channel changes along the section of interest but the volume flow rate or the discharge remains constant then the flow is described as *varied but steady flow*.

If in addition to depth the discharge also varies with time then the flow is described as *varied and unsteady.*

The flow could vary either gradually or rapidly depending upon the rate of change of depth along the length of the channel.

As mentioned earlier channels could have any shape or size. However manmade channels do conform to certain norms as distinct from natural channels or canals that transport water as in case of a river or a stream.

Most preferred shapes for manmade open channels are,

1. Rectangular
2. Triangular
3. Trapezoidal
4. Circular (Used in drainage systems).
5. Parabolic. (The shape of a shallow natural stream is parabolic)

Of these the trapezoidal channel has a larger flow area compared to its wetted perimeter and hence could be considered the most efficient. Their geometrical proportions are as laid out below.

12.1 RECTANGULAR SECTION

$$A = (b \times D)$$

$$WP = (b + 2D)$$

$$R = \frac{bD}{(b + 2D)}$$

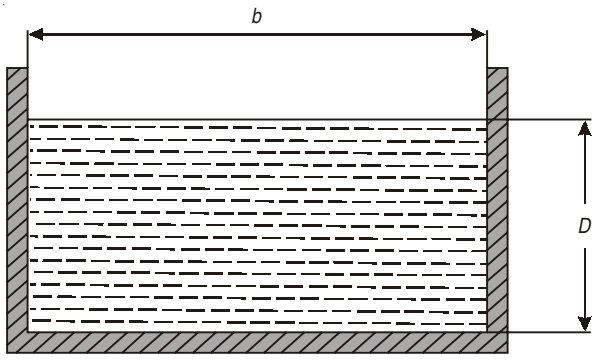


Fig. 12.2

Condition for most economical channel

$$\left(D = \frac{1}{2}b\right) \text{ and } \left(R = \frac{1}{2}D\right)$$

12.2 TRIANGULAR SECTION

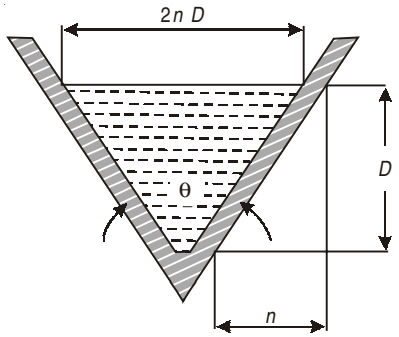


Fig. 12.3

$$A = (n \times D^2)$$

$$WP = 2D + \sqrt{1 + n}$$

$$R = \frac{nD}{2\sqrt{1 + n^2}}$$

Condition for most economical channel

$$\theta = 90°,$$

$$n = 1 \text{ and } R = \frac{D}{2\sqrt{2}}$$

12.3 TRAPEZOIDAL SECTION

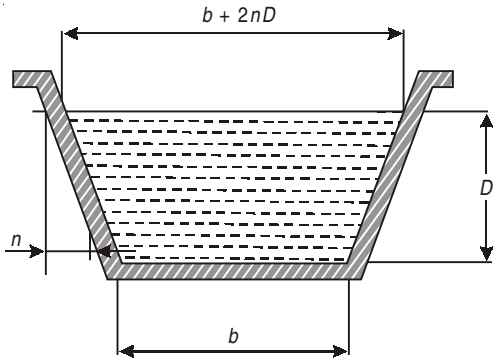


Fig. 12.4

$$A = (b + nD)D$$

$$WP = b + (2D)\sqrt{1 + n^2}$$

$$R = \frac{(b + nD)}{b + 2D\sqrt{1 + n^2}}$$

Condition for most economical channel

$$b = 2D\left[\sqrt{\left(\sqrt{1 + n^2}\right)} - n\right] \text{ and } R = \frac{D}{2}$$

If the side slope is specified in terms of an angle to the horizontal then tan

$$\theta = \sqrt{3}$$

or $$\theta = 60°$$

12.4 CIRCULAR SECTION

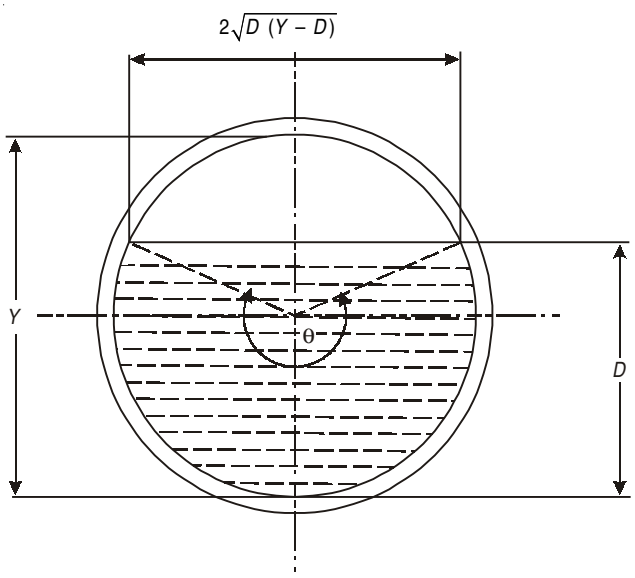


Fig. 12.5

$$A = \frac{[(\theta - \text{sine } \theta)\, y^2]}{8}$$

$$R = \left[\frac{(\theta - \text{sine } \theta)}{\theta}\right]\frac{y}{4}$$

$$WP = \theta\left(\frac{y}{2}\right)$$

Condition for maximum discharge;

$$D = 0.95\,y$$

and $\theta = 308.60°$

Condition for maximum velocity;

$$D = 0.811\text{m}$$

and $\theta = 257.5°$

Note: In case of triangular and trapezoidal shapes n is the horizontal distance corresponding to one unit of vertical distance. Rectangle could be considered as a trapezoid with $n = 0$. A = area, WP = wetted perimeter and R = Hydraulic radius. In case of circular section θ is measured in radians and where $D < \frac{y}{2}$, $\theta = 2\ \text{sine}^{-1}\left[\frac{1-2D}{y}\right]$ and where $D > \frac{y}{2}$, $\theta = 2\ \text{sine}^{-1}\left[\frac{2D}{y-1}\right]$. If $D = \frac{y}{2}$, $\theta = 0$.

The discharge through a circular channel is maximum up to the level $D = 0.95\ Y$. Beyond this level since the increase in wetted perimeter becomes disproportionate compared to the area of flow, the flow velocity and the hydraulic radius decreases. Consequently increase in area has no impact on the discharge. In short it can be said that a circular channel running full is discharging only as much as it would when running to a level = $0.95Y$.

Most economical section for a channel is the section that gives maximum discharge even when its wetted perimeter is the minimum. From the Chezy's equation it has already been established that $Q = A \times C \times \sqrt{(RS)}$, where the factor $[A \times C \times \sqrt{(R)}]$ represents the conveyance or the carrying capacity. Since A and C are constants for a given channel it is obvious that the conveyance capacity is directly proportional to the hydraulic radius R. Hydraulic radius in turn will be maximum, when the wetted perimeter is minimum. Based on this it has been established that of all the sections only semi-circular section meets this criteria.

Artificial channels are ordinarily constructed out of brick in cement mortar finished with concrete. For sewage lines concrete or cast iron pipes are the preferred choice. Storm water drains are constructed out of corrugated metal.

An expression for mean flow velocity was developed by Antoine Chezy and has come to be known as *Chezy's formula* which is obtained by equating the forces keeping the fluid in motion to the frictional resistance offered by the wetted perimeter.

Referring to Fig. (12.1),

$$\gamma \sin\theta \times AL = \tau \times WP \times L \quad \ldots(a)$$

where γ is the specific weight of water, A = wetted area of the water surface, τ is the shear stress, WP is the wetted perimeter and L is the length of the channel under consideration.

Since τ is proportional to the square of the velocity, τ can be expressed as being equal to kv^2, the value of k depending upon the material and flow surface roughness of the channel. Thus Eq. (a) can be written as,

$$\text{Thus } \gamma \sin\theta \times AL = kv^2 \times WP \times L$$

$$v = \sqrt{\frac{\gamma}{k}} \times \sqrt{\left(\frac{A}{WP}\right)} \sin\theta$$

Since $\frac{A}{WP}$ = Hydraulic radius R and sine θ represents the slope S the Chezy's equation can be further simplified to take the form,

$$v = C \times (RS) \quad \ldots(12.3)$$

Here, $$C = \sqrt{\frac{\gamma}{k}}$$

A channel of constant slope has a definite carrying capacity also called the *Conveyance* which is directly proportional to the discharge Q. Thus the carrying capacity of a channel is, $Q = A \times C \times \sqrt{(RS)}$, where the factor $[A \times C \times \sqrt{(R)}]$ represents the conveyance or the carrying capacity.

Since Chezy's constant $C = \sqrt{\frac{\gamma}{k}}$ doesn't produce accurate results because the value of k has not been properly recorded there are empirical formulae developed by other scientists to obtain the value of k accurately.

Bazin's Empirical formula to obtain the value of Chezy's constant C

$$C = \frac{(157.6)}{181} + \frac{k}{\sqrt{R}} \quad \ldots(12.4)$$

Values for Bazin's constant k which basically is indicative of the smoothness of the surface of the channel are listed in Table 12.1. Robert Manning an Irish engineer developed an empirical formula to accurately obtain the value of Chezy's constant and in the process developed his own equation for the flow velocity which has come to be known as the Manning's equation.

$$v = \frac{1}{n}\left(R^{\frac{2}{3}} S^{\frac{1}{2}}\right) \quad \ldots(12.5)$$

Table 12.1 *Values of Bazin's constant K*

1. Cement plaster or planed timber	0.11
2. Concrete, Brick or rough wood	0.21
3. Smooth masonry or rough brickwork	0.83
4. Clean and smooth excavated earth	1.54
5. Rough excavated earth	3.17
6. Earth canals in average condition	2.36

v is the mean velocity in m/sec, n is the Manning's constant which depends on the surface condition of the channel, R is the hydraulic radius in metres and S refers to the slope of the channel a dimensionless quantity being the ratio of the vertical drop h to length L. (Refer Fig. 12.1)

Manning's empirical formula to obtain Chezy's constant C

$$C = \frac{1}{n}\left(R^{\frac{1}{6}}\right) \qquad \ldots(12.6)$$

Values of Manning's constant n which is a function of the nature of the boundary surface differs widely from 0.010 to 0.100 depending upon the smoothness of the wetted perimeter of the channel and the material used to construct the channel. Their values as determined experimentally using different materials and surfaces are listed in Table 12.2. These values are average values and are accurate enough for purposes of design and analysis. However Manning's equation is applicable to turbulent flow regions only.

Table 12.2 *Values of manning constant n for various surfaces*

1. Glass, plastic and other similar smooth surfaces	$n = 0.010$
2. Unpainted steel or planed wood	$n = 0.012$
3. Coated cast iron	$n = 0.013$
4. Asphalt, trowel finished concrete, glazed brick	$n = 0.013$
5. Uncoated cast iron, sewer tile vitrified clay	$n = 0.014$
6. Brick in cement mortar, float finished concrete	$n = 0.015$
7. Formed unfinished concrete	$n = 0.017$
8. Clean excavated earth	$n = 0.022$
9. Corrugated metal	$n = 0.024$
10. Earth with light brush	$n = 0.050$
11. Earth with heavy brush	$n = 0.100$

In solving problems related to uniform flow through a channel one important parameter that needs to be considered is the *normal depth* (D_N) which is the depth at which the given discharge occurs in a channel. This depth is constant for channel of any configuration regardless of whether the top width is constant or increases with depth.

The standard open channel sections discussed hitherto cannot maintain constant velocity at all depths of flow. The flow velocity changes with depth because the hydraulic radius is different at different depths. Increase in flow velocity far beyond the designed values causes erosion of side walls of the channel

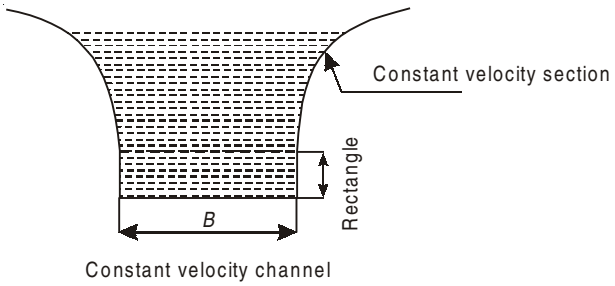


Fig. 12.6

and very low flow velocity results in silt deposit which in due course gets accumulated and cause obstruction to flow. In cases where constant velocity is desired at all depths of flow special side wall profiles are designed where the hydraulic radius remains constant at all depths of flow. Such sections are usually composite sections with the bottom portion designed for minimum discharge and the upper portion for constant velocity.

TYPICAL PROBLEMS AND SOLUTIONS

Problem 12.1 *A drainage pipe of inside diameter = 500 mm is running half full. Determine the hydraulic radius of the pipe.*

Solution

Hydraulic radius $$R = \frac{A}{WP}$$

$$= \left[\frac{\left(\frac{1}{2}\right) 0.7854\ (0.5^2)}{\left(\frac{1}{2}\right)\pi D}\right]$$

$$R = \frac{[0.196]}{\pi(0.500)} = 0.125 \text{ m} = 125 \text{ mm}$$

Problem 12.2 *Determine the hydraulic radius for the channel shown in Fig. 12.7.*

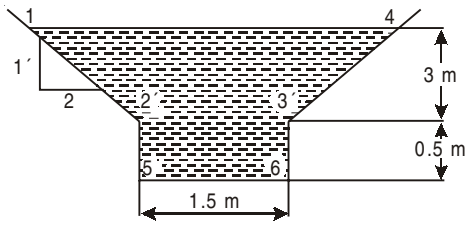


Fig. 12.7

$$\text{Hydraulic radius } R = \frac{A}{WP}$$

A = Area of trapezium 1, 2, 3, 4 + Area of rectangle 2, 3, 5, 6

$$A = (b + nD)\,D + (b \times d)$$
$$= [1.5 + 2(3)]3 + (1.5 \times 0.5)$$
$$= 23.25 \text{ m}^2$$

WP = wetted perimeter of trapezium 1, 2, 3, 4 + wetted perimeter of rectangle

$$WP = \left[b + (2D)\sqrt{1+n^2} - 1.5\right] + (b + 2d)$$

$$WP = \left[1.5 + (2\times 3)\sqrt{1+2^2} - 1.5\right] + (1.5 + 2 \times 0.5)$$
$$= 13.42 \text{ m}$$

$$R = \frac{23.25}{13.42}$$
$$= 1.732 \text{ m}$$

Problem 12.3 *A trapezoidal channel made of unfinished concrete is laid on a slope of 0.1 percent. The side walls make an angle of 45° and the bottom width = 4 m. If the discharge = 20 m^3/sec determine the depth of flow.*

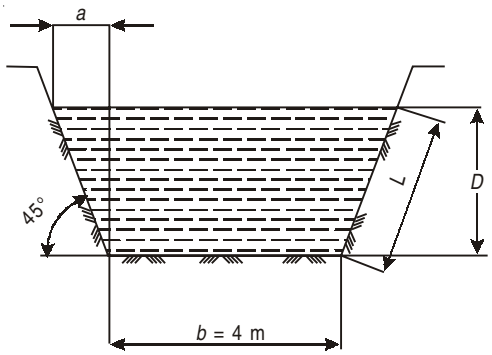


Fig. 12.8

Solution

Area of the trapezium $A = bD + aD$

$= 4D + (D \tan 45)(D)$

$A = 4D + D^2$

Wetted perimeter $WP = b + 2L = 4 + 2\left(\frac{a}{\text{sine } 45}\right)$

$= 3 + \frac{[2(D \tan 45)]}{\text{sine } 45}$

$WP = \frac{4 + 2D}{\text{sine } 45}$

Hydraulic radius $R = \frac{A}{WP}$

$= \frac{[(4D + D^2)]}{\left[\frac{4 + 2D}{\text{sine } 45}\right]}$ ...(a)

From Manning's Eq. (12.5)

$$v = \frac{1}{n}\left(R^{\frac{2}{3}} S^{\frac{1}{2}}\right)$$

In terms of discharge Q, the equation can be re-arranged as

$$Q = \left(\frac{1}{n}\right) A R^{\frac{2}{3}} S^{\frac{1}{2}} \quad \text{... (b)}$$

For unfinished concrete $n = 0.017$ from Table 12.2,

$S = 0.1\ \% = 0.001$

and $Q = 20\ \text{m}^3/\text{sec.}$

Substituting these values in (b) above,

$$20 = \left[\frac{1}{0.017}\right](4D + D^2)\left[\frac{4D + D^2}{\frac{(4 + 2D)}{\text{sine } 45^\circ}}\right]^{\frac{2}{3}} \times (0.001)^{\frac{1}{2}}$$

$D = 1.74$ m (through trial and error)

Problem 12.4 *Determine the slope at which a triangular ditch shown in Fig. 12.9 is laid if the discharge through the ditch is 0.170 m^3sec. Width of the ditch is 1.5 m and the pitch of the side walls, n = 2. Take Chexy's constant C = 50.*

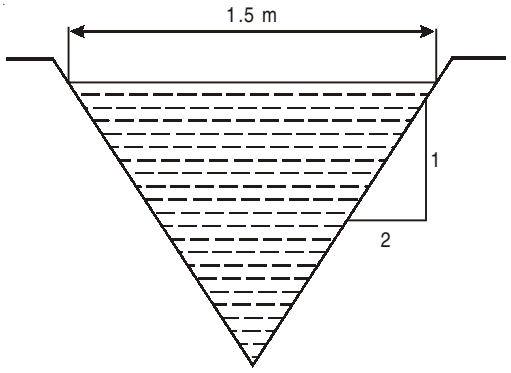


Fig. 12.9

In case of a triangle width of the channel $W = 2nD$ (Refer to geometrical proportions of channels).

Hence depth of the channel $D = \dfrac{W}{2n} = \dfrac{1.5}{2 \times 2} = 0.375$ m

Area of flow $A = n D^2$

$= 2 \times 0.375^2 = 0.281 \text{m}^2$

Wetted perimeter $WP = 2D\sqrt{1+n^2} = 2 \times 0.375\sqrt{1+2^2}$

$= 1.677$ m

Hydraulic radius $R = \dfrac{A}{WP} = \dfrac{0.281}{1.677} = 0.167$ m

Chezey's equation for Q,

$$Q = A \times C \times \sqrt{(RS)}$$

$$\sqrt{(RS)} = \frac{Q}{A \times C} = \frac{0.170}{0.281 \times 50} = 0.0120$$

$$RS = (0.0120)^2$$

$$S = \frac{(0.0120)^2}{R}$$

$$= \frac{(0.0120)^2}{0.167} = 8.76 \times 10^{-4}$$

The slope is 0.876 m for every 1000 m.

Problem 12.5 *A channel constructed out of clean excavated earth is laid on a slope of 1 m over a run of 1000 m. Determine the normal discharge for the configuration shown in Fig. 12.10. Take the Manning's constant n = 0.022.*

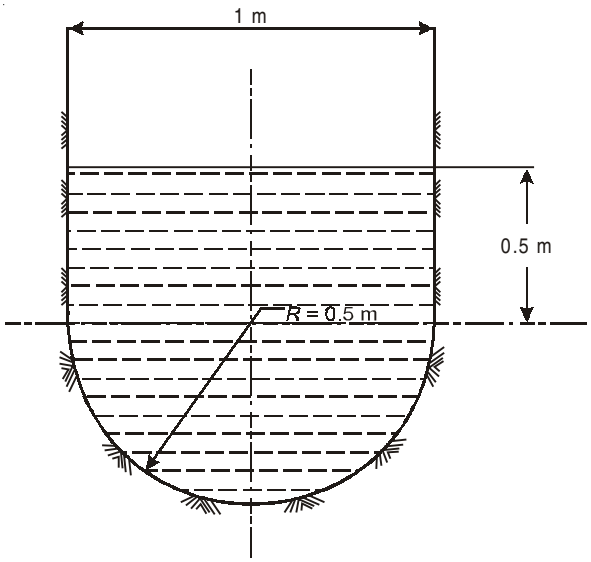


Fig. 12.10

Manning's discharge equation is,

$$Q = \left(\frac{1.00}{n}\right) A R^{\frac{2}{3}} S^{\frac{1}{2}} \qquad \ldots(a)$$

$$\text{Area } A = (1 \times 0.5) + \frac{(\pi \times 0.5^2)}{2}$$

$$= 0.892 \text{ m}^2$$

$$WP = 2(0.5) + (\pi \times 0.5)$$

$$= 2.570 \text{ m}$$

Hydraulic radius $$R = \frac{A}{WP}$$

$$= \frac{0.892}{2.570} = 0.347 \text{ m}$$

$$n = 0.022$$

and $$S = \frac{1}{1000} = 0.001$$

Substituting the above values in (a) above,

$$Q = \left(\frac{1.00}{0.022}\right) (0.892)\,(0.347)^{\frac{2}{3}}\,(0.001)^{\frac{1}{2}}$$

$$Q = 0.632 \text{ m}^3/\text{sec}$$

Problem 12.6 *A trapezoidal channel has a base width of 5 m and is laid on a slope of* $\frac{1}{1500}$. *The side slopes are 1 vertical to 2 horizontal. Manning's coefficient is 0.020. If the depth of flow is 1.5 m determine the volume flow rate. Compare this volume flow rate to that obtained by Chazey's equation. Use Manning's empirical formula to obtain Chazey's constant C.*

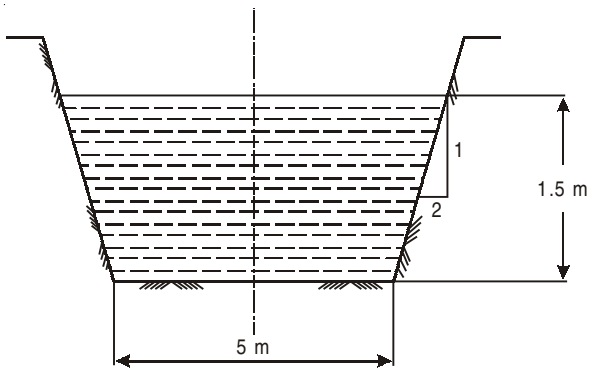


Fig. 12.11

$$A = (b + nD)D$$
$$= (5 + 2 \times 1.5)\ 1.5$$
$$= 12\ \text{m}^2$$

$$WP = b + 2D\sqrt{1+n^2}$$
$$= 5 + (2 \times 1.5)\sqrt{1+2^2}$$
$$= 11.70\ \text{m}$$

$$R = \frac{A}{WP}$$
$$= \frac{12}{11.70} = 1.025\ \text{m}$$

$$S = \frac{1}{1500} = (6.66 \times 10^{-4})$$
$$n = 0.020$$

Substituting these values in the Manning's discharge equation

$$Q = \left(\frac{1.00}{n}\right) AR^{\frac{2}{3}}S^{\frac{1}{2}}$$

$$Q = \left(\frac{1.00}{0.020}\right)(12)\,(1.025)^{\frac{2}{3}}\,(6.66 \times 10^{-4})^{\frac{1}{2}}$$

$$Q = 15.25\ \text{m}^3/\text{sec}$$

Manning's empirical formula to obtain Chezy's constant C from Eq. (12.6)

$$C = \frac{1}{n}\left(R^{\frac{1}{6}}\right)$$

$$C = \frac{1}{0.020}\left(1.025^{\frac{1}{6}}\right)$$

$$= 50.20 \text{ m}^{½} \text{ sec}^{-1}.$$

Chezey's equation for Q,

$$Q = A \times C \times \sqrt{(RS)}$$

$$= 12 \times 50.20 \times \sqrt{(1.025) \times (6.66 \times 10^{-4})}$$

$$Q = 15.74 \text{ m}^3/\text{sec}$$

Problem 12.7 *A trapezoidal channel is constructed out of clean and smooth excavated earth. If the slope of the bed is 1 in 1000, determine the most economical section for the channel that would discharge 0.75 m^3/sec. Use Bazin's empirical formula to obtain the value of Chezy's constant.*

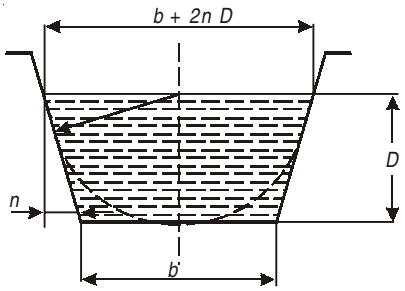


Fig. 12.12

Solution

Area of flow $A = (b + nD)D$

or $$= \left(\frac{A}{D} - nD\right) \quad \ldots(a)$$

$$WP = b + (2D)\sqrt{1+n^2}$$

$$WP = \left(\frac{A}{D} - nD\right) + (2D)\sqrt{1+n^2}$$

which can be re-arranged as,

$$WP = \frac{A}{D} + [(2)\sqrt{(1+n^2)} - n]\,D \quad \ldots(b)$$

For the most economical section WP must be the minimum for a given discharge. Treating A and n as constants we differentiate Eq. (b) with respect to D and equate it to zero. Thus,

$$\frac{d(WP)}{dD} = -\frac{A}{D^2} + [(2)\sqrt{(1+n^2)} - n] = 0$$

$$\frac{A}{D^2} = [(2)\sqrt{(1+n^2)} - n]$$

$$A = D^2[(2)\sqrt{(1+n^2)} - n]$$

but from (a) $A = (b + nD)D$ or $A = (bD + nD^2)$

$$(bD + nD^2) = D^2[(2)\sqrt{(1+n^2)} - n]$$

$$D^2\left(\frac{b}{D} + n\right) = D^2[(2)\sqrt{(1+n^2)} - n]$$

$$\frac{b}{D} = [(2)\sqrt{(1+n^2)} - n]$$

$$b = 2D[\sqrt{(1+n^2)} - n] \qquad \ldots(12.7)$$

(This is the condition for the most economical section that allows maximum discharge.)

Similarly it can be shown that for maximum efficiency side slope $n = \frac{1}{\sqrt{3}}$.

We now apply the above conditions to the given problem to obtain a configuration that would give maximum discharge.

From Eq. (12.7) for most economical section,

$$b = 2D[\sqrt{(1+n^2)} - n]$$

$$b = 2D\left[\sqrt{\left(1+\left(\frac{1}{\sqrt{3}}\right)^3\right)} - \frac{1}{\sqrt{3}}\right]$$

$$b = 2D\left[\sqrt{\left(1+\left(\frac{1}{3}\right)\right)} - \frac{1}{\sqrt{3}}\right] = 2D(0.577)$$

$$b = 1.1547D$$

$$A = (bD + nD^2)$$

$$= \left(1.1547D^2 + \frac{1}{3\,D^2}\right) = 1.732D^2$$

$$WP = b + (2D)\sqrt{1+n^2}$$

$$WP = 1.1547D + (2D)\sqrt{1+\left(\frac{1}{\sqrt{3}}\right)^2}$$

$$= 3.464D$$

Hydraulic radius $$R = \frac{A}{WP}$$

$$= \frac{1.732D^2}{3.464D} = 0.5D \qquad \ldots(c)$$

Before we substitute the above values in the Chezy's discharge equation, we obtain the value of Chezy's constant C from the Bazin's Empirical formula as given by Eq. (12.4)

$$C = \frac{(157.6)}{181} + \frac{k}{\sqrt{R}}$$

Bazin's constant k for a channel constructed out of clean and smooth excavated clay as given in Table 12.1 is 1.54.

$$C = \frac{(157.6)}{181} + \frac{1.54}{\sqrt{0.5D}}$$

$$= 0.865 + \frac{2.177}{\sqrt{D}}$$

Chezey's equation for Q,

$$Q = A \times C \times \sqrt{(RS)}$$

$$0.75 = 1.732D^2 \times \left(0.865 + \frac{2.177}{\sqrt{D}}\right) \times \sqrt{(0.5D \times 0.001)}$$

$$D = 3.4 \text{ m}$$

$$b = 1.1547 \times 3.4 = 3.92$$

Note. In Eq. (c) $R = \frac{D}{2}$ implies that for a channel of optimum proportions the sides and the base are tangential to a semi-circle drawn with its center on the free surface.

Problem 12.8 *A cast iron drain pipe of inside diameter 350 mm laid on a slope of 0.1% is running* $\frac{3}{4}$ *full. Determine the normal discharge.*

Solution

From Manning's discharge equation,

$$Q = \left(\frac{1.00}{n}\right) A\, R^{\frac{2}{3}} S^{\frac{1}{2}} \qquad \text{...(a)}$$

From Table 12.2 for cast Iron pipe $n = 0.013$

Slope $S = 0.1\% = 0.001$

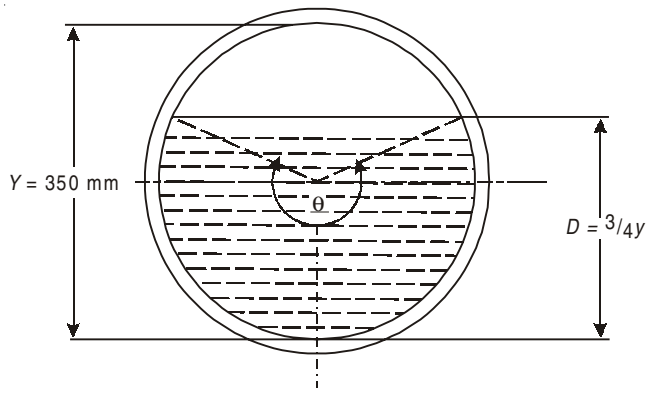


Fig. 12.13

Area $$A = \left[\frac{(\theta - \sin\theta)\, y^2}{8}\right]$$

...(Refer to circular section)

Since in this case $D > \frac{y}{2}$,

$$\theta = 2 \sin^{-1}\left[\frac{2D}{y} - 1\right]$$

$$= 2 \sin^{-1}\left[\frac{2(0.350 \times 0.75)}{0.350}\right] - 1$$

$$\theta = 2 \sin^{-1}[0.5]$$

$$= 2 \times 30° = 60°$$

$$= 1.047 \text{ radians.}$$

Area $$A = \left[\frac{(1.047 - \sin 60)\, 0.350^2}{8}\right]$$

$$A = \left[\frac{(1.047 - 0.8660)\, 0.350^2}{8}\right]$$

$$= 2.77 \times 10^{-3}\ \text{m}^2$$

$$WP = \theta\left(\frac{y}{2}\right)$$

$$= 1.047\left(\frac{0.350}{2}\right) = 0.183\ \text{m}$$

$$R = \frac{A}{WP}$$

$$= \frac{(2.77 \times 10^{-3})}{0.183} = 0.015\ \text{m}$$

Substituting these values in Eq. (a) above,

$$Q = \left(\frac{1.00}{0.013}\right)(2.77 \times 10^{-3})\,(0.015)^{\frac{2}{3}}\,(0.001)^{\frac{1}{2}}$$

$$Q = 4.095 \times 10^{-4}\ \text{m}^3/\text{sec}$$

Problem 12.9 *A trapezoidal channel has formed unfinished concrete surface and is laid on a slope that drops by 0.15 m in a length of 100 m. If the depth of the channel is 2 m, the base measures 3.5 m and the side walls slope are 3-horizontal to 2-vertical, determine the normal discharge, the Froude number.*

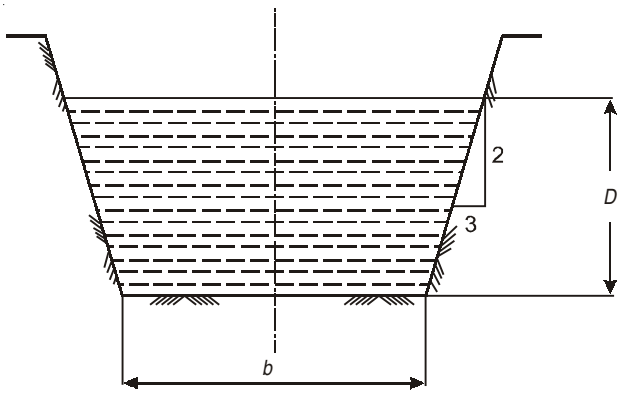


Fig. 12.14

The pitch of the channel $n = \frac{3}{2} = 1.5$

$$A = (b + n\,D)D$$

$$= (3.5 + 1.5 \times 2)\,2 = 13\ \text{m}^2$$

$$WP = b + 2D\sqrt{1+n^2}$$

$$= 3.5 + (2 \times 2)\sqrt{1+1.5^2} = 10.71 \text{ m}$$

$$R = \frac{A}{WP} = \frac{13}{10.71} = 1.214 \text{ m}$$

$$S = \frac{0.15}{100} = (1.5 \times 10^{-3})$$

Manning's constant for formed unfinished concrete $n = 0.017$

Substituting these values in the Manning's discharge Eq. (a)

$$Q = \left(\frac{1.00}{n}\right) A\, R^{\frac{2}{3}} S^{\frac{1}{2}} \qquad \ldots(a)$$

$$Q = \left(\frac{1.00}{0.017}\right) (13)\ (1.214)^{\frac{2}{3}}\ (1.5 \times 10^{-3})^{\frac{1}{2}}$$

$$Q = 33.06 \text{ m}^3/\text{sec}$$

From Eq. (12.2) Froude number is given by,

$$F_R = \frac{v}{\sqrt{gD_H}}$$

D_H is the hydraulic depth which is a ratio of the area of the channel to the breadth of the free surface of the fluid in the channel. Thus $D_H = \frac{A}{B}$ and v = the mean flow velocity at any cross section

$$v = \frac{Q}{A}$$

$$= \frac{33.06}{13} = 2.543 \text{ m/sec.}$$

$$B = b + 2nD$$

$$= 3.5 + 2\ (1.5)\ (2) = 9.5 \text{ m}$$

$$D_H = \frac{13}{9.5} = 1.37 \text{ m}$$

$$F_R = \frac{2.543}{\sqrt{9.81 \times 1.37}}$$

$$= 0.69 \text{ (because } F_R < 1 \text{ the flow is sub-critical)}$$

Problem 12.10 *A trapezoidal channel has to be built out of clean excavated earth. Determine the optimum basic dimensions of the channel that can discharge water at 4 m^3/sec with a flow velocity of 0.5 m/sec. The side slope 2 horizontal to 1 vertical.*

Solution

Conditions for most economical channel or optimum size are,

$$b = 2D\left[\sqrt{1+n^2} - n\right] \text{ and } R = \frac{D}{2}.$$

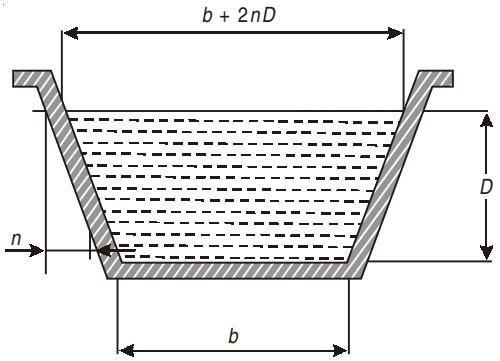


Fig. 12.15

$$n = 2$$

$$A = (b + nD)D = \frac{Q}{v} = \frac{4}{0.5} = 8 \text{ m}^2$$

$$R = \frac{A}{WP} = \frac{(b+nD)\,D}{\left[b+2D\sqrt{1+n^2}\right]} = \frac{D}{2}$$

$$\frac{8}{\left[b+2D\sqrt{1+n^2}\right]} = \frac{D}{2}$$

$$D\left[b+2D\sqrt{1+n^2}\right] = 16$$

$$D\left[b+2D\sqrt{1+2^2}\right] = 16$$

$$D[b + 4.472D] = 16$$

$$[Db + 4.472\,D^2] = 16 \quad \ldots(a)$$

$$A = (b + nD)\,D$$

$$= (bD + nD^2)$$

$$= (bD + 2D^2) = 8$$

or $$bD = (8 - 2D^2) \quad \ldots(b)$$

Substituting the value of bD in (a)

$$[(8 - 2D^2) + 4.472\,D^2] = 16$$

$$[(8 + 2.472\,D^2] = 16$$

$$D = 1.798 \text{ m}$$

$$b = 2D\left[\sqrt{(1+n^2)} - n\right]$$

……(one of the conditions for optimum section)

$$b = 2(1.798)\left[\sqrt{(1+2^2)} - 1\right] = 0.848 \text{ m}$$

The slope of the channel S is all that remains to be determined for which we use the Manning's equation.

$$Q = \left(\frac{1.00}{n}\right) A\, R^{\frac{2}{3}} S^{\frac{1}{2}}$$

Manning's constant n for clean excavated earth = 0.022

$$4 = \left(\frac{1.00}{0.022}\right)(8)\left(\frac{1.798}{2}\right)^{\frac{2}{3}} S^{\frac{1}{2}}$$

$$S = (1.39 \times 10^{-4})$$
$$= 0.0139\ \%$$

Problem 12.11 *A round pipe of inside diameter 1 m is laid on a slope of 1 in 1000. Determine the basic parameters of flow for optimum velocity and optimum discharge. Chezy's constant = 50.*

Solution

(i) *Optimum Velocity*

Flow velocity $v = C\sqrt{RS} = C\sqrt{\left(\frac{A}{WP}\right)S}$

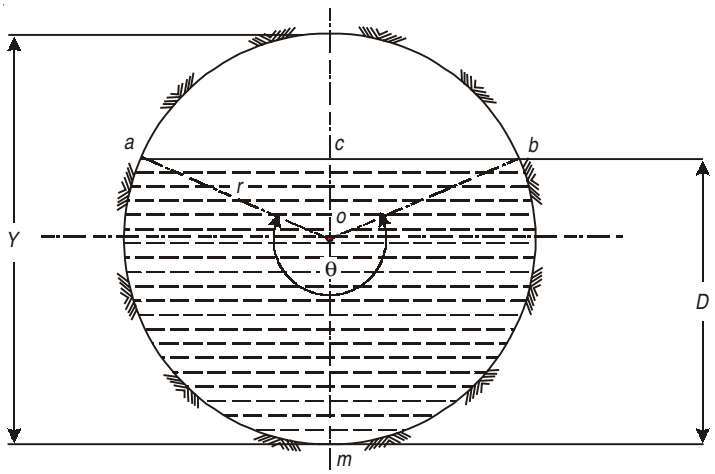


Fig. 12.16

Wetted perimeter $WP = r \times \theta$ …(a)

From this it follows that v is maximum when $\left(\frac{A}{WP}\right)$ is maximum.

By geometry of Fig. 12.16

$$\text{Area } A = \frac{\left[(\theta - \sin\theta)Y^2\right]}{8} \quad \ldots(a)$$

$$\text{Wetted perimeter } WP = \theta\left(\frac{Y}{2}\right) \quad \ldots(b)$$

$$\text{Hydraulic radius } R = \left[\frac{(\theta - \sin\theta)}{\theta}\right]\left(\frac{Y}{4}\right) \quad \ldots(b)$$

(1) **Maximum velocity and Depth of flow**

The condition for maximum velocity is that the bydraulic radius R must be maximum, that is, $\left(\frac{dR}{d\theta}\right) = 0$

$$\left(\frac{d}{d\theta}\right)\left[\frac{(\theta - \sin\theta)}{\theta}\right]\left(\frac{Y}{4}\right) = 0$$

From which, $\theta = 257.5° = 4.49$ rad.

Applying this condition to the given problem

$$\text{Area } A = \frac{\left[(\theta - \sin\theta)Y^2\right]}{8}$$

$$\text{Area } A = \frac{[(4.49 - \text{Sine } (189 + 77.5)^2]}{8}$$

$$\text{Area } A = \frac{[(4.49 - (-\text{sine } 77.5)]}{8}$$

$$\text{Area } A = \frac{[(4.49 + \text{sine } 77.5)]}{8}$$

$$\text{Area } A = \frac{[(4.49 + 0.9.976)]}{8}$$

$$\text{Area } A = \mathbf{0.683\ m^2}$$

From (b) above $WP = \theta\left(\frac{Y}{2}\right) = 4.49\left(\frac{1}{2}\right) = \mathbf{2.245\ m}$

From (c) above $R = \left(\frac{9.683}{2.245}\right) = \mathbf{0.30\ m}$

We can now determine the maximum flow velocity, $v = C\sqrt{RS}$

$$v = 50\sqrt{(0.30) \times \left(\frac{1}{1000}\right)} = \mathbf{0.866\ m/sec}$$

Referring to Fig. (12.16),

Depth of flow $$D = \left(\frac{Y}{2}\right) + OC$$

$$D = \left(\frac{1}{2}\right) + r\left[\cos 180 - \left(\frac{\theta}{2}\right)\right]$$

$$D = \left(\frac{1}{2}\right) - 0.5\left[\cos\frac{\theta}{2}\right]$$

$$D = \left(\frac{1}{2}\right) - 0.5\ [\text{Cos } 128.75]$$

$$D = \left(\frac{1}{2}\right) - 0.5\ [\cos 90 + 38.75]$$

$$D = \left(\frac{1}{2}\right) - 0.5\ [-\text{ sine } 38.75]$$

$$D = \left(\frac{1}{2}\right) + 0.5\ [0.626] = \mathbf{0.813\ m}$$

(2) **Values of *A*. θ, *R*** and **v** for **optimum discharge**

For optimum discharge $D = 0.95Y$

$$\frac{Y}{2} + OC = 0.95\,y$$

$$OC = 0.95Y - \left(\frac{Y}{2}\right) = (0.95 \times 1) - \left(\frac{1}{2}\right)$$

$$= \mathbf{0.450\ m}$$

$$r\left[\cos 180 - \left(\frac{\theta}{2}\right)\right] = 0.450\ \text{m}$$

$$\left[\cos 180 - \left(\frac{\theta}{2}\right)\right] = \left(\frac{0.450}{r}\right)$$

$$= \left(\frac{0.450}{0.5}\right) = 0.9$$

$$\left[-\cos\frac{\theta}{2}\right] = 0.90$$

By trial and error $\theta = 308.60 = 5.38$ rad

Applying this condition to the given problem

Applying this condition to the given problem

$$\text{Area } A = \frac{[(\theta - \text{sine } \theta)\,(y)^2]}{8}$$

$$\text{Area } A = \frac{[(5.38 - \text{sine } (308.6)\,(1)^2]}{8}$$

$$\text{Area } A = \frac{[(5.38 - \text{sine } (180 + 128.6)]}{8}$$

$$\text{Area } A = \frac{[(5.38 - (-\text{sine } 128.6°)]}{8}$$

$$\text{Area } A = \frac{[(5.38 + \text{sine } 128.6)]}{8}$$

$$\text{Area } A = \frac{[(5.38 + \text{sine } (90 + 38.6)]}{8}$$

$$\text{Area } A = \frac{[(5.38 + \cos 38.6)]}{8}$$

$$\text{Area } A = \mathbf{0.770\ m^2}$$

$$WP = \theta\left(\frac{y}{2}\right) - 5.38\left(\frac{1}{2}\right) = \mathbf{2.69\ m}$$

$$R = \frac{0.770}{2.69} = \mathbf{0.286\ m}$$

$$v = C\sqrt{RS} = 50\sqrt{(0.286) \times \left(\frac{1}{1000}\right)}$$

$$= \mathbf{0.845\ m/sec}$$

Problem 12.12 *A sewer when running $\frac{1}{4}$ full, has a flow velocity = 0.8 m/sec and when running full the discharge is 2 m^3/sec. The sewer is laid on a slope of 1 in 2000. Determine its diameter. Take Manning's constant n = 0.015.*

Solution

(i) *When running* $\frac{1}{4}$ *full*

Referring to the Fig. 12.17

$$\cos \alpha = \frac{\left(\frac{y}{4}\right)}{\left(\frac{y}{2}\right)} = 0.5.$$

Therefore $\alpha = 60°$

Area of the sector *nlmn* = A

$$= \frac{(\pi r^2\ 2a)}{360}$$

$$= \frac{(\pi y^2\ 120)}{4 \times 360} = 0.262\, y^2$$

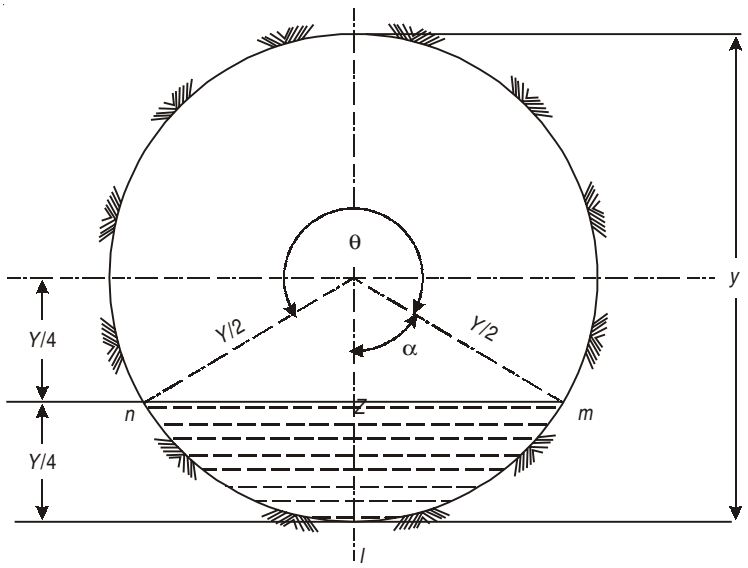


Fig. 12.17

Wetted perimeter = $(nLm) = WP$

$$= \frac{(\pi y\ 2a)}{360}$$

$$= \frac{(\pi\, y\, 120)}{360} = 1.047\, y$$

$$\text{Hydraulic radius } R = \frac{(0.262\, y^2)}{(1.047\, y)} = 0.250\, y$$

$$\text{Manning's equation for velocity} = v = \frac{1}{n}(R)^{\frac{2}{3}}\ (S)^{\frac{1}{2}}$$

$$v = \frac{1}{0.015}(0.250\, y)^{\frac{2}{3}}\ \left(\frac{1}{2000}\right)^{\frac{1}{2}}$$

$$= 0.582\ (y)^{\frac{2}{3}} \qquad \ldots(a)$$

$$\text{When flowing full } Q = A \times v$$

$$= 0.7854\, y^2 \times 0.582\ (y)^{\frac{2}{3}}$$

$$2 = 0.7854\, y^2 \times 0.582\ (y)^{\frac{2}{3}}$$

Diameter of the sewer $y = 1.74$ m

Problem 12.13 *It is desired that an existing rectangular channel 3 m wide × 1.5 m deep laid on a slope of 1 in 1000 be improved to handle twice the flow without affecting the slope and the depth of flow. Propose an alternative section that can meet this requirement at minimum cost. Cheyz's constant C = 50.*

Solution

Area of the rectangular channel $A = 3 \times 1.5$

$$= 4.5\ \text{m}^2$$

$$\text{Wetted perimeter } WP = 2(1.5) + 3 = 6\ \text{m}$$

$$\text{Hydraulic radius } R = \frac{4.5}{6} = 0.75$$

$$Q = AC\sqrt{RS}$$

$$= 4.5 \times 50\ \sqrt{0.75 \times \frac{1}{1000}}$$

$$= 6.16\ \text{m}^3/\text{sec}$$

If double this flow is required to be handled,

$$2Q = AC\ \sqrt{RS}$$

$$= A \times 50\ \sqrt{R\ \times \frac{1}{1000}}$$

$$2Q = AC\sqrt{RS}$$
$$= A \times 1.58\sqrt{R}$$
$$A\sqrt{R} = \frac{2Q}{1.58}$$
$$= \frac{(2 \times 6.16)}{1.58} = 7.8 \quad \text{...(a)}$$

A rectangular channel is more easily modified as a trapezoidal channel than any other section. Hence let us choose a trapezoidal section. Assume a side slope of 1.5 horizontal to 1 vertical tentatively.

Area of this section $A = (b + nD)D$

.....(Refer to geometrical properties of channel sections)

$$A = (3 + 1.5 \times 1.5)1.5$$
$$= 7.875 \text{ m}^2$$

Wetted perimeter $WP = b + 2D\sqrt{1+n^2}$

$$= 3 + 2 \times 1.5\sqrt{1+1.5^2} = 8.40 \text{ m}$$
$$R = \frac{7.875}{8.40} = 0.9375$$
$$A\sqrt{R} = 7.875 - 0.9375 = 7.62 \quad \text{...(b)}$$

Since (a) is almost equal to (b) a trapezoidal section of depth 1.5 m and base 3 m with side slopes 1.5 horizontal to 1 vertical is proposed. The proposed channel can handle a discharge,

$$Q = AC\sqrt{RS}$$
$$= 7.875 \times 50\sqrt{0.9375 \times \left(\frac{1}{1000}\right)}$$
$$= 12.05 \text{ m}^3/\text{sec}$$

QUESTIONS BANK

Section A

Review Questions

1. Mention some of the essential differences between closed conduit flow and open channel flow.
2. Define a prismatic channel and the nature of flow possible in a prismatic channel.
3. Define hydraulic depth, hydraulic radius and depth of flow.

4. Define the term conveyance as applied to open channels. When would the conveyance of a channel be maximum?
5. Define the term 'the most economical channel section'? Name the factors that characterize a section as the most economical section?
6. State the necessary and sufficient condition for the most economical trapezoidal section.
7. How do you detect that a velocity gradient exists in open channel flow? Explain how Reynold's number and Froude number together can define the condition of flow.
8. How do you define the slope of a channel. State the methods employed to characterize the slope.
9. Write down the several options available to you to determine the Chezy's constant C.
10. Define optimum depth of flow in respect of covered channels with free surface.

Section B

Multiple Choice Questions

Choose the right answer

1. Of all the open channel cross sections the best hydraulic section is the
   (a) rectangular section (b) trapezoidal section
   (c) semi-circular section (d) triangular section
2. The best trapezoidal section has a cross section which is
   (a) $\frac{1}{3}$ of a hexagon (b) $\frac{2}{3}$ of a hexagon
   (c) $\frac{1}{4}$ of a hexagon (d) $\frac{1}{2}$ of a hexagon
3. The most economical rectangular section is the one in which
   (a) $D = \frac{b}{2}$ (b) $D = 2b$
   (c) $b = \frac{D}{2}$ (d) None of these
4. In an open channel flow the hydraulic gradient line and the energy gradient line
   (a) Merge with the free surface of the liquid
   (b) HGL merges with EGL
   (c) HGL merges with the free surface
   (d) EGL merges with the free surface

5. The channel with the most economical cross section has
   (a) the least coefficient of friction
   (b) maximum hydraulic depth
   (c) minimum perimeter for a given area
   (d) None of these
6. Steady and uniform flow occurs when
   (a) the discharge remains constant but the depth varies
   (b) the discharge varies with time
   (c) both the depth and discharge remain constant
   (d) None of these
7. When Reynold's number is > 2000 and the Froude number is < 1 then the flow is characterized as
   (a) sub-critical and laminar
   (b) super-critical and laminar
   (c) super critical and turbulent
   (d) sub-critical and turbulent
8. In a triangular channel the width of the free surface of the liquid is
   (a) $W = 2nD$ (b) $W = nD^2$
   (c) $W = 2D\sqrt{1+n^2}$ (d) None of these

Section C

Problems

1. One half of a trapezoidal channel is used as a drainage line, with one side vertical. The other side has a side slope 2 horizontal to one vertical. The discharge through the channel is 10 m^3/sec. If the channel is laid on a slope of 0.2%, determine the most economical section for the channel. Manning's constant $n = 0.015$.

   (*Ans.* $D = 1.39$ m, $b = 1.72$ m, Area = 4.320 m^2)

2. Determine the discharge through a semi-circular channel of radius 300 mm. The depth of flow is equal to the radius of the channel and the channel is laid on a slope of 1 in 2000. Chezy's constant $C = 55$.

   (*Ans.* $Q = 0.141$ m^3/sec)

3. Prove that under maximum flow velocity condition the depth of flow is equal to 81 % of the diameter in case of flow through an open circular conduit. Determine the slope at which such a conduit having a diameter = 1 m is discharging 0.75 m/sec. $C = 60$. (*Ans.* $S = 0.1089$ %)

4. A square section open steel conduit is laid on a slope of 1 in 2000 in such a way that it stands vertical on its apex. If each side measures 0.5 m determine the volume flow rate under maximum discharge conditions. Take Manning's constant $n = 0.013$. (*Ans.* $Q = 0.107$ m^3/sec)

5. Discharge through a trapezoidal channel is 12 m^3/sec. The flow velocity = 2 m/sec. Determine the sizes for the most economical section given that side slopes are 3 horizontal to 2 vertical. Manning's constant n = 0.013. (*Ans.* S = 0.00214%)
6. Find the discharge through a triangular channel whose apex angle is 60° and laid on a slope of 1 in 1500. The depth of the channel is 1.5 m and C = 50. (*Ans.* Q = 1.027 m^3/sec)

Answers to Multiple Choice Questions

1. (c) **2.** (d) **3.** (a) **4.** (c)
5. (c) **6.** (c) **7.** (d) **8.** (a)

Chapter

13

Non-uniform Flow Through Open Channels

Flow in open channels except in case of prismatic channels, rarely if ever could be termed as uniform because of lateral contractions and variations in the slope at different sections. This chapter deals with non-uniform flow through open channels which includes both man-made and natural channels such as canals and rivers.

In uniform flow the total energy line and water surface are parallel to the bed slope. In non-uniform flow because of variations in the slope of the bed the total energy line is not parallel to the bed as the fluid is accelerating or decelerating at any particular point.

Whereas steady uniform flow occurs due to gravity, non-uniform flow is more spontaneous and occurs due to changes in width, depth or due to an obstruction in the flow path or due to waves and pressure surges. As such it becomes necessary to introduce some more definitions and concepts to enable the student to fully comprehend and analyze the flow in all its forms.

Since non-uniform flow is basically caused by variations in the depth of flow, depending upon whether this variation occurred suddenly or gradually the flow is classified as either *gradually varied flow or as a rapidly varied flow*. Obviously for a gradually varied flow to occur the variation in depth must occur over a specific length of the channel whereas in a rapidly varied flow the depth changes in a very short length of the channel.

13.1 CRITICAL DEPTH AND SPECIFIC ENERGY

Specific energy is the total energy both potential and kinetic that a unit weight of fluid flowing through a channel possess with respect to the channel bed. For channels of normal slopes where the channel falls by 1 m per 1000 m length (slope = 0.1%) the specific energy is given by

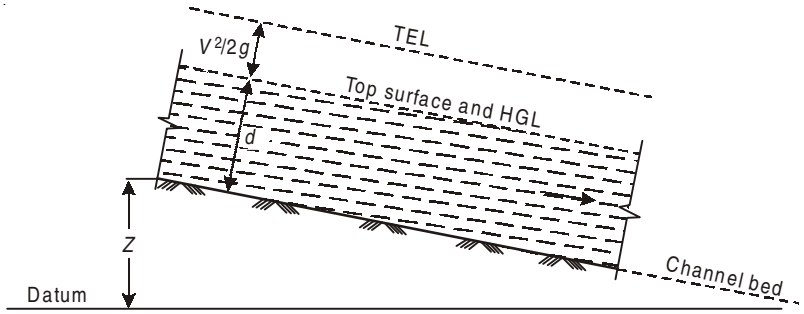


Fig. 13.1

$$E = D + \frac{v^2}{2g} \qquad \ldots(13.1)$$

For instance in case of a rectangular channel if the discharge per unit width of the channel is q then the flow velocity is $\left(\frac{q}{D}\right)$. The equation for specific energy can now be more explicitly written as

$$E = D + \left(\frac{q^2}{2g\,D^2}\right) \qquad \ldots(13.1a)$$

For a given discharge there may be flow at different depths and consequently specific energy at any particular reach could be minimum or maximum. For a given specific energy except for $E_{\min}$ there are two depths of flow known as alternate depths. The larger of these values corresponds to tranquil flow and the smaller corresponds to super-critical flows. (Fig. 13.2 b) The depth corresponding to minimum energy is considered as *critical depth* D_C. It is so called because it is the minimum specific energy that the fluid must possess for flow to occur. A mathematical expression which determines the critical depth can be obtained by differentiating Eq. (13.1 a) with respect to D and setting it equal to zero.

The critical depth (D_C) corresponding to minimum specific energy E is given by,

$$D_c = \left(\frac{q^2}{g}\right)^{\frac{1}{3}} \qquad \ldots(13.2)$$

The corresponding critical velocity v_c is given by,

$$v_c = \frac{q}{D_C} = \frac{q}{\left(\frac{q^2}{g}\right)^{\frac{1}{3}}} \qquad \ldots\text{from Eq. (13.2)}$$

$$v_c = (qg)^{\frac{1}{3}}$$

$$v^3_c = (qg) \qquad \text{(but } q = v_c \times D_c\text{) and hence}$$

$$v_c = \sqrt{gD_c} \qquad \ldots(13.3)$$

Figure 13.2 (a) is a graph of depth D versus specific energy E. It shows how the specific energy curve appears for a channel of particular cross section. Figure 13.2 (b) is a graph of discharge per unit width of the channel (q) versus the depth (D) which shows how the specific energy varies as the depth varies. The straight line passing through the origin at 45° represents the potential energy. Alternate depths of flow are indicated as 1 and 2.

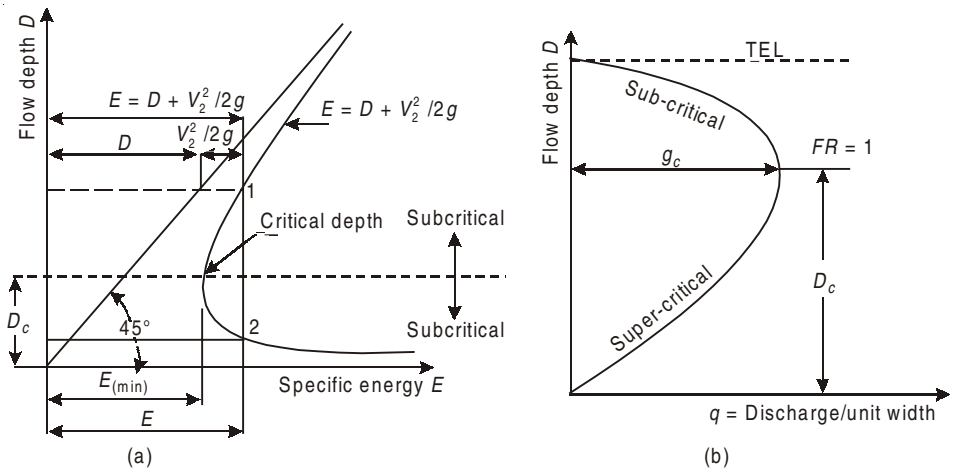


Fig. 13.2

In Fig. 13.3,

$$E = \left(D_1 + \frac{v^2}{2g}\right)$$

$$= \left(D_1 + \frac{v^2}{2g\,D_1^{\,2}}\right) \qquad \ldots(13.4)$$

From Eqs. (13.1) and (13.2) by eliminating q it can be shown that minimum specific energy is $1\frac{1}{2}$ times the critical depth. Thus,

$$E_{\min} = \left(\frac{3}{2}\right) D_c \qquad \ldots(13.5)$$

This also forms the condition for maximum discharge for a given value of specific energy.

From Eq. (13.2) Froude number $F_R = \dfrac{v}{\sqrt{gD_H}}$

D_H is the hydraulic depth which is a ratio of the area of the channel to the breadth of the free surface of the fluid in the channel. Thus $D_H = \dfrac{A}{b}$.

At critical depth $F_R = 1$, which means $v = \sqrt{gD_H}$

$$\frac{Q}{A} = \sqrt{gD_H}$$

or $$\left(\frac{Q}{A}\right)^2 = g\left(\frac{A}{b}\right)$$

or $$\frac{\left(\frac{Q}{A}\right)^2}{g\left(\frac{A}{b}\right)} = 1$$

or $$\left(\frac{Q^2}{g}\right) = (A^3{}_C / b_C) \qquad \ldots(13.6)$$

(b_C = Top width of the channel at critical depth)

Equation (13.2) is a straight forward equation to obtain the value of critical depth at any section in case of rectangular sections. In case of triangular and trapezoidal sections Eq. (13.2) can be used but with some slight modifications.

Hydraulic jump is a common phenomenon in open channel flows. It might occur naturally along the flow path or sometimes is designed to occur intentionally.

The velocity of water flowing over an Ogee weir or a spill way is usually in the super critical range. Such a high velocity would soon erode the spill way structure. The best remedy is to artificially create conditions for a hydraulic jump to occur. This is achieved by ensuring that the flow beyond the paved spillway (Apron) is in the sub-critical range or by introducing an obstruction to force the jump to occur. When high velocity water comes in contact with low velocity flow the energy contained in the fluid is suddenly dissipated. Because the flow velocity changes from super-critical to sub-critical the energy lost causes a wave to form by increasing the depth of flow many times over.

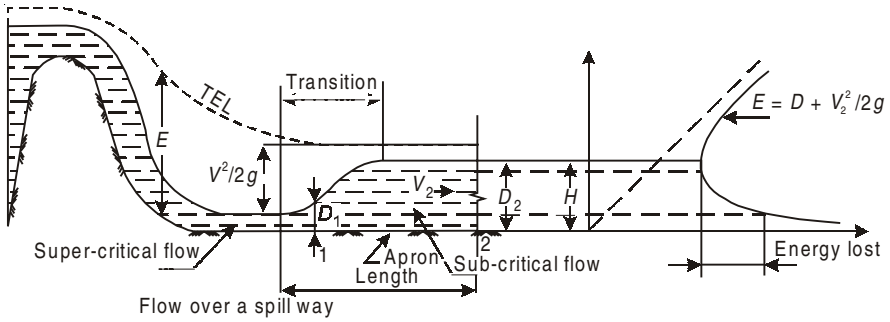


Fig. 13.3

Hydraulic jump can also occur at the toe of a sluice gate to a reservoir or when the slope of the channel bed suddenly changes. The only criteria for a hydraulic jump to occur, is that the flow before the jump must be in the supercritical range. The depth of the jump D_2 can be determined from the equation

$$D_2 = \left(\frac{D_1}{2}\right)\left[\sqrt{(1+8F^2R)}-1\right] \qquad \ldots(13.7)$$

The energy lost $$\Delta E = \frac{(D_2 - D_1)^3}{4D_1 D_2} \qquad \ldots(13.8)$$

F_R is Froude number at the upstream section.

From Eq. (13.2) the critical depth is

$$D_c = \left(\frac{q^2}{g}\right)^{\frac{1}{3}}$$

The jump must take place at a depth less than this value and to a depth greater than this value.

The velocity before the jump (v_1) and the velocity after the jump (v_2) can be established based on the corresponding Froude numbers. Since the depth before and after the jump are directly related to Froude number,

$$F_{R1} = \frac{v_1}{\sqrt{gD_{h1}}} \text{ and } F_{R2} = \frac{v_2}{\sqrt{gD_{h2}}} \qquad \ldots(13.10)$$

From continuity equation it can be established that F_{R2} is related to F_{R1} in the following manner.

$$F_{R2} = \frac{2\sqrt{2F_{R1}}}{\left[\left(\sqrt{1 + 8F_{R_1}^2}\right)-1\right]^{1.5}} \qquad \ldots(13.11)^*$$

* Applies for Rectangular Sections only.

13.2 TRANSITIONS

Non-uniform flow is a phenomenon attributable to the inevitable changes in cross section and depths of flow that exists in natural streams, canals and rivers. Even in man made channels, flow variations due to changes in cross section and slope cannot always be avoided. The best that can be done is to ensure that the change in the profile of the structure is smooth and gradual so that minimum energy loss occurs. The word *Transition* merely emphasizes that while analyzing the flow variations it may be assumed that the structural changes in cross section or depth of the channel is always smooth and gradual wherever such changes could be effected.

1. Due to changes in the slope of a river or canal bed critical flow conditions could occur because the flow rate is the maximum at critical depth. Consequently transition from sub-critical to super-critical flow occurs. (Refer Fig. 13.4).

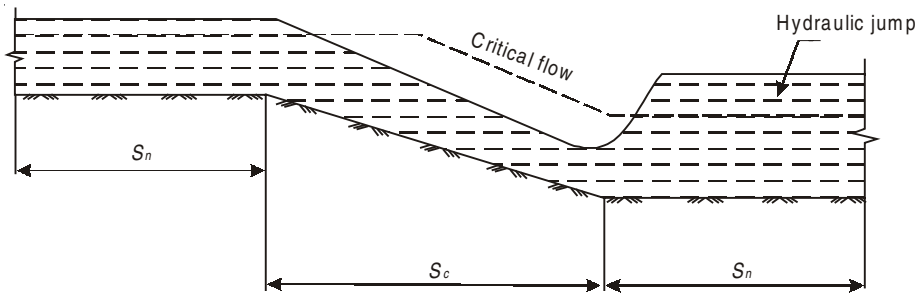


Fig. 13.4

2. At the entrance to a channel from a reservoir where a steep slope might exist the flow transition from sub-critical to supercritical would occur because tranquil flow condition prevails at the reservoir. (Fig. 13.5)

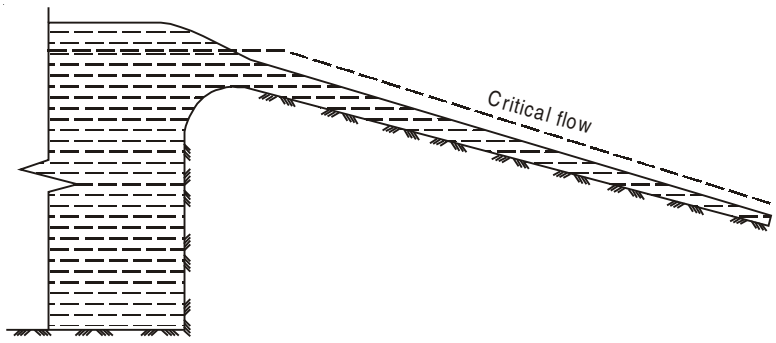


Fig.13.5

3. A transition might occur suddenly from sub-critical or tranquil to supercritical where the upstream slope is mild but at down stream it is a free fall.

A broad crested weir or spillway is a classic example. (Fig. 13.6).

Maximum discharge occurs over a broad crested weir because it is a free fall and hence the depth just prior to the fall is critical.

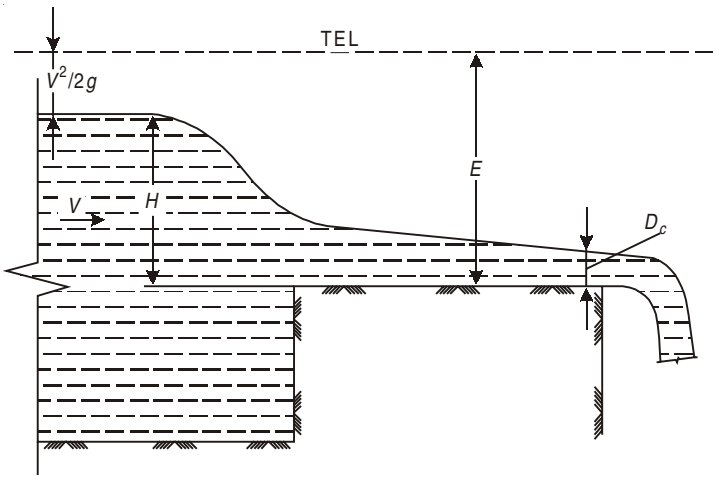


Fig.13.6

From Eq. (13.6) and referring to Fig. (13.6)

$$\left(\frac{Q^2}{g}\right) = \left(\frac{A_c^3}{b_c}\right)$$

$$\left(\frac{Q^2}{g}\right) = \frac{(D_c \times b_c)^3}{b_c}$$

$$Q^2 = g\left(D_c^3 \times b_c^2\right)$$

$$Q = \frac{\left(g\, D_c^3\right)^{1/2} \times b_c}{b_c} \qquad \text{Since } D_c = \frac{2}{3}\, E$$

$$Q = \left[g\left(\frac{2}{3}E\right)^3\right]^{1/2} \times b_c$$

$$Q = [1.705\,(E)^{1.5}]\, b_c \qquad \ldots (13.12)$$

Here E is the specific energy $\left(D_c + \frac{v^2}{2g}\right)$ measured above the crest of the weir, A_c = area of the channel, b_c = breadth of the channel. Since v is the upstream velocity.

If the height H is large compared to E, then $\frac{v^2}{2g}$ could be neglected as being too small to cause any significant difference. In such an instance Eq. (13.12) becomes,

$$Q = [1.705\,(H)^{1.5}]\,b_c \qquad ...(13.13)$$

Figure (13.7) represents a standing wave flume as given by Eq. (13.12)

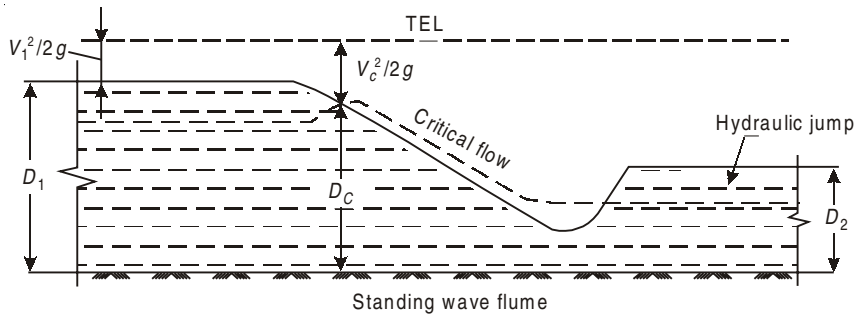


Fig. 13.7

A transition from tranquil to critical or super-critical flow might occur due to *lateral contractions.* (Fig. 13.8)

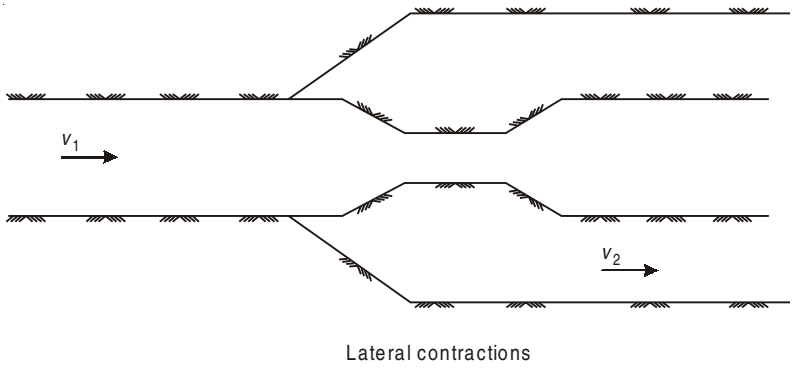


Fig. 13.8

When the width of the channel changes, while the slope and the discharge remains constant, the flow velocity suffers from the effect of lateral contraction. If the flow is tranquil or sub-critical the depth of flow will decrease whereas in case of super-critical flows the depth increases as the channel narrows. If the channel expands instead of contracting the depth of flow decreases and the transition is now from critical to tranquil.

A lateral contraction followed by an expansion results in the formation of a venturi flume similar to a venturimeter and is often taken advantage to measure

the flow through the channel provided the free surface is above the critical depth. (Fig. 13.9).

Referring to Fig. 13.9 and from the continuity equation

$$B_1 D_1 v_1 = B_2 D_2 v_2$$

$$v_1 = \left(\frac{B_2 D_2}{B_1 D_1}\right) v_2 \qquad ...(a)$$

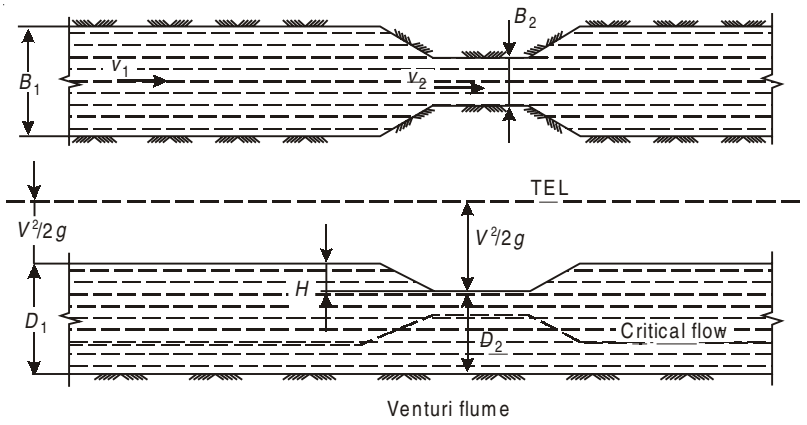


Fig. 13.9

If we neglect the losses $E_1 = E_2$.

$$D_1 + \frac{v_1^2}{2g} = D_2 + \frac{v_2^2}{2g}$$

$$D_1 - D_2 = \frac{v_2^2}{2g} - \frac{v_1^2}{2g} \qquad ...(b)$$

$$H = \frac{v_2^2}{2g} - \frac{\left[\left(\frac{B_2 D_2}{B_1 D_1}\right) v_2\right]^2}{2g}$$

$$H = \frac{v_2^2}{2g}\left[1 - \left(\frac{B_2 D_2}{B_1 D_1}\right)^2\right]$$

$$v_2 = \sqrt{\frac{2gH}{\left[1 - \left(\frac{B_2 D_2}{B_1 D_1}\right)^2\right]}} \qquad ...(13.14)$$

Since theoretical flow velocities are different from actual flow velocities,

coefficient of discharge which is of the order of 0.9 has to introduced in the discharge equation to obtain actual the discharge.

Even a hump in the channel bed often causes critical flow conditions and the formation of a standing wave (hydraulic jump).

Referring to Fig. 13.10 and applying the energy equation between sections 1 and 2,

$$\frac{v_1^2}{2g} + D_1 = \frac{v_2^2}{2g} + D_2 + Z_2 + \Delta E \qquad ...(13.15)$$

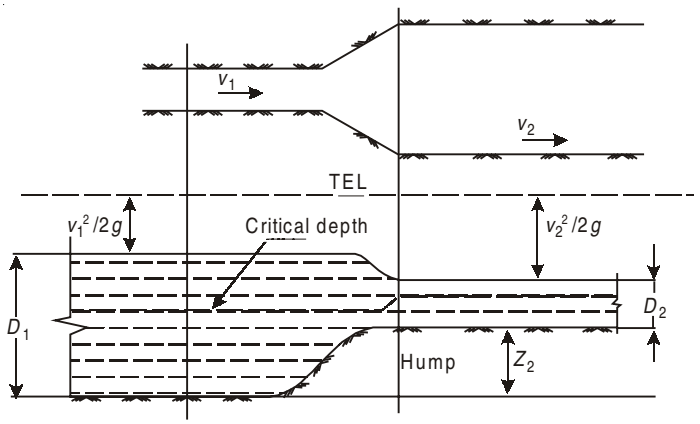


Fig.13.10

13.3 GRADUALLY VARIED FLOW

Hitherto in this chapter we discussed solutions to problems associated with non-uniform flows where the changes in depth or cross section of the channel were abrupt and hence could be considered as rapidly varied flow. In rapidly varied flows it is often necessary to consider the effect of acceleration of the fluid and the resulting change in momentum.

Gradually varied flow (GVF) is a non-uniform flow of a special kind in that the total energy line, the free water surface line and slope of the bed are not parallel to each other at any point as each line has its own slope. (Ref. Fig. 13.11). The basic feature of the gradually varied flow is the assumption that slope of the bed varies gradually over long distances so that it can be assumed the depth measured normal to the bed is equal to the vertical depth. Further, it is assumed that changes in area, depth, and surface roughness is not abrupt but occur gradually.

If it further assumed that the channel is rectangular, prismatic and of uniform roughness and the pressure distribution is hydro-static, it is possible to derive a mathematical expression that determines the depth of flow at desired locations from which a profile of the free water surface could emerge.

In deriving this expression it is assumed that either Chezy's equation or Manning's equation are as well applicable to steady non-uniform gradually varied flow as they are to steady uniform flows.

Applying the Bernoulli's equation between points 1 and 2 in Fig. 13.11.

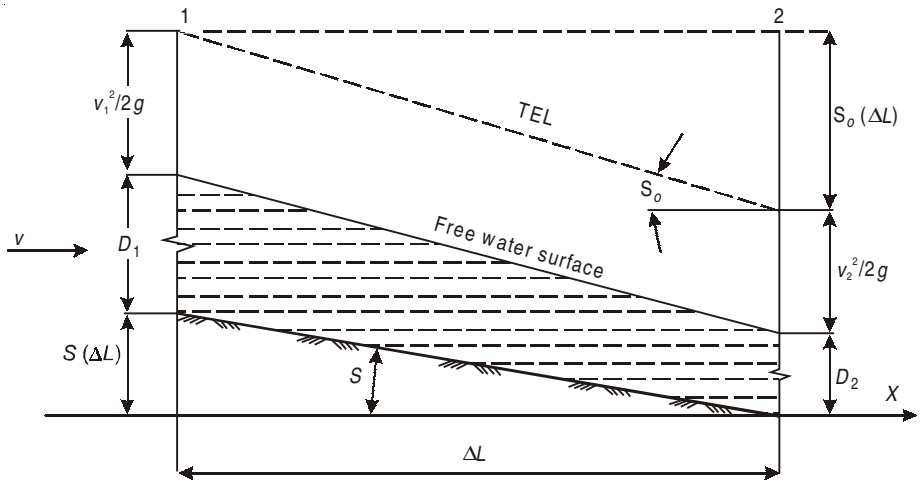


Fig. 13.11

$$(S \times \Delta L) + D_1 + \frac{v_1^2}{2g} = D_2 + \frac{v_2^2}{2g} + (S_0 \times \Delta L)$$

$$(S \times \Delta L) + D_1 + \frac{v_1^2}{2g} = (D_1 - \Delta D) + \frac{(v_1 + \Delta v)^2}{2g} + (S_0 \times \Delta L)$$

Expanding the terms within the bracket,

$$(S \times \Delta L) + D_1 + \frac{v_1^2}{2g} = (D_1 - \partial D) + \frac{(v_1^2 + 2v\partial v + \partial v^2)}{2g} + (S_0 \times \Delta L)$$

$$(S \times \Delta L) + D_1 + \frac{v_1^2}{2g} = (D_1 - \partial D) + \frac{v_1^2}{2g} + \frac{v\partial v}{g} + \frac{\partial v^2}{2g} + (S_0 \times \Delta L)$$

$$(S \times \Delta L) = -\partial D + \frac{v\partial v}{g} + \frac{\partial v^2}{2g} + (S_0 \times \Delta L)$$

Neglecting the term $\frac{\partial v^2}{2g}$ as being too small

$$(S \times \Delta L) = -\partial D + \frac{v\partial v}{g} + (S_0 \times \Delta L)$$

Divide throughout by ΔL

$$S = -\frac{\partial D}{\Delta L} + \frac{v\partial v}{g\ \Delta L} + S_0$$

$$\frac{\partial D}{\Delta L} = \frac{v\partial v}{g\ \Delta L} + (S_0 - S) \qquad \text{...(a)}$$

The discharge/unit width q,

$q = (D + \partial D)(v + \partial v)$ ... from continuity

$vD = (D + \partial D)(v + \partial v)$

$vD = vD + v\partial \mathrm{D} + D\partial v$

... again neglecting second order terms

Divide throughout by D and rearrange,

$$\partial v = -\frac{v\partial D}{D} \qquad \text{...(b)}$$

Substituting (b) in (a)

$$\frac{\partial D}{\Delta L} = -\frac{v(v\partial D)}{g\,D\Delta L} + (S_0 - S)$$

$$\frac{\partial D}{\Delta L} = (S_0 - S) - \frac{v^2(\partial D)}{g(D)\Delta L}$$

Multiplying throughout by $\frac{\Delta L}{\partial D}$

$$1 = \frac{\Delta L}{\partial D}(S_0 - S) - \frac{v^2}{g(D)}$$

$$1 + \frac{v^2}{g(D)} = \frac{\Delta L}{\partial D}(S_0 - S)$$

$$\frac{\partial D}{\Delta L} = \frac{(S_0 - S)}{\left[1 + \frac{v^2}{gD}\right]} \qquad \text{...(13.16)}$$

$$\frac{\partial D}{\Delta L} = \frac{(S_0 - S)}{1 + F_R^2} \qquad \text{...(13.17)}$$

(Since $F_R = \frac{v}{\sqrt{gD}}$)

$$\frac{\partial D}{\Delta L} = \frac{(S_0 - S)}{\left[1 + \left(\frac{D_c}{D}\right)^3\right]} \quad ...(13.18)$$

[(Since $v = \frac{q}{D}$), $\left[\frac{v^2}{gD} = \frac{q^2}{gD^3}\right]$ or $\left(\frac{v^2}{gD} = \left(\frac{D_c}{D}\right)^3\right)$

In the above equations D represents the depth of flow measured perpendicular to the channel bed and L represents the distance measured along the channel bed in the direction of flow. Hence $\frac{\partial D}{\Delta L}$ represents, rate of change of depth.

Consequently if $\frac{\partial D}{\Delta L} = 0$ the slope is zero which means the depth is constant, if $\frac{\partial D}{\Delta L}$ is > 0 the slope is positive ($S_0 < S$) indicating that the water level is increasing in the direction of flow (back water curve), if $\frac{\partial D}{\Delta L}$ is < 0 the slope is negative ($S_0 > S$) indicating that the water level is decreasing in the direction of flow. If the water level is increasing in the direction of flow then Eq. (13.17) gets modified, thus,

$$\frac{\partial D}{\Delta L} = \frac{(S_0 - S)}{\left[1 - \left(\frac{D_c}{D}\right)^3\right]} \quad ...(13.19)$$

Chezy's velocity equation $v = C\sqrt{RS_0}$

Discharge/unit width $q = DC\sqrt{RS_0}$ ...(a)

R is the hydraulic mean depth and S_0 is the slope of the energy gradient.

If D_N is the normal depth for a bed slope S then discharge per unit width

$$q = D_N C\sqrt{R_N S} \quad ... (b)$$

Since in case of GVF the energy gradient line (S_0) and the slope of the bed (S) are equal,

$$D\sqrt{D}S_0 = D_N \sqrt{D_N S} \text{ from which,}$$

$$D\sqrt{R}S_0 = D_N\sqrt{RS} \qquad \text{...(c)}$$

If the channel is assumed to be wide (The width of a wide channel is very large compared to its depth and hence depth factor is neglected while computing the wetted perimeter), $R = D$ and $R_N = D_N$. Substituting in (c)

$$D\sqrt{D}S_0 = D_N\sqrt{D_N S}$$

$$\left(\frac{S_0}{S}\right)^{1/2} = \left(\frac{D_N}{D}\right)^{3/2}$$

$$\left(\frac{S_0}{S}\right) = \left[\left(\frac{D_N}{D}\right)^{3/2}\right]^2$$

$$\left(\frac{S_0}{S}\right) = \left(\frac{D_N}{D}\right)^3$$

or
$$S_0 = S\left(\frac{D_N}{D}\right)^3 \qquad \text{...(d)}$$

From Eq. (13.19)
$$\frac{\partial D}{\Delta L} = \frac{(S_0 - S)}{\left[1 - \left(\frac{D_c}{D}\right)^3\right]} \qquad \text{...(e)}$$

Substituting (d) in (e)

$$\frac{\partial D}{\Delta L} = \frac{S\left(\frac{D_N}{D}\right)^3 - S}{\left[1 - \left(\frac{D_c}{D}\right)^3\right]}$$

$$\frac{\partial D}{\Delta L} = \frac{S\left[\left(\frac{D_N}{D}\right)^3 - 1\right]}{\left[1 - \left(\frac{D_c}{D}\right)^3\right]} \qquad \text{...(13.20)}$$

Based on these considerations *surface profiles*, *curves that define the profile of the free water surface for different conditions of slope,* each with definite characteristic are classified. Surface profiles are classified by a letter and a number.

The letter refers to type of bed slope whether mild (M) or adverse (A). The number refers to the number of profiles possible for a given slope.

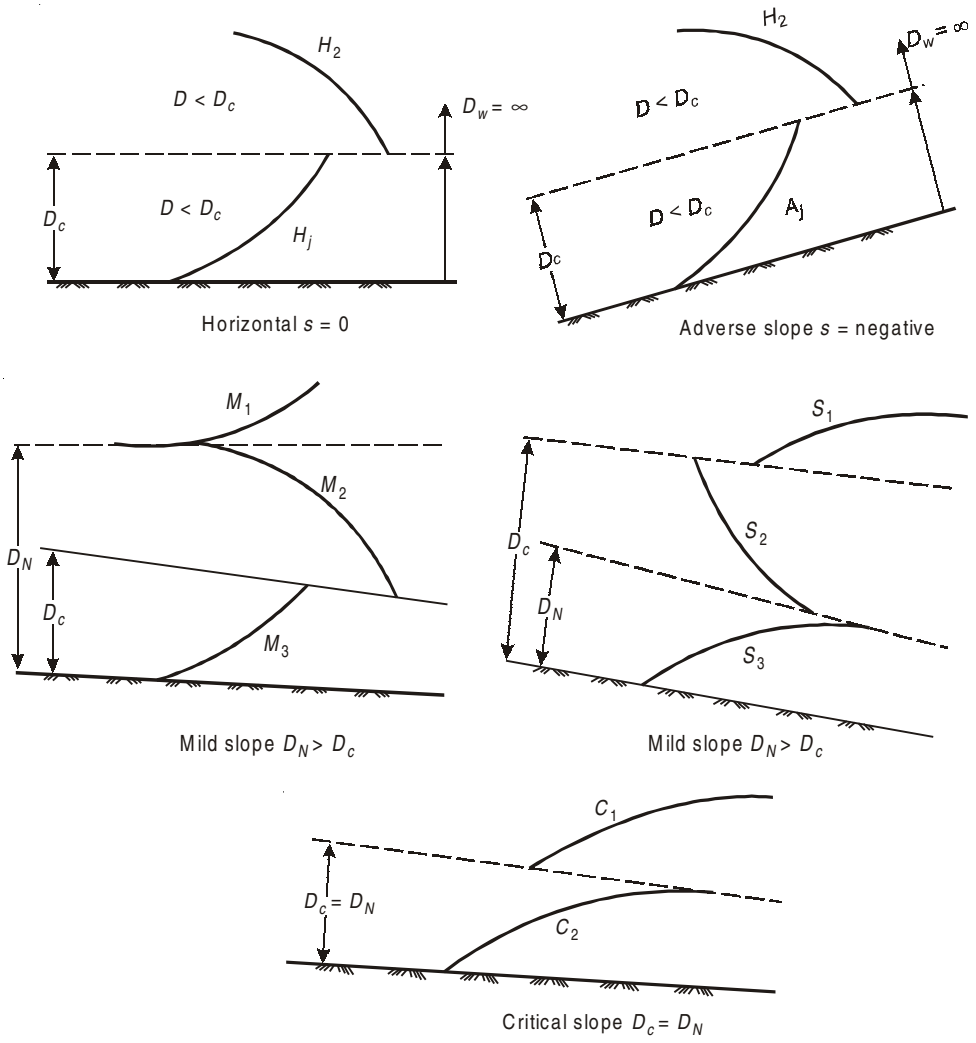


Fig. 13.11

The slope is classified as *horizontal* if ($S_0 = S = 0$). In this case the normal depth is infinite and the flow may be either above or below the critical depth.

If S increases in the direction of flow then S_0 is negative then the profile is said to be *adverse*. In this case there is no normal depth but the flow may be either above or below the critical depth.

It may be noted that for horizontal and adverse flows, uniform flow is not possible because in either case, normal depth is infinite. Hence only two profiles are possible in each case.

The profile is classified as *mild* if the slope is mild and when the normal depth D_N is greater than the critical depth D_c. Such profiles might occur for normal, below normal and critical depths.

When the normal depth D_N is less than the critical depth D_c the flow in the channel is rapid because the slope is steep which results in *steep profiles*.

The profile is termed *critical* when the normal depth and critical depths are equal. Corresponding to depths above and below the critical depth, only two profiles are possible.

If the depth of the water surface increases in the direction of flow, the profile is called *back water curve*. If the depth decreases the profile is termed *drop down curve*. Refer Fig. 13.12. The ratio of the difference in the specific energy levels ($E_2 - E_1$) and the difference in the slope between the energy line and the bed ($S - S_0$) determines the length of the backwater curve. ($D_2 - D_1$) is called the afflux.

Hydraulic jump is an ideal example to understand the importance of the study of water surface profiles. It is important to cause a hydraulic jump to occur on the apron of a spillway to prevent damage to the bed and side wall surfaces. This can be done only if we can determine where the jump would occur. One of the method adopted is to determine the conjugate depth, the depth beyond which the jump would occur, which is compatible with the surface profile upstream and downstream. Hydraulic jump would occur at the intersection of the depth conjugate to H_3 curve (Refer Fig. 13.12) and is made up of portions of both H_3 and H_2 curves because the jump lies in between.

Where, the slope of the channel changes from steep to mild, the jump may occur either at the upstream or downstream of the break. If the conjugate depth corresponding to the normal depth on the upstream slope of the jump is less than the normal depth down stream, the jump would occur on the upstream slope.

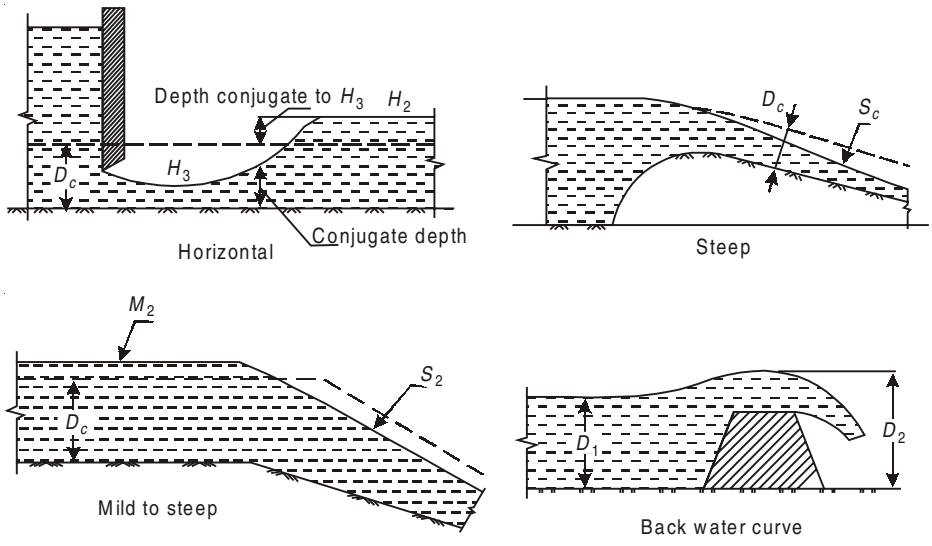


Fig.13.12

If this is greater than the normal depth down stream of the break, the jump must occur down stream as it cannot occur upstream.

In the case of GVF, the energy grade line which is the sine of the angle EGL makes with the horizontal, the HGL and the angle the channel bed makes with the horizontal are all different. While computing distance L between two sections having depths D_1 and D_2 computations are carried by what is called *step-by-step* method.

TYPICAL PROBLEMS AND SOLUTIONS

Problem 13.1 *A rectangular channel discharging 20 m^3/sec is 3 m wide × 1m deep at one section. If at the other section a hydraulic jump occurs determine the depth of flow at this section.*

Solution

From Eqs. (13.7) and (13.10)

$$D_2 = \left(\frac{D_1}{2}\right)\left[\sqrt{(1+8\,F_R^2)}-1\right] \qquad \ldots(a)$$

$$F_{R1} = \frac{v_1}{\sqrt{gD_{h1}}} \qquad \ldots(b)$$

D_{h1} is the hydraulic depth which is a ratio of the area of the channel to the breadth of the free surface of the fluid in the channel. Thus $D_{h1} = \dfrac{A}{b}$

$$D_{h1} = \frac{A}{b}$$

$$= \frac{(3 \times 1)}{3} = 1 \text{ m}$$

$$v_1 = \frac{Q}{A_1}$$

$$= \frac{20}{3 \times 1} = 6.66 \text{ m/sec}$$

$$F_R = \frac{6.66}{\sqrt{g \times 1}} = 2.12$$

(Satisfies the condition that the flow must be super-critical before the jump)

Substituting these values in Eq. (a)

$$D_2 = \left(\frac{1}{2}\right)\left[\sqrt{(1+8\,(2.12)^2}-1\right] = 2.54 \text{ m}$$

Problem 13.2 *A rectangular channel discharging 5 m^3/sec is 3 m wide × 0.5 m deep at one section. Determine whether a hydraulic jump is likely to occur. If a jump occurs at all then determine its height and head loss due to the jump.*

Solution

The necessary and sufficient condition for a hydraulic jump to occur is that the critical depth is more than the depth on the upstream side.

From Eq. (13.2)

$$\text{Critical depth } D_c = \left(\frac{q^2}{g}\right)^{\frac{1}{3}} \qquad \ldots(a)$$

q = discharge per unit width of the channel = $\frac{5}{3}$ = 1.66 m^3/sec

$$D_c = \left(\frac{1.66^2}{g}\right)^{\frac{1}{3}}$$

$$= 0.655 \text{ m}$$

Hydraulic jump occurs as $D_c > 0.5$ m

The depth of flow after the jump can be determined from Eqs. (13.7) and (13.10)

$$D_2 = \left(\frac{D_1}{2}\right)\left[\sqrt{(1+8F^2R)} - 1\right] \qquad \ldots(a)$$

$$F_{R1} = \frac{v_1}{\sqrt{gD_{h1}}} \qquad \ldots(b)$$

D_{h1} is the hydraulic depth which is a ratio of the area of the channel to the breadth of the free surface of the fluid in the channel. Thus $D_{h1} = \frac{A}{b}$

$$D_{h1} = \frac{A}{b}$$

$$= \frac{(3 \times 0.5)}{3} = 0.5 \text{ m}$$

$$v_1 = \frac{Q}{A_1} = \frac{5}{3} \times 0.5$$

$$= 3.33 \text{ m/sec}$$

$$F_R = \frac{3.33}{\sqrt{g}} \times 0.5 = 1.50$$

(Satisfies the condition that the flow must be super-critical before the jump)
Substituting these values in Eq. (a)

$$D_2 = \left(\frac{5}{2}\right)\left[\sqrt{(1+8\ (1.50)^2 - 1}\right]$$

$$= 0.839 \text{ m}$$

$$\text{Height of jump} = (0.839 - 0.5) = 0.339 \text{ m}$$

From Eq. (13.9) head loss due to the jump is given by,

$$H_L = \frac{(D_2 - D_1)^3}{(4D_1\ D_2)}$$

$$= \frac{(0.339)^3}{(4 \times 0.5 \times 0.839)} = 0.0232 \text{ m}$$

Problem 13.3 *Water is being discharged from a sluice gate 2 m wide. At point upstream the depth 0.5 m and the flow velocity is 15 m/sec. Assuming that a hydraulic jump would occur downstream, determine the Froude numbers before and after the jump, the flow velocity after the jump and the power dissipated.*

Solution

From Eq. (13.10)

$$F_{R1} = \frac{v_1}{\sqrt{gD_{h1}}}$$

$$F_{R1} = \frac{15}{\sqrt{g}\,(0.5)} = 6.77$$

From Eq. (13.11)

$$F_{R2} = \frac{2\sqrt{2}\,(6.77)}{\left[\left(\sqrt{1+8\times 6.77^2}\right)-1\right]^{1.5}} = 0.095$$

$$v_2 \times D_{h2} = v_1 \times D_{h1}$$

$$v_2 \times D_{h2} = (15) \times (0.5) = 7.5 \text{ m}^2/\text{sec}$$

or
$$D_{h2} = \frac{7.5}{v_2} \qquad \ldots(a)$$

$$F_{R2} = \frac{v_2}{\sqrt{g}Dh_2}$$

$$(F_{R2})^2 = \frac{v_2^2}{gDh_2} \qquad \ldots(b)$$

Substituting the value of D_{h2} from (a) in (b)

$$(F_{R2})^2 = \frac{v_2^3}{g(7.5)}$$

$$v_2^3 = (F_{R2})^2 \times g \times 7.5 = (0.095)^2 \times g \times 7.5 = 0.6640$$

$$v_2 = 0.872 \text{ m/sec}$$

$$D_{h2} = \frac{7.5}{0.872} = 8.60 \text{ m}$$

From Eq. (13.8) head loss due to the jump is given by,

$$\Delta E = \frac{(D_2 - D_1)^3}{(4D_1 D_2)}$$

$$= \frac{(8.60 - 0.5)^3}{(4 \times 0.5 \times 8.60)}$$

$$= 30.9 \text{ m.N/N}$$

Power dissipated = $\gamma Q \Delta E = (9.81 \times 1000)(W \times v_2 \times D_{h2}) \times 30.9$

W = width of the sluice gate = 2 m and $v_2 \times D_{h2} = 7.5$ m^2/sec

Power dissipated = $(9.81 \times 1000)(2 \times 7.5) \times 30.9 = 4.547$ MW

Problem 13.4 *The flow through a rectangular channel 3 m wide and 2.5 m deep is 12 m^3/sec. Determine the critical depth, critical velocity and the condition of the flow that is whether the flow is sub-critical or super-critical, laminar or turbulent. Take $v = 8.03 \times 10^{-7}$ m^2/sec.*

Solution

From Eq. (13.2) the critical depth is

$$D_c = \left(\frac{q^2}{g}\right)^{\frac{1}{3}}$$

Since q here represents the discharge per unit width $q = \frac{12}{3} = 4$ m

$$D_c = \left(\frac{4^2}{g}\right)^{\frac{1}{3}} = 1.17 \text{ m}$$

From Eq. (13.1) the specific energy E is given by

$$E = D + \frac{v^2}{2g} = 2.5 + \frac{1.6^2}{2g} = 2.63 \text{ m}$$

$$\left(v = \frac{Q}{A} = \frac{12}{3} \times 2.5 = 1.6 \text{ m/sec}\right)$$

From Eq. (13.3) the critical velocity is given by,

$$v_c = \sqrt{gD_c} = \sqrt{g \times 1.17} = 3.38 \text{ m/sec}$$

Condition of flow that is whether the flow is critical, sub-critical or super-critical can be determined from the Froude number (F_R),

$$F_R = v\sqrt{gD_H}$$

D_H is the hydraulic depth which is a ratio of the area of the channel to the breadth of the free surface of the fluid in the channel. Thus $D_H = \frac{A}{b}$

$$D_H = \frac{A}{b} = \frac{(3 \times 25)}{3} = 2.5 \text{ m}$$

$$F_R = \frac{2}{\sqrt{g \times 25}} = 0.40$$

Since $F_R < 1$, the flow is tranquil or sub-critical. The flow condition can also be assessed merely from the fact that $D_c < D$ and hence flow is sub-critical.

To determine whether the flow is laminar or turbulent we use the Reynolds number

$$R_E = \frac{VR}{v}$$

v is the mean flow velocity, R is the hydraulic radius and v is the kinematic viscosity.

$$R = \frac{A}{WP} = \frac{(3 \times 2.5)}{(3 + 2 \times 2.5)} = 0.9375 \text{ m}$$

$$v = 8.03 \times 10^{-7} \text{ m}^2 \text{/sec.}$$

$$R_E = \frac{(1.6) \times (0.9375)}{(8.03 \times 10^{-7})} = 1.875 \times 10^6$$

Hence the flow is turbulent but sub-critical.

Problem 13.5 *Calculate the critical depth corresponding to a flow of 5 m^3/ sec through a triangular channel of side slopes 2 horizontal to 1 vertical*

Solution

From Eq. (13.6),

$$\left(\frac{Q^2}{g}\right) = \left(\frac{A_c^3}{b_c}\right) \qquad \text{...(a)}$$

(b_c = Top width of the channel at critical depth)

For a triangular section $A_c = nD^2$ and $b_c = 2nD$

Substituting these values in (a)

$$\left(\frac{Q^2}{g}\right) = \frac{(nD^2)^3}{2nD}$$

$$\left(\frac{Q^2}{g}\right) = \left(\frac{n^2D^5}{2}\right)$$

$$D = \left[2\left(\frac{Q^2}{gn^2}\right)\right]^{1/5}$$

$$D_c = \left[2\left(\frac{5^2}{g \times 2^2}\right)\right]^{1/5}$$

$$D_c = 1.05 \text{ m}$$

Problem. 13.6 *Determine the minimum specific energy corresponding to a critical depth of 1.2 m for flow through a trapezoidal channel of normal slope with base width 3 m and side slope 1.5 horizontal to 1 vertical.*

Solution

From Eq. (13.1) the minimum specific energy corresponding to given critical depth is

$$E = D_c + \frac{v_c^2}{2g} \qquad \text{...(a)}$$

From Eq. (13.12),

$$\left(\frac{Q^2}{g}\right) = \left(\frac{A_c^3}{bc}\right)$$

But

$$v_c = \frac{Q^2}{A_c}$$

$$\frac{v_c^2}{2g} = \frac{Q^2}{2g(A_c)^2}$$

$$\frac{v_c^2}{2g} = \frac{A_c}{2(b_c)} \qquad \text{...(b)}$$

We can now eliminate $(v_c^2/2g)$ from Eqs. (a) and (b)

$$E = D_c + \frac{A_c}{2(b_c)} \qquad \text{...(c)}$$

For a trapezoidal channel $A_c = (b + n\,D_c)\,D_c$ and $b_c = (b + 2n\,D_c)$

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$$E = D_c + (b + n\,D_c)\,\frac{D_c}{2(b + 2n\,D_c)}$$

$$E = 1.2 + (3 + 1.5 \times 1.2)\,\frac{1.2}{2(3 + 2 \times 1.5 \times 1.2)}$$

$$E = 0.527 \text{ m}$$

Problem 13.7 *Determine the slope of the rectangular channel of base width 2m that will ensure critical flow at normal depth of 1.5 m. Take Manning's constant n = 0.015.*

Solution

From Eq. (13.6),

$$\left(\frac{Q^2}{g}\right) = \left(\frac{A_c^3}{b_c}\right) \qquad \text{...(a)}$$

$$A_c = 2 \times 1.5 = 3 \text{ m}^2$$

$$Q = \sqrt{g\left(\frac{A_c^3}{b_c}\right)} = \sqrt{g\left(\frac{3^3}{2}\right)} = 11.5 \text{ m}^3/\text{sec}$$

From Eq. (12.5)

$$Q = \left(\frac{1.00}{n}\right) A R^{\frac{2}{3}} S^{\frac{1}{2}}$$

$$R = \frac{A}{WP} = \frac{3}{(1.5 + 1.5 + 2)} = 0.6 \text{ m}$$

$$11.5 = \frac{(1.00)}{(0.015)}\,(3)\,(0.6)^{2/3}\,S^{1/2}$$

$$S^{1/2} = 11.5 \times \frac{(0.015)}{(3)(0.6)^{2/3}} = 0.080$$

$$S = 6.4 \times 10^{-3} = 0.64\ \%$$

Problem 13.8 *Determine the critical depth for a flow of 0.5 m^3/sec in a triangular channel of apex angle = 60°*

Solution From Eq. (13.6),

$$\left(\frac{Q^2}{g}\right) = \left(\frac{A_c^3}{b_c}\right) \qquad \text{...(a)}$$

For a triangular section $b_c = 2\,D_c \tan\frac{\theta}{2}$

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$$A_c = \left(\frac{1}{2} \times 2D_c \tan\frac{\theta}{2}\right) \times D_c$$

$$A_c = \left(D_c^2 \tan\frac{\theta}{2}\right)$$

Substitute the above values in (a)

$$\left(\frac{0.5^2}{g}\right) = \left(D_c^2 \tan\frac{\theta}{2}\right)^3 \left(2D_c \tan\frac{\theta}{2}\right)$$

$$\left(\frac{0.5^2}{g}\right) = D_c^5 \frac{(\tan 30)^2}{2}$$

$$\left(\frac{0.5^2}{g}\right) = D_c^5 \frac{(0.577)^2}{2}$$

$$D_c^5 = 0.153$$

$$D_c = 0.687 \text{ m}$$

Problem 13.9 *A rectangular channel discharging 20 m^3/sec is 3 m wide × 1 m deep at one section. If at the other section a hydraulic jump occurs determine the depth of flow at this section.*

Solution

From Eqs. (13.7) and (13.10)

$$D_2 = \left(\frac{D_1}{2}\right)\left[\sqrt{(1+8F_R^2)} - 1\right] \quad \text{...(a)}$$

$$F_{R1} = \frac{v_1}{\sqrt{gD_{h1}}} \quad \text{...(b)}$$

D_{h1} is the hydraulic depth which is a ratio of the area of the channel to the breadth of the free surface of the fluid in the channel. Thus $D_{h1} = \frac{A}{b}$

$$D_{h1} = \frac{A}{b} = \frac{(3\times 1)}{3} = 1 \text{ m}$$

$$v_1 = \frac{Q}{A_1} = \frac{20}{3} \times 1 = 6.66 \text{ m/sec}$$

$$F_R = \frac{6.66}{\sqrt{g \times 1}} \times 1 = 2.12$$

(Satisfies the condition that the flow must be supercritical before the jump)

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Substituting these values in Eq. (a)

$$D_2 = \left(\frac{1}{2}\right)\left[\sqrt{(1+8(2.12)^2}\right] = 2.54 \text{ m}$$

Problem 13.10 *A rectangular channel discharging 5 m^3/sec is 3 m wide × 0.5 m deep at one section. Determine whether a hydraulic jump is likely to occur. If a jump occurs at all then determine its height and head loss due to the jump.*

Solution The necessary and sufficient condition for a hydraulic jump to occur is that the critical depth is more than the depth on the upstream side.

From Eq. (13.2)

$$\text{Critical depth } D_c = \left(\frac{Q^2}{g}\right)^{1/3} \quad \text{...(a)}$$

q = discharge per unit width of the channel

$$= \frac{5}{3} = 1.66 \text{ m}^3/\text{sec}$$

$$D_c = \left(\frac{1.66^2}{g}\right)^{1/3} = 0.655 \text{ m}$$

Hydraulic jump occurs as $D_c > 0.5$ m

The depth of flow after the jump can be determined from Eqs. (13.7) and (13.10)

$$D_2 = \left(\frac{D_1}{2}\right)\left[\sqrt{(1+8F_R^2)-1}\right] \quad \text{...(a)}$$

$$F_{R1} = \frac{v_1}{\sqrt{gD_{h1}}} \quad \text{...(b)}$$

D_{h1} is the hydraulic depth which is a ratio of the area of the channel to the breadth of the free surface of the fluid in the channel. Thus $D_{h1} = \frac{A}{b}$

$$D_{h1} = \frac{A}{b} = \frac{(3\times 0.5)}{3} = 0.5 \text{ m}$$

$$v_1 = \frac{Q}{A_1} = \frac{5}{3} \times 0.5 = 3.33 \text{ m/sec}$$

$$F_R = \frac{3.33}{\sqrt{g \times 0.5}} = 1.50$$

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(Satisfies the condition that the flow must be super-critical before the jump)

Substituting these values in Eq. (a)

$$D_2 = \left(\frac{0.5}{2}\right)\left[\sqrt{(1+8(1.50)^2} - 1\right] = 0.839 \text{ m}$$

Height of jump = (0.839 – 0.5) = 0.339 m

From Eq. (13.8) head loss due to the jump is given by,

$$\Delta E = \frac{(D_2 - D_1)^3}{(4D_1D_2)}$$

$$= \frac{(0.339)^3}{(4 \times 0.5 \times 0.839)} = 0.0232 \text{ m}$$

Problem 13.11 *Discharge through a 4 m wide rectangular channel is 10 m^3/sec. The depth is uniform except for the existence of a hump as wide as the channel. If the height of the hump is 0.5 m determine the specific energy before the hump.*

Solution Critical depth D_c is given by (13.2)

$$D_c = \left(\frac{Q^2}{g}\right)^{1/3} \quad \text{...(a)}$$

$$q = \text{discharge per unit width} = \frac{10}{4} = 2.5 \text{ m}^3/\text{sec}$$

$$D_c = \left(\frac{2.5^2}{g}\right)^{1/3} = 0.860 \text{ m}$$

$$E_{\min} = \left(\frac{3}{2}\right)D_c \quad \text{... from Eq. (13.5)}$$

$$E_{\min} = \left(\frac{3}{2}\right) \times 0.860 = 1.29 \text{ m}$$

$$E - E_{\min} = \text{height of the hump} = 0.5 \text{ m}$$

$$E = E_{\min} + 0.5 = 1.29 + 0.5 = 1.79 \text{ m}$$

Problem 13.12 *Calculate the critical depth and the minimum specific energy for a discharge = 5 m^3/sec, through a rectangular channel 2 m wide. Determine the specific energy where the depth of flow is 0.5 m and the alternate depth for this energy.*

Solution Critical depth D_c is given by (13.2)

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$$D_c = \left(\frac{Q^2}{g}\right)^{1/3} \quad \text{...(a)}$$

q = discharge per unit width = $\frac{5}{2}$ = 2.5 m^3/sec

$$D_c = \left(\frac{2.5^2}{g}\right)^{1/3} = 0.860 \text{ m}$$

$$E_{\min} = \left(\frac{3}{2}\right) D_c \qquad \text{from Eq. (13.5)}$$

$$E_{\min} = \left(\frac{3}{2}\right) \times 0.860 = 1.29 \text{ m}$$

Where the depth of flow 0.5 m,

$$A = 2 \times 0.5 = 1 \text{ m}^2$$

$$v = \frac{5}{1} = 5 \text{ m/sec}$$

$$E = D + \frac{v^2}{2g} \text{ from Eq. (13.1)}$$

$$E = 0.5 + \frac{5^2}{2g} = 1.774 \text{ m}$$

For alternate depth corresponding to $E = 1.774$
Let D_A be the depth alternate to $D = 0.5$ m

$$E_2 = E = D_A + \frac{v_2^2}{2g}$$

(because the specific energy is the same at either depth)

$$D_A = E - \frac{v_2^2}{2g}$$

$$D_A = 1.774 - \frac{v_2^2}{2g} \quad \text{...(a)}$$

$A_1 \times v_1 = A_2 \times v_2$ ...from continuity equation

$Q = A_2 \times v_2$

$$5 = 2 \times D_A \times v_2 \qquad \text{Therefore } v_2 = \frac{2.5}{D_A} \quad \text{...(b)}$$

From (a) and (b)

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$$D_A = 1.774 - \frac{2.5^2}{2g \times D_A^2}$$

or $$D_A = 1.774 - \frac{0.318}{D_A^2}$$

Alternate depth corresponding to

$$E = 1.774$$

$$D_A = 1.65 \text{ m}$$

Problem 13.13 *Determine the discharge through a venturi flume whose breadth tapers down from 2 m at the entrance to 0.75 metres at the throat. The corresponding depths are 1.5 m and 1.25 m. Take* $C_d = 0.95$.

Solution

From Eq. 13.14

$$v_2 = \sqrt{\frac{2gH}{\left[1 - \left(\frac{B_2 D_2}{B_1 D_1}\right)^2\right]}} \qquad ...(13.16)$$

$$H = (1.5 - 1.25) = 0.25 \text{ m}$$

$$\left(\frac{B_2 D_2}{B_1 D_1}\right)^2 = \frac{(0.75 \times 1.25)}{(2 \times 1.5)^2} = 0.09$$

$$v_2 = \sqrt{\frac{2g \times (0.25)}{[1 - (0.09)]}} = 2.32 \text{ m/sec}$$

$$Q = C_d \times (B_2 D_2) \times v_2$$
$$= 0.95 \times (0.75 \times 1.25) \times 2.32 = 2.06 \text{ m}^3/\text{sec}$$

Problem 13.14 *Determine the two alternate depths of flow for a specific energy = 2 m and discharge/unit width = 1* m^3/sec.

Solution

From Eq. (13.1a)

$$E = D + \frac{v^2}{2g} \qquad ...(a)$$

$$E = D_1 + \frac{q^2}{2g\left(D_1^2\right)} \qquad ...(b)$$

$$D_1 = E - \frac{q^2}{2g\left(D_1^2\right)}$$

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$$D_1 = 2 - \frac{(1)^2}{2gD_1^2}$$

$$D_1 = 2 - \frac{0.0509}{D_1^2}$$

Alternate depth $D_1 = 1.988$ m (by trial and error)

Alternate depth D_2:

$$E = D_1 + \frac{v_1^2}{2g} \quad \text{from (a)}$$

$$2 = 1.988 + \frac{v_1^2}{2g} \qquad \ldots(c)$$

$$(2 - 1.988) = \frac{v_1^2}{2g}$$

or $$v_1 = \sqrt{2g(2-1.988} = 0.485 \text{ m/sec}$$

$$A_1 v_1 = A_2 v_2 \qquad \text{from continuity}$$

$$v_2 = \frac{A_1 v_1}{A_2} = \frac{[(b \times D_1)\ 0.485]}{(b \times D_2)}$$

$$v_2 = \frac{[(1.9888 \times 0.485)]}{D_2} = \frac{(0.9641}{D_2} \qquad \ldots(d)$$

$$E = D_2 + \frac{v_2^2}{2g} \qquad \ldots(e)$$

Substituting the value of v_2 in (e)

$$2 = D_2 + \frac{(0.9641)^2}{D_2^2 \times 2g}$$

$$2 = D_2 + \frac{(0.04737)}{D_2^2}$$

Alternate depth $D_2 = 0.157$ m (by trial and error)

Problem 13.15 *A transition occurs in the width of a rectangular channel from 8 m to 5 m and in the depth from 3 m to 3.5 m as shown in Fig. (13.11). Determine the discharge through the transition. Neglect energy loss.*

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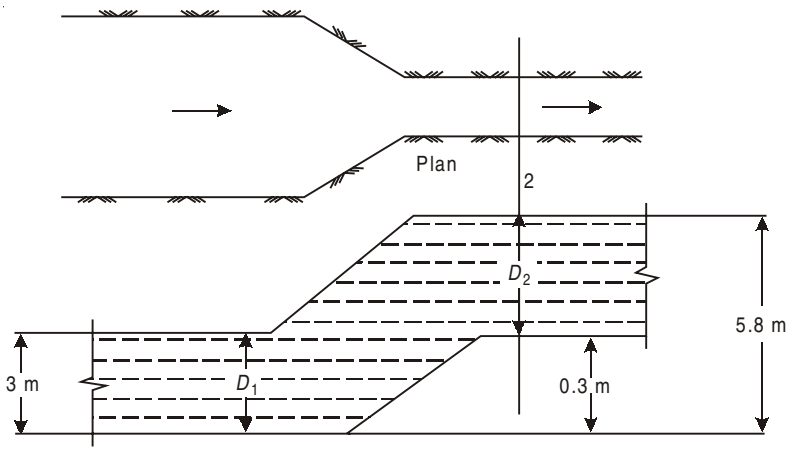


Fig. 13.13

Solution

From Eq. (15.3)

$$\frac{v_1^2}{2g} + D_1 = \frac{v_2^2}{2g} + D_2 + Z_2 + \Delta E$$

Since energy loss is neglected,

$$\frac{v_1^2}{2g} + D_1 = \frac{v_2^2}{2g} + (D_2 + Z_2)$$

$$\frac{Q^2}{(8\times 3)^2\ 2g} + 3 = \frac{Q^2}{(5\times 5.5)^2\ 2g} + 5.8$$

$$\frac{Q^2}{(29.35)} - \frac{Q^2}{(38.5)} = 5.8 - 3$$

$$\frac{Q^2}{(29.35)} - \frac{Q^2}{(38.5)} = 2.8$$

$$Q = 18.7 \text{ m}^3/\text{sec}$$

Problem 13.16 *It is proposed to effect a transition in a rectangular channel 3 m wide and 1.5 m deep. If the flow through the channel is 3 m^3/sec, determine the minimum width of contraction which will ensure that upstream flow condition remains unaffected. (Ref. Fig. 13.14)*

Solution Determine first the flow condition upstream; that is whether the flow is critical or otherwise. This can be ascertained by determining the Froude number.

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$$F_{R1} = \frac{v_1}{\sqrt{gD_{h1}}} \qquad \text{...(a)}$$

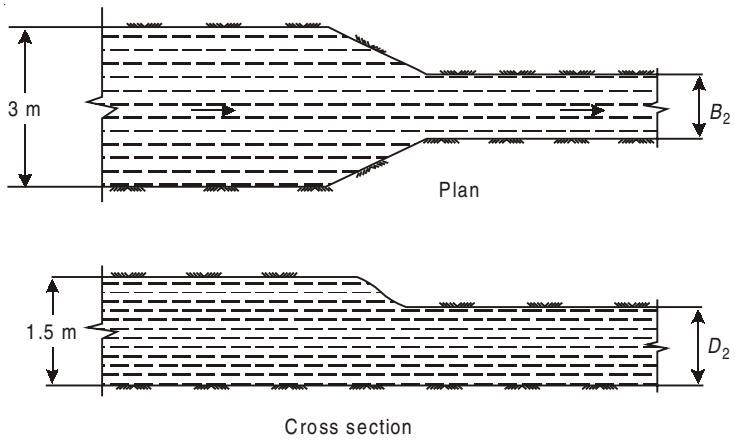


Fig. 13.14

D_{h1} is the hydraulic depth which is a ratio of the area of the channel to the breadth of the free surface of the fluid in the channel. Thus $D_{h1} = \frac{A}{b}$

$$D_{h1} = \frac{A}{b} = \frac{(3 \times 1.5)}{3} = 1.5 \text{ m}$$

$$v_1 = \frac{Q}{A_1} = \frac{3}{(3 \times 1.5)} = 0.666 \text{ m/sec}$$

$$F_{R1} = \frac{0.666}{\sqrt{g \times 1.5}} = 0.1736 \text{ (Tranquil flow)}$$

If this upstream flow condition should remain unaffected even after contraction then Froude number after contraction too must remain in the tranquil zone. This means $D_2 = D_c$.

$$D_c = \frac{2}{3} E \qquad \text{...(b)}$$

$$E = D + \frac{v^2}{2g}$$

$$E = \left[1.5 + \frac{0.666^2}{2g}\right] = 1.522 \text{ m}$$

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Therefore from (b) $D_c = \frac{2}{3}(1.522) = 1.015$ m

But from (13.2)

$$D_c = \left(\frac{q^2}{g}\right)^{1/3} \quad \text{...(c)}$$

$$1.015 = \left(\frac{q^2}{g}\right)^{1/3}$$

$$1.015 = \frac{(q)^{2/3}}{(g)^{1/3}}$$

$$1.015 = \frac{(q)^{2/3}}{2.140}$$

$$(q)^{2/3} = 1.015 \times 2.140 = 2.172$$

$$q = 3.20 \text{ m}^3/\text{sec per unit width of the channel}$$

But $Q = q \times B_2$.

Therefore minimum width $B_2 = \frac{Q}{q} = \frac{3}{3.20} = 0.9375$ m

Check : $F_{R2} = \frac{v_2}{\sqrt{gD_{h2}}}$

$$F_{R2} = \frac{3}{\frac{0.9375 \times 1.015}{\sqrt{g \times 1.015}}}$$

$= 0.998 < 1$ (Flow is tranquil)

Problem 13.17 *A rectangular channel contracts from 2.5 m to 1.2 m. At the zone of transition the channel bed suffers a hump = 0.1 m in height. If the discharge through the channel is 6.5 m^3/sec and the initial depth at the upstream section is 3 m, determine whether the upstream depth was affected and if so suggest a remedy.*

Solution For the flow to be continuous the specific energy at any section of the channel must be constant.

$$E_1 = E_2$$

$$E_1 = D_1 + \frac{v_1^2}{2g} = 3 + \frac{(6.5)^2}{(2.5 \times 3)^2 2g}$$

$$E_1 = 3.038 \text{ m}$$

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$$E_2 = D_2 + \frac{v_2^2}{2g} \quad \text{...(a)}$$

$$E_2 = E_{\min} + \text{height of the hump}$$

$$E_2 = E_{\min} + 0.1 \quad \text{...(b)}$$

$$E_{\min} = 1.5\, D_{c2} \quad \text{... from (13.5)}$$

But from (13.2)

$$D_{c2} = \left(\frac{q^2}{g}\right)^{1/3} = \left[\left(\frac{Q^2}{b_2^2 \times g}\right)\right]^{1/3}$$

$$= \left[\left(\frac{6.5^2}{1.2^2 \times g}\right)\right]^{1/3} = 2.99 \text{ m}$$

$$E_{\min} = 1.5\, D_{c2}$$

$$E_{\min} = 1.5 \times 2.99 = 4.486 \text{ m}$$

$$E_2 = E_{\min} + 0.1 \quad \text{...from (b)}$$

$$E_2 = 4.486 + 0.1 = 4.586 \text{ m}$$

Since $E_1 < E_2$ the remedy lies in increasing or decreasing the depth of flow D_1 such that $E_1 = E_2$. This means, $E_2 = D_1 + \frac{v_1^2}{2g}$

$$4.586 = D_1 + \frac{Q^2}{(b \times D_1)^2 \times 2g}$$

$$= D_1 + \frac{6.5^2}{(2.5 \times D_1)^2 \times 2g}$$

$$= D_1 + 0.3445\, D_1^2$$

Hence $D_1 + 0.3445\, D_1^2 = 4.586$

$D_1 = 2.48$ m (new depth at the upstream section)

Problem 13.18 *Find the rate of change of depth of water surface in a rectangular channel 15 m wide and 5 m deep. The slope of the channel bed 1 in 5000 and discharge = 40 m^3/sec. Manning's constant n = 0.017.*

Solution From Eq. (13.15) the rate of change of depth is given by,

$$\frac{\partial D}{\Delta L} = \frac{(S_0 - S)}{1 + \frac{v^2}{gD}} \quad \text{...(a)}$$

S_0 = the angle energy line makes with the horizontal which has to be determined from the Manning's equation.

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$$S = \frac{1}{5000}$$

$$= 2 \times 10^{-4}$$

$$v = \frac{Q}{A} = \frac{40}{(15 \times 5)}$$

$$= 0.533 \text{ m/sec.}$$

The Manning's velocity equation

$$v = \frac{1.00}{n} R^{2/3} S^{1/2} \qquad \text{...(b)}$$

Hydraulic radius $R = \frac{A}{WP}$

$$= \frac{(15 \times 5)}{(5 + 5 + 15)} = 3 \text{ m}$$

Substituting these values in (b)

$$0.533 = \left(\frac{1.00}{0.017}\right) 3^{2/3} S^{1/2}$$

$$S = S_0 = 1.898 \times 10^{-5}$$

Substituting this value in (a)

$$\frac{\partial D}{\Delta L} = \frac{\left(1.898 \times 10^{-5} - 2 \times 10^{-4}\right)}{\left(\frac{1 + 0.533^2}{g \times 5}\right)}$$

$$\frac{\partial D}{\Delta L} = \frac{(-1.81 \times 10^{-4})}{(5.79 \times 10^{-3})} = -0.0312$$

Since $\frac{\partial D}{\Delta L}$ is negative it indicates that the depth of water decreases in the direction of flow.

Problem 13.19 *Discharge through a rectangular channel 30 m wide is 50 m^3/sec. The angle that the channel bed makes with the horizontal = 0.113°. Identify the slope and designate the surface profiles accordingly. n = 0.017.*

Solution

From Eq. (13.2)

$$D_c = \left(\frac{q^2}{g}\right)^{1/3}$$

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$$D_c = \left[\frac{\left(\frac{50}{30}\right)^2}{g}\right]^{1/3} = 0.656 \text{ m}$$

The Manning's discharge equation

$$Q = \frac{1.00}{n} AR^{2/3} S_0^{1/2} \qquad \text{...(b)}$$

$$AR^{2/3} = \left[\left(\frac{Q \times n}{S_0^{1/2}}\right)\right]$$

$$S_0 = \text{sine } 0.113^\circ$$
$$= 1.97 \times 10^{-3}$$

$$AR^{2/3} = \frac{50 \times 0.017}{(1.97 \times 10^{-3})^{1/2}}$$

$$AR^{2/3} = 19.14$$

$$(30D)\left[\left(\frac{30D}{(30+2D)}\right)\right]^{2/3} = 19.14$$

Normal depth $D_N = 0.785$ m (by trial and error)

Since normal depth $D_N = 0.785$ m > critical depth $D_c = 0.656$ m the slope could be described as mild.

Curve in the region $D > 0.785$ is M_1 curve.
Curve in the region $0.785 > D > 0.656$ is M_2 curve.
Curve in the region $D < 0.656$ is M_3 curve.
] Ref. Fig. 13.11

Problem 13.20 *The depth of a wide channel with a slope 1 in 2000 varies from 2 m to 2.5 m over a length of 400m. If the discharge per unit width is 3 m^3/ sec, determine the Chezy's coefficient.*

Solution

$$\frac{\partial D}{\Delta L} = \left(\frac{2.5-2}{400}\right) = 1.25 \times 10^{-3}.$$

$$\frac{\partial D}{\Delta L} = \left(\frac{S_0 - S}{1 + \frac{v^2}{gD}}\right) \qquad \text{...(a)}$$

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$$S = \frac{1}{2000} = 5 \times 10^{-4}.$$

$$v = \frac{q}{D_m}$$

$$\text{Mean depth } D_m = \left(\frac{2.5+2}{2}\right) = 2.25 \text{ m}$$

$$v = \frac{3}{2.25} = 1.333 \text{ m/sec}$$

Substituting the relevant values in (a)

$$1.25 \times 10^{-3} = \left(\frac{S_0 - 5\times10^{-4}}{\left[1+\frac{(1.333)^2}{g\times2.25}\right]}\right)$$

$$1.25 \times 10^{-3} = \left(\frac{S_0 - 5\times10^{-4}}{[1.080]}\right)$$

$$S_0 = (1.25 \times 10^{-3}) \times (1.080) + (5 \times 10^{-4})$$

$$S_0 = 1.85 \times 10^{-3}$$

Chezy's velocity equation $v = C\sqrt{RS_0}$

$R = D_m = 2.25$ because the channel is described as wide. The width of a wide channel is very large compared to its depth and hence depth factor is neglected while computing the wetted perimeter.

$$1.333 = C\sqrt{(2.25\times1.85\times10^{-3})}$$

Chezy's coefficient $C = 20.66$

Problem 13.21 *A rectangular channel 4 m wide discharges 6 m^3/sec as the depth changes from 1.5 m at one section to 1 m at the other. Determine the slope of the energy line between these two sections. Assume GVF and take Chezy's coefficient C = 55.*

Solution Chezy's velocity equation $v = C\sqrt{RS_0}$

The equation can be re-arranged such that

$$S_0 = \frac{v^2}{C^2R}$$

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where S_0 = slope of the energy line at any section. The average slope between any two sections is given by,

$$S_A = \left(\frac{v_1^2}{C^2 R_1} + \frac{v_2^2}{C^2 R_2}\right) \Big/ 2$$

$$S_A = \left(\frac{v_1^2}{55^2 R_1} + \frac{v_2^2}{55^2 R_2}\right) \Big/ 2$$

$$v_1 = \frac{Q}{A_1} = \frac{6}{4} \times 1.5 = 1 \text{m/sec},$$

$$R_1 = \frac{A_1}{WP} = \frac{(4 \times 1.5)}{4 + 2(1.5)} = 0.857 \text{ m}$$

$$v_2 = \frac{Q}{A_2} = \frac{6}{4} \times 1.0 = 1.5 \text{ m/sec},$$

$$R_1 = \frac{A_2}{WP} = \frac{(4 \times 1)}{4 + 2(1)} = 0.666 \text{ m}$$

Substituting the relevant values in (a)

$$S_A = \left[\frac{1^2}{3025(0.857)} + \frac{1.5^2}{3025(0.666)}\right] \Big/ 2$$

$$S_A = 0.00075 \text{ m}$$

Problem 13.22 *After contracting below a sluice gate to a depth of 0.2 m, water discharges in to a wide horizontal channel with a velocity of 10 m/sec. If the height of the hydraulic jump is 1.5 m determine, the location of the depth conjugate to critical depth from the point of contraction. Manning's constant n = 0.020.*

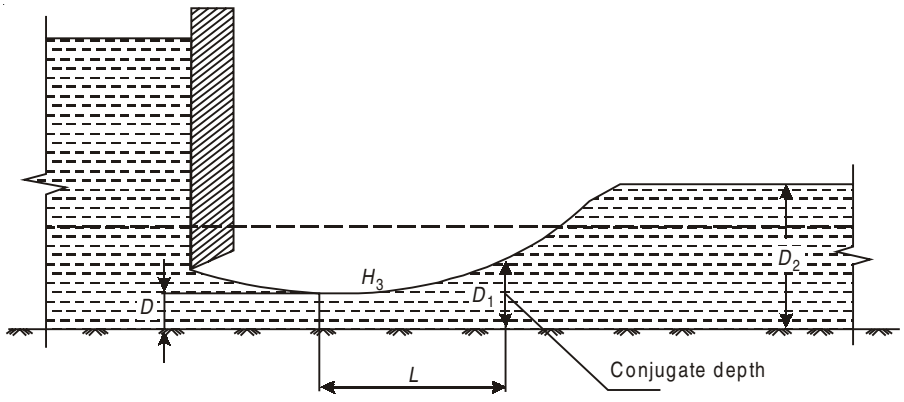


Fig. 13.15

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Solution Let D_2 be the height of the jump, D_1 be the conjugate depth before the jump and D be the depth at contraction. Let L be the distance from the point of contraction to conjugate depth. The depths are related to their corresponding Froude numbers in the following manner.

From Eq. (13.7)

$$D_2 = \left(\frac{D_1}{2}\right)\left[\sqrt{\left(1+8F_R^2\right)}-1\right] \qquad ...(a)$$

F_R is Froude number at the upstream section.

Let us apply the above equation to points between D and D_1.

$$D_1 = \left(\frac{D}{2}\right)\left[\sqrt{\left(1+8F_R^2\right)}-1\right] \qquad ...(b)$$

$$F_R = \frac{v}{\sqrt{gD}} = \frac{10}{\sqrt{g\times 0.2}} = 1.4$$

Substituting in (b) the relevant values,

$$D_1 = \left(\frac{0.2}{2}\right)\left[\sqrt{\left(1+8(1.4)^2\right)}-1\right]$$

$$D_1 = 0.308 \text{ m}$$

$$q = v \times D_1 = 10 \times 0.308 = 3.08 \text{ m}^3/\text{sec}$$

The distance L between two sections, having depths D and D_1, (in this case the H_3 profile) is determined by means of the step by step method and in a tabular format. First we arbitrarily select 5 steps corresponding to 5 depths (the higher the number of steps, the greater will be the accuracy) starting from 0.308 m in steps of 0.020 m and work backwards till we get to 0.228 m. ΔL corresponding to change in depth ΔD can be determined from the known conditions upstream if the flow upstream is tranquil and from the known conditions downstream if the flow down stream is shooting flow.

M_3–Profile $q = 3.08$ m^3/sec $n = 0.020$ $S = 0$

1 D_1 m	2 V m/sec	3 E m	4 ΔE m	5 S_0	6 S_A	7 ΔL m	8 L m
(a) 0.228	13.50	9.517		0.523			0
			–1.407		0.459		3.06
(b) 0.248	12.42	8.110		0.396			3.06
			–1.110		0.351	3.16	
(c) 0.268	11.50	7.00		0.306			6.26
			– 0.877		0.273	3.21	
(d) 0.288	10.70	6.123		0.240			9.47
			– 0.723		0.216	3.34	
(e) 0.308	10.0	5.4		0.192			12.81

$v = \frac{q}{D}$, $\Delta E = (b^3 - a^3)$ etc. $S_0 = v^2 n^2 / R^{1.33}$ $S_A = (a^5 + b^5) / 2$ etc. $\Delta L = \Delta E / (S - S_A)$, $L = \sum \Delta L$ $R = D$ (because the channel is wide)

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Problem 13.23 *The slope of a wide rectangular channel changes from 1 in 100 to 1 in 1500 and the discharge per unit width is 4 m^3/sec. Determine the position and the height of the hydraulic jump. Mannings coefficient n = 0.015.*

Solution This is an instance where the slope of the channel changes from steep to mild. The hydraulic jump can occur either upstream or down stream of

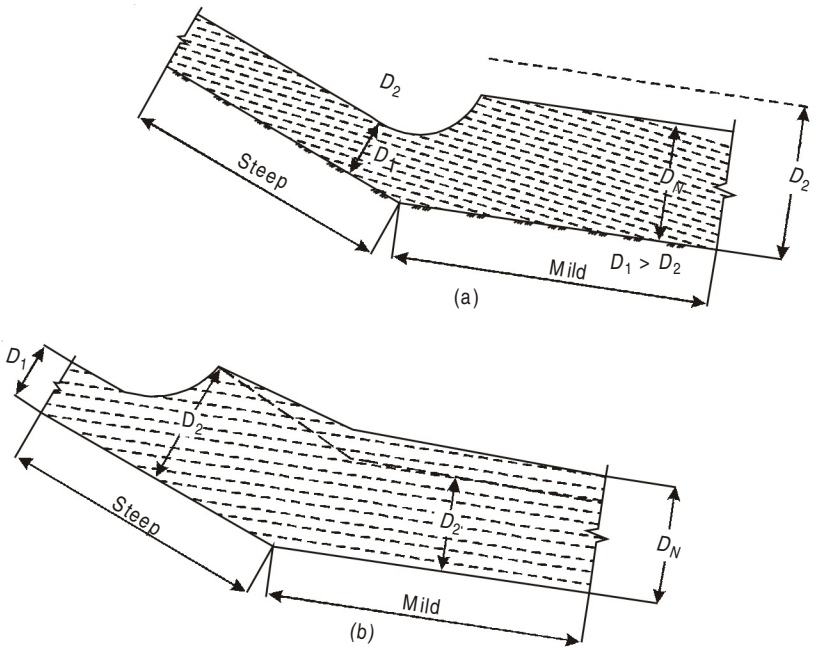


Fig.13.16

the break. It occurs upstream of the brake if the conjugate depth D_1 < normal depth D_N. (Fig. 13.16 b) If the conjugate depth D_1 > the normal depth D_N corresponding to mild slope, the jump would occur down stream of the break (Fig. 13.17 a). We have to therefore first determine the normal depth corresponding to each slope and the conjugate depth compatible with the surface profiles upstream and down stream.

(i) Mild slope (Fig. 13.16 a)

The Manning's discharge equation

$$Q = \frac{1.00}{n} AR^{2/3} S_0^{1/2} \qquad \text{...(b)}$$

$$AR^{2/3} = \left[\frac{(Q \times n)}{S_0^{1/2}}\right]$$

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$$(B \times D_N) R^{2/3} = \left[\left(\frac{Q \times n}{S_0^{1/2}}\right)\right]$$

$$(1 \times D_N) R^{2/3} = \left[\left(\frac{Q \times n}{S_0^{1/2}}\right)\right]$$

$$D_N \, R^{2/3} = \left[\frac{(4 \times 0.015)}{(1/1500)^{1/2}}\right] = 2.32$$

Since the channel is wide $R = D_N$

$$D_N (D_N)^{2/3} = 2.32$$

$$D_N = 1.657 \text{ m (corresponding to mild slope)}$$

(ii) Steep slope (Fig. 13.16 b)

The Manning's discharge equation

$$Q = \frac{1.00}{n} A R^{2/3} S_0^{1/2} \qquad \text{...(b)}$$

$$A R^{2/3} = \left[\left(\frac{Q \times n}{S_0^{1/2}}\right)\right]$$

$$(B \times D_N) R^{2/3} = \left[\left(\frac{Q \times n}{S_0^{1/2}}\right)\right]$$

$$(1 \times D_N) R^{2/3} = \left[\left(\frac{q \times n}{S_0^{1/2}}\right)\right]$$

$$D_N \, R^{2/3} = \frac{(4 \times 0.015)}{(1/100)^{1/2}} = 0.6$$

Since the channel is wide $R = D_N$

$$D_N (D_N)^{2/3} = 0.6$$

$$D_N = 0.736 \text{ m (corresponding to steep slope)}$$

(iii) Let us determine the conjugate depth D_1.

From Eq. (13.7) and referring to figure (Fig. 13.16 b)

$$D_2 = \left(\frac{D_1}{2}\right)\left[\sqrt{(1 + 8F_R^2)} - 1\right] \qquad \text{...(a)}$$

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F_R is Froude number at the upstream section.

Let us apply the above equation to points between D_1 and D_2.

$$F_R = \frac{v}{\sqrt{gD_1}} = \left(\frac{q}{D_1\sqrt{g \times D_1}}\right) = \left(\frac{4}{D_1\sqrt{g \times D_1}}\right)$$

$$F_R^2 = \left(\frac{4^2}{gD_1^3}\right) = \left(\frac{1.63}{D_1^3}\right) \quad \text{...(b)}$$

Substituting in (a) the relevant values,

$$0.736 = \left(\frac{D_1}{2}\right)\left[\sqrt{1+8\left(\frac{163}{D_1^3}\right)}-1\right]$$

$D_1 = 1.770$ m by trial and error (corresponding to steep slope).

Since $D_1 = 1.770$ m $> D_N = 0.736$ m hydraulic jump cannot occur upstream of the brake. Also, since $D_1 = 1.770$ m $> D_N = 1.657$m hydraulic jump occurs on the downstream of the break.

(iv) Height of the jump $H = 1.770 - 1.657 = 0.113$ m

Problem 13.24 *A wide canal is laid on a slope of 1 in 1600 and the normal depth of flow is 2 m. If due to lowering of the sluice the depth upstream increases to 3 m, determine the distance from the sluice where the depth is 2.7 m. Chezy's constant C = 55.*

Solution

From Eq. (13.19)

$$\frac{\partial D}{\Delta L} = S\frac{\left[\left(\frac{D_N}{D}\right)^3 - 1\right]}{\left[1-\left(\frac{D_C}{D}\right)^3\right]}$$

$$\frac{\Delta L}{\partial D} = \left(\frac{1}{S}\right)\left[1-\left(\frac{D_c}{D}\right)^3\right]\left[\left(\frac{D_N}{D}\right)^3 - 1\right]$$

$$\frac{\Delta L}{\delta D} = \left(\frac{1}{S}\right)\left[1-\left(\frac{D_c}{D}\right)^3\right]\left[\left(\frac{D_N}{D}\right)^3 - 1\right] \quad \text{...(a)}$$

$$D_N = 2 \text{ m}, \partial D = (2.7 - 2) = 0.7 \text{ m}, \text{ Slope } S = \frac{1}{1600}$$

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$$D_c = \left(\frac{q^2}{g}\right)^{1/3} \quad \text{... (b)}$$

Chezy's discharge equation $Q = A \times C\sqrt{RS_0}$

$$\frac{Q}{B} = q = D_N\left[55\sqrt{\frac{D_N}{1600}}\right]$$

$$q = 2\left[55\sqrt{\frac{2}{1600}}\right] = 3.89 \text{ m}^3/\text{sec}$$

$$D_c = \left(\frac{3.89^2}{g}\right)^{1/3} = 1.155 \text{ m}$$

Substituting the values in (a)

$$\Delta L = 0.7[1600]\left[1-\frac{1.155^3}{D^3}\right]\left[\left(\frac{2}{D}\right)^3 - 1\right]$$

$$\Delta L = (1120)\left[\left(1-\frac{1.155}{D}\right)^3\right]\left[\left(\frac{2}{D}\right)^3 - 1\right]$$

Now we divide the range from 2 m to 2.7 m in to 3 steps of 0.15 m each and arrange them in a tabular form starting from the cause of calculate the length for each step

1	*2*	*3*	*4*	*5*
Depth (*m*)	*Mean Depth* (*m*)	$1-\left(\frac{1.155}{D}\right)^3$	$\left(\frac{2}{D}\right)^3 - 1$	*L* (*m*)
2.700				
	2.612	0.913	– 0.550	1859.20
2.525				
	2.437	0.904	– 0.447	2265.00
2.350				
	2.262	0.866	– 0.308	3149.00
2.175				
	2.087	0.830	– 0.120	7,746.66
2.000				
				$\sum L$ = 15,019.86

At a distance of 15,019.86 m from the sluice the depth is 2.7 m

The negative sign under column is neglected it is merely indicative of the nature of the slope.

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QUESTIONS BANK

Section A

Review Questions

1. Mention some of the essential differences between steady uniform and non-uniform flow in open channels.
2. Define specific energy and critical depth. What is the effect of transition from super-critical to sub-critical flow?
3. What do you understand by the term 'alternative depths of flow' and at what point these depths coincide?
4. Draw a graph of q versus flow depth D and identify super-critical and sub-critical sections.
5. What does the 45° line from the origin represent in the graph of D versus E ?
6. Write down the specific energy equation for a rectangular channel in terms of discharge per unit width.
7. How is E_{min} related to critical depth and define the term hydraulic depth.
8. State the advantages and disadvantages of a standing wave and the value of Froude number when $v = \sqrt{gD}$.
9. What do you understand by the term 'transition'? Give an example where transition could occur from sub-critical to super-critical flow.
10. When does a Venturi-flume normally occur and state how a venture- flume could be an advantage.

Section B

Multiple Choice Questions

Choose the right answer

1. Flow in prismatic open channels is always
   (a) non-uniform
   (b) steady non-uniform
   (c) steady uniform
   (d) None of these
2. In non-uniform flow the total energy line is
   (a) parallel to water surface line
   (b) parallel to hydraulic gradient line
   (c) merges with the HGL
   (d) None of these

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3. Steady non-uniform flow occurs due to
   (a) an obstruction
   (b) variation in depth
   (c) waves and pressure surges
   (d) All of these
   (e) None of these
4. There are two depths of flow when
   (a) $E = E_{min}$ (b) $E = E_{max}$
   (c) $E \neq E_{min}$ (d) None of these
5. Critical depth is ≈ 2 m when q =
   (a) 9 m^3/sec (b) 10 m^3/sec
   (c) 11 m^3/sec (d) None of these
6. When $v_c = \sqrt{gD}$ the Froude number is equal to
   (a) 1 (b) < 1
   (c) > 1 (d) None of these
7. Hydraulic jump occurs due to flow transition from
   (a) super critical to sub critical
   (b) sub critical to super critical
   (c) super critical to critical (d) None of these.
8. A mild slope has
   (a) two surface profiles
   (b) three surface profiles
   (c) 4 surface profiles (d) None of these

Section C
Problems

1. A rectangular channel having a bed slope of 0.00015 and Manning's coefficient n = 0.020 contracts at a certain section from 3.5 m to 2 m. If the discharge is 2 m^3/sec, determine (i) the depth at the contracted portion (ii) just upstream of the contraction. (*Ans.* 1.156 m, 1.18 m)
2. Energy loss due to hydraulic jump through a rectangular channel is 0.66 m. The discharge/unit width is 3 m^3/sec. Determine the depths before and after the jump given that $3.5 > F_R > 2.5$. (*Ans.* 0.467 m, 1.76 m)
3. Energy loss due to hydraulic jump through a rectangular channel is 5 m and the ratio of sequent depths is 6 m. Determine the specific energy before the jump. (*Ans.* 8.16 m)

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4. Determine the discharge/unit width over a broad crested weir with a critical depth = 0.860 m. (*Ans.* 2.5 m^3/sec)
5. Using impulse momentum and the continuity equations prove that in case of a hydraulic jump the discharge and depths before and after the jump are related as under.

$$Q = D_1 D_2 \sqrt{\left(\frac{g}{3}\right)\left[\frac{(D_2^3 - D_1^3)}{(D_2^2 - D_1^2)}\right]}$$

6. The depth of water in a wide canal with a slope of 1 in 1000 increases from 1.5 m to 3 m at which point it discharges over a weir. Determine in four steps the distance from the weir where the depth is 2.2 m. $C = 60$.

(*Ans.* ≈ 3265.65 m)

Answers to Multiple Choice Questions

1. (c) **2.** (d) **3.** (d) **4.** (c)
5. (a) **6.** (a) **7.** (a) **8.** (b)

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Chapter

14

Turbulent Flow Through Pipes

In Chapter 11 (Flow Through Pipes) the relative roughness of the pipe and the consequent energy loss due to pipe wall friction was all that mattered to us while dealing with problems associated with turbulent flows ($R_E > 4000$). We were not concerned with the difference in velocity profiles between laminar and turbulent flows and how this variation affects flow conditions.

This chapter is devoted to understanding of how the random motion of fluid particles when superimposed on average motion create conditions that contribute to shear stresses which cause far greater energy losses than that due to viscous shear. Loss due to viscous shear can be substantially reduced by controlling the surface roughness of any closed conduit. Whereas shear stresses caused due to turbulent flow are not so easy to control.

In case of steady laminar incompressible flows the flow losses can be computed using straight forward equations developed by various investigators. The loss due to viscous shear is small as it is confined to a negligible length of the boundary. But in turbulent flow the shear stress is mainly due to momentum transfer since the fluid particles in adjacent layers have different velocities. The magnitude of shear stresses could therefore be significant and hence have to be taken into account. But no direct relationship that can account for the loss due to these stresses can be established since the very nature of turbulent flow is 'random and chaotic'. Since shear stresses contribute to losses to a large extent in turbulent flow the flow is often called *viscous flow*.

As we proceed further we recall what can be termed as semi-empirical equations developed by various investigators to solve practical problems. Since this chapter is exclusive to *turbulent flow through pipes* external flows are excluded. This chapter therefore deals with flow through closed conduits where the effect of the boundary covers the entire flow region.

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Figure 14.1 explains through relevant velocity profiles the transition from turbulent flow to laminar flow that occurs at the entrance to a reservoir with a well rounded mouth piece. At the entrance due to the action of wall shear stresses the velocity profile is almost uniform although the flow at this section is turbulent. This is because the chaotic motion at this section and consequent violent mixing of fluid particles causes velocity of fluid particles to change very rapidly from zero at the wall to maximum velocity throughout the entire section. Laminar flow is essentially made up of layers of fluid moving in tandem resulting in less momentum transfer between molecules compared to turbulent flow which results in a less uniform parabolic profile. In *laminar flow* the maximum velocity that occurs at the center and the local velocity can be defined through the equation

$$v_{max} = 2v\left[1-\left(\frac{R}{R_0}\right)^2\right] \qquad ...(14.1)$$

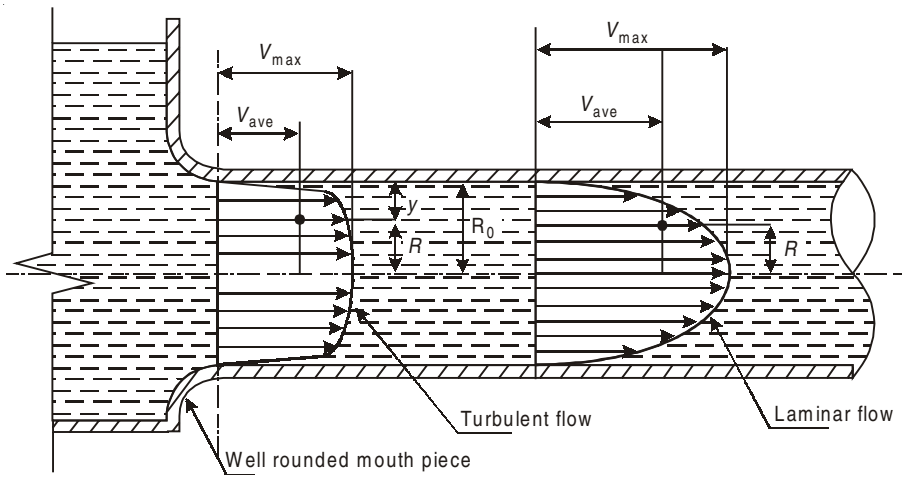


Fig. 14.1

14.1 TURBULENT FLOW RELATIONS AND VELOCITY DISTRIBUTION IN DIFFERENT REGIONS

Shear stress in turbulent flow is composed of viscous shear (μ) and shear stress due to eddy shear (η). The latter is in no way related to fluid property but depends on the magnitude of turbulence. Since exact analysis of velocity and pressure terms under turbulent flow conditions is complex, empirical equations based on dimensional analysis and practical experiments have been developed to asses the value of eddy shear. Of these, the *Prandtl's mixing length theory* is the one most widely used.

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Prandtl's mixing length theory is better appreciated if certain terms related to velocity and shear stress distribution in turbulent flow through pipes are defined.

Figure 14.2 illustrates the conditions during turbulent flow through a section of the pipe.

In the laminar sub layer region ($y \leq \delta$) shear stress, absolute viscosity and velocity distribution follow the Newton's law of viscosity.

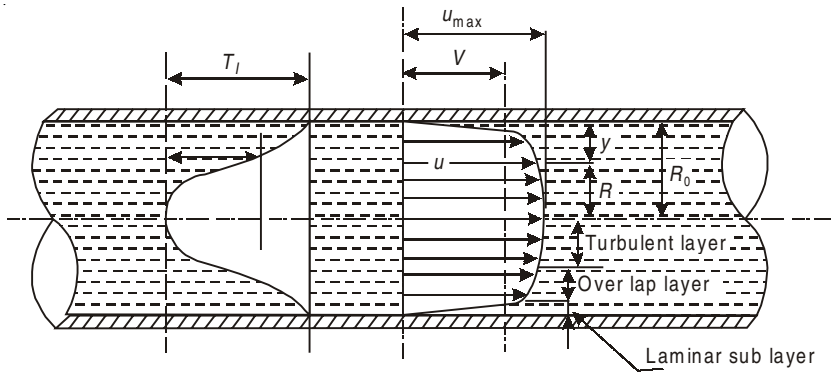


Fig. 14.2 *Shear and velocity distribution in turbulent flow*

$$\text{Thus } \tau_{Lam} = \mu \left(\frac{du}{dy}\right) \qquad \ldots(14.2)$$

$$\text{In the turbulent layer } \tau_{Tur} = \eta \left(\frac{du}{dy}\right) \qquad \ldots(14.3)$$

$$\text{Total shear stress } \tau_{Tur} = \mu \left(\frac{du}{dy}\right) + \eta \left(\frac{du}{dy}\right) \qquad \ldots(14.4)$$

Here η is referred to as Eddy viscosity whose value is determined from the Prandtl's mixing length theory. Eddy viscosity η is very large compared to viscous shear μ. It represents the coefficient of momentum transfer of molecules from regions of high concentration to regions where the concentration is low. Similar to Kinematic viscosity ν, there exists the term Kinematic eddy viscosity $\lambda = \left(\frac{\eta}{\rho}\right)$. Since λ depends on flow parameters alone, it is a more convenient form of expression.

In kinetic theory of gases we learnt that the gas molecule has a *mean free path,* which is the distance it covers before striking another molecule, Prandtl's mixing length theory assumes that a fluid particle too travels a distance L before a momentum change occurs in the new environment.

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This means that change in velocity is depending upon the extent of change in the components of local velocity in x and y directions (u' and v') between two points distance L apart.

$$u' \approx L\left(\frac{du}{dy}\right)$$

Prandtl further reasoned that there must be a correlation between these velocities similar to the correlation that exists between velocities at two different locations in the continuity equation. From which it emerged that u' is proportional to v' or,

$$u' \approx v' \approx L\left(\frac{du}{dy}\right)$$

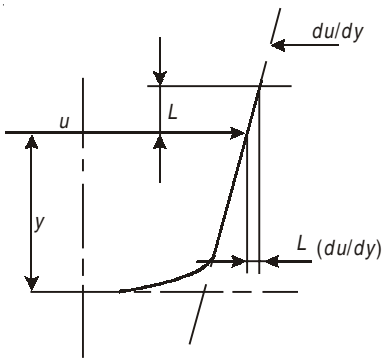


Fig. 14.3

$$(u' \times v') \approx \left[L\left(\frac{du}{dy}\right)\right] \times \left[L\left(\frac{du}{dy}\right)\right]$$

$$= \left[L^2\left(\frac{du}{dy}\right)^2\right]$$

$$\tau_{Tur} = \rho L^2\left(\frac{du}{dy}\right)^2 \quad \ldots(14.5)$$

$$\eta = \rho L^2\left(\frac{du}{dy}\right) \quad \ldots(14.6)$$

where, $L = ky$ ...(14.7)

This relationship between mixing length L and the wall distance y was proposed by Von Karman as it could not be obtained through Prandtl's equation. Karman

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reasoned that the mixing length L must be zero at the wall, because the violent nature of interchange of fluid particles that exists in the turbulent region is almost reduced to zero at the boundary. This reasoning gave raise to the equation,

$$L = k \frac{\left[\left(\frac{du}{dy}\right)\right]}{\left[\left(\frac{d^2u}{dy^2}\right)\right]} \qquad \ldots(14.8)$$

The value of k (Karman's universal constant) = 0.4 and is independent of both the boundary condition and Reynolds number and applies to all turbulent flow conditions. Shear stress in the laminar sub layer region is determined by,

$$\mu_* = \sqrt{\frac{\tau_0}{\rho}} \qquad \ldots(14.9)$$

The term, μ_* is called the shear velocity and the term $\sqrt{\frac{\tau_0}{\rho}}$ has the dimensions of velocity.

$$\tau_0 = \frac{(\rho v^2 f)}{2} \qquad \ldots(14.10)$$

τ = shear stress at any radius R

$$= \left(\tau_0 \frac{R}{R_0}\right) \qquad \ldots(14.10a)$$

(f = Darcy–Weisbach friction factor and v is the average velocity)

In the region $y \approx \delta$, the relationship between u and y is linear. Thus,

$$\left(\frac{u}{u*}\right) = \frac{(u*y)}{v} \qquad \ldots(14.11)$$

$$\frac{(u*y)}{v} \approx 5 \text{ and hence } \delta = 5\left(\frac{v}{u*}\right) \qquad \ldots(14.12)$$

The average velocity v is related to maximum velocity u_{max} in the following manner in case of turbulent flow through closed conduits.

$$v = u_{max} - \left(\frac{3u*}{2k}\right) \qquad \ldots(14.13)$$

In the overlap region the shear stress is substantially the same as at the laminar sub layer region and since dimensional similarity exists between mixing length L and y by comparing Eqs (14.8), (14.9) and (14.10) and by rearranging certain terms we obtain.

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$$\frac{du}{u*} = \left(\frac{1}{k}\right)\left(\frac{dy}{y}\right) \qquad \ldots(14.14)$$

or
$$\left(\frac{u}{u*}\right) = \left(\frac{1}{k}\right)(\log_e y + C) \qquad \ldots(14.15)$$

The value of C in Eq. 14.15 is determined based on the boundary condition.

For pipe flow At $y = R$, $u = u_{max}$, hence,

$$\left(\frac{u_{max}}{u*}\right) = \left(\frac{1}{k}\right)(\log_e R + C)$$

$$u_{max} = \left(\frac{u*}{k}\right)(\log_e R + C)$$

$$C = u_{max} - \left(\frac{u*}{k}\right)(\log_e R)$$

Substituting the value of C in Eq. (14.15)

$$\left(\frac{u}{u*}\right) = \left(\frac{1}{k}\right)\log_e y + u_{max} - \left(\frac{u*}{k}\right)(\log_e R)$$

$$u = \left(\frac{u*}{k}\right)\log_e y + u_{max} - \left(\frac{u*}{k}\right)(\log_e R)$$

$$u = u_{max} + \left(\frac{u*}{k}\right)(\log_e y - \log_e R)$$

$$u = u_{max} + \left(\frac{u*}{k}\right)\log_e\left(\frac{y}{R}\right) \quad \ldots(\text{Since } k = 0.4)$$

$$u = u_{max} + (2.5u_*)\log_e\left(\frac{y}{R}\right) \qquad \ldots(14.16)$$

Equation (14.16) is called *Prandtl's universal velocity distribution equation*

$$u - u_{max} = (2.5u_*)\log_e\left(\frac{y}{R}\right)$$

$$\frac{(u - u_{max})}{u*} = 2.5\log_e\left(\frac{y}{R}\right)$$

$$-\left[\frac{(u_{max} - u)}{u*}\right] = 2.5\log_e\left(\frac{y}{R}\right)$$

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$$\frac{(u_{max} - u)}{u*} = 2.5 \log_e \left(\frac{R}{y}\right)$$

$$\frac{(u_{max} - u)}{u*} = 2.5 \times 2.3 \log_{10} \left(\frac{R}{y}\right)$$

$$\frac{(u_{max} - u)}{u*} = 5.75 \log_{10} \left(\frac{R}{y}\right) \quad \ldots(14.17)$$

Equation (14.17) is called *the velocity deficit law* where $\frac{(u_{max} - u)}{u*}$ is the velocity deficit. $u = u_{max}$ when $y = R$.

While Eq. (14.11) applies to laminar sub-layer region, Eq. (14.16) gives satisfactory results in the overlap region and in the outer turbulent region velocity deficit law (14.17) applies.

Depending upon how smooth or rough the surfaces of the pipe through which the liquid is flowing the above equation do get modified since surface irregularities have not been considered while determining these equations.

Based both on the root mean square (RMS) value of the height of surface irregularities and the Reynolds number R_e the pipes are classified as either rough or smooth. In laminar flow conditions (R_e< 2000) surface roughness of the boundary is considered hydro-dynamically as smooth if the average height of surface irregularities is less than the laminar sub layer thickness δ. As R_e increases the thickness of the laminar sub layer decreases and eventually the laminar sub layer is completely overtaken by Eddy's formed due to turbulence. The boundary under these conditions is classified as *rough* hydro-dynamically.

As per the experimental data prepared by Nikuradse if the average height of the irregularities is ε then a boundary is considered as hydro-dynamically *Smooth* if $\left(\frac{\varepsilon}{\delta}\right) < 0.25$ and as *Rough* if $\left(\frac{\varepsilon}{\delta}\right) > 6$. In the overlapping region $\left(\frac{\varepsilon}{\delta}\right)$ lies in between 0.25 to 6. In terms of critical Reynolds number $\frac{(u*\varepsilon)}{v}$, a boundary is considered as *Smooth* if $\frac{(u*\varepsilon)}{v} < 3$, as Rough if $\frac{(u*\varepsilon)}{v} > 70$ and in the overlapping region $\frac{(u*\varepsilon)}{v}$ lies in the region between 3 – 70.

Based on these considerations the velocity distribution equations for turbulent flow through pipe have been redefined.

For both *smooth* and *rough* boundaries the relevant velocity distribution equation is

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$$\frac{(u_{max} - u)}{u*} = 5.75 \log_{10}\left(\frac{R}{y}\right) \quad \ldots(14.18)$$

If average flow velocity is considered instead of the maximum then Eq. (14.18) gets modified to the form,

$$\left(\frac{u - v}{u*}\right) = 5.75 \log_{10}\left(\frac{y}{R}\right) + 3.75 \quad \ldots(14.19)$$

For *turbulent flow* in pipe of radius R the local velocity at any distance y from the boundary is given by Eq. (14.16)

$$\left(\frac{u}{u*}\right) = \left(\frac{1}{k}\right)(\log_e y + C)$$

If the boundary is *Smooth* $\left(\frac{u}{u*}\right) = (5.75) \log \frac{(u* y)}{v} + 5.5 \quad \ldots(14.20)$

(For $R_E{}^* > 30$)

In terms of critical Reynolds number the equation can be written as

$$\left(\frac{u}{u*}\right) = (5.75) \log R_E{}^* + 5.5 \quad \ldots(14.20a)$$

If the boundary is *Rough* $\left(\frac{u}{u*}\right) = (5.75) \log\left(\frac{y}{\varepsilon}\right) + 8.5 \quad \ldots(14.21)$

The friction factor corresponding to the above equation

$$\frac{1}{\sqrt{f}} = 2 \log\left(\frac{R}{\varepsilon}\right) + 1.74 \quad \ldots(14.22)$$

D is the inside diameter of the pipe.

In turbulent flow the transition from smooth flow regime begins when,

$$\frac{\left(R_e \sqrt{f}\right)}{\left(\frac{R_0}{\varepsilon}\right)} = 17 \quad \ldots(14.22a)$$

Turbulent flow in Rough pipes begins when $\left(\frac{\varepsilon}{\delta}\right) > 6$ or when

$$\frac{\left(R_e \sqrt{f}\right)}{\left(\frac{R_0}{\varepsilon}\right)} > 400 \quad \ldots(14.22b)$$

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Turbulent flow in *Rough* pipes is independent of Reynolds number but is a function of $\left(\frac{\varepsilon}{\delta}\right)$ only.

For turbulent flow in terms of the *average velocity* and pipe radius,

$$\text{If the boundary is } Smooth \left(\frac{v}{u*}\right) = (5.75) \log \frac{(\mu * R)}{\upsilon} + 1.75 \qquad \ldots(14.23)$$

$$\text{If the boundary is } Rough \left(\frac{v}{u*}\right) = (5.75) \log \left(\frac{R}{\varepsilon}\right) + 4.75 \qquad \ldots(14.24)$$

The above logarithmic equations have a wide applicability but there exists another exponential empirical equation developed by Prandtl, called the Prandtl's $\frac{1}{7^{\text{th}}}$ power law. This law is applicable to values of $R_e < 100{,}000$. However this law is not valid at $y = 0$.

$$\left(\frac{u}{\mu_{\max}}\right) = \left(\frac{y}{R_0}\right)^{\frac{1}{7}} \qquad \ldots(14.25)$$

For values of $R_e > 100{,}000$ the value of the exponent decreases.

There exists a general form of logarithmic velocity distribution law for turbulent flow which relates local velocity (u) to average velocity (v) and considers the friction factor as well. It can be expressed either in terms of pipe radius R_0 and local radius R or in terms of distance from the pipe wall y and the pipe radius R_0 (Refer Fig. 14.2).

$$u = v[1 + 1.43\sqrt{f} + 2.15\sqrt{f} \log\left(1 - \frac{R}{R_0}\right) \qquad \ldots(14.26)$$

$$u = v[1 + 1.43\sqrt{f} + 2.15\sqrt{f} \log\left(\frac{y}{R_0}\right) \qquad \ldots(14.27)$$

These equations apply only where $y \neq 0$, and $R < R_0$ and at $y = 0$ or at $R = R_0$

$$v_{\max} = v\,(1 + 1.43\sqrt{f}\,) \qquad \ldots(14.28)$$

v is the average velocity and f is the friction factor which depends on the Reynolds number and the relative roughness of the pipe.

Either Chezy's or Darcy–Weisbach equations can be used alternately to compute turbulent flow losses. The flow losses between two sections of a closed conduit distance L apart is computed using energy equation and is given by,

$$\Delta L = \frac{(P_1 - P_2)}{\gamma} + Z_1 - Z_2 \qquad \ldots(14.29)$$

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In terms of shear stress τ_0 and the wetted perimeter WP

$$\Delta L = \frac{\tau_0 L(\text{WP})}{\gamma A} \quad \ldots(14.30)$$

Hence

$$h_f = 4f\left(\frac{L}{d}\right)\frac{v^2}{2g}$$

$$= \frac{\tau_0 L(\text{WP})}{\gamma A}$$

$$= \frac{(P_1 - P_2)}{\gamma} + Z_1 - Z_2 \quad \ldots(14.31)$$

If the conduit is a straight conduit then

$$(P_1 - P_2) = 4f\left(\frac{L}{d}\right)\frac{\rho v^2}{2} \quad \ldots(14.32)$$

The variation of frictional resistance (f) for different flow conditions is as under.

(1) Laminar flow

$$f = \frac{64}{R_e} \quad \ldots(14.33)$$

(2) Turbulent flow in smooth pipes ($4000 < R_e < 100000$)

$$f = \frac{0.0791}{(R_e)^{\frac{1}{4}}} \quad \ldots(14.34)$$

(3) Turbulent flow in smooth pipes when ($5 \times 10^4 < R_e < 4 \times 10^7$)

$$f = \frac{0.0008 + 0.05525}{(R_e)^{0.237}} \quad \ldots(14.35)$$

(4) The value of f for smooth turbulent ($10^4 < R_e < 10^5$) is given by equation

$$\left(\frac{1}{\sqrt{f}}\right) = 2\log(R_E\sqrt{f}) - 0.80 \quad \ldots(14.35a)$$

The above empirical formulae which determine friction factor (f) are all based on experiments conducted by Nikuradse using pipes of uniform sand grained roughness. However commercially available pipes display no such uniformity either in matters of degree of roughness or in matters of the composition of material of which they are made or the manufacturing methods employed. In order to overcome this anomaly the concept of *equivalent sand grained roughness or effective roughness factor* was introduced.

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Effective roughness factor $[\varepsilon_E]$ *is the roughness quotient of commercial quality pipes that gives the same value of* $[f]$ *in the turbulent region as that of a sand grained rough pipe originally used by Nikuradse.*

Values of $[\varepsilon_E]$ for different qualities of commercially available pipes are tabulated below.

TABLE 14.1

Material	$[\varepsilon_E]$ *mm*
1. Glass, Fiber glass, cold drawn copper and plastic etc,	0.0025
2. Wrought Iron/Steel tube	0.050
3. G.I tubes	0.150
4. Cast Iron	0.250
5. Pre-cast concrete tubes	0.30–3

From Nikuradse's experiments for turbulent flow in smooth pipes of any radius the value of $\left[\frac{(u* y)}{\upsilon}\right] \approx 11.6$ when $y = \delta$. Here y is the distance from the boundary and υ kinematic viscosity and δ is laminar sub-layer thickness.

The thickness of the laminar sublayer $\delta = \dfrac{11.6\,\upsilon}{u*}$ ...(14.36)

A relationship could be brought about between the pipe radius R_0 and laminar sub-layer thickness δ in terms of the Reynolds number R_E

$$\left(\frac{\delta}{R_0}\right) = \frac{11.6\,\upsilon}{R_0\,u*} = \frac{11.6\,\upsilon}{\left(\frac{D}{2}\right)\frac{\upsilon\sqrt{f}}{8}}$$

$$\left(\frac{\delta}{R_0}\right) = \frac{(65.6)}{R_E\sqrt{f}} \quad \text{...(14.37)}$$

$$\left(\frac{\varepsilon}{\delta}\right) = \left(\frac{\varepsilon}{R_0}\right)\left(\frac{R_0}{\delta}\right)$$

$$= \frac{(\varepsilon\, R_E\sqrt{f})}{(R_0 \times 65.6)} \quad \text{...(14.38)}$$

Colebrook proposed an empirical formula for commercial quality pipes which holds good for all the three regions of turbulent flow viz., laminar sub-layer, over-lapping and the outer turbulent core region.

$$\frac{1}{\sqrt{f}} + 0.869 \log_e \left[\frac{\left(\frac{\varepsilon}{D}\right)}{3.7} + \frac{2.253}{\left(R_e\sqrt{f}\right)}\right] = 0 \quad \text{... (14.39)}$$

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Swamee and Jain proposed an empirical equation to determine the friction factor in turbulent flow which holds good for values of $5 \times 10^3 < R_e < 1 \times 10^8$ and relative roughness values $\frac{D}{\varepsilon}$ ranging from 10^2 to 10^6.

$$f = \frac{0.25}{\left\{\log\left[\frac{1}{3.7}\left(\frac{D}{\varepsilon}\right) + \frac{5.74}{(R_e)^{0.9}}\right]\right\}^2} \qquad ...(14.40)$$

In summary, all of these methods which are meant to determine the friction factor at different flow and relative roughness conditions are tedious when compared to the ease of obtaining the same values directly from the *Moody's diagram*. However the student should be aware of the existence of these equations. They are however useful when simulating different conditions of flow during computer programming.

TYPICAL PROBLEMS AND SOLUTIONS

Problem 14.1 *At what distance r from the center of a tube of radius R the average velocity occurs in a laminar flow?*

Solution

Since the velocity distribution in laminar flow is a paraboloid its volume is one half of the circumscribing cylinder and hence the average velocity is one half of the maximum velocity.

$$\pi r^2 L = \frac{(\pi R^2 L)}{2}$$

$$r^2 = \frac{(R^2)}{2}$$

$$r = \frac{R}{\sqrt{2}} = 0.707 R$$

Problem 14.2 *Prove that in turbulent flow through a smooth pipe the maximum velocity is 5.5 times the shear velocity.*

Solution

(a) For smooth pipes from Eq. (14.20)

$$\left(\frac{u}{u*}\right) = (5.75) \log \frac{u* y}{\upsilon} + 5.5$$

...since for the present case $y = R$

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$$\left(\frac{u}{u*}\right) = (5.75)\log\frac{u*R}{\upsilon} + 5.5$$

...At the centerline $u = u_{max}$ and $R = 0$

Hence $u_{max} = 5.5\, u_*$

Problem 14.3 *In a turbulent flow through a pipe if the ratio of centerline velocity to mean velocity is 1.2 determine the friction factor.*

Solution

From Eq. (14.28) that centerline velocity and average velocity are related as under.

$$v_{max} = v\,(1 + 1.43\,\sqrt{f}\,)$$

$$\frac{v_{max}}{v} = (1 + 1.43\,\sqrt{f}\,)$$

$$1.2 = (1 + 1.43\,\sqrt{f}\,)$$

$$f = 0.0195$$

Problem 14.4 *At what flow velocity the roughness of a cast iron pipe of φ 250 mm will cause the flow to depart from hydro-dynamically smooth flow regime and enter the transition region and at what flow velocity can the flow be considered rough turbulent? $\upsilon = 1.14 \times 10^{-6}\ m^2/sec$.*

Solution

From Table 14.1 the effective roughness factor [ε_E]

for C.I. pipe is 0.250 mm $= 2.5 \times 10^{-4}$ m

In turbulent flow the transition from smooth flow regime begins when

$$\frac{\left(R_e\sqrt{f}\right)}{\left(\frac{R_0}{\varepsilon}\right)} = 17 \qquad \text{...from Eq.(14.22a)}$$

$$\left(R_e\sqrt{f}\right) = 17\left(\frac{R_0}{\varepsilon}\right)$$

$$= 17\left(\frac{0.125}{2.5 \times 10^{-4}}\right) = 8500$$

The value of f for smooth turbulent ($10^4 < R_e < 10^5$) is given by equation (14.35a)

$$\left(\frac{1}{\sqrt{f}}\right) = 2\log\left(R_e\sqrt{f}\right) - 0.80$$

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$$\left(\frac{1}{\sqrt{f}}\right) = 2\log(8500) - 0.80$$

$$\left(\frac{1}{\sqrt{f}}\right) = 2\log(8500) - 0.80 = 7.05$$

$$\sqrt{f} = \frac{1}{7.05} = 0.14$$

$$\left(R_e\sqrt{f}\right) = 8500$$

$$(R_e \times 0.14) = 8500$$

$$R_e = \frac{8500}{0.14} = 60{,}714$$

$$\left(\frac{vD}{\upsilon}\right) = 60{,}714$$

$$v = \frac{(\upsilon\ 60{,}714)}{D}$$

$$= (1.14 \times 10^{-6}) \times \frac{60{,}714}{0.250}$$

$$= 0.276 \text{ m/sec}$$

Hence the flow velocity cannot exceed 0.276 m/sec if the flow has to remain in the smooth turbulent regime.

Turbulent flow in *Rough* pipes begins when $\dfrac{\left(R_e\sqrt{f}\right)}{\left(\frac{R_0}{\varepsilon}\right)} = 400$

$$\left(R_e\sqrt{f}\right) = 400 \times \left(\frac{R_0}{\varepsilon}\right)$$

$$= \frac{400 \times 0.125}{(2.5 \times 10^{-4})}$$

$$= 200{,}000 \qquad \ldots(a)$$

From Eq. (14.22) for rough turbulent flow, the friction factor is given by,

$$\frac{1}{\sqrt{f}} = 2\log\left(\frac{R}{\varepsilon}\right) + 1.74$$

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$$\frac{1}{\sqrt{f}} = 2 \log \left(\frac{0.125}{2.5 \times 10^{-4}} \right) + 1.74$$

$$\frac{1}{\sqrt{f}} = 7.13$$

or $$\sqrt{f} = \frac{1}{7.13} = 0.14$$

$$\left(R_e \sqrt{f}\right) = 200{,}000 \quad \text{...from (a)}$$

$$R_e = \frac{200{,}000}{\sqrt{f}}$$

$$= \frac{200{,}000}{0.14} = 1.42 \times 10^6$$

$$\frac{vD}{\upsilon} = 1.42 \times 10^6$$

$$v = \frac{\left(1.42 \times 10^6\right)\upsilon}{D}$$

$$= \frac{\left(1.42 \times 10^6\right)(1.14 \times 10^{-6})}{0.250}$$

$$v = 5.68 \text{ m/sec.}$$

At this velocity the flow will be rough turbulent.

Problem 14.5 *If the average height of roughness of a rough pipe of inside diameter 500 mm is 0.4 mm, determine the power required to maintain a flow of water @ 0.75 m^3/sec through a length of 5 km.* $\upsilon = 1.14 \times 10^{-6}$ m^2/sec.

Solution

Let us first determine the Reynolds number which will decide whether the flow is laminar or turbulent.

$$R_e = \frac{vD}{\upsilon}$$

$$= \frac{QD}{A\upsilon}$$

$$= \frac{QD}{0.7854\, D^2 \upsilon}$$

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$$R_e = \frac{Q}{0.7854\, D\, \upsilon}$$

$$= \left(\frac{0.75}{0.7854 \times 0.5 \times 1.14 \times 10^{-6}}\right)$$

$R_e = 3.67 \times 10^6$. The flow is rough turbulent.

From Eq. (14.22) for rough turbulent flow, the friction factor is given by,

$$\frac{1}{\sqrt{f}} = 2 \log \left(\frac{R}{\varepsilon}\right) + 1.74$$

ε is given to be = 0.4 mm = 4×10^{-4} m

$$\frac{1}{\sqrt{f}} = 2 \log \left(\frac{0.25}{4 \times 10^{-4}}\right) + 1.74$$

$$\frac{1}{\sqrt{f}} = 7.33$$

$$\sqrt{f} = \left(\frac{1}{7.33}\right) = 0.068$$

$$f = 0.0186$$

Power required to maintain the flow $P = \gamma Q h_f$

Average velocity $$v = \left(\frac{0.75}{0.7854 \times 0.5^2}\right)$$

$$= 3.82 \text{ m/sec}$$

$$h_f = \frac{f L v^2}{2gD}$$

$$= \frac{(0.0186 \times 5000 \times 3.82^2)}{2 \times 9.81 \times 0.5}$$

$h_f = 138.33$ m

$P = \gamma Q h_f$

$= 9.81 \times 0.75 \times 138.33$

$= 1018$ kW

Problem 14.6 *In a turbulent flow through a pipe of diameter 350 mm, the difference between maximum velocity and the velocity as measured at 75 mm from the pipe wall was found to be 0.5 m/sec. Determine the discharge and the average height of roughness. f = 0.0150.*

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Solution

From Eq. (14.18)

$$\frac{u_{max} - u}{u*} = 5.75 \log_{10}\left(\frac{R}{y}\right)$$

$$\frac{(0.5)}{u*} = 5.75 \log_{10}\left(\frac{0.175}{0.0.075}\right)$$

Hence $u_* = 0.236$ m/sec

From Eq. (14.20)

$$\frac{(u - v)}{u*} = 5.75 \log_{10}\left(\frac{y}{R}\right) + 3.75.$$

(here, v is the average velocity)

At $y = R$, $u = u_{max}$

$$\frac{u_{max} - v}{u*} = 5.75 \log_{10}\left(\frac{R}{R}\right) + 3.75$$

$$\frac{(u_{max} - v)}{u*} = 3.75 \quad \ldots(a)$$

From Eq. (14.28)

$$v_{max} = v\,(1 + 1.43\,\sqrt{f}\,) \quad \ldots(b)$$

Substituting the value of v_{max} in (a)

$$\frac{[v\,(1+1.43\sqrt{f}) - v]}{0.236} = 3.75$$

$$\frac{[v + v(1.43 - \sqrt{0.0150}) - v]}{0.236} = 3.75$$

$$v\left(1.43\sqrt{0.0150}\right) = 3.75 \times 0.236$$

$$v = 5.05 \text{ m/sec.}$$

(i) $Q = 0.7854\,(0.350)^2 \times 5.05$

$= 0.485\ m^3/sec$

From Eq. (14.22) for rough turbulent flow, the friction factor is given by,

$$\frac{1}{\sqrt{f}} = 2 \log (R/\varepsilon) + 1.74$$

$$\frac{1}{\sqrt{0.0150}} = 2 \log (0.175/\varepsilon) + 1.74$$

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$$\log\left(\frac{0.175}{\varepsilon}\right) = \frac{\left(\frac{1}{\sqrt{0.0.075-1.74}}\right)}{2}$$

$$\log\left(\frac{0.175}{\varepsilon}\right) = 3.212$$

$$\left(\frac{0.175}{\varepsilon}\right) = 1629.3$$

$$\varepsilon = 0.00011 \text{ m}$$

$$= 0.107 \text{ mm (average height of roughness)}$$

Problem 14.7 *Water at 20° C flows through a smooth horizontal pipe with a mean velocity of 2.5 m / sec. If the ID of the pipe is 40 mm, determine the pressure gradient per unit length required to maintain the flow.*

Solution

From the tables, properties of water at 20°C

$$\gamma = 9.78 \text{ kN/m}^3$$

and $$\upsilon = 1\times10^{-6} \text{ m}^2\text{/sec.}$$

Let us first determine the Reynolds number.

$$R_e = \frac{vD}{\upsilon}$$

$$= \frac{2.5 \times 0.04}{1\times10^{-6}}$$

$$= 100{,}000$$

From (14.34) for turbulent flow in a smooth pipe ($4000 < R_e < 100000$)

$$f = \frac{0.0791}{(R_e)^{\frac{1}{4}}}$$

$$= \frac{0.0791}{(100{,}000)^{\frac{1}{4}}}$$

$$= 4.45 \times 10^{-3}$$

For a horizontal pipe the pressure gradient is given by

$$\frac{(P_2 - P_1)}{\gamma} = \frac{4 f L v^2}{2gD}$$

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$$(P_2 - P_1) = \gamma \left(\frac{4\, fLv^2}{2gD} \right)$$

$$(P_2 - P_1) = (9.78)\ \frac{[(4 \times 4.45 \times 10^{-3})\ 2.52]}{(2 \times 9.81 \times 0.040)}$$

$$(P_2 - P_1) = 1.38\ \text{kN/m}^2 \text{ per metre length of the pipe}$$

Problem 14.8 *Water at 20° C is being conveyed through a concrete pipe 600 mm diameter. If the Reynolds number is 90,000 determine the volume flow rate and the distance y from the pipe wall where the local velocity is equal to the average velocity. Assume* $\varepsilon_E = 0.4$.

Solution

From the tables, properties of water at 20°C

$$\gamma = 9.78\ \text{kN/m}^3$$

and $$\upsilon = 1 \times 10^{-6}\ \text{m}^2\text{/sec.}$$

Let us first determine the volume flow rate.

$$R_e = \frac{vD}{\upsilon}$$

$$90{,}000 = \frac{(v \times 0.6)}{1 \times 10^{-6}}$$

$$v = 0.15\ \text{m/sec}$$

$$Q = 0.15 \times 0.7854\ (0.600)^2$$

$$= 0.0424\ \text{m}^3\text{/sec}$$

Since it is a concrete pipe the flow may be assumed to be rough turbulent and from Eq. (14.22) for rough turbulent flow, the friction factor is given by,

$$\frac{1}{\sqrt{f}} = 2 \log \left(\frac{R}{\varepsilon} \right) + 1.74$$

$$\frac{1}{\sqrt{f}} = 2 \log \left(\frac{0.300}{0.0004} \right) + 1.74$$

$$f = 0.0178$$

From Eq. (14.27)

$$u = v \left[1 + 1.43\sqrt{f} + 2.15\sqrt{f} \log \left(\frac{y}{R_0} \right) \right]$$

$$0.15 = 0.15 \left[1 + 1.43\sqrt{0.0.0178} + 2.15\sqrt{0.0178} \log \left(\frac{y}{0.300} \right) \right]$$

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$$\left[1+1.43\sqrt{0.0.0178}+2.15\sqrt{0.0178}\log\left(\frac{y}{0.300}\right)\right] = 1$$

$$1.190 + 0.286 \log\left(\frac{y}{0.300}\right) = 1$$

$$0.286 \log\left(\frac{y}{0.300}\right) = -0.19$$

$$\log\left(\frac{y}{0.300}\right) = \frac{-0.19}{0.286} = -0.664$$

$$\left(\frac{y}{0.300}\right) = 0.216$$

$$y = 0.065 \text{ m}$$

Problem 14.9 *If the local velocity at 15 mm from the axis of a 50 mm diameter pipe conveying a liquid is 3 m/sec and the maximum velocity is 3.5 m/sec determine the volume flow rate.*

Solution

From Eq. (14.18)

$$\frac{(u_{max}-u)}{u*} = 5.75 \log_{10}\left(\frac{R}{y}\right)$$

$$\frac{(3.5-3)}{u*} = 5.75 \log_{10}\left(\frac{0.025}{0.025-0.015}\right)$$

$$\frac{(0.5)}{u*} = 5.75 \log_{10}\left(\frac{0.025}{0.025-0.015}\right)$$

$$\frac{(0.5)}{u*} = 2.288$$

$$u_* = 0.218 \text{ m /sec}$$

From Eq. (14.20)

$$\frac{(u-v)}{u*} = 5.75 \log_{10}\left(\frac{y}{R}\right) + 3.75.$$

At $y = R, u = u_{max}$

$$\frac{(u_{max}-v)}{u*} = 3.75$$

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$$\frac{(3.5 - v)}{0.218} = 3.75$$

$$v = 2.68 \text{ m /sec}$$

$$Q = 2.68 \times 0.7854\ (0.025)^2$$

$$= 0.0013 \text{ m}^3\text{/sec}$$

Problem 14.10 *Water at 20°C is flowing through a smooth pipe of radius 0.3 m @ 0.03 m³/sec. Determine (i) the wall shear stress (ii) Maximum velocity (iii) thickness of the laminar sub-layer.*

Solution

From the tables, properties of water at 20°C

$$\rho = 998 \text{ kg/m}^3$$

and $$\upsilon = 1 \times 10^{-6} \text{ m}^2\text{/sec.}$$

From Eqs. (14.9) and (14.10)

Shear stress in the laminar sub layer region is given by $u_* = \sqrt{\frac{\tau_0}{\rho}}$ ...(a)

(μ_* is called the shear velocity).

$$\tau_0 = \frac{\rho v^2 f}{2} \quad \text{...(b)}$$

$$v = \frac{0.03}{0.7854\ (0.6)^2}$$

$$= 0.106 \text{ m/sec}$$

Let us now determine the Reynolds number.

$$R_e = \frac{vD}{\upsilon}$$

$$= \frac{0.106 \times 0.6}{1 \times 10^{-6}}$$

$$= 63{,}600$$

From (14.34) for turbulent flow in a smooth pipe ($4000 < R_e < 100000$)

$$f = \frac{0.0791}{(Re)^{\frac{1}{4}}}$$

$$= \frac{0.0791}{(63{,}600)^{\frac{1}{4}}}$$

$$= 5 \times 10^{-3}$$

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From (b) $$\tau_0 = \frac{\rho v^2 f}{2}$$

$$\tau_0 = \frac{(998)\,(0.106)^2\,(5\times10^{-3})}{2}$$

$$= 0.028 \text{ N/m}^2$$

From (a) $$u_* = \sqrt{\frac{\tau_0}{\rho}}$$

$$u_* = \sqrt{\frac{0.028}{998}}$$

$$u_* = 0.0053 \text{ m/sec}$$

(a) For smooth pipes from Eq. (14.20)

$$\left(\frac{u}{u*}\right) = (5.75)\log\left(\frac{u*y}{\upsilon}\right) + 5.5$$

at $y = R$

$u = u_{max}$

$$\left(\frac{u_{max}}{u*}\right) = (5.75)\log\frac{(u\times R)}{\upsilon} + 5.5$$

$$\left(\frac{u_{max}}{0.0053}\right) = (5.75)\log\frac{(0.0053)\,(0.3)}{(1\times10^{-6})} + 5.5$$

$$\left(\frac{u_{max}}{0.0053}\right) = 23.90$$

$$u_{max} = 0.126 \text{ m/sec}$$

From Eq. (14.36)

The thickness of the laminar sub layer $\delta = \frac{11.6\,\upsilon}{u*}$

$$\delta = \frac{11.6\,(1\times10^{-6})}{0.126} = 9.2\times10^{-5} \text{ m}$$

$$= 0.092 \text{ mm}$$

Problem 14.11 *Water at 20°C is flowing through a smooth pipe of radius 0.3 m @ 0.35 m³/sec. Determine the point velocity and shear stress at 0.15 m from the center of the pipe.*

Solution

From the tables, properties of water at 20°C

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$$\rho = 998 \text{ kg/m}^3$$

and $$\upsilon = 1 \times 10^{-6} \text{ m}^2/\text{sec.}$$

$$v = \frac{0.35}{0.7854\ (0.6)^2}$$

$$= 1.237 \text{ m/sec}$$

$$R_e = \frac{vD}{\upsilon}$$

$$= \frac{1.237 \times 0.6}{1 \times 10^{-6}}$$

$$= 742{,}200$$

From (14.34) for turbulent flow in a smooth pipe ($4000 < R_e < 100000$)

$$f = \frac{0.0791}{(R_e)^{\frac{1}{4}}}$$

$$= \frac{0.0791}{(742{,}200)^{\frac{1}{4}}} = 2.69 \times 10^{-3}$$

From Eqs. (14.9) and (14.10)

Shear stress in the laminar sublayer region

$$\tau_0 = \frac{\rho v^2 f}{2}$$

$$\tau_0 = \frac{[(998)\ (1.237)^2\ (2.69 \times 10^{-3})]}{2}$$

$$= 2.05 \text{ N/m}^2$$

τ = shear stress at any radius R,

$$= \left(\frac{\tau_0 R}{R_0}\right)$$

From Eq. (14.10a)

τ at R = 0.15 m

$$= \frac{[(2.05) \times (0.15)]}{(0.30)}$$

$$= 1.026 \text{ N/m}^2$$

τ at 0.15 m from the center = 1.026 N/m^2

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$$\left(\frac{u}{u*}\right) = (5.75)\log\left(\frac{u*y}{\upsilon}\right) + 5.5$$

from Eq. (14.20)

$$u_* = \sqrt{\frac{\tau_0}{\rho}}$$

$$= \sqrt{\frac{2.05}{998}}$$

$$= 0.045 \text{ m/sec}$$

$$\left(\frac{u}{0.045}\right) = (5.75)\log\frac{(0.045)(0.15)}{1\times10^{-6}} + 5.5$$

$$\left(\frac{u}{0.045}\right) = 27.5$$

Point velocity u at a distance 0.15 m from the center = 1.238 m/sec.

Problem 14.12 *Prove that in a turbulent flow through a closed conduit the difference between the point velocity and the average velocity at any point is the same for both smooth and rough pipes.*

Solution

(i) For a smooth pipe from Eq. (14.20) the point velocity is given by,

$$\left(\frac{u}{u*}\right) = (5.75)\log\left(\frac{u*y}{\upsilon}\right) + 5.5$$

(ii) For a *smooth* pipe from Eq. (14.23) the average velocity is given by,

$$\left(\frac{v}{u*}\right) = (5.75)\log\frac{(u*R)}{\upsilon} + 1.75$$

The difference between (i) and (ii)

$$\left(\frac{u}{u*}\right) - \left(\frac{v}{u*}\right) = (5.75)\log\left(\frac{u*y}{\upsilon}\right) + 5.5 - \left\{(5.75)\log\frac{(u*R)}{\upsilon} + 1.75\right\}$$

$$\left(\frac{u}{u*}\right) - \left(\frac{v}{u*}\right) = \left\{(5.75)\log\left[\frac{(u*y)}{\upsilon}\right]\times\left[\frac{\upsilon}{(u*y)}\right] + 5.5\right\} - \left\{(5.75)\log\left[\frac{(u*R)}{\upsilon}\right]\times\left[\frac{\upsilon}{(u*y)}\right] + 1.75\right\}$$

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$$\left(\frac{u}{u*}\right) - \left(\frac{v}{u*}\right) = +5.5 - (5.75)\log\left[\frac{(u*R)}{\upsilon}\right] \times \left[\frac{\upsilon}{(u*y)}\right] - 1.75$$

$$\left(\frac{u}{u*}\right) - \left(\frac{v}{u*}\right) = -(5.75)\log\left[\frac{R}{y}\right] + 3.75$$

$$\left(\frac{u}{u*}\right) - \left(\frac{v}{u*}\right) = (5.75)\log\left[\frac{y}{R}\right] + 3.75 \qquad \ldots(a)$$

(iii) For a Rough pipe from Eq. (14.21) the point velocity is given by,

$$\left(\frac{u}{u*}\right) = (5.75)\log\left(\frac{y}{\varepsilon}\right) + 8.5$$

(iv) For a Rough pipe from Eq. (14.24) the average velocity is given by,

$$\left(\frac{v}{u*}\right) = (5.75)\log\left(\frac{R}{\varepsilon}\right) + 4.75$$

The difference between (iii) and (iv)

$$\left(\frac{u}{u*}\right) - \left(\frac{v}{u*}\right) = \left\{(5.75)\log\left(\frac{y}{\varepsilon}\right) + 8.5\right\} - \left\{(5.75)\log\left(\frac{R}{\varepsilon}\right) + 4.75\right\}$$

$$= \left\{(5.75)\log\left(\frac{y}{\varepsilon}\right)\left(\frac{\varepsilon}{y}\right) + 8.5\right\} - \left\{(5.75)\log\left(\frac{R}{\varepsilon}\right)\left(\frac{\varepsilon}{y}\right) + 4.75\right\}$$

$$\left(\frac{u}{u*}\right) - \left(\frac{v}{u*}\right) = 8.5 - \left\{(5.75)\log\left(\frac{R}{y}\right) + 4.75\right\}$$

$$\left(\frac{u}{u*}\right) - \left(\frac{v}{u*}\right) = 8.5 - (5.75)\log\left(\frac{R}{y}\right) - 4.75$$

$$\left(\frac{u}{u*}\right) - \left(\frac{v}{u*}\right) = (5.75)\log\left(\frac{y}{R}\right) + 3.75 \qquad \ldots(b)$$

(a) = (b) proves that the velocity distribution for both smooth and rough pipes is the same.

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Problem 14.13 *Determine the point velocity and the shear stress at a distance 125 mm from the wall of a smooth pipe of diameter 600 mm through which water at 20°C is being conveyed.* $R_e = 90{,}000$.

Solution

From the tables, properties of water at 20°C

$$\rho = 998 \text{ kg/m}^3$$

and $$\upsilon = 1\times10^{-6} \text{ m}^2/\text{sec}.$$

$$R_e = \frac{vD}{\upsilon}$$

$$90{,}000 = \frac{(v\times 0.6)}{1\times10^{-6}}$$

$$v = 0.15 \text{ m/sec}$$

From Eq. (14.34) for turbulent flow in a smooth pipe ($4000 < R_e < 100000$)

$$f = \frac{0.0791}{(R_e)^{\frac{1}{4}}}$$

$$= \frac{0.0791}{(90{,}000)^{\frac{1}{4}}}$$

$$= 4.56 \times 10^{-3}$$

From Eq. (14.27)

$$u = v\left[1+1.43\sqrt{f}+2.15\sqrt{f}\log\left(\frac{y}{R_0}\right)\right]$$

$$u = 0.15\left[1+1.43\sqrt{(4.56+10^{-3})}+2.15\sqrt{(4.56+10^{-3})}\log\left(\frac{0.125}{0.6}\right)\right]$$

$$u = 0.149 \text{ m/sec}$$

From Eqs. (14.9) and (14.10)

Shear stress in the laminar sublayer region $\tau_0 = \frac{\rho v^2 f}{2}$

$$\tau_0 = \frac{[(998)\,(0.15)^2\,(4.56\times10^{-3})]}{2}$$

$$= .051 \text{ N/m}^2$$

τ = shear stress at any radius R,

$$= \left(\frac{\tau_0 R}{R_0}\right)$$

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From Eq. (14.10 a)

$$\tau = \left[\left(\frac{\tau_0 (R_0 - y)}{R_0}\right)\right]$$

$$= 0.051 \left[\frac{(0.30 - 0.125)}{0.3}\right]$$

$$= 0.029 \text{ N/m}^2$$

Problem 14.14 *Determine the maximum permissible average height of surface roughness if water at 20°C has to flow through a pipe of 300 mm diameter at 3.5 m/sec with minimum loss of energy.*

Solution

From the tables, properties of water at 20°C

$$\rho = 998 \text{ kg/m}^3$$

and $$\upsilon = 1 \times 10^{-6} \text{ m}^2/\text{sec}.$$

From Eq. (14.22 a)

In turbulent flow the limiting value for the smooth flow regime is,

$$\frac{\left(R_e \sqrt{f}\right)}{\left(\frac{R_0}{\varepsilon}\right)} = 17 \qquad \ldots(a)$$

$$R_e = \frac{vD}{\upsilon}$$

$$= \frac{(3.5 \times 0.3)}{1 \times 10^{-6}} = 1.05 \times 10^6$$

From Eq. (14.35)

Turbulent flow in smooth pipes when $(5 \times 10^4 < R_e < 4 \times 10^7)$

$$f = 0.0008 + \frac{0.05525}{(R_e)^{0.237}}$$

$$f = 0.0008 + \frac{0.05525}{(1.05 \times 10^6)^{0.237}}$$

$$f = 2.86 \times 10^{-3}$$

Substituting the values in (a) and solving for average roughness ε

$$\frac{(1.05 \times 10^6)\sqrt{(2.86 \times 10^{-3})}}{\left(\frac{0.150}{\varepsilon}\right)} = 17$$

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$$\varepsilon = 4.54 \times 10^{-5}\ \text{m}$$
$$= 0.0454\ \text{mm}$$

Problem 14.15 *Difference in elevation between two oil reservoirs is 6.5 m and the two reservoirs are connected by a commercial quality steel pipe 500 m long. If the discharge is 60 L/s, determine the pipe ID. Obtain the friction factor from the Colebrook's expression* $\frac{1}{\sqrt{f}} = -0.869 \log_e \left[\frac{\left(\frac{e}{D}\right)}{3.7} + \frac{2.523}{\left(R_e \sqrt{f}\right)} \right]$.

$\mu = 0.055$ kg/m.s and $\gamma = 8.5$ kN/m^3.

Solution

$$Q = \frac{60}{1000} = 0.06\ \text{m}^3/\text{sec}$$

$$h_f = f \left(\frac{L}{d}\right) \frac{v^2}{2g}$$

(The various empirical equations enumerated are all based on $f = 4f$. So in this chapter we use the basic Darcy–Wisebach equation)

For non-horizontal pipes the following form of energy equation applies. Since the velocity head is constant,

$$h_f = \frac{(P_1 - P_2)}{\gamma} + (Z_1 - Z_2)$$

Since in this case $P_1 = P_2 = 0$

$$h_f = (Z_1 - Z_2) = 6.5\ \text{m}$$

$$\frac{f L v^2}{2gD} = 6.5 \qquad \ldots(a)$$

$$\frac{v^2}{2g} = \frac{(Q)^2}{2g(0.7854 D^2)^2}$$

$$= \frac{(Q)^2}{2g(0.7854^2) D^4}$$

$$= \frac{(0.06)^2}{2g(0.7854^2) D^4}$$

$$\frac{v^2}{2g} = \frac{(2.97 \times 10^{-4})}{D^4} \qquad \ldots(b)$$

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Substituting (b) in (a)

$$fL\,\frac{(2.97\times10^{-4})}{D^4(D)} = 6.5$$

$$fL\,\frac{(2.97\times10^{-4})}{D^5} = 6.5$$

$$D^5 = fL\,\frac{(2.97\times10^{-4})}{6.5}$$

$$= f(500)\,\frac{(2.97\times10^{-4})}{6.5}$$

$$D^5 = 0.0228\,f \qquad \ldots(c)$$

Let us assume $f = 0.045$

$$D^5 = 0.0228 \times 0.045$$

$$= 1.03 \times 10^{-3}.$$

$$D = 0.2526 \text{ m}$$

$$\gamma = 8.5 \text{ kN/m}^3$$

$$\rho = \frac{(8.5\times1000)}{9.81} = 866.5 \text{ kg/m}^3$$

$$v = \frac{Q}{A}$$

$$= \frac{0.06}{0.7854\,D^2}$$

$$= \frac{0.06}{0.7854\,(0.2526)^2}$$

$$= 1.2 \text{ m/sec}$$

$$\mu = \mathbf{0.055\ kg/m.s}$$

Reynold's number $R_E = \dfrac{(vD\rho)}{\mu}$

$$= \frac{(1.2)(0.2526)(866.5)}{0.055}$$

$$= 4775$$

For commercial quality steel pipe from Moody's chart

$$\varepsilon = 0.046 \text{ mm}$$

$$= 0.000046 \text{ m}$$

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Hence $$\frac{\varepsilon}{D} = \frac{0.000046}{0.2526}$$

$$= 0.000182.$$

Now we use the given Colebrook's formula to obtain the new value of f

$$\frac{1}{\sqrt{f}} = -0.869 \log_e \left[\frac{\left(\frac{e}{D}\right)}{3.7} + \frac{2.523}{\left(R_E \sqrt{f}\right)} \right]$$

$$\frac{1}{\sqrt{f}} = -0.869 \times 2.3 \log \left[\frac{\left(\frac{e}{D}\right)}{3.7} + \frac{2.523}{\left(R_E \sqrt{f}\right)} \right]$$

$$\frac{1}{\sqrt{f}} = -0.869 \times 2.3 \log \left[\frac{(0.000182)}{3.7} + \frac{2.523}{\left(4775\sqrt{f}\right)} \right]$$

$$\frac{1}{\sqrt{f}} = -0.869 \times 2.3 \log \left[\left(\frac{(0.0000491 + 0.000528)}{\sqrt{f}} \right) \right]$$

Assume f = 0.035 for first trial.

$$\frac{1}{\sqrt{0.035}} = -0.869 \times 2.3 \log \left[\left(\frac{(0.0000491 + 0.000528)}{\sqrt{0.035}} \right) \right]$$

$$5.345 = 5.08$$

Assume f = 0.04 for second trial.

$$\frac{1}{\sqrt{0.04}} = -0.869 \times 2.3 \log \left[\left(\frac{(0.0000491 + 0.000528)}{\sqrt{0.04}} \right) \right]$$

$$5 = 5.13$$

Assume f = 0.038 for third trial.

$$\frac{1}{\sqrt{0.038}} = -0.869 \times 2.3 \log \left[\left(\frac{(0.0000491 + 0.000528)}{\sqrt{0.038}} \right) \right]$$

$$5.129 = 5.115$$

f = 0.038 can be taken as the right value

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From Eq. (c) above,

$$D^5 = 0.0228 \times 0.038$$
$$= 0.0008664$$

Hence, $D = 0.2440$ m = 244 mm

This approach helps us solve the problem with four unknowns (f, D, v, R_E)

Problem 14.16 *It is proposed to reduce the effective roughness value* (ε_E) *of a badly damaged concrete pipe of 750 mm diameter from 6mm to 0.250 mm by providing a cast iron lining 15 mm thick. If the discharge is 2.5 m^3/sec, determine the extent of power saved per unit length of the pipe.* $\upsilon = 1 \times 10^{-6}$ *m^2/sec and* $\gamma = 9.79$ *kN/m^3.*

Solution

Power saved $P = \gamma Q h_{f\text{-}S}$ ...(a)

$$h_{f\text{-}S} \text{ per unit length} = \frac{f_1 v_1^2}{2gD_1} - \frac{f_2 v_2^2}{2gD_2} \quad \text{...(b)}$$

The subscripts 1 and 2 refer to values before and after the lining respectively. We will first determine $v_1 D_1$ and $v_2 D_2$ and the corresponding Reynold's numbers.

$$D_1 = 0.750 \text{ m}$$

and $D_2 = 0.750 - 0.30 = 0.720$ m

$$\left(\frac{\varepsilon_{E-1}}{D_1}\right) = \frac{0.006}{0.750} = 0.008$$

$$\left(\frac{\varepsilon_{E-2}}{D_2}\right) = \frac{0.00025}{0.730} = 0.000342$$

$$v_1 = \frac{Q}{A_1}$$

$$= \frac{2.5}{0.7854(0.750)^2}$$

$$= 5.65 \text{ m/sec}$$

$$v_2 = \frac{Q}{A_2}$$

$$= \frac{2.5}{0.7854(0.720)^2}$$

$$= 6.14 \text{ m/sec}$$

$$R_{E-1} = \frac{v_1 D_1}{\upsilon}$$

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$$= \frac{(5.65 \times 0.750)}{(1 \times 10^{-6})}$$

$$= 4.23 \times 10^6$$

$$R_{E-2} = \frac{v_2 D_2}{\upsilon}$$

$$= \frac{(6.14 \times 0.730)}{(1 \times 10^{-6})}$$

$$= 4.48 \times 10^6$$

We now use the Colebrooks equation to determine the f_1 and f_2.

$$\frac{1}{\sqrt{f}} + 0.869 \times 2.3 \log \left[\frac{\left(\frac{\varepsilon}{D}\right)}{3.7} + \frac{2.523}{\left(R_E \sqrt{f}\right)} \right] = 0$$

The student should solve for f_1 and f_2 by 'trial and error' using appropriate values of $\left(\frac{\varepsilon}{D}\right)$ and R_E. In the given case $f_1 = 0.035$ and $f_2 = 0.0155$ very nearly satisfy the above equation and hence can be taken as correct.

Substitute all of the values determined in Eq. (b) above

$$h_{f-S} \text{ per unit length} = \frac{(0.035)\,(5.65)^2}{2g(0.750)} - \frac{(0.0155)\,(6.14)^2}{2g(0.720)}$$

$$h_{f-S} \text{ per unit length} = 0.076 - 0.041$$

$$= 0.0346 \text{ m/metre length of the pipe.}$$

From Eq. (a) above,

$$\text{Power saved } P = \gamma Q h_{f-S}$$

$$= 9.79 \times 2.5 \times 0.0346$$

$$= 0.846 \text{ kW / unit length of the pipe}$$

Problem 14.17 *Using Prandtl's one seventh power law determine the point velocity at a distance 60 mm from the center of a smooth pipe of diameter 200 mm if the volume flow rate of water at 20° C through the pipe is 0.015 m^3/sec.*

Solution

From the tables, properties of water at 20°C

$$\rho = 998 \text{ kg/m}^3$$

and $$\upsilon = 1 \times 10^{-6} \text{ m}^2\text{/sec.}$$

$$V = \frac{Q}{A}$$

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$$= \frac{0.015}{0.7854\,(0.200)^2}$$

$$= 0.477 \text{ m/sec}$$

$$R_e = \frac{vD}{\upsilon}$$

$$= \frac{(0.477 \times 0.200)}{1\times10^{-6}}$$

$$= 9.5 \times 10^4$$

From (14.34) for turbulent flow in a smooth pipe ($4000 < R_e < 100000$)

$$f = \frac{0.0791}{(R_e)^{\frac{1}{4}}}$$

$$= \frac{0.0791}{(95000)^{\frac{1}{4}}}$$

$$= 4.5 \times 10^{-3}$$

From Eq. (14.28) the centerline velocity and average velocity are related as under.

$$v_{max} = v\,(1 + 1.43\,\sqrt{f}\,)$$

$$v_{max} = 0.477\,(1 + 1.43\,\sqrt{0.0045}\,)$$

$$v_{max} = 0.522 \text{ m/sec}$$

From Eq. (14.25) Prandtl's law applicable for values of R_e< 100,000.

$$\left(\frac{u}{\mu_{max}}\right) = \left(\frac{y}{R_0}\right)^{\frac{1}{7}} \qquad \ldots(a)$$

Here, R_0 = 100 mm

= 0.100 m

and y = 100 – 60

= 40 mm = 0.040 m

$$\left(\frac{u}{0.522}\right) = \left(\frac{0.040}{0.100}\right)^{\frac{1}{7}}$$

Point velocity u = 0.458 m/sec

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QUESTIONS BANK

Section A

Review Questions

1. Define turbulent flow. What is the region between R_E 2000 to R_E 4000 known as? Write down the appropriate velocity distribution equation for this region.
2. Explain how the Prandtl's mixing length theory helped in evaluating the apparent shear stress.
3. Factors f and $4f$ are alternately used in the Darcy– Weisbach equation. Explain where each factor is considered appropriate.
4. What does η in the equation $\tau = \eta \left(\frac{du}{dy}\right)$ represent? Define kinematic Eddy viscosity.
5. Define velocity deficit and critical velocity. What is the value of the proportionality constant in the relationship between mixing length and wall distance?
6. In turbulent flow through a smooth pipe write down the governing equations for determining the point velocity in the laminar sub-layer region and in the outer turbulent region.
7. Classify the three boundary layers as either rough, transition or smooth based on the critical Reynold's number and write down the relevant velocity distribution equation that applies to both smooth and rough boundaries.
8. On what factor does turbulent flow in rough pipe depends and when does the transition from smooth flow regime begins?
9. When is the turbulent flow considered as steady? State the advantage and the limitation of Prandtl's one seventh power law.
10. For steady incompressible flow through an inclined pipe express the irreversibilities in terms of head loss.

Section B

Multiple Choice Questions

Choose the right answer :

1. In turbulent flow the apparent shear is
   (a) greater than the viscous shear
   (b) less than the viscous shear
   (c) equal to the viscous shear
   (d) None of these

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2. In turbulent flow the main cause of shear is due to
   (a) mass transfer (b) energy transfer
   (c) momentum transfer (d) All of these
3. In the region $y \leq \delta$, the laminar sub-layer thickness is 5 times the ratio of
   (a) $\frac{u}{v}$ (b) $\frac{v}{u}$
   (c) $\frac{vs}{u*}$ (d) None of these
4. Turbulent flow in Rough pipes begins when $\left(\frac{\varepsilon}{d}\right)$ is
   (a) > 5 (b) < 6
   (c) < 12 (d) None of these
5. In steady and uniform turbulent flow the head loss due to friction depends on
   (a) pipe surface roughness
   (b) fluid density and viscosity
   (c) pipe length to diameter ratio
   (d) All of these
6. The Blasius formula for friction factor for turbulent flow in smooth pipes is valid for Reynold numbers up to
   (a) 10^5 (b) 10^6
   (c) 10^7 (d) None of these
7. Prandtl's one seventh power law is valid for
   (a) central turbulent core region
   (b) transition region
   (c) both central core and transition regions
   (d) None of these
8. Shear velocity is given by
   (a) $\sqrt{\frac{\rho}{\tau}}$ (b) $\sqrt{\frac{\rho}{\tau_0}}$
   (c) $\sqrt{\frac{\tau_0}{\rho}}$ (d) None of these
9. In the transition regime critical Reynold's number is
   (a) < 70 (b) > 70
   (c) > 60 (d) None of these

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10. For laminar flow in a tube of radius R the average velocity is
    (a) 0.493 R (b) 0.5 R
    (c) 0.293 R (d) None of these
11. The shear stress in turbulent flow in a fluid flowing through a circular section
    (a) remains constant
    (b) is zero at the center and increase linearly with radius
    (c) varies parabolically across the section
    (d) None of these

Section C

Problems

1. Water at 10° C is being pumped through a horizontal commercial quality pipe of diameter 60 mm @ 0.005 m^3/sec. Determine the pipe wall roughness if two pressure gauges mounted 50 m apart display 1500 kPa and 900 kPa. (*Ans.* $\varepsilon = 0.041$)
2. Determine the nature of the pipe wall surface if the surface roughness $\varepsilon = 0.2$ mm, while water at 20° C is flowing through the pipe. Shear stress = 5.5 Pa. (*Ans.* transition $R_{E^*} = 4.7$)
3. Water at 10°C is conveyed through a rough pipe of diameter 300 mm and length 600 m. If the discharge through the pipe is 0.05 m^3/sec, determine the head lost due to friction. Take absolute roughness ε = 0.45 mm. (*Ans.* 36.65 m)
4. In a turbulent flow through a 400 mm diameter pipe if the centerline velocity and the local velocity at a distance 100 mm from the wall are 3.5 m/sec and 3 m/sec respectively determine the wall shear stress if the fluid being conveyed is water at 10° C. (*Ans.* 83 N/m^2)
5. For a rough turbulent flow of water at 20°C through a concrete pipe whose effective roughness value is 1.5 mm determine the point velocity at a distance 150 mm from the center of the pipe. The diameter of the pipe 500 mm and the volume flow rate is 0.15 m^3/sec.
   (*Ans.* 0.814 m/sec)
6. For flow of water at 10°C if the point velocity at a distance 1 mm from the wall of the pipe is 0.735 m/sec and the average velocity is 0.930 m/sec identify the nature of the pipe wall surface. The diameter of the pipe is 50 mm. (*Ans.* Surface is smooth since ($R_{E^*} = 2.3$ which is < 3)

Answers to Multiple Choice Questions

1. (a) **2.** (c) **3.** (c) **4.** (d) **5.** (d) **6.** (b)
7. (a) **8.** (c) **9.** (a) **10.** (c) **11.** (c)

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Chapter

15

Boundary Layer Theory

In the previous chapter 14 while discussing turbulent flow through pipes the existence of three different forms of layer within the velocity profile of a turbulent flow of fluid was mentioned and the layers were individually identified as the laminar sub-layer, the transition layer and the turbulent core. But there exists a layer within these layers called the *Boundary layer* where all of the viscous effects due to flow is assumed to be concentrated. In this chapter we discuss extensively the qualitative and quantitative aspects of the boundary layer and how this layer develops through the transition from laminar to turbulent flow. This layer if not taken in to account will significantly affect our calculation of surface resistance. The concept of boundary layer was first introduced by Prandtl.

A flowing fluid suffers drag and consequently retardation in motion due to both pressure and skin friction forces. In this chapter we study exclusively the cause and effect of skin friction drag on the flow parameters and exclude the effect due to pressure (form drag).

Skin friction drag is confined to a small region of flow starting from the boundary. We are aware, particles of fluid adjacent to the boundary are stationary whereas those some distance away move with a certain velocity. It is evident therefore that there exists a velocity gradient in a direction perpendicular to the surface. The skin friction drag is confined to the region in between the 'no-slip' or the zero velocity zone and the free stream velocity zone. This zone is referred to as the *boundary layer*.

In terms of the local velocity (u) and the free stream velocity (U), the thickness (λ) of the boundary layer is defined as that distance from the surface where u = 99 percent of U. In other words the upper limit to this thickness is bound by the equation $u = 0.99U$. Refer to Figs. (15.1) and (15.2). This however is an arbitrary value and for exact values the thickness of boundary layer is more critically defined. Mathematical expressions are then derived consistent with each definition. Thus there came to be two definitions for boundary layer thickness.

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1. Displacement thickness (λ_D)
2. Momentum thickness (θ)

If U is the free stream velocity, and u is the velocity at any section within the boundary layer then displacement thickness is given by,

$$\lambda_D = \int_0^{\lambda} \left(\frac{1 - u}{U}\right) dy \qquad \ldots(15.1)$$

λ is the value of y at which $u = U$.

Equation (15.1) takes into consideration the loss in the mass flow rate suffered by the flowing fluid due to retardation at the boundary layer.

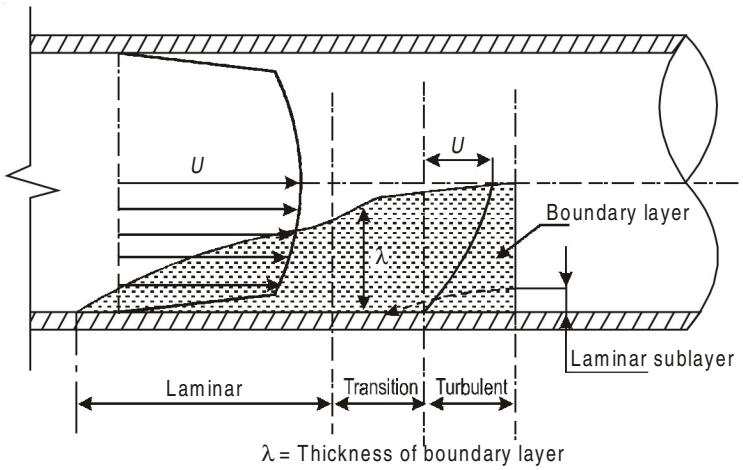


Fig. 15.1

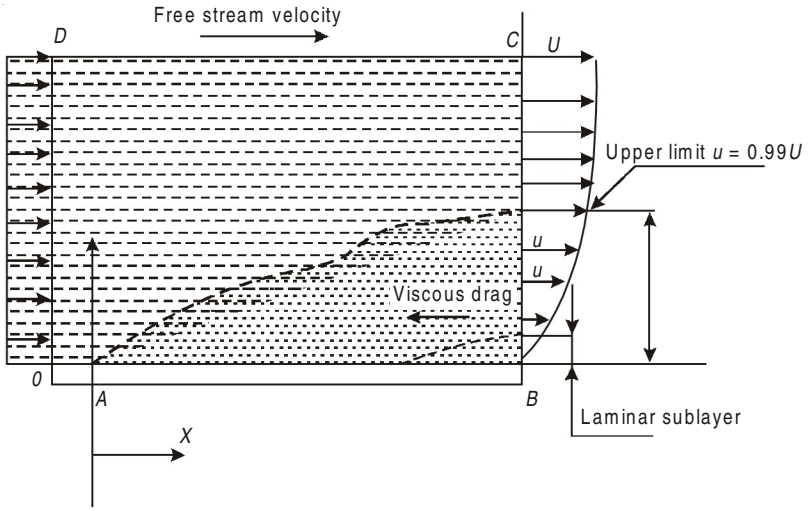


Fig. 15.2

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Momentum thickness (θ) is obtained by equating the momentum carried by an element of fluid of unit width, depth = $\sigma\theta$ and velocity U to the difference between the momentums an element of flowing fluid would have possessed in the absence of the boundary layer. Mathematically it is expressed as the result of,

$$(\rho\, U^2\, \theta) = \int_0^{\lambda} \rho(Us - u)u\, dy \qquad \ldots(15.2)$$

From which,

$$\theta = \int_0^{\lambda} \frac{u}{U}\left[1 - \frac{u}{U}\right] dy \qquad \ldots(15.2\ a)$$

Equation (15.2) takes into consideration the loss in momentum suffered by the flowing fluid due to retardation suffered at the boundary layer.

Instead of considering the loss in mass flow rate or the loss in momentum due to retardation at the boundary, if we consider the kinetic energy loss then it leads to yet an other definition called *Energy thickness* (λ_E).

$$\lambda_E = \int_0^{\lambda} \frac{u}{U}\left[1 - \frac{u^2}{U^2}\right] dy \qquad \ldots(15.2\ b)$$

The ratio $\frac{\lambda_D}{\theta}$ is called the shape factor and is denoted by the alphabet H and the ratio $\frac{y}{\lambda}$ is represented by η. Refer Fig. 15.3. $\ldots(15.2\ c)$

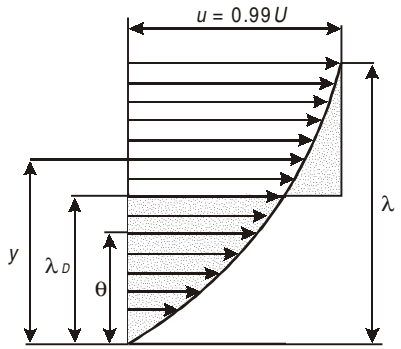


Fig. 15.3

Determination of shear stress at the boundary and the thickness of boundary layer involve complex viscous flow differential equations. But if we assume a certain velocity distribution for each flow pattern then momentum methods can be employed which give results that are almost the same as that obtained through more elaborate methods.

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Since in this chapter we are basically interested in the skin friction drag we are neglecting the drag due to pressure gradient. Hence it is more convenient if we consider flow over a flat plate instead of through a closed conduit as that way the pressure gradient $\left(\frac{dp}{dx}\right)$ can be rendered = 0. (Here x is length of the plate measured in the direction of flow from the leading edge.)

15.1 LAMINAR BOUNDARY LAYER

Referring to Fig. (15.2) the presence of laminar sub-layer within the boundary layer in the turbulent zone is supportive of the theory that the turbulence at the boundary is zero as it is a no slip region. Laminar sub-layer thickness δ is very small compared to the boundary layer thickness and hence its presence is only of academic interest and is generally overlooked for all practical purposes.

It has already been mentioned that prior assumption of a velocity profile is necessary for purposes of applying momentum methods to determine the boundary layer thickness.

Prandtl assumed the following velocity profile and applied the basic *conditions common to any velocity profile.*

$$\frac{u}{U} = \left[\frac{3}{2}\eta - \frac{\eta^3}{2}\right] \qquad \ldots \eta \text{ is the eddy viscosity}$$

The basic conditions are, at $y = 0$,

$u = 0$ and du/dy has a finite value and at $y = \lambda$,

$$u = U \text{ and } \frac{du}{dy} = 0$$

$$\frac{\lambda}{x} = 4.64\sqrt{\left(\frac{\upsilon}{Ux}\right)} = \frac{4.64}{\sqrt{R_x}} \qquad \ldots(15.3)$$

$$\lambda = \frac{4.64x}{\sqrt{R_x}} \qquad \ldots (15.3\text{ a})$$

$$\lambda = \frac{4.64}{\sqrt{\frac{\mu x}{\rho U}}} \qquad \ldots(15.3\text{ b})$$

$$\left(\text{because } R_x = \frac{\rho U x}{\mu}\right)$$

Equation (15.3 b) shows that the thickness of the boundary layer is directly proportional to the square root of the distance from the leading edge.

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Assuming other velocity distributions would not drastically alter the results obtained by Prandtl nor does Prandtl's result deviate too far from the actual mathematical results obtained by Blasius.

Blasius obtained the following equation for $\left(\frac{\lambda}{x}\right)$ through differential equations for viscous flow instead of assuming a velocity profile or applying momentum methods.

$$\frac{\lambda}{x} = \frac{5}{\sqrt{\left(\frac{\upsilon}{Ux}\right)}} = \frac{5x}{\sqrt{R_x}} \qquad \ldots(15.4)$$

In either case R_x depends on the length of the boundary x measured from the leading edge.

The maximum shear stress or the shear stress at the boundary is given by,

$$\tau_0 = \left(\frac{3}{2}\right)\left(\frac{\mu U}{\lambda}\right) \qquad \ldots(15.5)$$

Eliminating λ from Eqs. (15.2) and (15.3) we obtain an equation for τ_0 in terms of the measurable quantities.

$$\tau_0 = (0.322)\sqrt{\frac{\mu \rho U^3}{x}} \qquad \ldots(15.6)$$

The skin friction drag F_D over the plate of length L per unit width is obtained by integrating τ_0 over the limits 0 to L

$$F_D = (0.644)\sqrt{\mu \rho U^3 L} \qquad \ldots(15.7)$$

The skin friction drag F_D over the plate of length L per unit width can also be expressed in terms of the stagnation pressure and the coefficient of drag C_d

$$F_D = C_d \left[\frac{\rho U^3}{2}\right] L \qquad \ldots(15.8)$$

For the laminar boundary layer $C_d = \frac{1.328}{\sqrt{R_L}}$

where $$R_L = \frac{UL}{\upsilon} \qquad \ldots (15.9)$$

15.2 TURBULENT BOUNDARY LAYER

Properties of laminar boundary layer cited above are only of academic interest since the flow pattern encountered in practical engineering applications is almost

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always turbulent. Boundary layer becomes turbulent when Reynolds number lies in the region 5×10^5 to 1×10^6. The critical Reynolds number is 5×10^5 and it is believed that the transition from laminar to turbulent occurs at this number.

The major problem encountered while determining the properties of turbulent boundary layer was that in turbulent flow the shear stress is no longer, merely a function of viscosity and velocity gradient. Hence it was required that a new relation be found between the velocity distribution and the boundary layer.

It was at this stage Prandtl suggested that since the development of boundary layer for both closed conduits and flat plates are similar the velocity distribution equation applicable to pipe flow be considered as applicable to flat plates too. Prandtl's $\frac{1}{7^{\text{th}}}$ power law thus came to be the basis for developing turbulent boundary layer properties. Prandtl's universal velocity distribution equation for smooth pipes answers the need adequately but the calculations are involved compared to Prandtl's one seventh power law.

If we apply the Prandtl's $\frac{1}{7^{\text{th}}}$ power law $\left(\frac{u}{\mu_{max}}\right) = \left(\frac{y}{R_0}\right)^{\frac{1}{7}}$ to a flat plate subject to the same limitation of R_E the following corrections are required.

Since $u = U$ at $y = R$,

$$\left(\frac{u}{U}\right) = \left(\frac{y}{\lambda}\right)^{\frac{1}{7}}.$$

(Since $\lambda = R_0$ in the fully developed region) ...(15.10)

$$\left(\frac{u}{U}\right) = (\eta)^{\frac{1}{7}} \quad \eta \text{ is the eddy viscosity} \qquad ...(15.11)$$

The turbulent boundary layer properties valid for $5 \times 10^5 < R_E < 2 \times 10^7$ as determined by Prandtl and Blasius are,

$$\frac{\lambda}{x} = \frac{0.377}{\left(\frac{Ux}{\upsilon}\right)^{0.2}}$$

$$= \frac{0.377}{R_{Ex}^{0.2}} \qquad ...(15.12)$$

By comparing Eqs. (15.1) and (15.2) it can be said that λ increases more rapidly in the turbulent region.

The skin friction drag per unit width on one side of the plate is,

$$F_D = (C_D)\,\rho \left(\frac{U^2}{2}\right) L \qquad ...(15.13)$$

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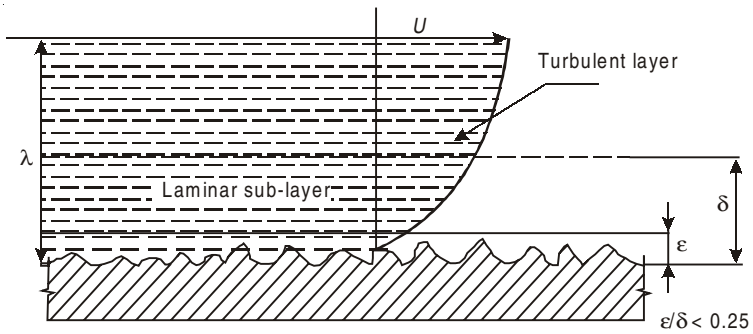
$$\text{The drag coefficient } C_D = \frac{0.072}{(R_L)^{0.2}} \qquad \ldots(15.14)$$

$$\text{The shear stress } \tau_0 = 0.029\,\rho U^2 \left(\frac{\upsilon}{U_x}\right)^{0.2} \qquad \ldots(15.14\text{ a})$$

If the R_E increases, the above equations have to be re-worked by assuming lower values for the exponent in the Prandtl's one-seventh power law.

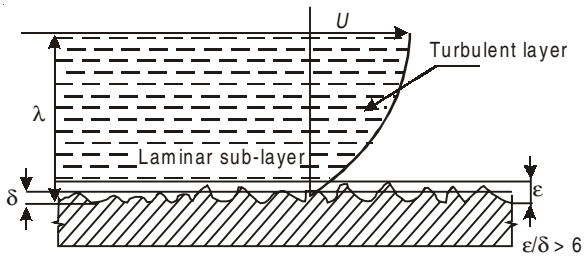
The same criteria, that was applied for defining a boundary as either smooth or rough in the previous chapter is applicable even here.

A boundary is hydro-dynamically *Smooth* if $\left(\frac{\varepsilon}{\delta}\right) < 0.25$ and as *Rough* if $\left(\frac{\varepsilon}{\delta}\right) > 6$. In the overlapping region $\left(\frac{\varepsilon}{\delta}\right)$ lies in between 0.25 to 6. When a plate is considered hydraulically smooth it indicates further losses cannot be prevented by reducing the surface roughness values. Here δ is the thickness of the laminar sub layer and ε is the roughness factor. Refer Figs. 15.4 and 15.5.



Surface roughness is well submerged within the laminar sub layer and hence is termed hydraulically smooth boundary

Fig. 15.4



Hydraulically rough boundary

Fig. 15.5

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The above equations are valid for turbulent flow within the range $R_E = 5 \times 10^5$ to 2×10^7. For values of R_E in the range of 2×10^7 to 2×10^9 *Schlichting* determined that the velocity at any point in the boundary layer varies as the logarithm of its distance from the boundary and the drag coefficient for these conditions is given by,

$$C_D = \frac{0.455}{(\log R_{EL})^{2.58}} \text{ and } \lambda = \frac{x\ (0.22)}{\sqrt{(R_L)^{\frac{1}{6}}}} \quad \ldots(15.15)$$

The above equations for laminar and turbulent boundary layer were developed assuming that the flow is either laminar or turbulent over the entire length of the plate. If we consider long plates then we have to consider the region at which the flow changes from laminar to turbulent which is called the *transition region* (Ref. Fig. 15.1) and develop appropriate equations for skin friction drag and the drag coefficient. If however the laminar boundary layer extends only for a short distance then the transition region can be neglected and the skin friction drag computed assuming turbulent flow for the entire length of the plate.

If we consider the transition region too then per unit width of the plate,

$$F_D = \left[\frac{\rho\ U^2}{2}\right] [C_D \text{ Turbulent} - C_D \text{ Transition} + C_D \text{ Laminar}]$$

$$F_D = \left[\frac{\rho\ U^2}{2}\right] \left[\frac{(0.072\ L)}{R_{EL}^{0.2}} - \frac{(0.072) L_X}{R_{ET}^{0.2}} + \frac{1.328\ L_X}{\sqrt{R_{EC}}}\right] \quad \ldots(15.16)$$

where, L is the length of the plate and R_{EL} is the corresponding Reynolds number $= \frac{UL}{\upsilon}$, L_X is the distance from the leading edge to the transition region and R_{ET} is the corresponding Reynolds number and R_{EC} is the critical Reynolds number $= 5 \times 10^5$.

$$\text{Since } \frac{(R_{EC})}{(R_{EL})} = \frac{(L_X)}{L}$$

$$(L_X) = \frac{(R_{EC})}{(R_{EL})} L$$

Substituting for (L_X) in Eq. (15.16) and solving for F_D gives,

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$$\frac{F_D}{\text{unit width}} = C_D \left[\rho \left(\frac{U^2}{2}\right) L\right]$$

$$\text{where, } C_D = \left[\frac{(0.072)}{(R_{EL})^{0.2}} - \frac{1700}{R_{EL}}\right] \quad \ldots(15.17)$$

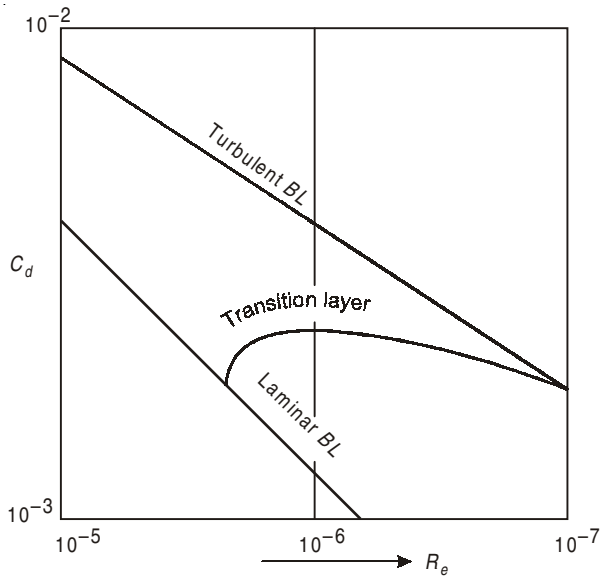
For Reynolds number between ($10^7 < R_{EL} < 10^9$) and for smooth boundary

$$\frac{F_D}{\text{unit width}} = C_D \left[\rho \left(\frac{U^2}{2}\right) L\right]$$

$$\text{where, } C_D = \left[\frac{0.455}{(\log R_{EL})^{2.58}} - \frac{1700}{R_{EL}}\right] \quad \ldots(15.18)$$

As we can see the term $\left(\frac{1700}{R_{EL}}\right)$ is very small and hence could even be neglected. Drag coefficients for completely rough boundaries have been developed by various investigators but the mathematics is too involved and is therefore outside the scope of this book.

Drag coefficients for various conditions of flow on a smooth plate is plotted with C_D on the Y axis and R_E on the X axis and the resulting graph is shown in Fig. (15.6).



Plot of (C_d) versus (R_e) for smooth plates

Fig. 15.6

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Laminar $$C_d = \frac{1.328}{\sqrt{R_L}}$$

where $R_L = \frac{UL}{\upsilon}$

$$\text{Transition } C_D = \left[\frac{0.455}{(\log R_{EL})^{2.58}} - \frac{1700}{R_{EL}}\right]$$

$$\text{Turbulent } C_D = \frac{0.072}{(R_L)^{0.2}}$$

15.3 VON KARMAN'S INTEGRAL MOMENTUM EQUATION

This equation is applicable for two dimensional incompressible steady flows along a flat plate and is based on the principle of conservation of momentum.

Von Karman opined that this principle is applicable to any section of the boundary layer, whether laminar or turbulent and obtained the equation by equating the skin friction drag to the product of rate of change of momentum and the mass of fluid affected by the boundary layer.

Consider the control volume *ODCB* in Fig. (15.3). Because the viscous drag due to shear is the only force resisting the flow, for unit width of the plate normal to the paper the viscous drag is given by,

$-F_D$ = momentum efflux through *CD* in the *x*-direction – momentum influx through *OD* in the *x*-direction + net volume influx through *OD* and *CD*.

Momentum efflux through *CD* in the *x*-direction = $\rho \int_0^{\delta} (u)^2 \, dy$

Momentum influx through *OD* in the *x*-direction = $\rho (U)^2 \delta$

By continuity net volume influx through *OD* and *CD* must be equal to the volume efflux through *DC*.

$$\text{Volume efflux through } DC = \int_0^{\delta} (U - u) dy$$

Momentum of volume efflux through $DC = U \rho \int_0^{\delta} (U - u) dy$

$$-F_D = \rho \int_0^{\delta} (u)^2 dy - \rho (U)^2 \delta + U \rho \int_0^{\delta} (U - u) dy \quad \ldots(a)$$

Here δ is the height of the control volume in the *y*-direction (*OD*) so chosen as to ensure that the velocity in the *x*-direction remains undisturbed.

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If we re-arrange Eq. (a) above,

$$-F_D = \int_0^{\delta} (u)^2 \, dy + U \rho \int_0^{\delta} (U-u) dy - \rho (U)^2 \delta$$

From which after simplifying we obtain,

$$-F_D = \rho \int_0^{\delta} u (U-u) dy$$

or
$$F_D = -\rho \int_0^{\delta} u (U-u) dy \quad \ldots(15.19)$$

From Eq. (15.2)

$$(\rho U^2 \theta) = \int_0^{\delta} \rho (Us - u) dy$$

Comparing the above two equations we obtain,

$$F_D = \rho U^2 \theta \quad \ldots(15.20)$$

$$\tau_0 = \frac{d}{dx} (F_D)$$

$$= \frac{d}{dx} \rho U^2 \theta$$

$$\tau_0 = \rho U^2 \left[\frac{d\theta}{dx} \right] \quad \ldots(15.21)$$

$$\frac{\tau_0}{\rho U^2} = \left[\frac{d\theta}{dx} \right] \quad \ldots(15.22)$$

$$\frac{\tau_0}{\rho U^2} = \frac{d}{dx} \left[\int_0^{\delta} \frac{u}{U} (1 - uU) dy \right] \quad \ldots(15.23)$$

Because from Eq. (15.2 a) θ is the momentum thickness = $\int_0^{\delta} \frac{u}{U} \left(\frac{1-u}{U} \right) dy$

For incompressible flows the boundary layer thickness for smooth plates between 160 – 200 mm long is of the order of 0.75 mm at $R_E = 10^6$.

While we have defined the drag coefficient (C_D) for the total drag (F_D) on one side of the plate of length L in terms of the stagnation pressure, there is another coefficient called the *Coefficient of skin friction* (C_f) which is a ratio of the shear stress (τ_0) to the stagnation pressure. Thus,

$$\text{Coefficient of skin friction } C_f = \frac{\tau_0}{\left[\frac{1}{2} (\rho A U^2) \right]} \quad \ldots(15.23 \text{ a})$$

Here A represents the area of the plate. It may be noted that the values of drag coefficients also depends, just the same was as the boundary layer thickness λ, on the velocity profile chosen although this variation is marginal.

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15.4 BOUNDARY LAYER : SEPARATION AND CONTROL

The boundary layer should ideally stick to the solid boundary as the flow continues. The thickness of the boundary layer increases till the local velocity equals 0.99 of free stream velocity. This increase was possible because the fluid layer close to the surface was able to overcome the frictional drag at the cost of its own kinetic energy. This eventually causes the velocity of the layer close to the boundary to gradually decrease and at some point the kinetic energy loss is so severe that the boundary layer separates from the solid surface unable to overcome the resistance offered by the solid boundary. This phenomenon is called *boundary layer separation* and the point at which this occurs is called the point of separation.

Nowhere this phenomenon is more evident than during flow past a converging-diverging boundary. Along a flat plate the boundary layer sticks to the boundary regardless of the length of the plate because the pressure gradient

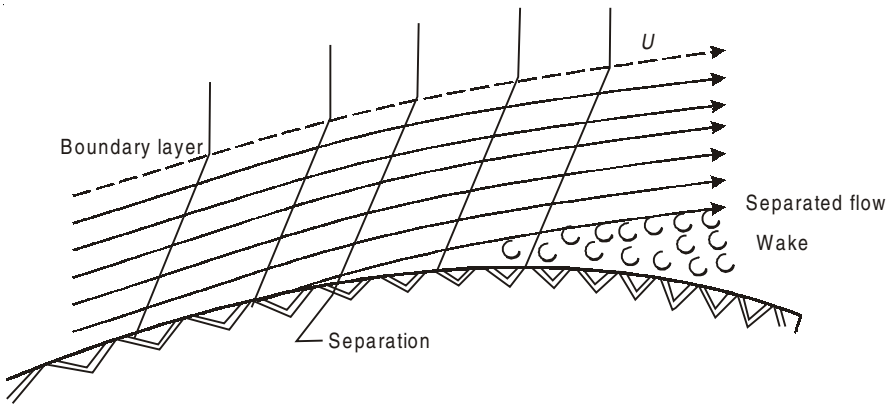


Fig. 15.7

is zero. But in a converging-diverging boundary as the pressure increases the boundary layer thickens downstream. (Refer Fig.15.7). The adverse pressure gradient in combination with boundary shear causes a decrease in momentum of the boundary layer to such an extent that at some point that the layer separates from the solid body. In the separated portion called the wake reverse flow occurs resulting in the formation of eddies.

The best design procedure adopted to eliminate the adverse effect of boundary layer separation is to ensure at the design stage itself that the point of separation is moved as far downstream of the boundary as possible by profiling the boundary curvature appropriately. This in other words is called stream-lining the curvature. This overcomes the form drag.

Sand blasting the boundary causes a rough surface leading to early formation of the turbulent layer. Greater momentum transfer within the turbulent boundary layer requires larger adverse pressure gradient for the separation to occur. This overcomes the skin friction drag.

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Other measures adopted to control the effect of boundary layer separation are,

- By providing suction slots at the point of separation which sucks away the retarded fluid.
- By imparting additional kinetic energy to the fluid from an external source.
- By providing guide blades at appropriate locations.

The point of separation of the boundary layer is determined from the condition that,

If $\left[\frac{\partial u}{\partial y}\right]_{y=0}$ is negative it indicates that the boundary layer has separated and on the contrary if it is positive it indicates that the boundary layer is stable and is sticking to the boundary.

If $\left[\frac{\partial u}{\partial y}\right]_{y=0}$ is 0 then it indicates that the flow is on the verge of separation.

TYPICAL PROBLEMS AND SOLUTIONS

Problem 15.1 *The velocity profile for the boundary layer on a flat plate is* $\frac{u}{U} = (\eta - \eta^2)$. *Determine the shape factor H if* $\eta = \frac{y}{\lambda}$, *where* λ *is the nominal thickness of the boundary layer.*

Solution

Since the shape factor H is the ratio of displacement thickness (λ_D) to momentum thickness (θ) we have to first determine the values of both λ_D and θ.

From Eq. (15.1)

$$\lambda_D = \int_0^{\lambda} \left(\frac{1-u}{U}\right) dy$$

It is given that $\eta = \frac{y}{\lambda}$,

$$d\eta = \frac{dy}{\lambda}$$

or $$dy = \lambda\,(d\eta)$$

If $$y = 0,$$

$$\eta = 0$$

and if $$y = \lambda,\ \eta = 1$$

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$$\lambda_D = \lambda \int_0^1 \left(\frac{1-u}{U}\right) dy$$

$$\lambda_D = \lambda \int_0^1 [1 - f(\eta)]\, d\eta$$

When $\frac{u}{U} = f(\eta) = \eta$

$$\lambda_D = \int_0^1 [1 - (\eta - \eta^2)]\, d\eta$$

$$\lambda_D = \lambda \left\{\eta - \left(\frac{\eta^2}{2} - \frac{\eta^3}{3}\right)\right\}_0^1$$

$$\lambda_D = \lambda \left\{1 - \left(\frac{1}{2} - \frac{1}{3}\right)\right\}$$

$$\lambda_D = \left(\frac{5\lambda}{6}\right)$$

From Eq. (15.2 a) momentum thickness

$$\theta = \int_0^{\lambda} \frac{u}{U}\left(\frac{1-u}{U}\right) dy$$

$$\theta = \lambda \int_0^1 (\eta - \eta^2)\ [1 - \eta + \eta^2]\, d\eta$$

$$\theta = \lambda \int_0^1 (\eta - 2\eta^2 + 2\eta^3 - \eta^4)\ d\eta$$

$$\theta = \lambda \int_0^1 \left[\left(\frac{\eta^2}{2} - \frac{2\eta^3}{3} + \frac{\eta^4}{2} - \frac{\eta^5}{5}\right)\right]_0^1$$

$$\theta = \left(\frac{2}{15}\right)\lambda$$

Shape factor $H = \frac{\lambda_D}{\theta}$

$$= \frac{\left(\frac{5\lambda}{6}\right)}{\left(\frac{2}{15}\right)\lambda} = 6.25$$

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Problem 15.2 *Given that the velocity distribution in the laminar boundary layer is* $\frac{u}{U} = (2\eta - \eta^3)$, *where* $\eta = \frac{y}{\lambda}$, *verify whether this distribution satisfies the requisite boundary conditions.*

Solution

The requisite boundary conditions are,

(a) Essential conditions: At $y = 0$,

$$u = 0$$

and $$\frac{\partial^2 u}{\partial y^2} = 0$$

(b) Desirable condition : $$\frac{\partial u}{\partial y} = 0$$

(c) Essential conditions : At $y = \lambda$,

$$u = U$$

(d) Desirable condition : $$\frac{\partial u}{\partial y} = 0 \text{ and } \frac{\partial^2 u}{\partial y^2} = 0$$

The given equation is,

$$\frac{u}{U} = (2\eta - \eta^3) \text{ ,where } \eta = \frac{y}{\lambda},$$

$$u = U(2\eta - \eta^3)$$

$$u = U\left[\left(\frac{2y}{\lambda}\right) - \left(\frac{y}{\lambda}\right)^3\right]$$

Because at $y = 0$,

$u = 0$, part of condition (a) is met.

$$\frac{\partial u}{\partial y} = U\left[\left(\frac{2}{\lambda}\right) - \left(\frac{3y^2}{\lambda^3}\right)\right]$$

Since at $y = 0$, $\frac{\partial u}{\partial y} \neq 0$ condition (b) is not met

$$\frac{\partial^2 u}{\partial y^2} = -U\left[\left(\frac{6y}{\lambda^3}\right)\right]$$

Since at $y = 0$,

$\frac{\partial^2 u}{\partial y^2} = 0$, another part of condition (a) is met.

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Since at $y = \lambda$,

$$u = U \text{ but } \frac{\partial u}{\partial y} \neq 0 \text{ and } \frac{\partial^2 u}{\partial y^2} \neq 0$$

Since only one essential condition is met out of the four the given equation does not represent the velocity distribution for laminar boundary layer.

Problem 15.3 *Determine the displacement thickness, momentum thickness and the energy thickness if the boundary layer velocity distribution for turbulent flow is* $\frac{u}{U} = \left(\frac{y}{\lambda}\right)^{\frac{1}{7}}$.

Solution

From Eq. (15.1) displacement thickness

$$\lambda_D = \int_0^{\lambda}\left(1-\frac{u}{U}\right)dy$$

$$\lambda_D = \int_0^{\lambda}\left[1-\left(\frac{y}{\lambda}\right)^{\frac{1}{7}}\right]dy$$

$$\lambda_D = \left[y-\frac{7}{8}\left(\frac{y^{\frac{8}{7}}}{\lambda^{\frac{1}{7}}}\right)\right]_0^{\lambda}$$

$$\lambda_D = \left[\lambda-\frac{7}{8}\left(\frac{\lambda^{\frac{8}{7}}}{\lambda^{\frac{1}{7}}}\right)\right]$$

$$\lambda_D = \left(\frac{\lambda}{8}\right)$$

$$\theta = \int_0^{\lambda}\frac{u}{U}\left(1-\frac{u}{U}\right)dy$$

From Eq. (15.2 a)

$$\theta = \int_0^{\lambda}\left(\frac{y}{\lambda}\right)^{\frac{1}{7}}\left[1-\left(\frac{y}{\lambda}\right)^{\frac{1}{7}}\right]dy$$

$$\theta = \int_0^{\lambda}\left[\left(\frac{y}{\lambda}\right)^{\frac{1}{7}}-\left(\frac{y}{\lambda}\right)^{\frac{2}{7}}\right]dy$$

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$$\theta = \left[\frac{7}{8}\frac{\left(y^{\frac{8}{7}}\right)}{\lambda^{\frac{1}{7}}} - \frac{7}{9}\frac{\left(y^{\frac{9}{7}}\right)}{\lambda^{\frac{2}{7}}}\right]_0^{\lambda}$$

$$\theta = \left[\frac{7}{8}\frac{\left(\lambda^{\frac{8}{7}}\right)}{\lambda^{\frac{1}{7}}} - \frac{7}{9}\frac{\left(\lambda^{\frac{9}{7}}\right)}{\lambda^{\frac{2}{7}}}\right]$$

$$\theta = \left(\frac{7}{72}\right)\lambda$$

From Eq. (15.2 b) energy thickness λ_E is given by,

$$\lambda_E = \int_0^{\lambda} \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right) dy$$

$$\lambda_E = \int_0^{\lambda} \left(\frac{y}{\lambda}\right)^{\frac{1}{7}}\left[1 - \left(\frac{y}{\lambda}\right)^{\frac{2}{7}}\right] dy$$

$$\lambda_E = \int_0^{\lambda} \left[\left(\frac{y}{\lambda}\right)^{\frac{1}{7}} - \left(\frac{y}{\lambda}\right)^{\frac{3}{7}}\right] dy$$

$$\lambda_E = \left[\frac{7}{8}\left(\frac{y^{\frac{8}{7}}}{\lambda^{\frac{1}{7}}}\right) - \frac{7}{10}\left(\frac{y^{\frac{10}{7}}}{\lambda^{\frac{1}{7}}}\right)\right]_0^{\lambda}$$

$$\lambda_E = \left(\frac{7}{40}\right)\lambda$$

Problem 15.4 *The equation for boundary layer velocity distribution* $\frac{u}{U} = \left(\frac{y}{\lambda}\right) - \left(\frac{y}{\lambda}\right)^2$. *Determine the shape factor H.*

Solution

Since the shape factor H is the ratio of displacement thickness (λ_D) to momentum thickness (θ) we have to first determine the values of both λ_D and θ.

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From Eq. (15.1)

$$\lambda_D = \int_0^{\lambda}\left(1-\frac{u}{U}\right)dy$$

$$\lambda_D = \int_0^{\lambda}\left[1-\left(\frac{y}{\lambda}\right)+\left(\frac{y^2}{\lambda^2}\right)\right]dy$$

$$\lambda_D = \left[y-\left(\frac{y^2}{2\lambda}\right)+\left(\frac{y^3}{3\lambda^2}\right)\right]_0^{\lambda}$$

$$\lambda_D = \left[y-\left(\frac{y^2}{2\lambda}\right)+\left(\frac{y^3}{3\lambda^2}\right)\right]$$

$$= \frac{5\lambda}{6}$$

From Eq. (15.2 a) momentum thickness

$$\theta = \int_0^{\lambda}\frac{u}{U}\left(1-\frac{u}{U}\right)dy$$

$$\theta = \int_0^{\lambda}\left\{\left(\frac{y}{\lambda}\right)-\left(\frac{y}{\lambda}\right)^2\left[1-\left(\frac{y}{\lambda}\right)+\left(\frac{y}{\lambda}\right)^2\right]\right\}dy$$

$$\theta = \int_0^{\lambda}\left\{\left(\frac{y}{\lambda}\right)-\left(\frac{y^2}{\lambda^2}\right)\left[1-\left(\frac{y}{\lambda}\right)+\left(\frac{y^2}{\lambda^2}\right)\right]\right\}dy$$

$$\theta = \int_0^{\lambda}\left\{\left(\frac{y}{\lambda}\right)-\left(\frac{y^2}{\lambda^2}\right)+\left(\frac{y^3}{\lambda^3}\right)-\left(\frac{y^2}{\lambda^2}\right)+\left(\frac{y^3}{\lambda^3}\right)-\left(\frac{y^4}{\lambda^4}\right)\right\}dy$$

$$\theta = \int_0^{\lambda}\left\{\left(\frac{y}{\lambda}\right)-\left(\frac{2y^2}{\lambda^2}+\frac{2y^3}{\lambda^3}\right)-\frac{y^4}{\lambda^4}\right\}dy$$

$$\theta = \left\{\left(\frac{y^2}{2\lambda}\right)-\frac{2y^2}{3\lambda^2}+\frac{2y^4}{\lambda^5}-\frac{y^5}{5\lambda^4}\right\}_0^{\lambda}$$

$$\theta = \left\{\left(\frac{y^2}{2\lambda}\right)-2\left(\frac{y^2}{3\lambda^2}\right)+\frac{2y^4}{4\lambda^3}-\left(\frac{y^5}{5\lambda^4}\right)\right\}$$

$$\theta = \left(\frac{2}{15}\right)\lambda$$

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Shape factor $$H = \frac{\theta}{\lambda_D} = \frac{\left(\frac{2}{15}\right)\lambda}{\frac{5\lambda}{6}} = \frac{4}{25}$$

Problem 15.5 *Determine the boundary layer thickness, shear stress and the coefficient of drag for the velocity distribution in the laminar boundary layer that Prandtl assumed.*

Solution

Prandtl assumed the following velocity profile,

$$\frac{u}{U} = \left(\frac{3}{2}\right)\eta - \frac{\eta^3}{2} \quad \ldots \eta \text{ is the eddy viscosity}$$

$$= \left(\frac{y}{\lambda}\right) \text{with the basic condition that at } y = 0,$$

$$u = 0 \text{ and } \frac{du}{dy} \text{ has a finite value and at } y = \lambda,$$

$$u = U \text{ and } \frac{du}{dy} = 0$$

$$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\lambda}\right) - \frac{1}{2}\left(\frac{y}{\lambda}\right)^3 \qquad \ldots(a)$$

From Eq. (15.23) the shear stress (τ_0) is related to free stream velocity (U) as

$$\frac{\tau_0}{\rho U^2} = \frac{d}{dx}\left[\int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right)dy\right]$$

$$\frac{\tau_0}{\rho U^2} = \frac{d}{dx}$$

$$\int_0^{\lambda}\left\{\frac{3}{2}\left(\frac{y}{\lambda}\right) - \frac{1}{2}\left(\frac{y}{\lambda}\right)^3\left[1 - \frac{3}{2}\left(\frac{y}{\lambda}\right) - \frac{1}{2}\left(\frac{y}{\lambda x}\right)^3\right]dy\right\}$$

After simplifying, integrating and applying the limits,

$$\frac{\tau_0}{\rho U^2} = \left(\frac{39}{280}\right)\frac{d\lambda}{dx}$$

$$\tau_0 = \left(\frac{39}{280}\right)\frac{\rho U^2 d\lambda}{dx} \qquad \ldots(b)$$

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$$\tau_0 = \mu\left(\frac{du}{dy}\right) \text{ at } y = 0 \qquad \ldots(c)$$

But from (a)

$$u = U\left[\frac{3}{2}\left(\frac{y}{\lambda}\right) - \frac{1}{2}\left(\frac{y}{\lambda}\right)^3\right]$$

$$\frac{du}{dy} = U\left[\left(\frac{3}{2\lambda}\right) - \left(\frac{3y^2}{2\lambda^3}\right)\right] \qquad \ldots(d)$$

Apply the value of $\left(\frac{du}{dy}\right)$ obtained in (d) in Eq. (c)

$$\tau_0 = \mu U\left[\frac{3}{2\lambda} - \frac{3y^2}{2\lambda^3}\right]$$

at

$$y = 0, \tau_0 = \left(\frac{3\mu U}{2\lambda}\right) \qquad \ldots(e)$$

Equating, (b) and (e)

$$\left(\frac{39}{280}\right)\frac{\rho U^2 d\lambda}{dx} = \left(\frac{3\mu U}{2\lambda}\right)$$

From which,

$$\lambda\,(d\lambda) = \left(\frac{420}{39}\right)\left(\frac{\mu}{\rho U}\right)dx$$

Integrating both sides,

$$\frac{\lambda^2}{2} = \left(\frac{420}{39}\right)\left(\frac{\mu}{\rho U}\right)x + C$$

Since when $x = 0$,

$\lambda = 0$,

Hence $C = 0$,

Therefore,

$$\frac{\lambda^2}{2} = \left(\frac{420}{39}\right)\left(\frac{\mu}{\rho U}\right)x$$

$$\lambda = \sqrt{2\left(\frac{420}{39}\right)\left(\frac{\mu}{\rho U}\right)x}$$

$$= 4.64\sqrt{\left(\frac{\mu}{\rho U}\right)x}$$

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$$\lambda = \frac{(4.64\ x)}{\sqrt{R_{EX}}} \qquad \ldots(f)$$

Coefficient drag C_D = Force on one side of the plate $\dfrac{(F_D)}{\left(\dfrac{\rho A U^2}{2}\right)}$

where ρ is density, A is the area of the plate and U is the free stream velocity.

Hence $$C_D = \frac{(F_D)}{\left(\dfrac{\rho A U^2}{2}\right)} \qquad \ldots(g)$$

$$\frac{F_D}{\text{unit width of the plate}} = \int_0^L \tau_0 dx \qquad \ldots(h)$$

0

From (e) above $$\tau_0 = \left(\frac{3\mu U}{2\lambda}\right)$$

$$\tau_0 = \left[\frac{3\mu U}{2}\frac{(4.64\ x)}{\sqrt{R_{EX}}}\right] = \frac{\left(\dfrac{0.323\mu U}{x}\right)}{\sqrt{R_{EX}}} \qquad \ldots(i)$$

Substituting the value of τ_0 in (h)

$$\frac{F_D}{\text{unit width of the plate}} = \int_0^L \frac{\left(\dfrac{0.323\mu U}{x}\right)}{\sqrt{R_{EX}}\,(dx)}$$

$$\frac{F_D}{\text{unit width of the plate}} = 0.323 \int_0^L \left(\frac{\mu U}{x}\right)\frac{\sqrt{\dfrac{(\rho U)}{\mu}}}{\mu}\,(dx)$$

$$\frac{F_D}{\text{unit width of the plate}} = 0.323\ (\mu U)\left(\frac{\left(\sqrt{\rho U}\right)}{\mu}\right)\int_0^L \left(x^{-1}\right)\left(x^{\frac{1}{2}}\right)dx$$

$$\frac{F_D}{\text{unit width of the plate}} = 0.323\ (\mu U)\left(\frac{\left(\sqrt{\rho U}\right)}{\mu}\right)\int_0^L \left(x^{-\frac{1}{2}}\right)dx$$

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$$\frac{F_D}{\text{unit width of the plate}} = 0.323\,(\mu U)\left(\frac{\left(\sqrt{\rho U}\right)}{\mu}\right)\left[\frac{\left(x^{\frac{1}{2}}\right)}{\frac{1}{2}}\right]_0^L$$

After applying the limits and simplifying

$$\frac{F_D}{\text{unit width of the plate}} = (0.323)\,(2)\,(\mu U)\left(\frac{\left(\sqrt{\rho U}\right)L}{\mu}\right)$$

$$\frac{C_D}{\text{unit width of the plate}} = \frac{0.646\,(\mu U)\left(\frac{\sqrt{(\rho U)L}}{\mu}\right)}{\left(\frac{1}{2}\rho L U^2\right)}$$

$$\frac{C_D}{\text{unit width of the plate}} = \frac{1.292}{\sqrt{(\rho U)L}} = \frac{1.292}{\left(\sqrt{R_{EL}}\right)}$$

Problem 15.6 *Determine the boundary layer thickness and surface shear stress if the velocity distribution for the laminar boundary layer is defined by* $\frac{u}{U} = \sin\left(\frac{\lambda y}{2\lambda}\right)$. *Assume smooth flow on a flat plate.*

Solution

From Eq. (15.23) the shear stress (τ_0) is related to free stream velocity (U) as

$$\frac{\tau_0}{\rho U^2} = \frac{d}{dx}\left[\int_0^{\delta}\frac{u}{U}\left(1-\frac{u}{U}\right)dy\right]$$

$$\frac{\tau_0}{\rho U^2} = \frac{d}{dx}\int_o^{\lambda}\sin\left(\frac{\pi y}{2\lambda}\right)\left[1-\sin\left(\frac{\pi y}{2\lambda}\right)\right]dy$$

$$\frac{\tau_0}{\rho U^2} = \frac{d}{dx}\int_0^{\lambda}\sin\left(\frac{\pi y}{2\lambda}\right)\left[-\sin^2\left(\frac{\pi y}{2\lambda}\right)\right]dy$$

$$\frac{\tau_0}{\rho U^2} = \frac{d}{dx}\int_0^{\lambda}\sin\left(\frac{\pi y}{2\lambda}\right)-\left[1-\cos^2\frac{\frac{\pi y}{2\lambda}}{2}\right]dy$$

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$$\frac{\tau_0}{\rho U^2} = \frac{d}{dx}\int_0^{\lambda} \sin\left(\frac{\pi y}{2\lambda}\right) - \left[\frac{1-\cos\frac{\pi y}{\lambda}}{2}\right] dy$$

After integration and application of limits,

$$\frac{\tau_0}{\rho U^2} = \left[\frac{(4-\pi)}{2\pi}\right]\frac{d\lambda}{dx} \quad \ldots(a)$$

But $$\frac{u}{U} = \sin\left(\frac{\pi y}{2\lambda}\right) \quad \ldots(\text{Given})$$

$$\frac{\partial u}{\partial y} = U\left[\cos\left(\frac{\pi\, y}{2\,\lambda}\right)\right]\left(\frac{\pi}{2d\lambda}\right)$$

But $$\tau_0 = \mu\left[\frac{\partial u}{\partial y}\right]_{y=0}$$

Therefore, $$\tau_0 = \mu U\left[\cos\left(\frac{\pi\ 0}{2\ \lambda}\right)\right]\left(\frac{\pi}{2\lambda}\right)$$

$$\tau_0 = \left(\frac{\mu U\pi}{2\lambda}\right) \quad \ldots(b)$$

Equating (a) and (b)

$$\left(\frac{\mu U\pi}{2\lambda}\right) = \rho\, U^2\left[\frac{(4-\pi)}{2\pi}\right]\frac{d\lambda}{dx}$$

$$\left(\frac{2\lambda}{\mu U\pi}\right) = \left(\frac{1}{\rho\, U^2}\right)\left[\frac{2\pi}{(4-\pi)}\right]\left(\frac{dx}{d\lambda}\right)$$

$$(2\lambda)\, d\lambda = \left(\frac{\mu U\pi}{\rho\, U^2}\right)\left[\frac{2\pi}{(4-\pi)}\right] dx$$

$$(\lambda)\, d\lambda = \frac{\left(\frac{\mu\pi^2}{\rho U}\right)}{(4-\pi)}\, dx$$

$$(\lambda)\, d\lambda = 11.50\left(\frac{\mu}{\rho U}\right) dx$$

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$$\int (\lambda)\, d\lambda = 11.50 \int \left(\frac{\mu}{\rho U}\right) dx$$

$$\left(\frac{\lambda^2}{2}\right) = 11.50 \left(\frac{\mu x}{\rho U}\right) + C$$

When $x = 0$, $\lambda = 0$ and hence $C = 0$

$$(\lambda^2) = 23 \left(\frac{\mu x}{\rho U}\right)$$

$$\lambda = \sqrt{23\left(\frac{\mu x}{\rho U}\right)}$$

$$\lambda = \sqrt{4.795\left(\frac{\mu x}{\rho U}\right)x}$$

$$\lambda = \frac{4.795x}{\sqrt{R_{Ex}}}$$

Problem 15.7 *A flat plate 200 mm wide × 700 cm long is fully immersed longitudinally in still-waters (25°C) and is being dragged @ 9 m/sec. Find the thickness of the boundary layer, total drag on the plate and the shear stress at the trailing edge.*

Solution

We have to first determine whether the flow is laminar or turbulent. The free stream velocity $U = 9$ m/sec. Kinematic viscosity of water at 25°C = 0.897×10^{-6} m^2/sec

Hence $$R_{EL} = \frac{UL}{\upsilon}$$

$$= \frac{(9 \times 0.7)}{0.897 \times 10^{-6}}$$

$$= 7 \times 10^6$$

Since $7 \times 10^6 > R_{E\,(\text{Critical})} = 5 \times 10^5$ the boundary layer is turbulent.

From Eq. (15.13)

$$\frac{F_D}{\text{unit width}} = (C_D)\, \rho \left(\frac{U^2}{2}\right) L$$

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where, $C_D = C_D = \dfrac{0.072}{(7 \times 10^6)^{0.2}}$

$$C_D = 0.00307$$

$$F_D = C_D \left[\rho\left(\frac{U^2}{2}\right)(L \times B)\right]$$

$$= 0.00307\left[998\left(\frac{9^2}{2}\right)(0.7 \times 0.2)\right]$$

$$= 17.375 \text{ N (Drag on one side of the plate)}$$

$$\text{Total Drag} = 2 F_D$$

$$= 2 \times 17.375 = 34.75 \text{ N}$$

The turbulent boundary layer Eq. (15.12) valid for $5 \times 10^5 < R_E < 2 \times 10^7$ as determined by Prandtl and Blasius can be used to determine the boundary layer thickness λ.

$$\frac{\lambda}{x} = \frac{0.377}{\left(\frac{Ux}{\upsilon}\right)^{0.2}}$$

$$= \frac{0.377}{R_{Ex}^{0.2}}$$

Here x represents the length of the plate.

$$\frac{\lambda}{0.7} = \frac{0.377}{(7 \times 10^6)^{0.2}}$$

$$= 0.016$$

$$\lambda = 0.011 \text{ m}$$

From Eq. (15.14 a)

$$\text{The shear stress } \tau_0 = 0.029\, \rho\, U^2 \left(\frac{\upsilon}{Ux}\right)^{0.2}$$

At the trailing edge $x = L$ and hence shear stress at the trailing edge is,

$$\tau_0 = 0.029\,(998.2)\,(9)^2 \left(\frac{0.897 \times 10^{-6}}{9 \times 0.7}\right)^{0.2}$$

$$= 100 \text{ N/m}^2$$

Problem 15.8 *A flat plate 3 m wide × 20 m long is fully immersed longitudinally in still-waters (20°C) and is being dragged @ 9 m/sec. Find the position of the transition point of boundary layer and the total drag on the plate*

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and the drag on 3 m of the plate from the leading edge.

Solution

$$U = 9 \text{ m/sec},$$

$$\upsilon \text{ for water at } 20°C = 1.007 \times 10^{-6} \text{ m}^2/\text{sec}.$$

Substituting the values in equation below,

$$R_{EL} = \frac{UL}{\upsilon}$$

$$= \frac{(9 \times 20)}{1.007 \times 10^{-6}}$$

$$= 1.78 \times 10^{8}$$

(Indicates turbulent boundary layer)

Let the transition point be L_T. The critical Reynolds number R_{Ec} when the transition occurs is (5×10^5)

$$R_{Ec} = \frac{UL_T}{\upsilon} \quad \ldots(a)$$

$$(5 \times 10^5) = \frac{(9)\, L_T}{1.007 \times 10^{-6}}$$

$$L_T = 0.0559 \text{ m}$$

Drag on one side of the plate

$$\frac{F_D}{\text{unit width}} = (C_D)\, \rho\, \frac{(U^2)}{2\, L}$$

From Eq. (15.18)

For Reynolds number between $(10^7 < R_{EL} < 10^9)$

$$C_D = \left[\frac{0.455}{(\log R_{EL})^{2.58}} - \frac{1700}{R_{EL}}\right]$$

Neglecting the last term as being too small

$$C_D = \left[\frac{0.455}{(\log R_{EL})^{2.58}}\right]$$

$$C_D = \left[\frac{0.455}{(\log 1.78 \times 10^8)^{2.58}}\right]$$

$$= 0.00196$$

$$\text{Total drag } F_D = (C_D)\, \rho\, U^2\, L \times B$$

$$F_D = (0.00196)\, (998.2)\, (9)^2\, (20) \times (3)$$

$$= 9{,}508 \text{ N}$$

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Drag on 3 m of the plate from the leading edge

$$\text{Total drag at 3 m} = F_D = (C_D)\ \rho\ U^2\ L \times B$$

$$F_D = (0.00196)\ (998.2)\ (9)^2\ (3) \times (3)$$

$$= 1{,}426 \text{ N}$$

Problem 15.9 *A smooth stationary plate (5 m × 25 m) is held longitudinally in a stream of water flowing with a velocity of 5 m/sec. Determine the thickness of the boundary layer at the transition point and at the trailing edge. Also determine the shear stress at the trailing edge.* $\rho = 1000\ kg/m^3$ *and* $\upsilon = 10^{-6}\ m^2/sec.$

Solution

$$U = 5 \text{ m/sec},$$

$$\upsilon \text{ for water} = 1 \times 10^{-6} \text{ m}^2/\text{sec}.$$

Substituting the values in equation below,

$$R_{EL} = \frac{UL}{\upsilon}$$

$$= \frac{(5 \times 25)}{1 \times 10^{-6}}$$

$$= 1.25 \times 10^{8}$$

(Indicates turbulent boundary layer)

To determine the thickness of the boundary layer at the transition point we have to first determine its length from the leading edge.

Let the transition point be L_T. The critical Reynolds number R_{Ec} when the transition occurs is (5×10^5)

$$R_{Ec} = \frac{UL_T}{\upsilon} \qquad \ldots(a)$$

$$(5 \times 10^5) = \frac{(5)\ L_T}{1\ \times\ 10^{-6}}$$

$$L_T = 0.1 \text{ m}$$

Thickness at the point of transition that is, the laminar boundary layer thickness is given by Prandtls Eq. (15.3 a)

$$\lambda = \frac{4.64x}{\sqrt{R_x}}$$

Here $x = L_T$

$= 0.1$ m and $R_x = 5 \times 10^5$

$$\lambda = \frac{4.64\ (0.1)}{\sqrt{5 \times 10^5}}$$

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Thickness of boundary layer at the transition point $\lambda = 0.000656$ m

From Eq. (15.15) since $R_{EL} = 1.25 \times 10^8$ which is $> 10^7$

Thickness of the boundary layer at the trailing edge is given by,

$$\lambda = \frac{x\ (0.22)}{(R_L)^{\frac{1}{6}}}$$

Here, $x = 25$ m

$$\lambda = \frac{25\ (0.22)}{(1.25 \times 10^8)^{\frac{1}{6}}}$$

$$\lambda = 0.245 \text{ m}$$

From Eq. (15.14 a)

$$\text{The shear stress } \tau_0 = 0.029\ \rho U^2 \left(\frac{\upsilon}{Ux}\right)^{0.2}$$

At the trailing edge $x = L$ and hence shear stress at the trailing edge is,

$$\tau_0 = 0.029(1000)(5)^2 \left(\frac{1 \times 10^{-6}}{5 \times 5}\right)^{0.2}$$

$$= 24 \text{ N/m}^2$$

Problem 15.10 *A cylinder of the shape of a bullet is being towed through sea waters in fully immersed state @ 20 km/hour. The cylinder is 1.5 m in diameter and is 10 m long. Determine the total surface resistance.* $\rho = 1000$ *kg/m*^{3} *and* $\upsilon = 1.519 \times 10^{-6}$ *m*^{2}*/sec.*

Solution

$$20 \text{ km/hour} = \frac{(20 \times 1000)}{3600}$$

$$= 5.55 \text{ m/sec} = U$$

$$R_{EL} = \frac{UL}{\upsilon} = \frac{(15.55 \times 10)}{1.519 \times 10^{-6}}$$

$$= 3.65 \times 10^7$$

(Indicates highly turbulent boundary layer)

From Eq. (15.18)

For Reynolds number between ($10^7 < R_{EL} < 10^9$)

$$C_D = \left[\frac{0.455}{(\log R_{EL})^{2.58}} - \frac{1700}{R_{EL}}\right]$$

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$$C_D = \left[\frac{0.455}{(\log\ 3.65\times10^7)^{2.58}} - \frac{1700}{3.65\times10^7}\right]$$

$$C_D = [0.0025 - 0.0000465]$$

$$= 0.00245$$

$$\text{Total drag } F_D = \left[(C_D)\ \rho \left(\frac{U^2}{2}\right) L\times B\right]$$

$$F_D = (C_D)\ \rho \left(\frac{U^2}{2}\right) (\text{surface area})$$

$$F_D = (C_D)\ \rho \left(\frac{U^2}{2}\right) (\lambda DL)$$

$$F_D = (0.00245)\ (1000) \left(\frac{5.55^2}{2}\right) (\lambda \times 1.5 \times 10)$$

$$= 1778\ \text{N}$$

Problem 15.11 *For the following velocity profiles determine the state of the boundary layer.*

(*i*) $$\frac{u}{U} = \left(\frac{y}{\lambda}\right)^{\frac{1}{7}}$$

(*ii*) $$\frac{u}{U} = 2\left(\frac{y}{\lambda}\right) - 2\left(\frac{y}{\lambda}\right)^3$$

Solution

(i) $$\frac{u}{U} = \left(\frac{y}{\lambda}\right)^{\frac{1}{7}}$$

$$u = U\left(\frac{y}{\lambda}\right)^{\frac{1}{7}}$$

$$\frac{\partial u}{\partial y} = U\left[\left(\frac{1}{7}\right)\left(\frac{y}{\lambda}\right)^{\frac{1}{7}} - 1\right]\left(\frac{y}{\lambda}\right)$$

Since at $\left[\frac{\partial u}{\partial y}\right]_{y=0} = 0$ the boundary layer is on the verge of separation.

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(ii)
$$\frac{u}{U} = 2\left(\frac{y}{\lambda}\right) - 2\left(\frac{y}{\lambda}\right)^3$$

$$u = 2U\left[\left(\frac{y}{\lambda}\right) - \left(\frac{y}{\lambda}\right)^3\right]$$

$$\frac{\partial u}{\partial y} = 2U\left[\left(\frac{y}{\lambda}\right)\times\left(\frac{y}{\lambda}\right) - 3\left(\frac{y}{\lambda}\right)^2\left(\frac{1}{\lambda}\right)\right]$$

Since at $\left[\frac{\partial u}{\partial y}\right]_{y=0} = \left(\frac{2U}{\lambda^2}\right)$ is a positive value the boundary layer is stable.

QUESTIONS BANK

Section A

Review Questions

1. What do you understand by the term 'Boundary layer'? When is the boundary layer considered as laminar boundary layer?
2. Define the various ways in which boundary layer thickness is measured. Derive an expression for momentum thickness in terms local velocity u and free stream velocity U.
3. For the laminar boundary layer that Prandtl assumed, state the boundary conditions that must be satisfied.
4. Define critical Reynolds number and state the factors on which critical Reynolds number depends.
5. Compare the rate of variation in thickness in the laminar boundary layer to that of the turbulent boundary layer.
6. Justify the statement that the shear stress on a smooth plate is a direct function of the velocity gradient at the surface of the plate.
7. Explain why transition occurs at much lower values of R_E in rough plates as compared to smooth and polished plates.
8. For what range of R_E Prandtl's velocity distribution equation and the logarithmic velocity distribution equation becomes applicable?
9. Comment on the determination of turbulent boundary layer thickness in rough plates.
10. Explain the phenomenon of boundary layer separation and how the form drag and the skin friction drag are controlled to overcome boundary layer separation.

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Section B

Multiple Choice Questions

Choose the right answer

1. The velocity distribution in a boundary layer is given by $\frac{u}{U} = 3\left(\frac{y}{1}\right) - 2\left(\frac{y}{1}\right)^2$. The displacement thickness λ_D of the boundary layer is

   (a) $\left(\frac{3}{1}\right)$ (b) $\left(\frac{1}{8}\right)$

   (c) $\left(\frac{1}{6}\right)$ (d) None of these

2. Laminar boundary layer thickness varies as

   (a) $x^{-\frac{1}{2}}$ (b) $x^{\frac{1}{7}}$

   (c) $x^{\frac{1}{2}}$ (d) None of these

3. Turbulent boundary layer thickness varies as

   (a) $x^{-\frac{1}{5}}$ (b) $x^{\frac{1}{5}}$

   (c) $x^{\frac{4}{5}}$ (d) None of these

4. If the boundary layer thickness is λ, the displacement thickness is λ_D and the momentum thickness is θ then the shape factor H is

   (a) $\left(\frac{\lambda}{\lambda_D}\right)$ (b) $\left(\frac{\theta}{\lambda_D}\right)$

   (c) $\left(\frac{\lambda_D}{\theta}\right)$ (d) None of these

5. The boundary layer becomes turbulent when R_E ranges between
   (a) 300,000 – 500,000
   (b) 500,000 – 1,000,000
   (c) 700,000 – 1,500,000
   (d) None of these

6. In the study of boundary layer development it is essential that we lay emphasis on

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(a) dependence of shear stress on velocity gradient
(b) dependence of shear stress on skin friction drag
(c) formation of eddy viscosity due to random nature of motion of fluid particles
(d) All of these (e) None of these

7. Nominal thickness of the boundary layer is

(a) $\frac{U}{u} = 0.99$
(b) is the same as displacement thickness
(c) is the same as momentum thickness
(d) None of these

8. The displacement thickness of a boundary layer
(a) is the distance through which the main flow is shifted
(b) is more than the momentum thickness
(c) is defined as, $\lambda_D = \int_0^{\lambda}\left(1-\frac{u}{U}\right)dy$
(d) All of these (e) None of these

9. Thickness of the turbulent boundary layer ($5 \times 10^5 < R_E < 2 \times 10^7$) is

(a) $\frac{4.64x}{\sqrt{R_x}}$ (b) $\frac{4.5x}{\sqrt{R_x}}$

(c) $\frac{0.377\ x}{R_{Ex}^{0.2}}$ (d) None of these

10. The appropriate equation for determination of C_D for flows in the range (2×10^7 to 2×10^9) is

(a) $\frac{x\ (0.22)}{\sqrt{(R_E)^{\frac{1}{6}}}}$ (b) $\frac{0.455}{\sqrt{(\log R_E)^{2.58}}}$

(c) $\frac{0.022}{\sqrt{(\log R_E)^{2.58}}}$ (d) None of these

11. Shear stress at the wall at ($y = 0$) is stated as

(a) $\tau_0 = \mu\left[\frac{\partial u}{\partial y}\right]_{y=0}$ (b) $\tau_0 = \rho\left[\frac{\partial u}{\partial y}\right]_{y=0}$

(c) $\tau_0 = u\left[\frac{\partial u}{\partial y}\right]_{y=0}$ (d) None of these

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12. The values of λ, C_D and C_f that Blasius obtained were

(a) $\left[\left(\frac{5.48x}{\sqrt{Re_x}}\right)\left(\frac{0.73}{\sqrt{Re_x}}\right)\left(\frac{1.46}{\sqrt{Re_x}}\right)\right]$

(b) $\left[\left(\frac{5.84x}{\sqrt{Re_x}}\right)\left(\frac{0.686}{\sqrt{Re_x}}\right)\left(\frac{1.372}{\sqrt{Re_x}}\right)\right]$

(c) $\left[\left(\frac{5.00x}{\sqrt{Re_x}}\right)\left(\frac{0.664}{\sqrt{Re_x}}\right)\left(\frac{1.328}{\sqrt{Re_x}}\right)\right]$

(d) None of these

Section C
Problems

1. Determine the shape factor for the velocity profile $\frac{u}{U} = (\eta)^{\frac{1}{7}}$ where, $\eta = \left(\frac{y}{1}\right)$. (*Ans.* 1.285)

2. The velocity distribution equation for the laminar boundary layer is $\frac{u}{U} = \sin\left(\frac{\pi y}{2\lambda}\right)$. Assuming that the flow is over a smooth plate, determine from the first principles
(i) the boundary layer thickness
(ii) the shear stress at the boundary.

3. Derive from first principles the equation $\frac{\tau_0}{\rho U^2} = \left(\frac{\partial\theta}{\partial x}\right)$ where θ is momentum thickness, U is free stream velocity and τ_0 is the shear stress.

4. For the velocity profile $\frac{u}{U} = 2\left(\frac{y}{\lambda}\right) - \left(\frac{y}{\lambda}\right)^2$ derive the boundary layer thickness, coefficient of drag force C_D and the coefficient of skin friction drag C_f.

5. For the velocity profile $\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\lambda}\right) - \frac{1}{2}\left(\frac{y}{\lambda}\right)^3$ determine the boundary

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layer thickness, coefficient of drag force and the skin friction drag and compare the values obtained with the Blasius exact solution.

6. Air flows over a smooth plate with a velocity = 10 m/sec. If $R_E = 2.5 \times 10^5$ find the length of the laminar boundary layer and the maximum thickness of the boundary layer. Assume cubic velocity profile. ν for air $= 1.5 \times 10^{-5}$ m^2/sec. (*Ans.* $L = 0.375$ m) and ($l = 0.00348$ m)

7. A flat plate 40 m long and 10 m wide is towed along in sea-waters ($\upsilon = 1.519 \times 10^{-6}$ m^2/sec and $\rho = 1020$ kg/m^3) with a velocity = 2 m/sec. Determine the skin friction drag on the plate assuming $\left(\frac{\varepsilon}{\delta}\right) = 0.20$.

(*Ans.* 34.68 kN)

8. A smooth plate 2 m wide × 5 m long is held submerged in oil stream flowing with a velocity = 3 m/sec. Determine the boundary layer thickness and the shear stress at the edge of the laminar boundary layer and the thickness, the shear stress at the trailing edge, the total drag and the resistance offered by the plate. ($\rho = 850$ kg/m^3 and $\upsilon = 1.35 \times 10^{-5}$).

(*Ans.* $\lambda_L = 0.016$ m $\tau_{0L,} = 3.42$ N/m^2 $\lambda_{,} = 0.1145$ $\tau_{0,} = 2.42$ N/m^2 and $F_{DT} = 344.25$ N and $P = 1.03$ kW)

Answers to Multiple Choice Questions

1. (c)	**2.** (c)	**3.** (c)	**4.** (c)	**5.** (b)	**6.** (d)
7. (d)	**8.** (d)	**9.** (c)	**10.** (b)	**11.** (a)	**12.** (c)

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Chapter

16

Flow Around Submerged Bodies (Drag and Lift)

When we talk of submerged bodies one glaring example that comes to the mind of even a layman is the sub-marine. However it takes an engineer or a scientist to explain that an aircraft in flight or a train or an automobile, a kite flying high in air or even the man jumping out of a plane assisted by a parachute and a host of other objects too can be considered as submerged bodies. This is because a submerged body is any *moving body immersed in a fluid and exposed to the forces exerted by the fluid in which it is immersed.* All of the forces around a submerged body in motion are too complex to assess individually but two principal resultants of all these forces have been identified and defined. These are the *Drag and the lift*.

As already explained in the previous chapter the drag on a body has two components; the *form drag or the pressure drag and the skin friction drag.* In the previous chapter we were not concerned with the form drag as we were not particular at all about the shape of the body. So we merely confined ourselves to skin friction drag. In this chapter the effect of pressure drag on the body in motion is analyzed. In addition while discussing about 'Lift' and air-foil section that is designed to cause lift, we will introduce a third component to the drag.

Skin friction drag cannot be effectively reduced as it depends on the exposed surface area, velocity and the density of the fluid, all of which are need based. However form drag can be effectively controlled if not altogether eliminated by streamlining the shape of the body so that flow of fluid around it is smooth and unobstructed.

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While determining the form drag using the basic equation $F_D = \left[C_D \left(\frac{\rho v^2}{2} \right) A \right]$ we must remember that C_D here, is the form drag coefficient and area A is a characteristic of the body in motion. When you are looking straight at an automobile it is the 'front end two dimensional area' that is of concern and it is this area that is used to compute the form drag. Even while calculating the Reynolds number of the flow in the equation $R_E = \left(\frac{vD}{\upsilon} \right)$, D refers to the diameter of the body and not to the diameter of the flow passage area.

Among the living creatures the fish is a perfect example of a streamlined body. Form drag or the pressure drag is substantially reduced due to its streamlined shape and the only force it has to overcome is the skin friction drag. Coefficient of drag C_D for a body shaped similar to a fish such as an air-ship also called the blimp is ≈ 0.04 whereas for a sphere moving in air it is ≈ 0.5, for the same value of R_E. Even in case of an air-craft or a submarine it is the skin friction drag that is of concern because the form drag is considerably reduced due to the shape of the fuselage in case of an air-craft or the hull in case of a submarine. Both have more or less the same streamlined cross section throughout. All other factors remaining the same the C_D for the fuselage of an aircraft however is less than that for a submarine only because the density of water is higher.

It is necessary at this stage that we compare the flow pattern around a sphere to the flow pattern around a streamlined body. Figures 16.1 and 16.2 are intended to help us comprehend the effect and purpose of streamlining a body.

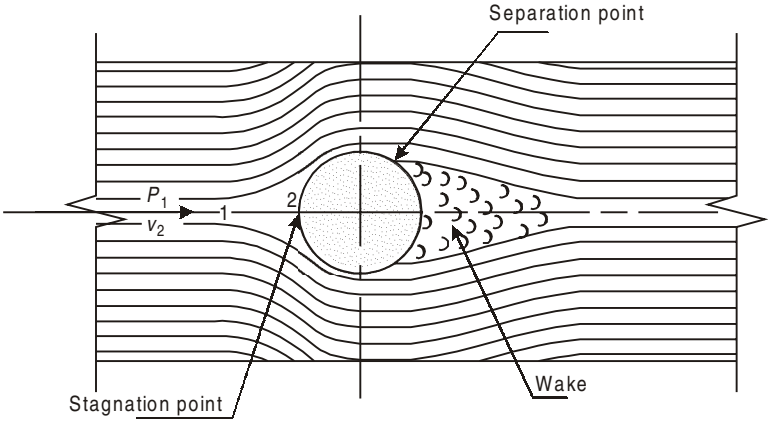


Fig.16.1

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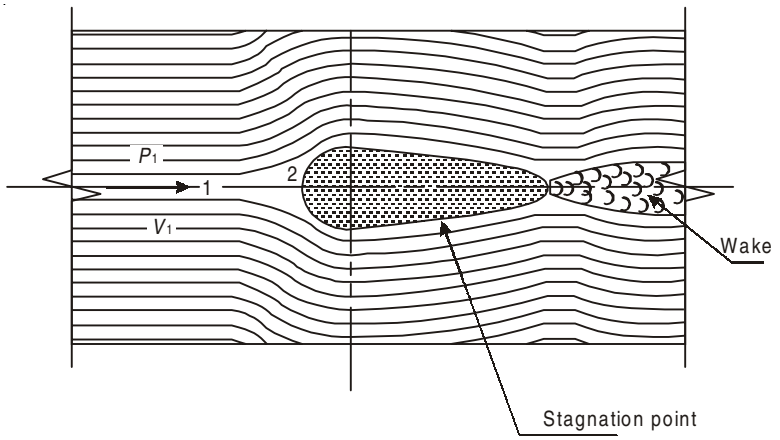


Fig.16.2

We have in the previous chapter discussed separation of the boundary layer and the formation of the wake. The point where the fluid stream comes to rest is the stagnation point and at this point the pressure is much higher than the static pressure.

In the equation $F_D = \left[C_D\left(\frac{\rho v^2}{2}\right)A\right]$ the term $\left(\frac{\rho v^2}{2}\right)$ is the defined as the dynamic pressure. The stagnation pressure is higher than the static pressure to the extent of the dynamic pressure. This means $P_S = P + \left(\frac{\rho v^2}{2}\right)$ where P_S is the stagnation pressure. It can be seen that the whole purpose of streamlining a body is to shift the point of stagnation as far down stream as possible so that the size of the turbulent wake is lower as the formation of turbulent takes place far away from the useful boundary.

The whole purpose of streamlining a body is to reduce the form drag on the body by shaping it up in such a way that its surface to a large extent coincides with the fluid flow lines. On the other hand a square or a rectangular plate placed across a fluid flow stream obstructs the flow completely and consequently induces maximum drag. Such a body is called a *Bluff body* as opposed to *Streamlined body*.

Variation of drag coefficients for various bodies at different Reynolds numbers have been experimentally determined and are listed in Table 15.2 . However the drag coefficients for a few interesting bodies such as a sphere and a cylinder are analyzed below.

A sphere is chosen to demonstrate how the pressure drag coefficient for a smooth sphere drops sharply as the Reynolds number increases. At values of

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$R_E < 0.15$ (highly viscous fluid) C_D is as high as 100 while at values of $R_E \approx 10$, C_D drops to ≈ 4. For values of R_E between $10^3 - 10^5$ the values of C_D ranges from 0.35 – 0.45. At Reynolds number beyond 2×10 there is a sharp drop in C_D for spheres from 0.4 – 0.17. This is because as the boundary layer shifts from laminar to turbulent the point where the separation occurs moves further back causing a decrease in the size of the wake. This is the principle reason why the golf balls are dimpled. Dimples effect a shift from laminar to turbulent boundary layer early in the flight path causing an abrupt decrease in the value of C_D at low values of R_E which in turn increases the distance covered by a ratio $\approx$ 1:3.

For cylinders $C_D \approx 60$ for R_E as low as 0.1 and at $R_E \approx 1$, the value of C_D drops to 10. For values of R_E between $10^3 - 10^5$ the values of C_D ranges from 0.95 – 1.45.

16.1 STOKES LAW AND TERMINAL VELOCITY

A sphere falling through a highly viscous fluid has a value of $R_E < 1$ and hence viscous drag or the friction drag component constitutes the main drag as no separation occurs and no wake forms as the layers remain attached to the surface. C.G Stokes established a general relationship between drag coefficient (C_D) and Reynolds number (R_E) and developed a distinct equation for drag force on a sphere with $R_E < 0.2$. Stokes equation for computing the total drag force on a smooth sphere with $R_E < 0.2$ is,

$$F_D = 3 \pi \mu DU \qquad ...(16.1)$$

He further established that skin friction drag forms $\frac{2}{3}$ of this total drag force, the rest being the pressure drag component. Thus skin friction drag on a sphere falling through a highly viscous fluid is,

$$F_D = 2 \pi \mu DU \qquad ...(16.2)$$

By equating Eq. (16.1) to the general form of the drag force equation

$$F_D = \left[C_D \left(\frac{\rho v^2}{2} \right) A \right]$$

$$3 \pi \mu DU = \left[C_D \left(\frac{\rho v^2}{2} \right) A \right].$$

Since A in this case the cross sectional area and not the surface area,

$$3 \pi \mu DU = \left[C_D \left(\frac{\rho v^2}{2} \right) A \right] \left(\frac{\pi D^2}{4} \right)$$

from this it emerged that,

$$C_D = \left(\frac{24}{R_E} \right) \qquad ...(16.3)$$

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This equation called the Stokes law applies to spherical forms and for values of R_E less than 0.2. At the maximum value of R_E, C_D for a sphere falling through a highly viscous fluid is as high as 120. It drops to a value as low as 0.2 for values of $R_E > 10^5$.

For values of R_E in the range $(1 < R_E < 100)$ C_D can be approximated using the C. W. Oseen's law where the inertial effects have also been partly accounted for.

$$C_D = \left(\frac{24}{R_E}\right)\left\{1+\left(\frac{3}{16}\right)R_E\right\}^{\frac{1}{2}} \qquad ...(16.4)$$

For values of R_E in the range $(0.2 < R_E < 500)$ C_D can be approximated using Allen flow law,

$$C_D = \left(\frac{18.5}{R_E}\right)^{0.6} \qquad ...(16.4\ a)$$

For values of R_E in the range $(500 < R_E < 10^5)$ C_D can be determined using Newton flow,

$$C_D = 0.44 \qquad ...(16.4\ b)$$

The magnitude of the form drag coefficient depends on the form, Reynolds number, surface-roughness of the body and on the presence of other bodies in the vicinity. As the density of air is lower at higher altitudes than at sea level drag on the body moving through air also changes accordingly.

Stokes law is useful for determining the viscosity of a fluid using an instrument called the falling ball viscometer. Its other uses are for purposes of de-silting the reservoir and for separating the metal chips from the coolant in machine tool operations.

The final velocity that a body attains while falling through a viscous fluid from a state of rest is called the *Terminal velocity of a body also called the settling velocity*. In case of a sphere falling through a viscous fluid from a state of rest the terminal velocity is determined as follows.

After the body has attained its terminal velocity it is once again in a state of rest and is therefore in equilibrium which means,

$$W_S - F_B - F_D = 0 \qquad ...(16.5)$$

Here, W_S, F_B, and F_D are the weight of the sphere acting $\downarrow$, the buoyant force acting $\uparrow$ and the drag force acting $\uparrow$.

$$W_S = \gamma\,(\text{Volume of the sphere}) = \gamma_S\left(\frac{\pi D^3}{6}\right)$$

$$F_B = \gamma\,(\text{Volume of the fluid}) = \gamma_F\left(\frac{\pi D^3}{6}\right)$$

$$F_D = 3\,\pi\,\mu\,\upsilon D \quad \text{From (16.1)}$$

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Substituting these values in Eq. (16.5) and after simplifying, the terminal velocity v is given by,

$$v = \left[\frac{(\gamma_s - \gamma_F)D^2}{18\mu}\right] \qquad ...(16.6)$$

In terms of coefficient of drag,

$$v^2 = \left(\frac{4}{3}\right)\left[\frac{(\gamma_s - \gamma_F)D}{\rho\, C_D}\right] \qquad ...(16.6\ a)$$

Equation (16.6) is valid for Stokes flow $R_E < 1$ and $C_D = \left(\frac{24}{R_E}\right)$

16.2 FLOW PAST STATIONARY CYLINDERS

A cylinder is a good example of a two dimensional body if conditions can be created in a laboratory such that the effect of flow on the ends is prevented. Such a cylinder could be imagined as a cylinder of infinite length.

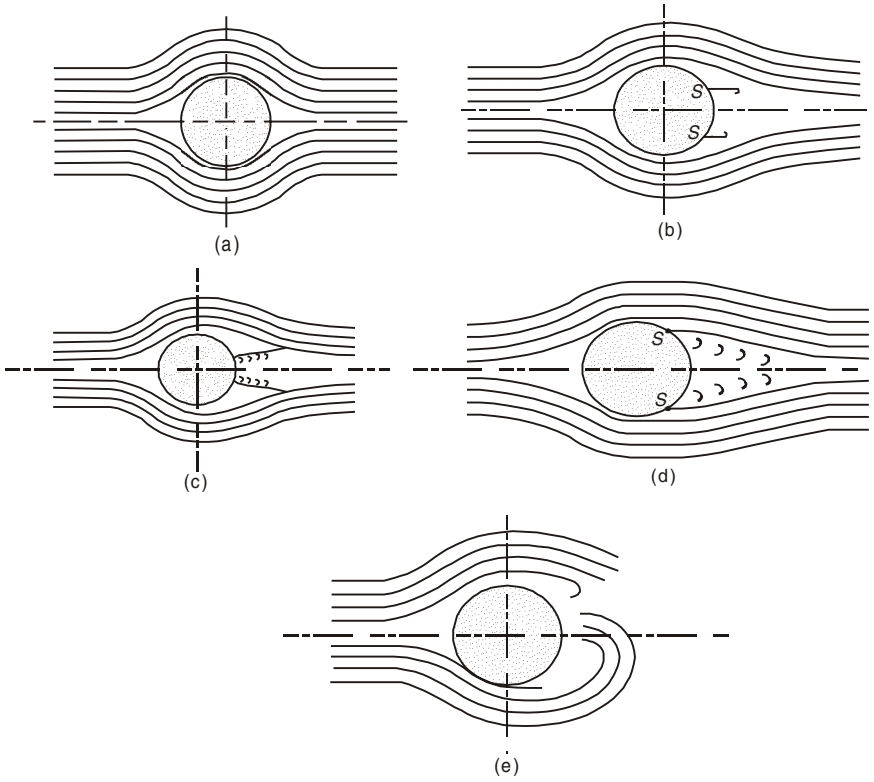


Fig.16.3

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Flow pattern and drag coefficient around such a cylinder is discussed at different velocities. At values of $R_E \left(\frac{UD}{\upsilon}\right) < 1$, the streamlines are symmetrical about the axis (Fig. 16.3 a) and the drag is essentially due to viscous effect. The form drag has negligible effect. The values of C_D could be as high as 100. At $2 < R_E < 30$ the separation of boundary layer occurs in the wake in the form of two vortices which rotate in a direction opposite to each other. (Fig. 16.3 b)

At $90 < R_E < 10^3$ number of vortices rotating in opposite direction increases. (Fig. 16.3 c)

At $R_E > 2 \times 10^5$ the wake size dramatically reduces causing a reduction in the number of vortices, the wake now is purely turbulent and the drag is mainly pressure drag. (Fig. 16.3 d)

Further increase in R_E causes elongation and between $90 < R_E < 120$ the eddies start oscillating leading to an eventual break up from the cylinder in straight formation called rows. Since they shed from either side of the cylinder (Fig. 16.3 e) two characteristic rows vortices in the wake are formed. Von Karman showed mathematically which was later confirmed experimentally that $\frac{y}{x} = 0.281$ where y is the vertical distance between the rows in which the now separated eddies are formed and x is the horizontal distance between two separated eddies. The shredded eddies are eventually washed away by the stream.

This periodic vortex shedding induces forced vibrations or lift pulsations on the cylinder which affects the natural frequency of the body leading to a phenomenon called *resonance*. The process is described by the *Strouhal number* $S_t = \left(\frac{fD}{v}\right)$.

The critical Strouhal number is = 0.2 for a wide range of Reynolds numbers. The so called *singing* of telephone wires and electrical cables is due to resonance caused by the lateral wind. Again the flutter of aero-foils and the collapse of suspension bridges are attributed to resonance. The frequency of vortex shedding is given by the empirical formula

$$f_V = 0.20 \left[1 - \frac{20}{R_E}\right] \text{ for values } 250 < R_E < 2 \times 10^5 \qquad ...(16.7)$$

The values of C_D for cylinders unlike spheres, depends not only on the values of R_E but on their $\frac{L}{D}$ ratios as well as can be seen from Table 16.1. Where $\frac{L}{D}$ ratio is taken to be ∞ drag coefficient could vary from 1 – 1.2 for values of R_E in between $10^4 - 10^5$.

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16.3 SPINNING SPHERES AND CYLINDERS

Ping pong balls, golf balls, cricket balls are all real time examples of spinning bodies in motion through a quiescent fluid such as air. The spinning of spheres such as these have an effect on the separation points of boundary layers that causes a wake which is asymmetrical to the axis of motion. (Fig. 16.4). This asymmetry creates a force in a direction perpendicular to its axis

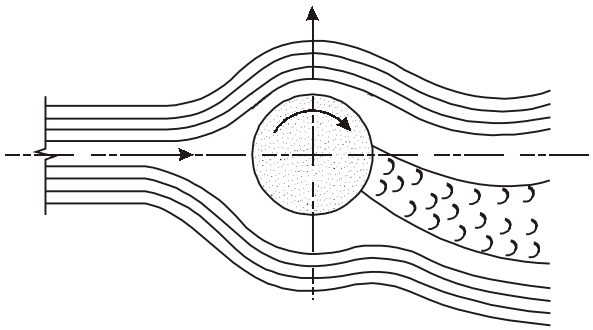


Fig.16.4

mainly due to reduced pressure at the upper surface as compared to the surface below. It is this difference in pressure that causes lift. A rotating or a spinning body has a coefficient of lift (C_L) in addition to drag coefficient (C_D). For a spinning sphere the drag coefficient varies from 0.55 to 0.62 and the lift coefficient varies from 0.1 to 0.4 for values of $R_E = 10^5$. The coefficient of lift depends on the diameter of the sphere and the angular velocity (ω).

16.4 SPINNING CYLINDERS

Flow past a stationary cylinder has already been discussed. Here we are interested in analyzing the effect of super-imposing this state on a cylinder which is spinning on its axis.

During flow past a stationary cylinder the flow lines are symmetrical about the axis of the cylinder as can be observed from Fig. 16.3 a. The peripheral velocity $u = 2U \sin \theta$, where U is the flow velocity and e is the angle between the axis of the cylinder and the upstream stagnation point.

If the cylinder is spinning on its axis a constant circulation (Γ) is imparted to the cylinder. The flow pattern is concentric stream lines around the cylinder. The peripheral velocity is $u = \dfrac{\Gamma}{\pi D}$.

When we super-impose one of these motions on the other the combined effect leads to summation of these two velocities and the peripheral velocity on the surface of the sphere at any point is,

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$$u = 2U \sin \theta + \frac{\Gamma}{\pi D} \qquad ...(16.8)$$

Figure 16.5 illustrates graphically the effect of this super-imposition.

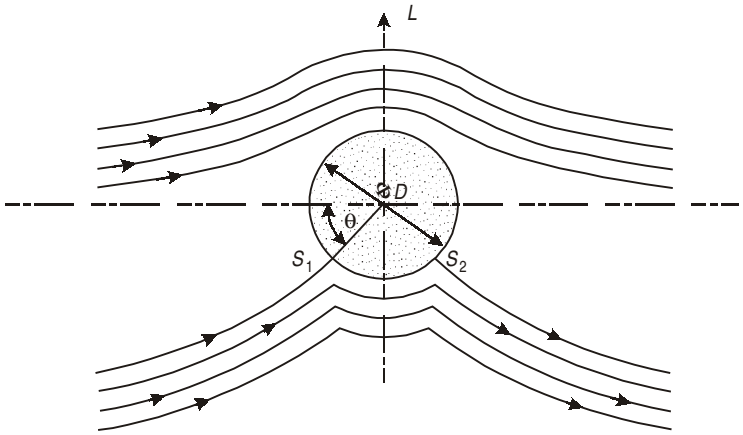


Fig. 16.5

The flow pattern around a rotating cylinder is such that the direction of tangential velocity at any point in the upper half is opposite to that of the tangential velocity at any point in the bottom half. When such a condition is superimposed on a flow pattern where the streamlines are in the same direction both in the top and in the bottom as is the case during flow past a stationary cylinder, the net effect is reduced velocity in the lower half causing an increase in pressure at the lower half and reduced pressure at the upper half due to increased velocity. Such a pressure distribution induces a resultant force which is the force that causes lift. This phenomenon that causes lift in rotating bodies is called the *Magnus effect*.

Kutta-Joukowski derived a mathematical relationship between the lift force (L_F) on a rotating cylinder and the circulation $\mathbf{\Gamma}$ (also called theoretical boundary layer circulation) imparted to it.

= circumference × peripheral velocity. ($\mathbf{\Gamma}$ = circumference × peripheral velocity)

Thus $$L_F = \rho\, LU\, \mathbf{\Gamma} \qquad ...(16.9)$$

where L is the length of the cylinder, $\mathbf{\Gamma}$ is the circulation imparted in m^2/s, ρ is the density of the fluid in kg /m^3 and U is the fluid velocity in m/sec.

Stagnation points are points where the peripheral velocity (u) at any point on the surface of the cylinder is zero. Hence these points could be located simply by setting Eq. (16.8) to zero.

Hence $$2U \sin \theta + \frac{\Gamma}{\pi D} = 0$$

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$$\text{sine } \theta = -\frac{\Gamma}{2U\pi D} \qquad ...(16.10)$$

In case of a stationary cylinder $\Gamma = 0$, hence $\sin \theta = 0$, which means $\theta = 0$, or $\theta = 180$ which is why the stagnation points are located on the axis.

In case of the cylinder shown in Fig. (16.5) $\Gamma < 2\,U\pi D$ hence $\theta < 90^\circ$, or $\theta > 180$ and hence the stagnation points are located below the axis.

Where $\Gamma = 2\,U\pi D$ stagnation points coincide at the bottom on the vertical axis since in this case $\sin \theta = 1$ and hence $\theta = 90^\circ$ or 270°.

Where $\Gamma > 2\,U\pi D$ stagnation points do coincide but do not occur on the cylinder surface but lie on the flow stream since $\sin \theta = -1$ is unacceptable.

The pressure at any point on the surface of the cylinder and the consequent lift force exerted can be determined by applying the Bernoulli's equation to any two points 1 and 2, these points being the upstream edge of the flow and any point on the surface of the cylinder respectively.

Thus, $$p_1 + \frac{1}{2}\rho U^2 = p_2 + \frac{1}{2}\rho u^2$$

Since from Eq. (16.8) $u = 2U \sin \theta + \dfrac{\Gamma}{\pi D}$

$$p_1 + \frac{1}{2}\rho U^2 = p_2 + \frac{1}{2}\rho\left[2U \sin\theta + \left(\frac{\Gamma}{\pi D}\right)\right]^2$$

Solving for difference in pressure that is generating lift ,

$$p_2 - p_1 = \frac{1}{2}\rho\left(U^2 - 1\right)\left[2U \sin\theta + \left(\frac{\Gamma}{\pi D}\right)\right]^2 \qquad ...(16.11)$$

Coefficient of lift (C_L)

From Eq. (16.9)

The lift force $L_F = \rho\, LU\, \mathbf{\Gamma}$ ...(a)

Lift is also given by $L_F = \frac{1}{2}\rho C_L U^2 A$

Lift is also given by $L_F = \frac{1}{2}\rho C_L U^2 (L \times D)$ ...(b)

Equating (a) and (b) we get an expression for C_L .

$$C_L = \frac{2\Gamma}{UD} = \frac{2(\pi D u)}{Du} = \frac{[2\pi u]}{U} \qquad ...(16.12)$$

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16.5 LIFT AND DRAG ON AIR-FOIL SECTIONS

Sections particularly created to increase lift and decrease drag on bodies submerged in air are called *Air-foil* sections. If such sections were used on bodies submerged in water, then they are termed as *Hydro-foil* sections. In general any circular arc such as sphere or a cylinder about which we have already discussed is a lift producing body.

Aircrafts carry heavy payloads but nevertheless take off and fly at great heights defying gravity. While the engines are meant for generating the forward propulsive force required to overcome drag, it is the wings on either side of the air-craft, their projected area and their cross section that is helping the aircraft to take off and stay afloat. Let us examine how this is achieved by analyzing the forces acting on an Air-foil when moving in still air. But before that let us define some important definitions associated with lift and drag on an air foil.

- *Leading edge of an air-foil* is that edge facing the direction of flow of air or the upstream edge and *trailing edge* is the down stream edge.
- *Camber line* is the center line of the cross section of the air-foil.
- *Chord line* is the straight line passing through the leading and trailing edges.
- *Chord length* is the straight line distance between the leading and trailing edges.
- *Aspect ratio* is the ratio of span (b) to mean chord length (c).

Thus $$AR = \frac{b}{c} = \frac{b \times (b \times c)}{A \times c} = \frac{b^2}{A}$$

- *Angle of attack or angle of incidence* (α) is the angle that the chord line makes with the direction of motion of the free stream velocity U.

As the air-craft gathers speed on the ground the velocity of the air stream on the upper surface of the air-foil section increases thus decreasing the pressure on the upper surface. At the same time the velocity of the air-stream at the lower surface of the airfoil decreases causing corresponding increase in pressure. This difference in pressure between the upper and the lower surface acts on the exposed lower surface area of the wing and creates the force required to lift the air-craft of the ground at the point the aircraft attains take off velocity.

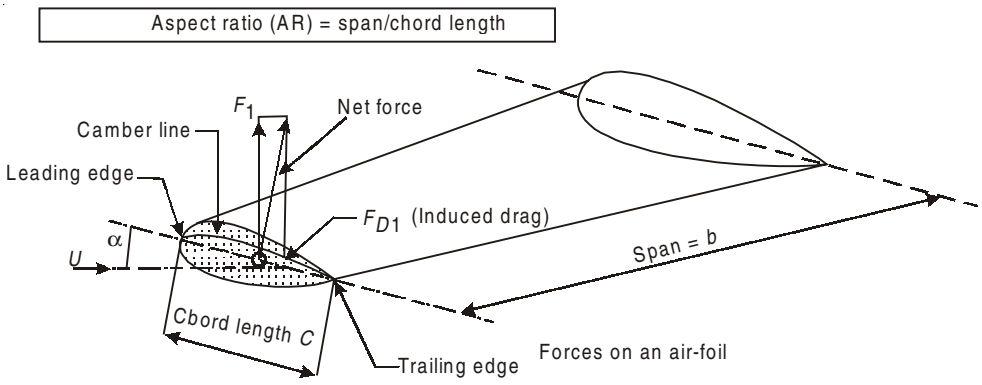


Fig.16.8

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It is important to appreciate the importance of the angle of attack (α). Even at small angles of attack the velocity of the fluid flowing through the bottom surface of an air-foil reduces thus increasing the pressure whereas the reverse effect is caused at the upper surface thus creating favourable conditions for clockwise and anti-clockwise vorticity within the boundary layers. Circulation around the air-foil is now possible and since the clockwise vorticity over the top surface is more than that at the bottom surface the resulting circulation too is clockwise.

The circulation around an air-foil is given by

$$\Gamma = \pi CU \text{ sine } \alpha \qquad ...(16.14)$$

The lift force

$$F_L = C_L \left(\frac{\rho U^2}{2}\right) A \qquad ...(16.15)$$

Here C_L is the coefficient of lift, whose value depends on the shape of the air-foil and angle of attack α. Flow Reynolds number, surface roughness of the air-foil section, air turbulence etc. also are factors affecting lift.

Since the chord length of an air-foil section is not constant through the span length but keeps changing *aspect-ratio is convenient to define the area of an air-foil at any point on the span.* Aspect ratio also helps in achieving uniformity while comparing different wing sections.

In addition to skin friction drag and pressure drag, a third component called the *induced drag* also contributes to the total drag on an air-foil. The net force on an airfoil acts perpendicular to the chord line and the lift force and the induced drag are obtained by resolving the net force in the vertical and horizontal directions respectively.

Thus

$$F_{DI} = C_{DI} \left(\frac{\rho U^2}{2}\right) A \qquad ...(16.16)$$

In terms aspect ratio (AR) and assuming an elliptical distribution around the air-foil C_{DI} the coefficient of induced drag

$$C_{DI} = \frac{(C_L)^2}{\pi(\text{AR})} \qquad ...(16.16\text{ a})$$

Since C_L the coefficient of lift $= \dfrac{F_L}{\left(\frac{\rho U^2}{2}\right)} A$ ...(c)

$$\frac{C_L}{F_L} = \frac{1}{\left(\frac{\rho U^2}{2}\right)} \qquad ...(d)$$

From (b) and (c)

$$\frac{C_{DI}}{F_{DI}} = \frac{C_L}{F_L} \qquad ...(16.17)$$

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$\frac{C_L}{F_D} = \frac{F_L}{C_D}$ is called the lift to drag ratio of an air-foil. Higher values of this ratio, indicates that lift is maximum and drag is minimum which is the purpose of an air-foil.

By equating the lift force (Eq. 16.15) and circulation (Eq. 16.14) it can be shown that the coefficient of lift depends on the angle of attack.

$$\text{Thus } C_L = \frac{(2\pi\rho\, CU^2\, F_L \sin\alpha)}{(\rho\, CU^2\, F_L)} = 2\pi \sin\alpha \qquad ...(16.18)$$

Lift and total drag coefficients for an air-foil section are obtained from Table 16.1 which is a graph of angle of attack versus lift coefficients.

TYPICAL PROBLEMS AND SOLUTIONS

Problem 16.1 *A 150 mm square bar placed horizontally is moving with a velocity = 7 m/sec through an air stream. If the length of the bar is 2 m, determine the drag force. ν for air =1.5 × 10^{-5} m^2/sec.*

Solution The drag force $F_D = \left[C_D\left(\frac{\rho v^2}{2}\right)L \times B\right]$

Since the drag coefficient (C_D) depends on the Reynolds number, we must first determine the Reynolds number for the flow.

$$R_E = \left(\frac{vL}{\upsilon}\right)$$

$$= \frac{(7 \times 0.15)}{1.5 \times 10^{-5}} = 7 \times 10^4$$

(L is the length parallel to the flow = $\frac{150}{1000}$ = 0.15 m)

For a square cylinder for values of R_E between 10^4 to 10^5 C_D from Table 16.2 is 1.60.

υ for air =1.5 × 10^{-5} m^2/sec. The corresponding value of ρ = 1.204 kg/m^3

$$F_D = \left[\frac{1.60(1.204 \times (7)^2}{2} 2 \times 0.15\right] = 14.15 \text{ N}$$

Problem 16.2 *A steel ball of the shape of a sphere is falling freely through a highly viscous fluid. If the diameter of the ball is 8 mm and speed of fall is 3 mm/s determine the drag on the sphere and the coefficient of drag. $\upsilon = 2 \times 10^{\%4}$ m^2/s and $\mu = 2.5 \times 10^{-3}$ N.s/m^2.*

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Solution
$$R_E = \frac{vD}{\upsilon} = \frac{\left[\left(3\times10^{-3}\right)\left(8\times10^{-3}\right)\right]}{\left(2\times10^{-4}\right)} = 0.12$$

Since in this case $R_E < 0.2$ viscous drag is considered the main drag as the pressure drag is almost negligible.

Hence from Eq. (16.1)

$$F_D = 3\,\pi\,\mu\,DU$$
$$= 3\,\pi\,(2.5\times10^{-3})\,(8\times10^{-3})\,(3\times10^{-3})$$
$$= 5.65\times10^{-7}\ \text{N}$$

From Eq. (16.3)

$$C_D = \left(\frac{24}{R_E}\right) = \frac{24}{0.12} = 200$$

Problem 16.3 *A steel ball 5 mm in diameter is dropped in water. Calculate the net force acting on the steel ball if its velocity is 1.5 m/s. Steel weighs = 76.9 × 10 ^{3} N/m^3, γ_w= 9790 N/m^3 and* $\upsilon = 1\times10^{-6}$ m^2/s

Solution The forces resisting the fall are the force due to buoyancy and the force due to viscous drag, both of which are acting in a direction which opposes the downward force due to the weight of the ball.

The downward force F_w = [Volume of the steel ball (V) × specific weight of the steel ball γ)

$$V = \frac{\pi}{6}\left(D^3\right)$$

where, D is the diameter of the ball = (5×10^{-3}) m.

$V = \{0.523\,(5\times10^{-3})^3\} = 6.5\times10^{-8}$ m^3.

$\gamma = 76.9\times10^3$ N/m^3

$F_W = (6.5\times10^{-8})\,(76.9\times10^3) = 5\times10^{-3}$ N↓

Force due to buoyancy is lifting the ball up and is equal to F_B = (Specific weight of water) × (Volume of fluid displaced).

Specific weight = 9790 N/m^3

Volume of fluid displaced = the volume of the sphere = (6.5×10^{-8}) m^3

$$F_B = 9790\times(6.5\times10^{-8})$$
$$= 4.136\times10^{-4}\ \text{N}\uparrow$$

$$F_D = C_D\left(\frac{\rho U^2}{2}\right)A \qquad \text{...(a)}$$

$$R_E = \frac{vD}{\upsilon} = \frac{1.5\times\left(5\times10^{-3}\right)}{\left(1\times10^{-6}\right)} = 7500$$

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Since the value of $R_E = 7500$, we will use Eq. (16.4 b) and take the value of $C_D = 0.44$. Substituting all the relevant values in Eq. (a) above,

$$F_D = C_D \frac{\rho U^2}{2} A$$

(here A is the cross sectional area of the sphere)

$$F_D = 0.44\left[\frac{(9790)(1.5)^2}{2}\right](0.7854 \times 0.005^2)$$

$$= 0.095 \text{ N}$$

The net force = $F_w - F_D - F_B$

$(5 \times 10^{-3}) - (0.095) - (4.136 \times 10^{-4}) = -0.095$ N↓

Problem 16.4 *A steel ball of diameter 2.5 mm is dropped in glycerin. If the specific gravity of the fluid is 1.26, determine the terminal velocity of the ball. Steel weighs = 76.9 × 10 ^{3} N/m^3. μ = 0.96 Pa.s.*

Solution From Eq. (16.6) the terminal velocity is given by,

$$v = \left[\left(\frac{\gamma_s - \gamma_F}{18\ \mu} D^2\right)\right] \quad \text{...(a)}$$

$$\gamma_F = (1.26 \times 9.81 \times 1000)$$

$$= 12360 \text{ N/m}^3$$

Substituting the values in (a)

Terminal velocity $$v = \frac{\left[(76.9 \times 10^3) - (12360)\right](0.0025)^2}{18(0.96)}$$

$$= 0.023 \text{ m/s}$$

Check whether R_E is < 0.2 as otherwise Eq. (a) above is not applicable

$$R_e = \frac{\rho UD}{\mu} = \frac{1260 \times 0.023 \times 0.0025}{0.96} = 0.075$$

$$\rho = 1.26 \times 1000 = 1260 \text{ kg/m}^3.$$

Since $R_E < 0.2$, the assumption of Stokes law was correct.

Problem 16.5 *An air stream directed vertically upwards with a velocity = 7 m/sec can support a sphere of diameter 100 mm. Determine the weight of the ball. For air take $\rho = 1.20$ kg/m^3 and $\upsilon = 1.5 \times 10^{-5}$ m^2/s*

Solution Weight of the sphere = Drag force since no other forces are indicated.

Drag force $$F_D = C_D \frac{\rho U^2}{2} A$$

(here A is the cross sectional area of the sphere)

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The only unknown factor in the above equation is C_D.

$$R_E = \frac{vD}{\upsilon}$$

$$= \frac{(7\times 0.1)}{(1.5\times 10^{-5})}$$

$$= 4.666 \times 10^4.$$

For this value of R_E the drag coefficient = 0.44 (from Eq. 16.4 b)

Therefore $$F_D = C_D \left(\frac{\rho U^2}{2}\right) A$$

$$= \frac{0.44\left[(1.22)(7)^2\right]}{2}\left[0.7854\times(0.1)^2\right]$$

Therefore F_D = weight of the sphere = 0.103 N

Problem 16.6 *Determine the settling velocities of a mixture of coarse clay whose diameter = 5 × 10^{-6} m and coarse sand whose average diameter ≈ 1 × 10^{-3} m in water. $\gamma_W = 9790\,N/m^3$ and specific gravities of sand and clay are 2.5 and 1.5 respectively. Take $\mu = 10^{-3}\,N.s/m^2$.*

Solution From Eq. (16.6) the terminal/settling velocity is given by,

$$v = \frac{\left[(\gamma_s - \gamma_F)D^2\right]}{18\mu} \quad ...(a)$$

Let us first determine the settling velocity of clay particles. Since it is well-known that settling velocities of clay particles are extremely small, we will assume that Stokes Law applies and later check for the correctness of our assumption.

γ_s = *sg* of clay × 9.81 × 1000

= 1.5 × 9.81 × 1000 = 14,715 N/m^3

$$v = \frac{[(14{,}715 - 9790)\,(5\times 10^{-6})^2]}{18\,(10^{-3})}$$

$$= 6.84 \times 10^{-6} \text{ m/s}$$

Let us now check for R_E to ensure that it is < 1

$$R_E = \frac{\rho UD}{\mu}$$

$$= \left[\frac{(9790)}{9.81}\,\frac{(6.84\times 10^{-6})(5\times 10^6)}{(10^{-3})}\right]$$

$$= 3.4 \times 10^{-4}$$

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Since $R_E < 1$ Stoke's law applies.

Now for the sand particles since they are larger in size compared to clay particles let us assume a Reynolds number to start with and later check on the correctness of our assumption. Let us assume $R_E = 75$ and apply Eq. (16.4 a) to obtain the discharge coefficient.

$$C_D = \frac{(18.5)}{(R_E)^{0.6}} = \frac{(18.5)}{(75)^{0.6}} = 1.387$$

We may now apply Eq. (16.6 a)

$$v^2 = \left(\frac{4}{3}\right)\left[\frac{(\gamma_s - \gamma_w)D}{(\rho C_D)}\right]$$

$$\gamma_S = sg \text{ of sand particle} \times 9.81 \times 1000$$
$$= 2.5 \times 9.81 \times 1000$$
$$= 24{,}525 \text{ N/m}^3$$

$$v^2 = \left[\frac{(\gamma_s - \gamma_w)D}{(\rho C_D)}\right]$$
$$= 0.014$$
$$v = 0.118 \text{ m/s}$$

Now Check for Reynolds Number

$$R_E = \left(\frac{\rho UD}{\mu}\right)$$

$$= \left(\frac{9790}{9.81}\right)\frac{(0.118)(1\times10^{-3})}{10^{-3}} = 117$$

Since our assumption $R_E = 75$ is wrong, we may have to solve the problem by trail and error by assuming different values of R_E each time. The student should attempt this as an exercise.

Problem 16.7 *Assuming Stokes law applies, determine the maximum size of sand particles that will settle in water.* $\gamma_W = 9790\,N/m^3$ *and specific gravity of sand is 2.5. Take* $\mu = 10^{\%\,3}\,N.s/m^2$.

Solution From Eq. (16.6) the terminal /settling velocity is given by,

$$v = \left[\frac{(\gamma_s - \gamma_F)D}{18\mu}\right] \qquad ...(a)$$

The limitation of Stoke's law is $R_E = \dfrac{\rho UD}{\mu} = 1$

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$$D = \frac{\mu}{\rho U} \qquad \text{...(b)}$$

Substituting (b) in (a)

$$v = \left[\frac{(\gamma_s - \gamma_F)\left(\frac{\mu}{\rho U}\right)^2}{18\mu}\right]$$

$$v = \left[\frac{(\gamma_s - \gamma_F)\left(\frac{\mu}{\rho v}\right)^2}{18\mu}\right] \text{ (Since } v \text{ is same as } U\text{)}$$

$$v^3 = \left[\frac{(\gamma_s - \gamma_F)\left(\frac{\mu}{\rho}\right)^2}{18\mu}\right]$$

γ_S = sg of sand particle × 9.81 × 1000
= 2.5 × 9.81 × 1000
= 24,525 N/m^3

$$\rho_w = \frac{9790}{9.81} = 998 \text{ kg/m}^3.$$

$$v^3 = \left[(24525 - 9790)\left(\frac{10^{-3}}{998}\right)^2\right]$$

$$v^3 = \left(18 \times 10^{-3}\right)$$

$= 8.2 \times 10^{-7}$

$v = 9.3 \times 10^{-3}$ m/s

From (b) $D = \dfrac{(10^{-3})}{998\ (9.3 \times 10^{-3})}$

$= 1 \times 10^{-4}$ m = 0.1 mm

Problem 16.8 *The drag resistant coefficient and the drag resistant area of a passenger car are 0.44 and 1.5 m^2. Determine the power required to overcome the drag force when the car is travelling at 70 km/hour. ρ_{air} = 1.25 kg/m^3.*

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Solution Power required is $F_D \times v$ ...(a)

$$F_D = C_D\left(\frac{\rho U^2}{2}\right)A \quad \text{...(b)}$$

(Here A is the drag resistant area which in case of a passenger car is the frontal projected area)

$$F_D = \frac{0.44\left[1.25\left(\frac{70\times1000}{(60\times60)}\right)^2\right]1.5}{2}$$

$$= 156 \text{ N}$$

Substituting the value of F_D in (a)

$$P = \frac{156(70\times1000)}{60\times60} = 3033 \text{ W}$$

$$= 3.033 \text{ kW}$$

Problem 16.9 *Four hemispherical cups are mounted on extension rods as shown in Fig. 16.6 at a distance of 200 mm from their axis of rotation. Diameter of each cup is 75 mm. Determine the torque required to overcome the drag if the cups are rotated at 60 rad / mt in air. Neglect the resistance offered by the extension rods. For air take* $\mu = 1.20$ *kg/m*^{3} *and* $\upsilon = 1.5 \times 10^{-5}$ *m*^{2}*/s.*

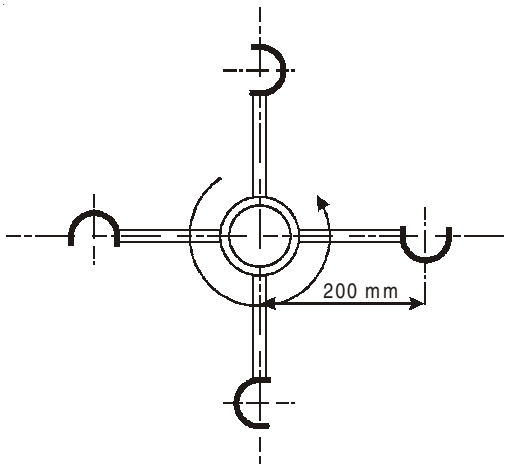


Fig. 16.6

Solution Since all the four cups are mounted with their open ends facing the direction of rotation, each at the same distance from the axis of rotation and their diameters being the same, the total resistance offered is the sum of individual resistances.

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Tangential velocity of the cup = $v_T = \omega \times r$, where r is the distance from the center of rotation to the center of the cup = 200 mm = 0.2 m and ω is given to be = 60 radians /mt = 1 rps.

Therefore, $v_T = \omega \times r = 1 \times 0.2 = 0.2$ m/sec

$$F_D = C_D \left(\frac{\rho U^2}{2} \right) A \qquad \text{...(a)}$$

(Here A is the frontal area of the cup)

From Table II C_D for a hemispherical cup of this orientation = 1.35

$$F_D = 1.35 \left[\frac{(1.20)(0.2)^2}{2} \right] 0.7854 (0.075)^2$$

$$= 1.43 \times 10^{-4} \text{ N}$$

This is the drag force on one cup.

Drag force on 4 cups = $4\ (1.43 \times 10^{-4}) = 5.72 \times 10^{-4}$ N

Torque required = $F \times r = (5.72 \times 10^{-4}) \times 0.2 = 1.145 \times 10^{-5}$ N

Problem 16.10 *If two hemispherical cups each of diameter 150 mm are placed one opposite the other as shown in Fig. 16.7 determine the torque required to prevent the cups from rotation against a wind velocity = 50 km/hour. For air take $\rho = 1.20$ kg/m^3 and $\upsilon = 1.5 \times 10^{-5}$ m^2/s. Neglect bearing friction and assume that the rods are free to rotate.*

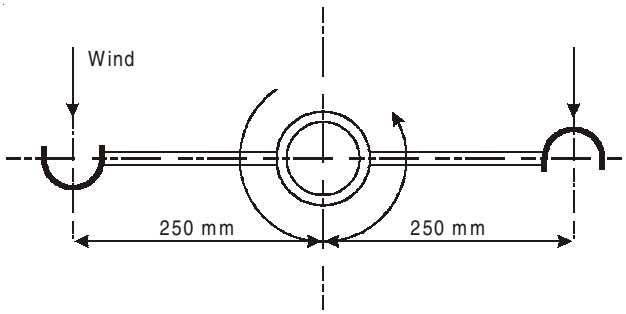


Fig.16.7

Solution If the two cups were placed in the same direction then the net drag force is zero and consequently there would be no rotation. But because the cups are placed one opposite the other and are exposed to same wind speed, there would a net drag force proportional to the difference in their discharge coefficient.

From Table 16.2 the discharge coefficient for cup 1, $C_{D1} = 1.35$ and for cup 2 $C_{D2} = 0.41$

$$\text{Wind velocity } v = \frac{50 \times 1000}{60 \times 60} = 13.88 \text{ m/s}$$

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$$\text{The drag force} = C_D \left(\frac{\rho U^2}{2}\right) A$$

$$\text{The net drag force} = (C_{D1} - C_{D2}) \left(\frac{\rho U^2}{2}\right) A$$

$$\text{The net drag force} = \frac{(1.35 - 0.41)\,(1.2)\,(13.88)^2}{2}\,(0.7854\,(0.150)^2)$$

$$= 1.92 \text{ N}$$

Counter torque required = Net torque × lever arm = 1.92 × (0.250) = 0.48 N.m

Problem 16.11 *A man and a hemispherical parachute together weigh 1500 N. If the terminal velocity of the man jumping from this parachute is equivalent to a free fall from a height of 3 m, determine the diameter of the parachute required to achieve this. For air take* $\rho = 1.20$ *kg/m*^{3}.

Solution The drag force on the parachute = total downward force = 1500 N.

Since the concave portion is facing the direction of air flow $C_D = 1.35$ from Table 16.2.

Since the terminal velocity (v) is equivalent to a free fall from a height of 3 m,

$$v = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 3} = 7.67 \text{ m/s}$$

$$\text{The drag force} = C_D \left(\frac{\rho U^2}{2}\right) A$$

$$1500 = \frac{1.35\,[(1.2)\,(7.67)^2]}{2}\,(0.7854\,(D)^2)$$

$$1500 = 37.42\,(D)^2$$

Diameter of the parachute $D = 6.33$ m

Problem 16.12 *Determine the position of stagnation points and the lift force on a vertical circular rotating cylinder of diameter 500 mm × 5 m long. The air stream velocity = 20 m/sec and the RPM = 200. For air take*$\rho = 1.20$ *kg/m*^{3}.

Solution From Eq. (16.10) $\sin\theta = -\left(\frac{\Gamma}{2U\pi D}\right)$ ...(a)

Circulation $\Gamma = \pi D \times v_p = \left[\pi\,(0.50)\,\frac{(\pi \times 0.50 \times 200)}{60}\right.$

$$\Gamma = 8.22 \text{ m}^2/\text{s}$$

Substituting this value in (a)

$$\text{sine } \theta = -\left[\frac{8.22}{2(20)\,\pi(0.50)}\right] = -0.1308$$

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$$\theta = -\sin 7.5^\circ$$
$$\theta = \sin(180 + 7.5) = 187.5^\circ \text{ and } \sin(360 - 7.5) = 352.5^\circ$$

Stagnation points are located at 187.5° and 352.5° from the horizontal axis of the cylinder.

From Eq. (16.9)

The lift force $L_F = \rho LU\Gamma$ ...(a)

$$L_F = \rho LU\Gamma = (1.2)(5)(20)(8.22) = 986.4 \text{ N}$$

Problem 16.13 *A kite in the form of air-foil has an effective area = 0.1m² and weighs 0.5N. The chord attached to the kite makes an angle of 45° to the horizontal. Find the wind speed and the tension in the string if the drag coefficient is 0.15 and lift coefficient = 0.3. For air take ρ = 1.20 kg/m³.*

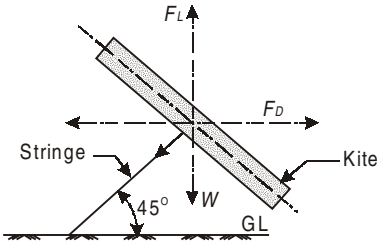


Fig. 16.9

Solution Referring to Fig. (16.9)

Drag $F_D = C_D \left(\frac{\rho U^2}{2}\right) A$

Drag $F_D \equiv T \sin 45^\circ$

i.e., $C_D \left(\frac{\rho U^2}{2}\right) A = T \sin 45^\circ$

$$\frac{(0.15)\,[(1.25)\,(U^2)]\,0.1}{2} = T\,(0.707)$$

$$0.009375\,(U^2) = 0.707\,T \quad \text{...(a)}$$

$$F_L = C_L \left(\frac{\rho U^2}{2}\right) A$$

$$F_L \equiv W + T \cos 45^\circ$$

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i.e., $$C_L \left(\frac{\rho U^2}{2}\right) A = W + T \cos 45^\circ$$

$$\frac{(0.30)\ [(1.25)\ (U^2)]\ 0.1}{2} = 0.5 + T\,(0.707)$$

$$0.01875\,(U^2) = 0.5 + 0.707\,T$$

$$0.01875\,(U^2) - 5 = 0.707\,T \qquad \text{...(b)}$$

From (a) and (b)

$$0.009375\,(U^2) = 0.01875\,(U^2) - 5$$

$$0.01875\,(U^2) - 0.009375\,(U^2) = 5$$

$$0.009375\,(U^2) = 5$$

Wind speed U = 23 m/s

From (a)

$$0.009375\,(U^2) = 0.707\,T$$

$$0.009375\,(23^2) = 0.707\,T$$

Tension T = 7 N

Problem 16.14 *A kite in the form of air-foil has an effective area = 0.1 m^2 and weighs 0.5 N. If the wind speed is 23 m/s find the angle of inclination of the chord to the horizontal and the tension in the string if the drag coefficient is 0.15 and lift coefficient = 0.3. For air take $\rho = 1.20\ kg/m^3$.*

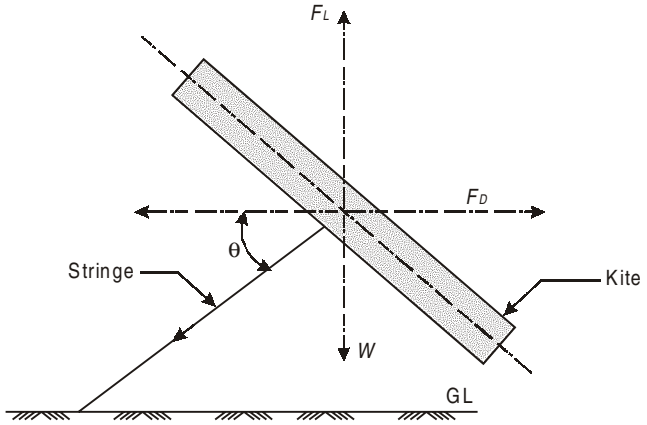


Fig. 16. 10

Referring to the Fig. (16.10)

Drag $$F_D = C_D \left(\frac{\rho U^2}{2}\right) A$$

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Drag $F_D \equiv T \sin\theta$

i.e., $C_D \left(\frac{\rho U^2}{2}\right) A = T \sin\theta$

$$\frac{(0.15)\,[(1.25)\,(U^2)]\,0.1}{2} = T \sin\theta$$

$$0.009375\,(23^2) = T \sin\theta$$

$$4.96 = T \sin\theta \quad \ldots(a)$$

$$F_L = C_L \left(\frac{\rho U^2}{2}\right) A$$

$$F_L \equiv W + T\cos\theta$$

i.e., $C_L \left(\frac{\rho U^2}{2}\right) A = W + T\cos\theta$

$$\frac{(0.30)\,[(1.25)\,(23^2)]\,0.1}{2} = 0.5 + T\cos\theta$$

$$0.01875\,(U^2) = 0.5 + T\cos\theta$$

$$0.01875\,(23^2) - 5 = T\cos\theta$$

$$0.01875\,(529) = T\cos\theta$$

$$9.9 = T\cos\theta \quad \ldots(b)$$

From (a) and (b)

$$\frac{\sin\theta}{\cos\theta} = \frac{4.96}{9.9} = \tan\theta = 0.50$$

Angle of inclination of the chord to the horizontal $\theta = 26.56^\circ$

From (b)

Tension in the string $T = \frac{9.9}{\cos 26.56^\circ} = 11$ N

Problem 16.15 *Determine the weight of an aeroplane cruising at 930 km/hour if the effective wing area is 40 m^2 and the coefficient of lift = 0.45. For air take $\rho = 0.45$ kg/m^3.*

Solution Weight of the plane = Lift force $F_L = C_L \left(\frac{\rho U^2}{2}\right) A$

$$= \frac{0.45\left[(0.45)(U^2)\right]}{2}(40)$$

Weight of the plane $= 4.05\,U^2 = \frac{4.05 \times 930 \times 1000}{(60 \times 60)^2} = 270281$ N

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Problem 16.16 *An aeroplane weighing 25 kN is cruising at 900 km / hour. If the effective area of the wing is 30 m^2 and the coefficient of drag is 0.16, determine the coefficient of lift, the thrust developed by the engines and the power required. For air take $\rho = 0.45\ kg/m^3$.*

Solution

$$\text{Weight of the plane} = \text{Lift force } F_L = C_L \left(\frac{\rho U^2}{2}\right) A$$

$$= C_L \left[\left(\frac{0.45(U^2)}{2}\right)\right](40)$$

$$25 \times 1000 = C_L \frac{\left[(0.45)\left(\frac{900 \times 1000}{3600}\right)^2\right]}{2}(40)$$

$$25000 = C_L(562500)$$

$$C_L = 0.044$$

$$\text{Thrust developed by the engines} = \text{drag force} = F_D = C_D \left(\frac{\rho U^2}{2}\right) A$$

$$\text{Thrust} = C_D \left(\frac{\rho U^2}{2}\right) A$$

$$= 0.16 \frac{\left[(0.45)\left(\frac{900 \times 1000}{3600}\right)^2\right]}{2}(40)$$

$$\text{Thrust} = 0.16 \times (562500) = 90{,}000 \text{ N}$$

$$\text{Power required} = \text{Thrust} \times \text{velocity}$$

$$= \frac{90{,}000 \times (900 \times 1000)}{3600} = 22{,}500 \text{ kW}$$

Problem 16.17 *The cruising speed of an aircraft is 500 km/h and its weight is 40 kN. The span and chord measures 12 m and 3 m respectively. If the coefficient of pressure drag is 0.015 assuming an elliptical distribution around the airfoil determine the total thrust developed by the engines. For air take $\rho = 0.45\ kg/m^3$.*

Solution

$$\text{Total thrust} = \text{The total drag force} = F_D = C_D \left(\frac{\rho U^2}{2}\right) A \qquad \ldots(a)$$

Here C_D is the total drag coefficient = Induced drag coefficient due to lift + pressure drag coefficient.

From Eq. (16.16 a)

If we assume an elliptical distribution around the air-foil,

$$C_{DI} \text{ the coefficient of induced drag} = \frac{(CL)^2}{\pi(AR)} \qquad \ldots(b)$$

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$$F_L = C_L \left(\frac{\rho U^2}{2}\right) A$$

$$40 \times 1000 = C_L \left[\frac{(0.45)\left(\frac{500 \times 1000}{3600}\right)^2}{2}\right](12 \times 3)$$

$$40{,}000 = C_L\,[156250]$$

$$C_L = 0.256$$

Substituting this value in (b)

$$C_{DI} = \frac{(CL)^2}{\pi(AR)} = \frac{(0.256)^2}{\pi\left(\frac{12}{3}\right)} = 0.0052$$

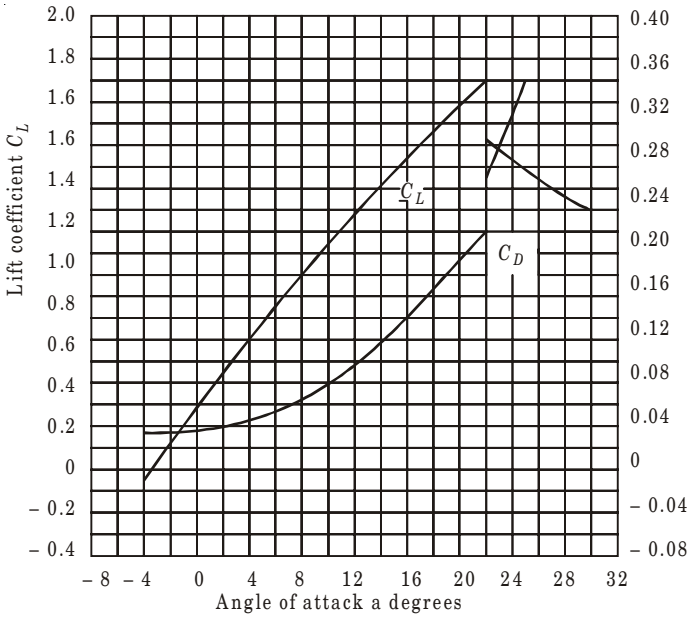
The total drag coefficient C_D = Induced drag coefficient due to lift + pressure drag coefficient.

$$C_D = 0.0052 + 0.015 = 0.020$$

From (a) total thrust $= F_D = 0.020 \dfrac{\left[(0.45)\left(\frac{500 \times 1000}{3600}\right)^2\right]}{2}(12 \times 3)$

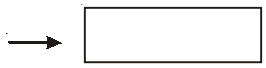
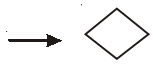
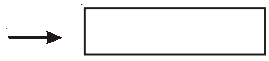
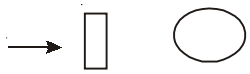
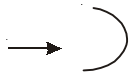
Total thrust $= F_D = 3125$ N

TABLE 16.1



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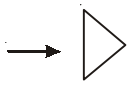
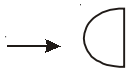
TABLE 16.2 *Drag coefficient for some common bodies for R_2 between 10^4 to 10^4*

Body form	*Direction of flow and orientation*		C_n
1. Cylinder		$\frac{L}{D}-1$	0.91
		$\frac{L}{D}-2$	0.85
		$\frac{L}{D}-4$	0.87
		$\frac{L}{D}=7$	0.99
2. Square cylinder			1.60
3. Rectangular plate		$\frac{L}{b}=1$	1.16
		$\frac{L}{b}=4$	1.17
		$\frac{L}{b}=8$	1.23
		$\frac{L}{b}-12.5$	1.34
		$\frac{L}{b}-\infty$	2.00
4. Circular disc			1.11
5. Elliptical cylinder		$\frac{L}{D}=2$	$\frac{0.6}{0.46}$
		$\frac{L}{D}-4$	0.32
		$\frac{L}{D}-8$	$\frac{0.29}{0.20}$
6. Square cylinder			2.0
7. Semi tublar			2.3
			1.12

Contd.

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8. Triangular		Included Angle 120° Included Angle 60° Included Angle 30°	2.0 2.2 1.8
		Included Angle 120° Included Angle 60° Included Angle 30°	1.72 1.30 1.00
9. Hemispherical cup			0.41
			1.35

QUESTIONS BANK

Section A

Review Questions

1. In what ways submerged bodies differ from floating bodies? Give a few examples for submerged bodies.
2. What other factors affect the performance of an air foil of infinite length as compared to an air foil of finite length?
3. State the importance of angle of attack and aspect ratio.
4. Under what conditions does a particle moving with a velocity v will either keep moving up or remain suspended or will fall down?
5. Explain Von Karman's vortex street and the causes for the so called 'singing' of telephone wires.
6. Explain an airfoil section and how it is designed to cause lift while moving through still air.
7. What is the generation of lift by a spinning cylinder in a fluid field called?
8. What does Kutta–Joukowski equation as applicable to a circular cylinder relate to?
9. Write down the limiting Reynolds number for Stokes flow, Allen flow and Newton's flow.
10. Explain pressure drag, Skin-friction drag and the induced drag and write down the governing equations for determining each drag.

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Section B

Multiple Choice Questions

Choose the right answer

1. In the equation $\left[C_D\left(\frac{\rho V^2}{2}\right)A\right]$ the term $\left(\frac{\rho V^2}{2}\right)$ refers to
   (a) dynamic pressure (b) stagnation pressure
   (c) static pressure (d) All of these
2. Purpose of streamlining a body is to
   (a) Move the stagnation points as close to the leading edge of an air-foil as possible
   (b) Equalize pressure distribution on the top and bottom surfaces of an air-foil
   (c) Move the stagnation points as far closer to the trailing edge as possible
   (d) None of these
3. Net force on an air-foil in a direction perpendicular to the chord line is
   (a) $(F_L + F_D)$ (b) $(F_L - F_D)$
   (c) $\left(\sqrt{F_L + F_D}\right)$ (d) None of these
4. Coefficient of pressure drag (C_P) depends on
   (a) local pressure near the leading edge
   (b) the pressure near the trailing edge
   (c) velocity near the trailing edge
   (d) All of these
5. Aspect ratio is
   (a) $\left[\frac{(\text{Span})^2}{\text{Area}}\right]$ (b) $\left[\frac{(\text{Span})^2}{\text{Chord length}}\right]$
   (c) $\left[\frac{\text{Chord length}}{\text{Span}}\right]$ (d) None of these
6. Lift force is
   (a) due to angle of attack
   (b) due to the pressure difference between the top and the bottom surface of an air-foil
   (c) a component of the net force acting in a direction perpendicular to the direction of the fluid stream
   (d) All of these

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7. Lift experienced by a spinning sphere is due to
   (a) circulation imparted to it
   (b) the pressure difference between the top and bottom halves
   (c) shift in the separation pints on the body resulting in asymmetrical positioning of wake
   (d) All of these
8. The C_D for a sphere obeying Stokes law and falling through a fluid with a value of $R_E = 0.15$ is
   (a) 120 (b) 140
   (c) 160 (d) None of these
9. For Stokes flow drag is equal to
   (a) $3 \mu \pi DU$ (b) $2 \mu \pi DU$
   (c) $\mu \pi DU$ (d) None of these
10. Strouhal number is
   (a) $\left(\frac{fD}{V}\right)$ (b) $\left(\frac{fD}{\mu}\right)$
   (c) $\left(\frac{fD}{\upsilon}\right)$ (d) None of these
11. Terminal velocity is useful for
   (a) timing the rate of fall of a sphere
   (b) for determining the mean diameter of spherical particles
   (c) for determining the viscosity of the fluid
   (d) All of these

Section C

Problems

1. Determine the counter force required to hold at rest a circular disk of diameter 20 m when a jet of air stream strikes normal to it at 30 m/s. For air take $\rho = 1.20$ kg/m^3. $C_D = 1.10$. (*Ans.* 186.60 kN)
2. A rod 50 mm in diameter and 2 m long is placed perpendicular to an air stream of velocity 2 m/sec and held firmly at one end. Calculate the maximum bending moment on the rod. For air take $\rho = 1.20$ kg/m^3. (*Ans.* 0.5356 Nm)
3. Determine the terminal velocity of an Aluminum sphere of diameter 70 mm falling freely in castor oil. $\gamma_{Al} = 26.5$ kN/m^3. (*Ans.* 0.007 m/s)
4. A flat square plate of area 2 m^2 is held inclined at an angle θ to the direction of a jet of air stream whose velocity is 10 m/sec. If the coefficients of lift and drag are 0.5 and 0.1 respectively, determine the angle of inclination and the resultant force on the plate. For air take $\rho = 1.20$ kg/m^3. (*Ans.* $\theta = 78.7°$ and $R = 61.2$ N)

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5. The thrust required by airplane during take off and to maintain a level flight at a cruising speed of 880 km/h is100 kN. If the angle of attack is 15° determine the weight of the aircraft and the power required.

(*Ans.* W = 373 kN and P = 91,177 kW)

6. Determine the lift and the position of stagnation points for a circular cylinder φ 2 m × 10 m long, rotating at 200 r.p.m when placed with its axis vertical to a stream of water jet of velocity 12 m/sec. ρ for water = 1000 kg/m^3.

(*Ans.* 4.73 kN, 195.13°, 344.87°)

7. Derive an expression for the theoretical coefficient of lift for a rotating circular cylinder in terms of its peripheral velocity and air flow velocity. Give reasons why the experimental values differ from theoretical values.

Answers to Multiple Choice Questions

1. (a) **2.** (c) **3.** (d) **4.** (d) **5.** (a) **6.** (c)
7. (c) **8.** (c) **9.** (a) **10.** (a) **11.** (d)

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Chapter

17

Compressible Fluid Flow

For all practical purposes liquids are considered incompressible and they are therefore out of purview of this chapter. It is well-known that all gases including air are highly compressible and the effect of their compressibility is dealt with in this chapter. Even here, if the changes in their temperature or density are small they are treated as incompressible and a discussion on such gases is excluded. In this chapter our discussions are confined mainly to gases which undergo a substantial change of state due to changes in their density or temperature or both. The yardstick used to identify such gases is their Mach numbers. *Mach number is a ratio of the velocity of a fluid or the velocity of a body moving in that fluid to the velocity of sound waves generated at that point in the fluid.*

For instance when a piston compresses a gas contained in a cylinder a disturbance is created and the time it takes for this disturbance to be transmitted depends upon the number of molecules contained with in a specific distance. Solids are closely packed and hence the disturbance is transmitted instantaneously. In liquids this disturbance is transmitted faster than in gases. As density and hence the pressure changes occur in a gas the molecules come closer together and enable transmission of that disturbance. This disturbance travels at the velocity of sound (C) and is equal to

$$C = \sqrt{\frac{dp}{d\rho}} \qquad \ldots(17.1)$$

Thus Mach number $M = \frac{V}{C}$,

where C is the velocity of sound .... (17.1 a)

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$$\text{For liquids } C = \sqrt{\frac{K}{\rho}}$$

For gases if the process is adiabatic

$$C = \sqrt{kRT}$$

$$\text{If the process is Isothermal } C = \sqrt{RT} \quad \ldots(17.2)$$

Here K = Bulk modulus, ρ = density, R = universal gas constant in J/kg K and T is the absolute temperature in degree Kelvin and k is the adiabatic exponent whose value for air is 1.4.

If $M < 0.3$, the gas can be treated as incompressible, if M lies between 0.3 to 0.8 the flow is termed *subsonic*, if M lies between 0.8 to 1.2 the flow is termed *transonic*, if M lies between 1.2 to 3 the flow is termed *supersonic* and if M is > 3 the flow is termed *hypersonic*.

$$\text{For an ideal gas } p = \rho RT \quad \ldots(17.3)$$

or $$pv = RT$$

where v is the specific volume $\left(\frac{\text{volume}}{\text{unit mass}}\right) = \frac{1}{\rho}$

In its differential form the above equation is written as,

$$\left[\left(\frac{dp}{p}\right) - \frac{d\rho}{\rho} - \frac{dT}{T}\right] = 0$$

$$R = 287 \text{ J/kg}$$

C_P and C_V are specific heat at constant pressure and specific heat at constant volume and both have the same units as R

$$K = C_P - C_V \quad \ldots(17.4)$$

$$C_P = \left[\frac{kR}{k} - 1\right]$$

and $$C_V = \left[\frac{R}{k} - 1\right] \quad \ldots(17.4\text{ a})$$

$$\frac{C_P}{C_V} = k = 1.4 \text{ for air} \quad \ldots(17.5)$$

$$\text{For a perfect gas } pv^k = \text{constant}$$

$$= p_1 v_1^{\,k} = p_2 v_2^{\,k} \quad \ldots(17.6)$$

If the conservation of mass is considered, $m = \rho A v$ = constant $\ldots(17.7)$

where, m is the mass flow rate.

From the steady flow energy equation,

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$$\left(\frac{1}{2}\right)v_1^2 + H_1 + q - W = \left(\frac{1}{2}\right)v_2^2 + H_2 \qquad \ldots(17.8)$$

Here, q and w are the heat added and the work done per unit mass respectively.

In addition either the Euler's equation

$$\left[\frac{dp}{\rho} + vdv + gdz = 0\right] \qquad \ldots(17.9)$$

can be used if the flow is frictionless or, the momentum equation, if friction and other forces are present. Since (ρ) is not constant in compressible fluid flow it is not taken out of the integral sign while integrating Eq. (17.9).

A change of state in a gas may occur under different conditions such as constant volume, constant pressure, and constant temperature or under adiabatic conditions in which no heat gets out or gets into the system during the process.

Under constant volume (*Isochoric*) conditions, work done $W = 0$,

$$Q = C_V(T_2 - T_1) \qquad \ldots(17.10)$$

Under constant pressure (*Isobaric*) conditions, work done $W = p\,(v_2 - v_1)$,

$$Q = C_P(T_2 - T_1) \qquad \ldots(17.11)$$

Under constant temperature (*Isothermal*) conditions, since work done is equal to heat added,

$$W = Q = p_1 v_1 \ln\left(\frac{v_2}{v_1}\right) \qquad \ldots(17.12)$$

Under constant pressure (*Adiabatic*) conditions since no heat is added or extracted from the system,

$$Q = 0$$

$$\text{Work done } W = \frac{(p_1v_1 - p_2v_2)}{k} - 1 \qquad \ldots(17.13)$$

The above equation is applicable for frictionless adiabatic process also called reversible adiabatic process or isentropic process for which the value of $k = 1 = \frac{C_P}{C_V}$. Where $k \neq 1$, the process is called polytropic where friction and other forces are considered.

All of the above equations can be used in their present form or in their original form as presented in chapters dealing with incompressible flows except for Bernoulli's equations which needs to be re-worked taking in to consideration the effect of compressibility. Bernoulli's equation has to be considered separately for Isothermal and adiabatic processes. Since in earlier chapters there were no density or temperature changes thermodynamic properties of gases such as enthalpy, internal energy and entropy was not discussed. But under compressible flow they form an important factor and hence are included.

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Enthalpy $H = U + pv$ where, U is the internal energy ...(17.14)

Change in enthalpy $\Delta h = C_P \Delta T$ ...(17.14 a)

Internal energy is the heat energy stored in a gas and as per Joules law U is a function of T only and hence pressure or volume changes are not taken into account. Since the absolute energy content in any substance is difficult to quantify, we are contend to measure the change in internal energy which answers all our needs.

Entropy is a measure of the amount of energy available in a gas for conversion into work.

Thus for Isothermal process where T is constant,

$$\text{Entropy} = (s_2 - s_1)$$

$$= \frac{Q}{T}$$

$$= C_P \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{P_2}{P_1}\right) \quad ...(17.15)$$

For adiabatic process where $Q = 0$,

$$\text{Entropy} = (s_2 - s_1) = 0 \quad ...(17.16)$$

17.1 BERNOULLI'S EQUATION FOR COMPRESSIBLE FLUIDS UNDER GOING FRICTIONLESS ADIABATIC PROCESS

If we integrate Euler's Eq. (17.9) we have,

$$\int \frac{dp}{\rho} + \frac{v_2}{2} + gz = C \quad ...(17.17)$$

Since the process is adiabatic $\left(\frac{p}{\rho^k}\right) = C = \gamma$

or $$\rho = \left[\frac{p}{\gamma}\right]^{\frac{1}{k}} \quad ...(17.18)$$

By substituting Eq. (17.18) in (17.17) and integrating we obtain,

$$\left(\frac{k}{k-1}\right)\frac{p}{\rho} + \frac{v^2}{2} + gz = C \quad ...(17.19)$$

If we consider two points 1 and 2 along the stream line then,

$$\left[\frac{k}{k-1}\right]\left[\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right] + \left[\frac{v_1^2}{2} - \frac{v_2^2}{2}\right] + gz = 0 \quad ...(17.20)$$

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17.2 BERNOULLI'S EQUATION FOR COMPRESSIBLE FLUIDS UNDER GOING FRICTIONLESS ISOTHERMAL PROCESS

From Eq. (17.17)

$$\int \frac{dp}{\rho} + \frac{v^2}{2} + gz = C \qquad \ldots(a)$$

Since for Isothermal flow $\left[\frac{p}{\rho}\right]$ = another constant C_1

and hence $[\rho] = \left(\frac{p}{C_1}\right)$

$$\int \frac{dp}{\left(\frac{p}{C_1}\right)} + \frac{v^2}{2} + gz = C$$

$$\int \frac{dp(C_1)}{\rho} + \frac{v^2}{2} + gz = C$$

$$(C_1) \int \frac{dp}{\rho} + \frac{v^2}{2} + gz = C$$

$$(C_1) \ln \frac{p}{\rho} + \frac{v^2}{2} + gz = C$$

$$\left(\frac{p}{\rho}\right) \ln p + \frac{v^2}{2} + gz = C$$

or between two stations,

$$\left(\frac{p_1}{\rho_1}\right) \ln p_1 + \frac{v_1^2}{2} + gz_1 = \left(\frac{p_2}{\rho_2}\right) \ln p_2 + \frac{v_2^2}{2} + gz_2$$

Since a change in elevation is insignificant for gases and also since

$$\rho = \frac{p}{RT}$$

$$\frac{\left(v_2^2 - v_1^2\right)}{2} = 2RT \ln \frac{p_1}{p_2} \qquad \ldots(17.21)$$

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17.3 STEADY ADIABATIC FRICTIONLESS FLOW THROUGH A CONVERGING AND DIVERGING NOZZLE

For gases the continuity equation $a_1 v_1 = a_2 v_2$ which indicates that decrease in velocity will be made up by an increase in area of cross section to keep the flow rate constant is not applicable the same way as it does for liquids since the density is also a variable. For a horizontal flow through a duct Euler's Eq. (17.9) reduces to the form,

$$\frac{dp}{\rho} + v dv = 0 \qquad \ldots(a)$$

But from Eq. (17.1)

$$C = \sqrt{\frac{dp}{d\rho}}$$

or $$C^2 = \frac{dp}{d\rho}$$

or $$dp = C^2 d\rho$$

Substituting this value of dp in (a)

$$C^2 \frac{d\rho}{\rho} + v dv = 0 \qquad \ldots(b)$$

From the law of conservation, $\rho A v = C$

or differentiation of $\rho A v = 0$

or $d(\rho A v) = 0$

or $d\rho (A v) + (A v) d\rho = 0$

or $\rho A (dv) + \rho v (dA) + A v (dp) = 0$

Divide throughout by $\rho A v$ and after re-arranging,

$$\frac{(dp)}{\rho} + \frac{(dA)}{A} + \frac{(dv)}{v} = 0 \qquad \ldots(17.22)$$

After comparing Eqs. (17.22) and (b above) and after simplifying,

$$\frac{(v dv)}{C^2} - \frac{dv}{v} - \frac{dA}{A} = 0$$

Dividing throughout by $\frac{dv}{A}$,

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$$\frac{(dA)}{dv} = \frac{A}{v}\left[\left(\frac{v^2}{C^2}\right) - 1\right] \qquad \text{since } \frac{v}{C} = M \text{ (Mach \#)}$$

$$\frac{(dA)}{A} = \frac{dv}{v}\,[M^2 - 1] \qquad \ldots(17.23)$$

The above equation establishes a relationship between area, velocity and flow Mach number for flow through convergent divergent nozzles. A similar equation that establishes a relationship between changes in pressure, area and flow Mach number is,

$$\frac{(dp)}{dA} = \left(\frac{\rho v^2}{A}\right)\left[\frac{1}{(1 - M^2)}\right] \qquad \ldots(17.24)$$

For a *Convergent* nozzle between two points 1 and 2, if $\frac{dv}{v} > 0$, $\frac{dA}{A} < 0$ and $dp < 0$ the flow may be considered as *subsonic*. ($M < 1$)

For a *Divergent* nozzle between two points 1 and 2, if $\frac{dv}{v} < 0$, $\frac{dA}{A} > 0$ and $dp > 0$ the flow may be considered as *subsonic*. ($M < 1$)

For a *Convergent* nozzle between two points 1 and 2, if $\frac{dv}{v} > 0$, $\frac{dA}{A} > 0$ and $dp > 0$ the flow may be considered as *Supersonic*. ($M > 1$)

For a *Divergent* nozzle between two points 1 and 2, if $\frac{dv}{v} > 0$, $\frac{dA}{A} > 0$ and $dp < 0$ the flow may be considered as *Supersonic* ($M > 1$).

In case of a *Sonic flow* ($M = 1$) since flow passage area is constant

$$\frac{dA}{A} = 0 \text{ and } \frac{dv}{v} = 0$$

and dp is indeterminate for isentropic flows.

17.4 SHOCK WAVES AND WAVE PROPAGATION

In a Stationary fluid, due to its elastic properties even small pressure changes are propagated at the speed of sound ($M = 1$). The disturbances due to these pressure changes are spherical circles concentric to the source of disturbance (S) and are equidistant from each other. That is the distance travelled by second wave is twice the distance travelled by the first and the distance travelled by the third wave is thrice the distance travelled by the first etc. Refer Fig. (17.1a)

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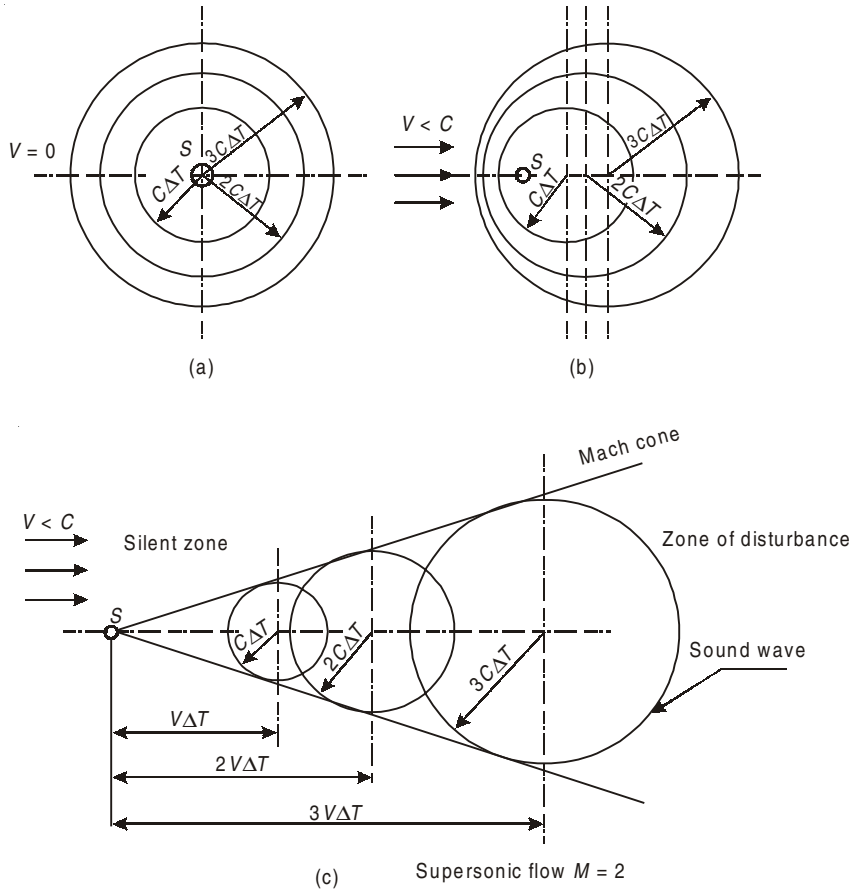


Fig. 17.1

If Δt is the time lapse between each wave then the distance travelled by the first wave is $C\Delta t$, the distance travelled by the second wave is $2C\Delta t$ and by the third wave it is $3C\Delta t$ and so on. Here c is the velocity of sound.

If instead of being stationary the fluid was moving with a velocity v (where $v < c$) the moving fluid gets ahead of the waves and the waves though are concentric spherical circles, they are not concentric to the source of disturbance (S) any more. The absolute velocity at which the waves are now propagated depends upon the direction of fluid motion whether forward ($v + C$) or backward ($v - C$) Fig. (17.1b). When velocity of the fluid v becomes greater than the velocity of sound (where $v > c$) and hence $M > 1$ the fluid stream moves ahead of the spheres faster than they are generated since $v\Delta t > C\Delta t$ and results in the formation of a cone called the Mach cone. The cone angle $\theta = \sin^{-1}\left(\frac{1}{M}\right)$, Fig. 17.1 (c).

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Mass flow through an orifice discharging from a huge reservoir containing gas

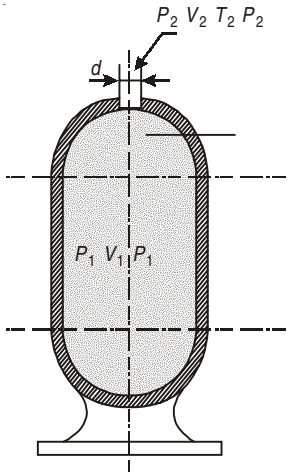


Fig. 17.2

From Eq. (17.20)

$$\left[\frac{k}{k}-1\right]\left[\frac{p_1}{\rho_1}-\frac{p_2}{\rho_2}\right]+\left[\frac{v_1^2}{2}-\frac{v_2^2}{2}\right]+gz = 0 \qquad \ldots(a)$$

Neglecting the elevation factor since the fluid involved is gas and the velocity factor since $v_1 = 0$ and $v_2 = v$ the above equation reduces to the form,

$$\left[\frac{k}{k}-1\right]\left[\frac{p_1}{\rho_1}-\frac{p_2}{\rho_2}\right] = \left[\frac{v_2}{2}\right] \qquad \ldots(b)$$

Since for adiabatic flow

$$(p_1 v_1^{\,k} = p_2 v_2^{\,k}) = \left(\frac{p_1}{p_1^{\frac{}{k}}} = \frac{p_2}{p_2^{\frac{}{k}}}\right)$$

$$= \left(\rho_2 = \frac{p_1}{p_1^{\frac{}{k}}}\right)$$

$$= \left(\rho_2 = \rho_1 r^{\frac{1}{k}}\right) \qquad \ldots(17.25)$$

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where $r = \left(\frac{p_2}{p_1}\right)$ and $\left(\frac{p_2}{\rho_1}\right) = \left(\frac{p_1}{\rho_1}\right)\left[r^{\left[\frac{k-1}{k}\right]}\right]$

If we substitute this value in Eq. (b) it would result in an equation of the form,

$$\left[\frac{k}{k-1}\right]\left(\frac{p_1}{\rho_1}\right)\left[1-r^{\left[\frac{k-1}{k}\right]}\right] = \frac{v^2}{2}$$

From which after simplifying we obtain an expression for average velocity of flow,

$$v = \sqrt{2\left[\frac{k}{k-1}\right]\left(\frac{p_1}{\rho_1}\right)\left[1-r^{\left[\frac{k-1}{k}\right]}\right]} \quad \ldots(17.26)$$

In terms of mass flow/unit time = $(Av\rho)$

$$m = A\sqrt{\left[\frac{2k}{k-1}\right](p_1\rho_1)\left[r^{\frac{2}{k}} - r^{\left[\frac{k+1}{k}\right]}\right]} \quad \ldots(17.27)$$

When calculating the actual discharge the coefficient of discharge must be taken in to account. Here $r = \left(\frac{p_2}{p_1}\right)$.

In terms of weight flow rate the above equation takes the form,

$$W = A_2\sqrt{\left[\frac{2gk}{k-1}\right](p_1\gamma_1)\left[r^{\frac{2}{k}}\right] - \left[r^{\left(k+\frac{1}{k}\right)}\right]} \quad \ldots(17.28)$$

where $r = \left(\frac{p_2}{p_1}\right)$ is greater than > the critical pressure ratio.

If $r = \left(\frac{p_2}{p_1}\right)$ is less than < the critical pressure ratio then the flowing equation may be used to compute the weight flow rate.

$$W = A_2\sqrt{\left[\frac{2g\,k}{k+1}\right](p_1\gamma_1)\left[2(k+1)\right]^{\frac{2}{k-1}}} \quad \ldots(17.28\,a)$$

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The condition for maximum discharge v_m is given by,

$$v_m = \sqrt{[\gamma]\left(\frac{p_2}{\rho_2}\right)} = C \qquad \ldots(17.29)$$

where, C is the velocity of sound at the exit to the orifice. If instead of an orifice a mouth piece such as a *converging diverging nozzle* was fitted then the area A in Eq. (17.27) will be based on the throat diameter of the nozzle instead of the orifice diameter.

From Eq. (17.27) it is clear that the pressure difference $(p_1 - p_2)$ increases as the pressure ratio r decreases and consequently there is an increase in weight flow rate. This increase however reaches its maximum value and the point at which this reaches a maximum value is called *critical pressure ratio* and is given by,

$$(r)_c = \left[\frac{2}{(k+1)}\right]^{\frac{k}{(k-1)}} \qquad \ldots(17.30)$$

When the critical pressure ratio is reached the flow velocity at the throat of a nozzle is equal to the speed of sound in gas $\left[C = \frac{(kg\ p_2)}{\gamma_2}\right]$. This is also the maximum velocity of flow through a converging nozzle. Supersonic flows cannot be obtained through mainly a converging nozzle.

17.5 STAGNATION PRESSURE, STAGNATION DENSITY AND STAGNATION TEMPERATURE

Stagnation point is the point where the relative velocity (v_S) is zero on the body immersed in a fluid. This velocity is converted in to pressure head (p_S). Consequently their will be variations in other properties such as density and temperature which are designated as ρ_S and T_S respectively.

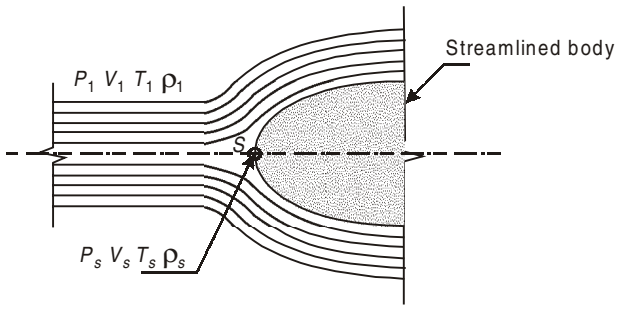


Fig. 17.3

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Figure 17.3 elaborates on these points. S is the stagnation point and (p_S), ρ_S and T_S are the pressure, density and temperature corresponding to the stagnation point.

Let at some section A–A the relative velocity, pressure, density and temperature be v_1, p_1, ρ_1 and T_1.

Applying Bernoulli's equation between section A–A and the stagnation point it can be shown that the stagnation pressure p_S is related to free stream pressure p_1 in the following manner.

$$p_S = p_1 \left\{ \left[1 + \frac{(k-1)}{2} \right] (M_1)^2 \right\}^{\frac{(k)}{k-1}} \qquad \ldots(17.31)$$

Here $\left[M_1 = \frac{v_1}{C_1} \right]$. The same way, it can be shown that the *stagnation density* is related to density of the free stream fluid in the following manner.

$$\rho_S = \rho_1 \left\{ \left[1 + \frac{(k-1)}{2} \right] (M_1)^2 \right\}^{\frac{(k)}{k-1}} \qquad \ldots(17.32)$$

Correction factors for compressibility need to be addressed here.

If we apply the binomial theorem to Eq. (17.29) and expand and simplify by noting $\gamma = \left[C^2 \frac{\rho}{p} \right]$ the following equation would emerge.

$$(p_S - p_1) = \frac{(\rho_1 v_1^2)}{2} \left[1 + \left(\frac{M_1^2}{4} \right) + \frac{(2-\gamma)}{24} (M_1)^2 \right] + \ldots. \qquad \ldots(17.33)$$

If $\left[(p_S - p_1) = \frac{(\rho_1 v_1^2)}{2} \right]$ the effect of compressibility is neglected because the pressure difference $(p_S - p_1)$ is insignificant. This would be case if the flow Mach number is < 0.2. At higher values of Mach numbers the effect of compressibility must be taken into account.

Stagnation temperature may be expressed the same way as,

$$T_S = T_1 \left[1 + \frac{(k-1)}{2} \right] (M_1)^2 \qquad \ldots(17.34)$$

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17.6 FLOW THROUGH A LAVAL NOZZLE

Laval nozzle so named after a Swedish scientist is designed to produce supersonic flows. It is basically a convergent-divergent nozzle which ensures that the flow is sub-sonic in the converging region, sonic in the throat region and supersonic in the diverging region. The flow is governed by the pressure difference between the inlet and the outlet. There will therefore be no flow as long the inlet pressure is equal to the outlet pressure or the pressure $p_3 = p_1$.

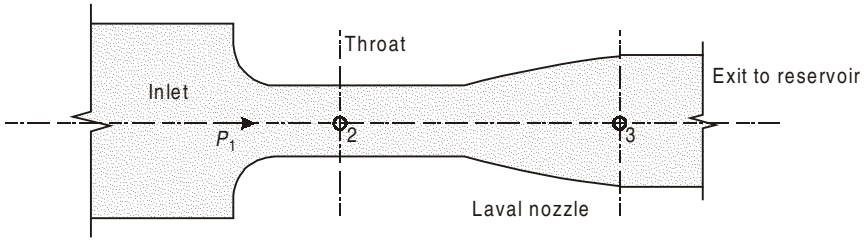


Fig. 17.4

For sub-sonic flow to occur pressure at the throat must be greater than the critical pressure. That is, $p_2 > 0.528\, p_1$.

At the throat since $M = 1$, $p_2 = 0.528\, p_1$ and the flow will remain sub-sonic on either side of the throat and isentropic as well.

Upon further reduction of pressure at the outlet although the flow at the throat is sub-sonic the flow at the divergent section will now be super-sonic resulting in the formation of a shock wave at some point on the divergent section. The flow velocity now decelerates to $M < 1$ down stream of the shock wave while the flow upstream remains supersonic.

The shock wave forms basically due to the exit pressure being less than the back pressure. The shock waves could be either normal or oblique to the direction of flow. Small differences between the back pressure and the pressure at the exit produce oblique shock waves whereas large differences cause normal shock waves. Oblique shock waves also form when a supersonic flow is forced to change direction by a solid boundary.

For a Laval nozzle the area ratios between the throat (A_T) where the $M = 1$ and the area A at any section where M is either greater or less than1 is given by,

$$\frac{A}{A_T} = \frac{1}{M}\left[2 + \frac{(k-1)\,M^2}{k+1}\right]^{\frac{(k+1)}{2\,(k-1)}} \qquad \ldots(17.35)$$

For air since $k = 1$, the maximum mass flow rate is given by the equation,

$$m_{\max} = 0.686\, A_T \sqrt{p_1 \rho_1} \qquad \ldots(17.36)$$

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17.7 NORMAL SHOCK WAVE THROUGH A DIVERGENT NOZZLE

A normal shock wave is similar to a hydraulic jump that occurs in open channel flow. The flow is supersonic before the shock and subsonic after the shock. Area across the shock wave is taken to be constant since the thickness of the wave itself is very small. A normal shock wave propagated in a diffuser causes deceleration of supersonic flow to sub-sonic flow in the zone of disturbance which results in an increase in their temperature, pressure and density. An expression could be obtained that determines the extent of variation of these properties.

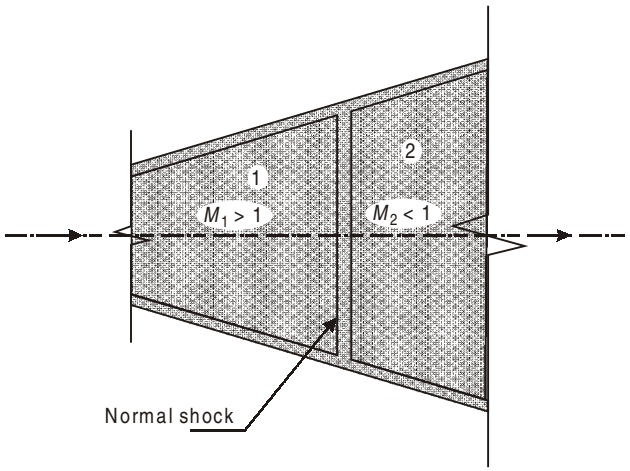


Fig. 17.5

Let p_1, ρ_1, T_1, v_1 and p_2, ρ_2, T_2, v_2 are the pressure, density, temperature and velocity before and after the shock respectively.

If the flow is considered adiabatic then the governing equations are,

Continuity equation $\rho_1 v_1 = \rho_2 v_2$ since the flow passage area is constant …(a)

The momentum equation $(p_2 - p_1) = \rho_1 v_1^2 - \rho_2 v_2^2$ ... (b)

The Bernoulli's Eq. (17.19)

$$\left[\frac{k}{k-1}\right]\frac{p}{\rho} + \frac{v^2}{2} + gz = C \qquad \ldots(c)$$

From Eqs. (a) and (b), we obtain *Rankine line equation*.

$$p_1 + (\rho_1 v_1)^2 = p_2 + (\rho_2 v_2)^2 \qquad \ldots(17.37)$$

Similarly from Eqs. (a) and (c) we obtain the *Fanno line equation*.

$$\left[\frac{k}{k-1}\right]\frac{p_1}{\rho_1} + \frac{(\rho_1 v_1)^2}{2\rho_1^2} = \left[\frac{k}{k-1}\right]\frac{p_2}{\rho_2} + \frac{(\rho_2 v_2)^2}{2\rho_2^2} \qquad \ldots(17.38)$$

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For instance, changes in stagnation pressure could be obtained by the equation,

$$\frac{p_2}{\rho_1} = \frac{(1+kM_1^2)}{(1+kM_2^2)} \quad \ldots(17.39)$$

Changes in temperature is given by,

$$\frac{T_2}{T_1} = 1 + \left[\frac{(k-1)}{2}\right]M_1^2 + \left[\frac{(k-1)}{2}\right]M_2^2 \quad \ldots(17.40)$$

Changes in velocity and density is given by,

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2}$$

$$= \frac{(k+1)M_1^2}{\left[(k-1)M_2^2+2\right]} \quad \ldots(17.40\text{ a})$$

Changes in Mach # is obtained by,

$$M_2^2 = \frac{(k-1)M_1^2+2}{\left(2kM_1^2\right)-(k-1)} \quad \ldots(17.41)$$

Entropy changes across a shock wave is given by,

$$S_2 - S_1 = C_V \text{ In}\left[\left(\frac{p_2}{p_1}\right)\left[\frac{\rho_2}{\rho_1}\right]^k\right] \quad \ldots(17.42)$$

Here subscripts 1 and 2 refer to points upstream of the shock wave and downstream of the shock wave respectively.

Stagnation temperature remains constant across a normal shock but stagnation pressure and density changes occur.

17.8 VELOCITY MEASUREMENT USING PITOT-STATIC TEST TUBE

While dealing with incompressible flow in earlier chapters we have discussed pitot-static tube used for measurement of fluid velocity. There, all that required to be determined was difference between the static head and the stagnation head $(h_1 - h_2)$, so that Torricelli's equation $v = \sqrt{2g(h_1 - h_2)}$ can be used directly to calculate the velocity. But while dealing with compressible fluids the actual pressure difference should be multiplied by the compressibility correction factor as determined in Eq. (17.33).

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$$(p_S - p_1) = \frac{(\rho_1 v_1^2)}{2}\left[1 + \left(\frac{M_1^2}{4}\right) + \frac{(2-\gamma)}{24}(M_1)^2\right].....$$

$\frac{(\rho_1 v_1^2)}{2}$ is the term that equals the reading obtained from the pitot test tube and the rest of the terms in the binomial equation represents the compressibility correction factor.

TYPICAL PROBLEMS AND SOLUTIONS

Problem 17.1 *Determine the value of universal gas constant R, C_P and C_V in heat units, given that the gas concerned is diatomic and its molecular weight is 29.*

Solution

$$R = (8314)\ \text{kJ/kmol}$$

$$\frac{K}{29} = 286.7\ \text{J/kg. K}$$

$$\text{Since one kcal} = 4812\ \text{J}$$

$$R \text{ in heat units} = \frac{286.7}{4812}$$

$$= 0.0595\ \text{kcal / kg. K}$$

$$\text{Since gas is diatomic } \frac{C_P}{C_V} = 1.4 = k$$

$$C_P = \frac{kR}{k-1}$$

$$= \frac{1.4\ (0.059)}{(1.4-1)}$$

$$= 0.206\ \text{kcal / kg. K}$$

$$C_V = \frac{R}{k-1}$$

$$= \frac{(0.059)}{(1.4-1)}$$

$$= 0.1475\ \text{kcal/kg. K}$$

Problem 17.2 *Determine the specific weight of air at a pressure of 500 kPa gauge and 25° C. Assume R = 286.7 J/kg. K.*

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Solution

Since we are determining the specific weight, we have to convert R into weight units.

$$\text{Since one } J = 1 \text{ Nm},$$

$$R = \frac{(286.7 \times 1)}{9.81}$$

$$R = 29.2 \text{ Nm/N.K}$$

$$\gamma = \frac{p_{(abs)}}{RT_{(abs)}}$$

$$= \frac{(500 \times 1000 + 101.3 \times 1000)}{29.2\ (25 + 273)}$$

$$\gamma = 69 \text{ N/m}^3$$

Problem 17.3 *Determine the velocity of the bullet fired in air at 20° C given that Mach cone angle is 30°. Assume all other values not given.*

Solution

Assuming adiabatic expansion,

$$\text{Sonic velocity } C = \sqrt{kRT}$$

Since the bullet is fired in air R may be assumed = 286.7 J/kg. K and k may be assumed = 1

$$T = (20 + 273) = 293 \text{ K}$$

$$\text{Sonic velocity } C = \sqrt{1.4\ (286.7)\ (293)} = 343 \text{ m/s}$$

$$\text{Mach cone angle sine } \theta = \left(\frac{1}{M}\right) = \left(\frac{1}{\frac{v}{C}}\right) = \left(\frac{C}{v}\right) = \left(\frac{343}{v}\right)$$

$$\text{sine } 30 = \left(\frac{343}{v}\right)$$

$$0.5 = \left(\frac{343}{v}\right)$$

$$v = 686 \text{ m/s}$$

Problem 17.4 *A sphere (ρ = 1750 kg/m^3) falling freely through an air attains the velocity of sound. Determine its diameter if the density of air at 20 C is 1.225 kg/m^3 and the drag coefficient = 0.45. Assume suitable values where required.*

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Solution

R for air may be assumed = 286.7 J/kg. K

The ratio of specific weights k may be assumed = 1.4

$$T = 273 + 20 = 293 \text{ K}$$

$$\text{Sonic velocity } C = \sqrt{kRT}$$

$$= \sqrt{1.4\,(286.7)\,(293)} = 343 \text{ m/s}$$

= velocity of the sphere

Since for a sphere falling freely Weight of the sphere = the drag force,

$$\text{(Specific weight)} \times \text{(volume of the sphere)} = C_D \times \left(\frac{1}{2}\rho A v^2\right)$$

$$(1750 \times 9.81) \times \frac{\pi D^3}{6} = (0.45) \times \frac{1}{2}\,(1.225 \times 0.7854 D^2 \times 343^2)$$

$$D = 2.3 \text{ m}$$

Problem 17.5 *Gas is flowing at the rate of 1.5 kg/s through a stepped pipe. If at the section where the diameter is 50 mm, the pressure is 3 × 10⁵ Pa (abs) and the temperature is 25° C and at the section where the diameter is 30 mm the pressure is 5 × 10⁵ Pa (abs) determine the velocities at these sections assuming Isothermal conditions. Assume R = 286.7 J/kg K.*

Solution

At the section where the diameter D_1 = 50 mm = 0.05 m,

$$\text{Area } A_1 = 0.7854\,(0.05^2)$$

$$= 0.00196 \text{ m}^2$$

and $$\text{Temperature } T_1 = (25 + 273)$$

$$= 298 \text{ K}$$

and $$\text{Pressure } p_1 = 3 \times 10^5 \text{ Pa}$$

At the section where the diameter D_2 = 30 mm = 0.03 m,

$$\text{Area } A_2 = 0.7854\,(0.03^2)$$

$$= 0.00070 \text{ m}^2$$

and $$\text{Pressure } p_2 = 5 \times 10^5 \text{ Pa}$$

From the characteristic gas equation $\rho_1 = \frac{p_1}{RT_1} = \frac{(3 \times 10^5)}{(286.7)\,(298)}$

$$\rho_1 = 3.5 \text{ kg/m}^3$$

At section 2, $$\rho_2 = \frac{p_2}{RT_2} = \frac{(5 \times 10^5)}{(286.7)\,(298)}$$

$$= 5.85 \text{ kg/m}^3$$

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(Here since isothermal conditions are assumed $T_1 = T_2$)

Mass flow rate (m) is given to be = 1.5 kg/s

From the continuity equation, $\rho_1 A_1 v_1 = \rho_2 A_2 v_2$

i.e., $1.5 = \rho_2 A_2 v_2$

$= 5.85 \times 0.00070 \times v_2$

Hence $v_2 = 366.3$ m/s

Similarly, $1.5 = \rho_1 A_1 v_1$

$= 3.5 \times 0.00196 \times v_1$

Hence $v_1 = 218.65$ m/s

Problem 17.6 *Air stored in a receiver at a pressure of 10×10^5 Pa (gauge) and at a temperature of 25° C is fitted with a nozzle whose throat diameter is 5 mm. Determine the weight flow rate through the nozzle, the exit velocity and the Mach number if the pressure outside is 1 atmosphere. Assume R = 286.7J/kg.K.*

Solution

For air ($k = 1.4$) the critical pressure ratio is determined from Eq. (17.30)

$$(r)_c = \left[\frac{2}{(k+1)}\right]^{\frac{k}{(k-1)}} = \left[p_2 / p_1\right]$$

$$p_2/p_1 = \left[\frac{2}{(1.4+1)}\right]^{\frac{1.4}{1.4-1}} = 0.528$$

$$\text{The actual pressure ratio} = \frac{P_{atm}}{p_{1(abs)}}$$

$$= \frac{10^5}{[(10 \times 10^5) + 10^5]} = 0.090$$

(1 atm $\approx 10^5$ Pa)

As the actual pressure ratio is less than the critical pressure ratio we use Eq. (17.28 a) for computing the flow rate.

$$W = A_2 \sqrt{\left[\frac{2gk}{k+1}\right](p_1\ \gamma_1)\left[\frac{2}{(k+1)}\right]^{\frac{2}{k-1}}} \quad \ldots(a)$$

In Eq. (a) above the unknowns must be determined.

A_2 = area at the throat of the nozzle

$= 0.7854\,(0.005)^2$

$= 1.96 \times 10^{-5}$ m^2.

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Since we are computing the weight flow rate we have to express the value of R in terms specific to the weight flow rate. Thus, the given value of R for air = 286.7 J/kg. $K \equiv 29.2$ N.m/N.K.

$$\gamma_1 = \frac{p_1}{RT_1}$$

$$= \frac{[(10 \times 10^5) + 10^5]}{(29.2)\,(273 + 25)}$$

$$= 126.4 \text{ N/m}^3$$

Substituting these values in Eq. (a) above,

$$W = 1.96 \times 10^{-5} \sqrt{\left[2\frac{(9.81)\,(1.4)}{(1.4+1)}\right](11 \times 10^5)\,(126.4)\left[\frac{2}{(1.4+1)}\right]^{\frac{2}{1.4-1}}}$$

$$W = 0.49 \text{ N/s}$$

$$\text{Sonic velocity } C = \sqrt{(k \times R \times T_2)} \qquad \ldots \text{(b)}$$

$$\frac{T_2}{T_1} = \frac{2}{(k+1)}$$

$$= \frac{2}{(1.4+1)} = 0.833$$

Therefore $T_2 = 0.833\, T_1$

Substituting in (b)

$$C = \sqrt{(k \times R \times T_2)}$$

$$= \sqrt{(1.4) \times (286.7) \times (0.833 \times 298)}$$

Sonic velocity $c = 315.65$ m/s

The flow velocity can be computed using Eq. (17.26)

$$v = \sqrt{\left[\frac{2k}{k-1}\right]\left(\frac{p_1}{\rho_1}\right)\left[1 - r^{\frac{(k-1)}{k}}\right]}$$

$$v = \sqrt{\left[\frac{2(1.4)}{1.4-1}\right]\left(\frac{11 \times 10^4}{12.87}\right)\left[1 - 0.090^{\frac{(1.4-1)}{1.4}}\right]}$$

Here, $\rho_1 = \frac{p_1}{RT_1}$

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$$= \frac{\left[(11\times10^5)\right]}{(286.7)(25+273)}$$

$$= 12.87 \text{ kg/m}^3$$

$$v = 544.75 \text{ m/s}$$

$$\text{Mach number } M = \frac{v}{C}$$

$$= \frac{544.75}{315.65} = 1.72$$

Problem 17.7 *Air flows through a Venturi meter of inlet diameter 100 mm and throat diameter 30 mm at a pressure of 100 Pa and 80 Pa respectively. The density of air at the inlet is 1.3 kg / m³. Determine, the volume flow rate. Assume adiabatic conditions.*

Solution

Applying Bernoulli's Eq. (17.20) to points between the inlet and the throat considering them as points 1 and 2 respectively,

$$\left[\frac{k}{k-1}\right]\left[\frac{p_1}{\rho_1}-\frac{p_2}{p_2}\right]+\left[\frac{v_1^2}{2}-\frac{v_2^2}{2}\right]+gz = 0$$

If we ignore the elevation factor z then the above equation reduces to the form,

$$\left[\frac{k}{k-1}\right]\left[\frac{p_1}{\rho_1}-\frac{p_2}{p_2}\right]+\left[\frac{v_2^2}{2}-\frac{v_1^2}{2}\right] \quad \ldots(a)$$

From the continuity equation $\rho_1 A_1 v_1 = \rho_2 A_2 v_2$

$$v_2 = \frac{(\rho_1 A_1 v_1)}{(\rho_2 A_2)} \quad \ldots(b)$$

$$\rho_2 = \rho_1 \left(\frac{p_2}{p_1}\right)^{\frac{1}{k}}$$

$$= \left(\frac{80}{100}\right)^{\frac{1}{1.4}}$$

$$= 1.106 \text{ kg/m}^3$$

$$A_1 = 0.7854\,(0.1)^2$$

$$= 0.007854 \text{ m}^2$$

and

$$A_2 = 0.7854\,(0.03)^2$$

$$= 0.000706 \text{ m}^2$$

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$$v_2 = \frac{(1.3 \times 0.007854 \times v_1)}{(1.106 \times 0.000706)}$$

Substituting the values of ρ_2, A_1 and A_2 in Eq. (b)

$$v_2 = (13.07\, v_1)$$

Substituting the value of v_2 in (a) with all other relevant values,

$$\left[\frac{1.4}{1.4-1}\right]\left[\frac{100}{1.3} - \frac{80}{1.106}\right] = \left[\frac{(13.07\, v_1)^2}{2} - \frac{v_1^2}{2}\right]$$

$$16.1 = \frac{(v_1)^2}{2}\,[13.07^2 - 1]$$

$$16.1 = (v_1)^2\,(84.9)$$

$$(v_1)^2 = \frac{16.1}{(84.9)}$$

$$v_1 = 0.435 \text{ m/s}$$

$$\text{Volume flow rate} = A_1 \times v_1$$

$$= 0.007854 \times 0.435$$

$$= 3.42 \times 10^{-3}\ \text{m}^3/\text{s}$$

Problem 17.8 *Compressed nitrogen gas is stored in a large receiver fitted with a nozzle of throat diameter 15 mm. The pressure inside the receiver is 1500 kN/m² (gauge) and the temperature is 30°C. Determine the mass flow rate if the gas is released to another closed tank at a controlled pressure of (1). 500 kN/m² and (2) 1000 kN/m². Neglect the discharge coefficient. Assume R = 286.7 J/kg. K.*

Solution

Since Nitrogen is diatomic k may be assumed = 1.4

Critical pressure ratio is determined from Eq. (17.30)

$$(r)_c = \left[\frac{2}{(k+1)}\right]^{\frac{k}{(k-1)}}$$

$$(r)_c = \left(\frac{p_2}{p_1}\right) = \left[\frac{2}{(1.4+1)}\right]^{\frac{1.4}{1.4-1}} = 0.528$$

The actual pressure ratio $\left(\frac{p_2}{p_1}\right) = \frac{500 \times 1000}{[(1500 \times 1000)]} = 0.333$

(1 atm ≈ 10^5 Pa)

(1) When the exit pressure = 500 kN/m²

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Since actual pressure ratio is less than the critical pressure ratio,

Mass flow rate $m_{max} = A_2 \rho_2 v_2$ ...(a)

A_2 = area at the throat of the nozzle

$= 0.7854\,(0.015)^2$

$= 1.76 \times 10^{-4}$ m^2.

$$\rho_2 = \frac{p_2}{RT_2} = \frac{(500 \times 1000 + 10^5)}{(286.7)(303)} = 6.9 \text{ kg/m}^3$$

$$T_2 = T_1 \left(\frac{2}{k+1}\right)$$

$$= 303 \left(\frac{2}{1.4+1}\right) = 252.5$$

Substituting these values in (a)

$$m_{max} = (1.76 \times 10^{-4})\,(6.9)\,\sqrt{kRT_2}$$

(Since the discharge velocity is sonic velocity)

$$m_{max} = (1.76 \times 10^{-4})\,(6.9)\,\sqrt{1.4 \times (286.7) \times (252.5)}$$

$= 0.386$ kg/sec

(2) When the exit pressure = 1000 kN/m^2,

The actual pressure ratio $= \frac{p_2}{p_1} = \frac{(1000)}{(1500)} = 0.666$

Since actual pressure ratio is more than the critical pressure ratio,

We use Eq. (17.27) to determine the mass flow rate.

If we neglect the discharge coefficient the mass flow/unit time is given by equation

$$m = A \sqrt{\left(\frac{2k}{k-1}\right)\left(\frac{p_1}{\rho_1}\right)\left[r^{\frac{2}{k}} - r^{\left(\frac{k+1}{k}\right)}\right]} \quad \text{...(b)}$$

where $r = \frac{p_2}{p_1}$

$$\rho_1 = \frac{p_1}{RT_1}$$

$$= \frac{(1500 \times 1000 + 10^5)}{(286.7)\,(303)}$$

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$$= 18.4 \text{ kg/m}^3$$

$$A = (1.76 \times 10^{-4})$$

$$p_1 \times \rho_1 = (1500 \times 1000 + 10^5) \times 18.4$$

$$= 29440000.$$

If we substitute these values in Eq. (b)

$$m = 1.76 \times 10^{-4} \sqrt{\left(\frac{2(1.4)}{1.4-1}\right)(29440000)\left[(0.666)^{\frac{2}{1.4}} - (0.666)^{\left(\frac{1.4+1}{1.4}\right)}\right]}$$

$$m = 0.618 \text{ kg/s}$$

Problem 17.9 *Weight flow rate of air through a convergent divergent nozzle fitted to a huge tank is 15 N/sec. Pressure and the temperature at the convergent section is 500 kN/m² and 80°C respectively. If the nozzle discharges to a receiving tank at 100 kN/m², determine the diameter of the nozzle at the throat and at the exit and the Mach Numbers.*

Solution

Since the nozzle exit pressure = atmospheric pressure it may be assumed that the expansion is complete and hence the velocity at the throat is sonic velocity. M can therefore be straight away taken =1 at the throat.

Pressure at the throat $p_2 = p_1 \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}$

$$p_2 = (500)\left(\frac{2}{1.4+1}\right)^{\frac{1.4}{1.4-1}} = 264.14 \text{ kN/m}^2$$

$$\frac{T_2}{T_1} = \frac{2}{k+1} = \frac{2}{1.4+1} = 0.833$$

$$T_2 = 0.833 \times (80 + 273) = 294 \text{ K}$$

$$v_2 = C_2$$

$$= \sqrt{kRT_2}$$

$$= \sqrt{1.4 \times 286.7 \times (294)} = 343.5 \text{ m/s}$$

Weight flow rate $W = \rho_2 A_2 v_2$ ...(a)

$$\rho_2 = \frac{p_2}{RT_2}$$

$$= \frac{(264.14 \times 1000)}{286.7 \times (294)}$$

$$= 3.13 \text{ kg/m}^3$$

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Substituting these values in (a)

$$15 = 9.81(3.13) \times 0.7854\,(d_2)^2 \times 343.5$$

Diameter at the throat $d_2 = 0.0425$ m

$= 42.5$ mm

To determine the Mach number at the exit we have to first determine v_3 and C_3. From Eq. (17.26)

$$v = \sqrt{\left[\frac{2k}{k-1}\right]\left(\frac{p_1}{\rho_1}\right)\left[1 - r^{\frac{(k-1)}{k}}\right]}$$

$$v_3 = \sqrt{\left[\frac{2k}{k-1}\right](RT_1)\left[1 - \left(\frac{p_3}{p_1}\right)^{\frac{(k-1)}{k}}\right]}$$

$$v_3 = \sqrt{\left[\frac{2(1.4)}{1.4-1}\right](286.7 \times 353)\left[1 - \left(\frac{100}{500}\right)^{\frac{(1.4-1)}{1.4}}\right]}$$

$$v_3 = 505 \text{ m/sec}$$

$$M_3 = \frac{v_3}{C_3} = \frac{505}{\sqrt{kRT_3}} \qquad \ldots(b)$$

$$\frac{T_3}{T_2} = \left(\frac{p_3}{p_2}\right)^{\frac{k-1}{k}}$$

$$= \left(\frac{100}{264.14}\right)^{\frac{1.4-1}{1.4}} = 0.758$$

$$T_3 = 294 \times 0.758$$

$$= 222.9 \text{ K}$$

From (b), $$M_3 = \frac{505}{\sqrt{1.4 \times 286.7 \times 222.9}} = 1.68$$

To determine the area of the nozzle at the exit, we use the same method adopted to obtain the diameter at the throat.

Weight flow rate $W = \rho_3 A_3 v_3$ …(c)

$$\rho_3 = \frac{p_3}{RT_3}$$

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$$= \frac{(100 \times 1000)}{286.7 \times (222.9)}$$

$$= 1.56 \text{ kg/m}^3$$

Substituting these values in (a)

$$15 = 9.81(1.56) \times 0.7854\,(d_3)^2 \times 505$$

Diameter at the throat $d_2 = 0.0425 \text{ m} = 0.049 \text{ m}$

$= 49.7 \text{ mm}$

Problem 17.10 *Pitot tube fixed to an airplane records a stagnation pressure = 80 kN/m² (abs.). If the air density and the temperature at the altitude at which the plane is flying is 0.088 kg / m³ and – 56.5° C respectively, determine the velocity of the plane and the stagnation temperature. Assume values not given.*

Solution

Air pressure at the altitude at which it is flying

$$p = T \times \rho R$$

$$= (273 - 56.5) \times 0.088 \times 286.7$$

$$= 5.462 \text{ kPa}$$

$$\text{Sonic speed } C = \sqrt{\frac{(kp)}{\rho}}$$

$$= \sqrt{\frac{(1.4 \times 5462)}{0.088}} = 294.7 \text{ m/s}$$

Stagnation pressure p_s is given by Eq. (17.31)

$$p_S = p_1 \left\{ \left[1 + \frac{(k-1)}{2} \right] (M_1)^2 \right\}^{\frac{(k)}{k-1}}$$

Here $\left[M_1 = \frac{v_1}{C_1} \right]$

$$p_S = 5462 \left\{ \left[1 + \frac{(1.4-1)}{2} \right] (M_1)^2 \right\}^{\frac{(1.4)}{1.4-1}}$$

$$80 \times 1000 = 5462\,\{1.2\,(M_1)^2\}^{3.5}$$

$$M_1 = 1.34$$

Velocity of the plane $= v$

$$= M_1 \times C_1$$

$$= 1.34 \times 294.7$$

$= 394.7 \text{ m/s}.$

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Stagnation temperature is given by Eq. (17.34)

$$T_S = T_1\left[1+\frac{(k-1)}{2}\right](M_1)^2\Bigg]$$

$$= 216.5\left\{\left[1+\frac{(1.4-1)}{2}\right](1.34)^2\right\}$$

$$= 466.5 \text{ K}$$

Problem 17.11 *Air is discharged from a large reservoir fitted with a Laval nozzle under high pressure to create supersonic flow conditions. The diameter at the throat is 0.8 m and the Mach number at the exit where the diameter is constant is 5. Determine the diameter at the exit.*

Solution

For a Laval nozzle the area ratios between the throat (A_T) where the $M = 1$ and the area A at any section where M is either greater or less than1 is given by Eq. (17.35)

$$\frac{A}{A_T} = \frac{1}{M}\left[2+\frac{(k-1)M^2}{k+1}\right]^{\frac{(k+1)}{2(k-1)}}$$

$$\frac{A}{0.7854(0.8)^2} = \frac{1}{5}\left[2+\frac{(1.4-1)5^2}{1.4+1}\right]^{\frac{(1.4+1)}{2(1.4-1)}}$$

Area at the exit A = 12.56 m^2

Diameter at the exit = 4 m

Problem 17.12 *If the air density and the temperature at the altitude at which an air plane is flying is 0.088 kg/m^3 and – 56.5° C respectively and if the speed of the plane is 1000 km/hour determine the pressure, density and temperature at the stagnation point.*

Solution

$$P = \rho RT$$

$$= 0.088 \times 286.7 \times (273 - 56.5) = 5462 \text{ Pa}$$

Sonic velocity $C = \sqrt{kRT}$

$$= \sqrt{1.4 \times 286.7 \times 216.5} = 294.78 \text{ m/s}$$

$$v = \frac{(1000 \times 1000)}{3600} = 277.7 \text{ m/s}$$

$$M = \frac{v}{C} = \frac{277.7}{294.78} = 0.942$$

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From Eq. (17.31) the stagnation pressure is,

$$p_S = p_1 \left\{ \left[1 + \frac{(k-1)}{2} \right] (M_1)^2 \right\}^{\frac{(k)}{k-1}}$$

$$p_S = 5462 \left\{ \left[1 + \frac{(1.4-1)}{2} \right] (0.942)^2 \right\}^{\frac{(1.4)}{1.4-1}}$$

$$p_S = 6808 \text{ Pa}$$

From Eq. (17.32) the stagnation density is,

$$\rho_S = \rho_1 \left\{ \left[1 + \frac{(k-1)}{2} \right] (M_1)^2 \right\}^{\frac{(k)}{k-1}}$$

$$\rho_S = 0.088 \left\{ \left[1 + \frac{(1.4-1)}{2} \right] (0.942)^2 \right\}^{\frac{(1.4)}{1.4-1}}$$

$$\rho_S = 0.106 \text{ kg/m}^3$$

Stagnation temperature is given by Eq. (17.34)

$$T_S = T_1 \left[1 + \frac{(k-1)}{2} \right] (M_1)^2 \Bigg]$$

$$= 216.5 \left\{ \left[1 + \frac{(1.4-1)}{2} \right] (0.942)^2 \right\} = 230.5 \text{ K}$$

Problem 17.13 *An airplane is flying at a speed corresponding to Mach 1.5. The temperature of the surrounding air at this altitude is –56.5° C. Determine the speed of the plane in kilometers / hour. Assume suitably values not given.*

Solution

$$\text{Sonic speed } C = \sqrt{kRT}$$

$$= \sqrt{1.4 \times 286.7 \times (273 - 56.5)}$$

$$= 294.8 \text{ m/s}$$

$$\text{Speed of the plane } v = M \times C$$

$$= 1.5 \times 294.8$$

$$= 442.2 \text{ m/s} = 1591 \text{ km/hr}$$

Problem 17.14 *At an elevation of 15,000 m from mean sea level the temperature of the surrounding air is – 56.5°. A sonic boom was heard from a*

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plane flying at this altitude at M = 2 by an observer while the plane was sighted directly overhead. Determine the distance the plane would have travelled from the point of observation at the instance the sonic boom was heard.

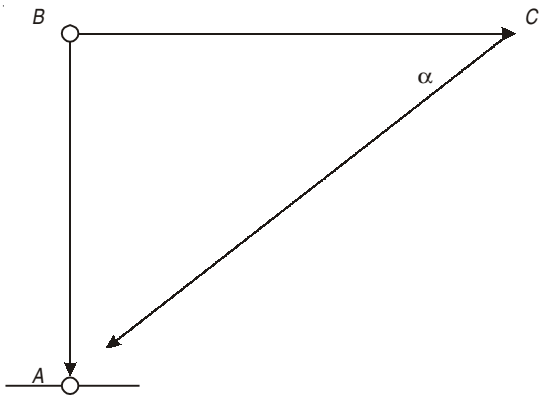


Fig. 17.6

Solution

A is the position of the observer. Distance AB = 15,000 m. BC is the distance the plane would have travelled = $v \times t$. α is the angle of elevation of the plane with respect to the observer.

$$\text{Velocity of the plane} = C \times M$$

$$= \left(\sqrt{1.4 \times 286.7 \times (273 - 56.5)} \times 2\right)$$

$$= 589.5 \text{ m/s}$$

$$\left[\sin \alpha = \frac{1}{M} = \left(\frac{1}{2}\right) = 0.5\right] \qquad [\text{Therefore } \alpha = 30°]$$

$$\tan \alpha = \frac{AB}{BC}$$

$$= \frac{15000}{v \times t}$$

$$\tan 30° = \frac{15000}{(589.5) \times t}$$

$$0.577 = \frac{25.445}{t}$$

The time lapse after the plane was sighted was t = 44 sec

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The distance (D) the plane would have travelled after the sonic boom was heard,

$$D = 589.5 \times 44$$
$$= 25996.2 \text{ m or } 26 \text{ km}$$

Problem 17.15 *Determine the flow conditions before and after a normal shock that occurs at twice the speed of sound if the air pressure and density at the point the shock originates is 54 kPa and 0.7364 kg/m³.*

Solution

From Eq. (17.39) pressure change across a normal shock is,

$$\frac{p_2}{p_1} = \frac{(1 + k\ M_1^2)}{(1 + k\ M_2^2)} \qquad \text{... (a)}$$

Here $M_1 = 2$ and the value of M_2 is obtained using Eq. (17.41)

$$M_2^2 = \frac{(k-1)\ M_1^2 + 2}{(2\ k\ M_1^2) - (k-1)}$$

$$M_2^2 = \frac{(1.4-1)\ 2^2 + 2}{(2 \times 1.4 \times 2^2) - (1.4-1)}$$

$$M_2 = 0.577$$

Substituting the values in (a) above,

$$\frac{p_2}{54} = \frac{(1 + 1.4 \times 2^2)}{(1 + 1.4 \times 0.577^2)}$$

$$\text{from which } p_2 = 243 \text{ kPa}$$

Changes density is given by Eq. (17.40 a)

$$\frac{\rho_2}{\rho_1} = \frac{(k+1)\ M_1^2}{[(k-1)\ M_2^2 + 2]}$$

$$\frac{\rho_2}{0.7364} = \frac{(1.4+1)\ 2^2}{[(1.4-1)\ 0.577^2 + 2]}$$

$$\rho_2 = 3.31 \text{ kg/m}^3$$

Changes in temperature is given by Eq. (17.40)

$$\frac{T_2}{T_1} = \frac{1 + \left[\frac{(k-1)}{2}\right] M_1^2}{1 + \left[\frac{(k-1)}{2}\right] M_2^2} \qquad \text{... (b)}$$

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$$T_1 = \frac{p_1}{\rho_1 R}$$

$$= \frac{(54 \times 1000)}{0.7364 \times 286.7}$$

$$= 255 \text{ K}$$

Substituting in (b) the relevant values,

$$\frac{T_2}{255} = \frac{1 + \left[\frac{(1.4-1)}{2}\right] 2^2}{1 + \left[\frac{(1.4-1)}{2}\right] 0.577^2}$$

$$T_2 = 430 \text{ K}$$

QUESTIONS BANK

Section A

Review Questions

1. Distinguish clearly between Isothermal, Adiabatic and polytropic processes.
2. What does a decrease in pressure ratio actually indicate? Define critical pressure ratio. What will be the velocity at the throat of a converging diverging nozzle when the critical pressure ratio is reached?
3. Define sub-sonic, sonic, supersonic velocity and Mach number. Write down the governing equation for speed of sound in any gas in terms of pressure and specific weight.
4. Explain the principle difference between a convergent-divergent nozzle and the Laval nozzle?
5. A hydraulic jump converts super-critical flow to a sub-critical flow. Name the condition under which something similar occurs during compressible flows.
6. Explain how a disturbance in a compressible fluid is propagated starting from a fluid which is stationary.
7. Define stagnation point. Up to what value of Mach number the compressibility effects are neglected ?
8. Starting from the first principles, obtain individual expressions for all the stagnation properties.
9. Explain how the mass rate of flow varies with the pressure ratio in gases.
10. Define shock strength and explain the conditions under which an oblique shock wave occurs.

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Section B

Multiple Choice Questions

Choose the right answer

1. Continuity equation for compressible flow is
   (a) $a_1v_1 = a_2v_2$ (b) $ra_1v_1 = ra_2v_2$
   (c) $rav = C$ (d) None of these
2. For isothermal process
   (a) $C = \frac{K}{\rho}$ (b) $\sqrt{RT}$
   (c) $\sqrt{rRT}$ (d) None of these
3. For a constant volume process work done is given by
   (a) $p(v_2 - v_1)$ (b) $\frac{(p_1v_2 - p_2 v_2)}{(k - 1)}$
   (c) 0 (d) None of these
4. Mach number is the square root of the ratio of
   (a) the kinetic energy in a fluid to its potential energy
   (b) inertia force to elastic force
   (c) elastic force to inertia force
   (d) None of these
5. For a transonic flow M lies between
   (a) 0.8 – 1.2 (b) 1 – 1.2
   (c) 1 – 1.5 (d) None of these
6. Mach cone angle sin α is
   (a) $\frac{v}{C}$ (b) $\frac{C}{v}$
   (c) $\sqrt{\frac{C}{v}}$ (d) None of these
7. Stagnation point is the point on an immersed body where the
   (a) pressure is zero (b) temperature = 0
   (c) velocity is zero (d) All of these
8. For a sub sonic flow in a convergent nozzle
   (a) $p_2 < p_1$ (b) $p_2 = p_1$
   (c) $p_2 > p_1$ (d) None of these

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9. Value of critical pressure ratio for all di-atomic gases is
   (a) 0.685 (b) 0.520
   (d) 0.525 (e) None of these
10. Normal shock wave occurs when
   (a) subsonic flow suddenly changes to supersonic flow
   (b) when flow is suddenly obstructed due to a bend
   (c) when supersonic flow is suddenly reduced to sonic flow
   (d) None of these

Section C

Problems

1. Given that the molecular weight of oxygen is 32 and carbon dioxide is 44 determine the value of universal gas constant for each.
2. If 10 kg of Oxygen is compressed from 200 kPa at 30° C to 400 kPa at 75° C determine the change in entropy. Assume values not given suitably.
   (*Ans.* – 563.15 J/K)
3. Air is flowing through a stepped pipe from section 1 to section 2 with a velocity of 250 m/s. At section 1 the pressure and temperature is 90 kPa (abs) and 35° C and at section 2 the pressure is 120 kPa (abs). Find the velocity of the fluid at section 2 for isentropic conditions. Assume values not given suitably. (*Ans.* v_2 = 99.8 m/s)
4. Determine, the compressibility correction factor for flow of air given that the Mach no. is 1.5. (*Ans.* 1.69)
5. Determine the speed of an airplane flying at an altitude of 20,000 m where the temperature is % 56.50° C given that M = 1.5. Assume values not given suitably. (*Ans.* 442 m/s)
6. If air at 100° C is discharged from a large tank to atmosphere through a convergent nozzle, determine the minimum pressure in the tank required to produce sonic velocity in the nozzle. Also compute the sonic velocity.
   (*Ans.* 195.6 kPa and 386.9 m/s)
7. An observer on the ground hears the sonic boom 10 seconds after sighting the plane directly overhead. Determine the velocity of the plane if the plane is flying at an altitude of 10,000 m where the temperature is % 49.9° C. (*Ans.* 313.6 m/s)

Answers to Multiple Choice Questions

1. (c)	**2.** (b)	**3.** (c)	**4.** (b)	**5.** (a)
6. (b)	**7.** (c)	**8.** (a)	**9.** (e)	**10.** (c)

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Chapter

18

Dimensional Analysis and Similitude

Dimensional analysis is a technique used to investigate engineering problems especially those involved in fluid mechanics by comparing theoretical results with those obtained experimentally. Dimensional analysis is not possible unless all variables involved in engineering problems are first grouped as dimensionless parameters so that both sides of an equation describing a physical situation is technically correct in all of its forms. For instance although $2 \times 2 = 4$ mathematically makes sense it doesn't make sense at all from the dimensional stand point unless you qualify it with units employed to measure the algebraic numbers so that it leaves none in doubt that you are multiplying metres by metres and not metres by feet.

An equation describing a physical phenomenon is correct only if both sides of the equation are *dimensionally homogeneous*.

Let us examine what it does mean. In the equation $F = ma$, in SI units F is measured in N (Newtons), mass in kg and acceleration a has units, m/s^2. Thus in terms of its units employed the equation can be written as $N = kg \times m/s^2$. Here the units employed are homogeneous since $kg \times m/s^2$ is nothing but N so that both sides of the equation are well balanced. In terms of its dimensions it can be written as $N = MLT^{-2}$. Here M represents mass, L represents metres and T represents the time.

There are several quantities that are non-dimensional such as, strain, friction factor, Froude number, Mach number, Reynolds number, Strouhal number, Relative humidity, Angular displacement, Ratio of specific heats and many more. All of these are ratios whose units are the same and hence they cancel each other out leaving behind merely an arithmetical number.

Only three dimensions mass, length and time (MLT) are treated as fundamental dimensions. All others are derived from these three. Whether it is length, area or

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volume they all can be represented by one dimension L as (L), (L^2) and (L^3) respectively.

We have previously mentioned angular displacement is non-dimensional. Let us examine how it came to be so.

$$\text{Angular displacement} = \text{Arc length (metres)/ Radius (metres)} = \frac{L}{L} = 1.$$

Mass density has no dimension of its own but its dimension can be derived from the three fundamental dimensions.

$$\text{Mass density} = \text{Mass/Volume} = \left(\frac{kg}{L \times L \times L}\right) = kg/m^3$$

or $$kg\ m^{-3} = ML^{-3}.$$

The same way, even the dimensions for power could be derived from the fundamental dimensions.

$$\text{Power} = \text{work done/unit of time}$$

$$= \frac{\text{Force} \times \text{Distance}}{\text{Time}}$$

$$= \frac{(N \times L)}{T} = \frac{(MLT^{-2} \times L)}{T} = ML^2\ T^{-3}.$$

When it comes to *thermodynamic properties* where temperature (Θ) and consequently heat energy (H) is involved they are grouped separately under the HMLTΘ system and not under the MLT system.

Here 'H' is the quantity of heat required to raise mass 'm' of a substance of specific heat 'c' through a temperature difference = Θ.

Hence H = cmΘ or Θ = Hm^{-1} if we neglect '*c*' as a dimensionless quantity. In terms of dimensions under the HMLTΘ system it can be written as [Θ]. Since heat if considered as [MΘ] can be measured in calories and in joules if considered as [$ML^2\ T^{-2}$].

The dimensions of both the fundamental units and the derived units under the MLT system and under the HMLTΘ system are tabulated in Table 18.1. In dimensional analysis we make series of assumptions that lead to an approximate result as compared to algebraic analysis which is precise. Nevertheless the approximate results can be quantified and unknown factors that still remain can be determined experimentally. The dimensions are independent of the system of units employed for measurement. For instance specific weight has units N/m^3 under the SI system and under the FPS system has unit's lb/ft^3 but both have the same dimension [$ML^{-2}\ T^{-2}$].

18.1 USES OF DIMENSIONAL ANALYSIS

Dimensional analysis helps us in ensuring that an equation or a relationship between two physical situations is dimensionally homogeneous.

Dimensional analysis makes it easy for us to convert from one unit to the other.

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Dimensional analysis helps us to determine the dimensions of any physical quantity. The value of any unknown constant can be determined experimentally by comparing the solution obtained by dimensional analysis with the experimental values.

TABLE 18.1 *Dimensions [MLT] of important variables under SI and FPS systems*

Quantity	*SI*	*FPS*	*Dimension*
1. Length	m	ft	L
2. Area	m^2	ft^2	L^2
3. Volume	m^3	ft^3	L^3
4. Second Moment of Area	m^4	ft^4	L^4
5. Strain		– Ratio –	
6. Linear Velocity	m/s	ft/sec	LT^{-1}
7. Angular Velocity		Angle/Time	T^{-1}
8. Linear Acceleration	m/s	ft/s	LT^{-2}
9. Angular Acceleration		—	T^{-2}
10. Volume Flow Rate	m^3/s	ft^3/sec	$L^3\ T^{-1}$
11. Mass		–[F/g]–	M
12. Force		–[m × g]–	MLT^{-2}
13. Mass Density	kg/m^3	lb/ft^3	ML^{-3}
14. Specific Weight	N/m^3	lb/ft^3	$ML^{-2}\ T^{-2}$
15. Specific Gravity		–[Ratio]–	
16. Pressure/Stress	N/m^2	lb/ft^2	$ML^{-1}\ T^{-2}$
17. Modulus of Elasticity	N/m^2	lb/ft^2	$ML^{-1}\ T^{-2}$
18. Impulse	N/s	lb/sec	MLT^{-1}
19. Mass Moment of Inertia	$kg\text{-}m^2$	$lb\text{-}ft^2$	ML^2
20. Linear Momentum	kg-m/s	lb-ft/sec	MLT^{-1}
21. Angular Momentum	$kg\text{-}m^2/s$	$lb\text{-}ft^2/sec$	ML^2T^{-1}
22. Work/Energy	Nm	lb-ft	$ML^2\ T^{-2}$
23. Power	Nm/s	lb-ft/sec	$ML^2\ T^{-3}$
24. Dynamic Viscosity		—	$ML^{-1}\ T^{-1}$
25. Kinematic Viscosity		—	$L^2\ T^{-1}$
26. Surface Tension		—	$M\ T^{-2}$
27. Temperature		–[Θ]–	$L^2\ T^{-2}$
28. Enthalpy		–[H] –	$ML^2\ T^{-2}$
29. Entropy		$–[H\Theta^{-1}]–$	M
30. Coefficient of Thermal Expansion		$–[\Theta^{-1}]–$	$L^{-2}\ T^{2}$

Contd.

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Contd.

	Quantity	*SI*	*FPS*	*Dimension*
31.	Specific Heat		$-[HM^{-1}\Theta^{-1}]-$	
32.	Ratio of Specific Heat		–Ratio–	
33.	Universal Gas Constant (R)		$-[L^2T^{-2}\Theta^{-1}]-$	
34.	Coefficient of Heat Transfer		$-[HL^{-2}T^{-1}\Theta^{-1}]-$	
35.	Mechanical Equivalent of Heat		$-[H^{-1}ML^2T^{-2}]-$	

* The quantities shown in brackets from serial numbers 27 to 35 relate to dimensions of thermodynamic properties under HMLT Θ system.

There are two commonly employed theorems governing the dimensional analysis of problems. These are,

1. Rayleigh's method. This method is good if the number variables are < 4 or 5.
2. *Buckingham's π theorem.* Under the Rayleigh method a dependent variable such as x is expressed in terms of independent variables such as x_1, x_2, x_3, x_4 as under.

$$x = f(x_1, x_2, x_3, x_4) \quad \ldots(a)$$

In order to maintain dimensional homogeneity between RHS and LHS of equation (a) we raise the powers of the independent variables so that the equation (a) now takes the form,

$$x = k\,(x_1^{\,a}, x_2^{\,b}, x_3^{\,c}, x_4^{\,d}) \quad \ldots(b)$$

Here k is a non-dimensional constant.

The dimensions of each of the quantities on either side of the equation are first written down and the sum of the exponents of these dimensions, are then equated to obtain a dimensionless group controlling a phenomenon.

Since Rayleigh's method is simpler compared to Buckingham's π theorem let us consider it first and solve a few problems.

18.1.1 Buckingham's π Theorem

This is also called dimensional analysis by group method. The theorem states *that ' if there are n variables in a dimensionally homogeneous group and if these variables contain m primary dimensions then the variables can be grouped in to (n – m) non-dimensional parameters. Buckingham's π theorem is so called because π is non-dimensional.*

This method though more popular and often simpler than other methods is suitable only for conditions that involve MLT. Where thermodynamic properties like heat and temperature are involved this method may be cumbersome as it increases the number of variables involved.

The theorem is best understood by solving some practical problems but the following guidelines have to be borne in mind while solving problems.

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The *m* primary dimensions earlier referred to might have variables that are repeated. It should be ensured that these repeated variables do not form dimensionless groups among themselves and that no two repeated variables have the same dimensions. The repeated variables should be so chosen that each variable represents one important fluid property such as linear dimension, density, viscosity etc.

18.4 SIMILITUDE

A prototype of an aircraft or a ship is the full scale version of a developmental effort which is ready for testing and field trails. Students should appreciate that the performance of an air-foil or hydro-foil sections of the prototype cannot be tested under actual conditions unless at great costs. It is for this reason a 'mock up' or a scale model of the actual prototype is first built and tested under actual conditions. The scale model though is a miniature version of the prototype, is geometrically, dynamically and kinematically similar to the prototype. *Similitude is in short a method adopted to assess the performance of the prototype from the scale models which are similar in all respects to the prototype except for overall size.*

In addition to economical advantage, laboratory testing of models offer other advantages too. Models can redesigned to suit engineering requirements any number of times at short notice, the performance of the prototype during field trails can be assessed far in advance and models can provide easy answers to some of the most complicated theoretical problems. Models are invariably built and studied particularly in aircraft, ship building and hydraulic machinery industry such as those manufacturing water, steam or gas turbines, pumps and compressors. They are also helpful in civil and structural engineering works to study the effect of wind forces on tall structures.

Geometric similarity. This means that geometric properties such as Length, Breadth, Height, Diameter, area and volume of both the prototype and the model must exhibit the following relationship.

$$\frac{L_m}{L_P} = \frac{B_m}{B_P} = \frac{H_m}{H_P} = \frac{D_m}{D_P} = \text{scale ratio } L_R \quad \ldots(18.1)$$

Similarly areas and volumes of the prototype and the model must exhibit the following relationship.

$$\frac{A_m}{A_P} = (\text{scale ratio})^2 = [L_R]^2 \quad \ldots (18.2)$$

$$\frac{V_m}{V_P} = (\text{scale ratio})^3 = [L_R]^3 \quad \ldots (18.3)$$

Kinematic similarity. This means the conditions of motion such as the direction of motion, velocity and acceleration of both the prototype and the model should exhibit the following similarities in addition to being geometrically similar.

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$$\frac{v_m}{v_P} = \text{Velocity ratio} = v_R \qquad \ldots(18.4)$$

$$\frac{a_m}{a_P} = \text{Acceleration ratio} = a_R \qquad \ldots(18.5)$$

Dynamic similarity. If in addition to geometric and kinematic similarities if the forces that the prototype and the scale model are exposed to are similar regardless of the nature of the force, whether it is gravity (g), viscosity (v), surface tension (s) elasticity (e) or inertial (I) then the two systems are assumed to exhibit dynamic similarity.

For instance if it is gravity the following relationship must exist

$$\frac{F_{gm}}{F_{gP}} = \text{constant.}$$

Likewise if it is inertial the following relationship must exist

$$\frac{F_{Im}}{F_{IP}} = \text{constant}$$

In general therefore for dynamic similarity the two systems must exhibit the following relationships.

$$\frac{F_{gm}}{F_{gP}} = \frac{F_{Im}}{F_{IP}} = \frac{F_{Vm}}{F_{VP}} = \frac{F_{Sm}}{F_{SP}} = \text{constant} \qquad \ldots(18.6)$$

We are already familiar with dimensionless flow parameters such as Reynolds number, Froude number, and Mach number. Two more important parameters exit that are relevant for purposes of examining the similarities between the model and the prototype. These are, the *Weber number* (W) and the *Euler number* (E).

Weber number $W = \left[\sqrt{\frac{F_I}{F_S}}\right]$. It is basically a relationship between inertial forces to surface tension.

Euler number $E = \left[\sqrt{\frac{F_I}{F_S}}\right]$. It is basically a relationship between inertial forces to pressure forces.

Weber number is useful while analyzing capillary action and atomization whereas Euler's number helps in analyzing problems where a pressure gradient exits between two flow points. It is mandatory that for flow similarities to exist each of these dimensionless flow parameters must be constant for both the scale model and the prototype. For instance $(R_E)_{\text{model}} = (R_E)_{\text{Proto}}$.

(where specifically not mentioned the subscript (m) refers to the model and (P) to prototype)

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Since Reynolds number is a ratio of inertial forces to viscous force it is used where viscous forces are predominant such as in completely submerged bodies, flow through closed conduits and where viscous flow is involved.

Thus, $$\left[\frac{v\,D}{\upsilon}\right]_m = \left[\frac{v\,D}{\upsilon}\right]_P \quad \ldots(18.7)$$

where gravity is predominant Froude number (F_R) which is a ratio of inertial force to gravity is used where resistance due to wave forces are involved, in open channel flows and in hydraulic structures such as spill ways and off shore structures.

Thus, $$\left[\frac{v}{\sqrt{gL}}\right]_m = \left[\frac{v}{\sqrt{gL}}\right]_P \quad \ldots(18.8)$$

In addition, we have the time ratio (T_R), acceleration ratio (a_R).

$$T_R = \left[\frac{L_R}{v_R}\right] \text{ and } a_R = \left[\frac{v_R}{T_R}\right] \quad \ldots(18.9)$$

TYPICAL PROBLEMS AND SOLUTIONS

Problem 18.1 *Derive an expression for the drag force on a sphere moving through a fluid medium by dimensional analysis.*

Solution

The drag force on a sphere is a function of its diameter, velocity, density of the fluid medium and the dynamic viscosity.

Thus $F_D = f(v, D, \rho\, \mu)$ Applying Rayleigh's logic we now raise the powers of the independent variables

$$F_D = k\,(v^a, D^b, \rho^c\, \mu^d)$$

Now we write down the dimensions of each of these quantities

$$MLT^{-2} = k\,[(LT^{-1})^a, (L)^b, (ML^{-3})^c\,(ML^{-1}\,T^{-1})^d] \quad \ldots(a)$$

1. Compare the exponents of M on either side of the Eq. (a)

$$M = 1 = c + d$$

2. Compare the exponents of L on either side of the Eq. (a)

$$L = 1 = a + b - 3c - d$$

3. Compare the exponents of T on either side of the Eq. (a)

$$T = -2 = -a - d$$

or $$a + d = 2$$

Since there are four variables and three equations the only way to find a solution is by expressing the any three variables in terms of the fourth variable making sure that we select the most important quantity as the fourth variable. Obviously in the present case dynamic viscosity is the most important quantity and so we express the other three in terms of dynamic viscosity.

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Re-arranging the values of MLT above in terms of d

$$a = 2 - d$$
$$b = 2 - d$$
$$c = 1 - d$$

Substituting the values of a, b, c in the equation for F_d,

$$F_D = k\,[(v)^{2-d}, (D)^{2-d}, (\rho)^{1-d}\,(\mu)^{d}]$$
$$F_D = k\,[(v)^2\,(D)^2\,(\rho)\,[\,D^{-d} \times v^{-d} \times \rho^{-d} \times \mu^{d}]$$
$$F_D = k\,[(v)^2\,(D)^2\,(\rho)\left[\frac{\mu}{D\,v\,\rho}\right]^{d}$$
$$F_D = [(v)^2\,(D)^2\,(\rho)]\,\varphi\left[\frac{\mu}{D\,v\,\rho}\right]^{d}.$$

Problem 18.2 *Derive an expression for the pressure drop in a smooth horizontal tube by dimensional analysis. Assume laminar flow.*

Solution

The pressure drop (Δp) through a horizontal tube depends on the length of the pipe (L), the inside diameter of the tube (d), the viscosity of the fluid (μ) and the flow velocity (v). Therefore,

$$\Delta p = k\,[(L)\,(d)\,(\mu)\,(v)]$$
$$\Delta p = k\,[(L)^a\,(d)^b\,(\mu)^c\,(v)^d]$$

In terms of MLT,

$$ML^{-1}T^{-2} = k\,[(L)^a\,(L)^b\,((ML^{-1}T^{-1})^c\,(LT^{-1})^d] \quad \ldots(a)$$

Compare the exponents of ML and T on either side of the Eq. (a)

[M = 1 = c], [L = –1 = $a + b - c + d$] and [T = –2 = $- c - d$]

Since [c = 1], and [$c + d$ = 2] it means d = 1

Since [–1 = $a + b - c + d$] it means [$a = -(b + 1)$]

Substituting the values in Eq. (a)

$$\Delta p = k\,[(L)^{-(b+1)}\,(L)^b\,((ML^{-1}T^{-1})\,(LT^{-1})]$$
$$\Delta p = k\,[(L)^{-(b+1)}\,(L)^b\,((ML^{-1}T^{-1})\,(LT^{-1})]$$
$$\Delta p = k\,[(L)^{-(b+1)}\,(L)^b\,(\mu)\,(v)]$$
$$\Delta p = k\left[\frac{(\mu)\,(v)}{L}\right]\left(\frac{D}{L}\right)^{b}$$
$$\Delta p = \left[\frac{(\mu)\,(v)}{L}\right]\varphi\left(\frac{D}{L}\right) \text{ is the required expression.}$$

Problem 18.3 *Using dimensional analysis, derive an expression for the skin friction drag experienced during motion by a ship floating in sea waters.*

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Solution The skin friction drag (F_D) experienced by a ship in motion is a function of its length (L), velocity (v), density of sea water (ρ), the dynamic viscosity (μ) of the fluid and the acceleration due to gravity (g)

Thus $F_D = f\ (v, L, \rho, \mu \text{ and } g)$. Applying Rayleigh's logic we now raise the powers of the independent variables.

$F_D = k\ (v^a, L^b, \rho^c\ \mu^d g^e)$ Now we write down the dimensions of each of these quantities

$$MLT^{-2} = k[(LT^{-1})^a, (L)^b, (ML^{-3})^c\ (ML^{-1}\ T^{-1})^d\ (LT^{-2})^e] \quad \ldots(a)$$

1. Compare the exponents of M on either side of the Eq. (a)
$$M = 1 = c + d$$
2. Compare the exponents of L on either side of the Eq. (a)
$$L = 1 = a + b - 3c - d + e$$
3. Compare the exponents of T on either side of the Eq. (a)
$$T = -2 = -a - d - 2e$$

In this case there are five variable but only three equations. Values of two of the variables should therefore be expressed in terms of the other three. This means we determine the value of b, c and a in terms of d and e.

$$c = (1 - d)$$
$$a = (2 - d - 2e) \text{ and } b = (2 - d + e)$$

Substituting the values of c, a and b in equation,

$$F_D = k\ (v^a, L^b, \rho^c\ \mu^d g^e)$$
$$F_D = k\ [(v)^{2-d-2e}, (L)^{2-d+e}, (\rho)^{1-d}\ (\mu)^d (g)^e]$$
$$F_D = k\ [(v)^2\ (L)^2\ (\rho)]\ [(\rho)^{-d}\ (\mu)^d L^{-d+e}\ (v)^{-d-2e}\ (g)^e]$$
$$F_D = k\ [(v)^2\ (L)^2\ (\rho)]\ [(\rho)^{-d}\ (\mu)^d L^{-d}\ (v)^{-d}]\ [(L)^e\ (v)^{-2e}\ (g)^e]$$
$$F_D = k\left\{[(v)^2(L)^2(\rho)]\left[\left(\frac{\mu}{L\rho v}\right)^d\right]\left[\left(\frac{Lg}{v^2}\right)^e\right]\right\}$$
$$F_D = [v^2\ L^2\ \rho]\ \varphi\left[\left(\frac{\mu}{L\rho v}\right)\right]\left(\frac{Lg}{v^2}\right)$$

is the required expression.

Problem 18.4 *Let us try to solve Problem (3) using Buckingham's π theorem.*

Solution
$$F_D = \varphi\ (v, L, \rho, \mu \text{ and } g) \quad \ldots(a)$$
$$0 = \varphi\ (F_D\ v, L, \rho, \mu \text{ and } g) \quad \ldots(b)$$

Since there are 6 variables in all $n = 6$.

Now let us write the dimensions for each of the 6 variables.

$[F_D = MLT^{-2}]$, $[v = LT^{-1}]$, $[L = L]$, $[\rho = ML^{-3}]$, $[\mu = ML^{-1}\ T^{-1}]$ and $[g = LT^{-2}]$

Fundamental dimensions however are 3 (MLT).

Hence $(n - m) = (6 - 3) = 3$

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= number of dimensionless terms or π terms.

Equation (b) may re-written as,

$$\varphi_1 (\pi_1, \pi_2, \pi_3) = 0$$

The next step involves selection of variables that are repeated as per the criteria already mentioned. Number of variables to be selected are 3 since the given variables are 3 dimensional consisting of only (MLT). Since F_D is a dependent variable it should be left out. Again the three variables must be so selected that each represents one property viz, geometric (length or diameter), flow (velocity or volume flow rate) and fluid property (density or viscosity). Let us choose L v and ρ as repeating variables whose dimensions L, LT^{-1} and ML^{-3} satisfy the condition that together they should contain all the three basic dimensions and yet not form a dimensionless group on their own.

$$\pi_1 = [M^0 L^0 T^0] = F_D L^a v^b \rho^c$$
$$[M^0 L^0 T^0] = [MLT^{-2}] [L]^a [LT^{-1}]^b [ML^{-3}]^c$$

Now we can equate the powers of MLT

$$\left.\begin{array}{l} 1+c=0 \\ 1+a+b-3c=0 \\ -2-b=0 \end{array}\right] \text{ which yields } c = -1,\ b = -2 \text{ and } a = -2$$

$$\pi_1 = F_D L^{-2} v^{-2} \rho^{-1}$$

$$\pi_1 = \left[\frac{F_D}{L^2 v^2 \rho^1}\right] \qquad \ldots(a)$$

Similarly,

$$\pi_2 = g L^a v^b \rho^c$$
$$[M^0 L^0 T^0] = [LT^{-2}] [L]^a [LT^{-1}]^b [ML^{-3}]^c$$

$$\left.\begin{array}{l} c = 0 \\ 1+a+b-3c=0 \\ -2-b=0 \end{array}\right] \text{ which yields } a = 1,\ b = -2 \text{ and } c = 0$$

Hence $$\pi_2 = g L v^{-2} \rho^0 = \left[\frac{g L}{v^2}\right] \qquad \ldots(b)$$

Similarly,

$$\pi_3 = \mu L^a v^b \rho^c$$
$$[M^0 L^0 T^0] = [ML^{-1} T^{-1}] [L]^a [LT^{-1}]^b [ML^{-3}]^c$$

$$\left.\begin{array}{l} 1+c=0 \\ 1+a+b-3c=0 \\ -2-b=0 \end{array}\right] \text{ which yields } a = -1,\ b = -1 \text{ and } c = -1$$

Hence $$\pi_3 = \mu L^{-1} v^1 \rho^{-1}$$

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$$= \left[\frac{\mu}{L v \rho}\right] \qquad \ldots(c)$$

Hence, $$\left[\frac{F_D}{L^2 v^2 \rho}\right] = f\left[\frac{gL}{v^2}, \frac{\mu}{L v \rho}\right]$$

$$\left[\frac{F_D}{L^2 v^2 \rho}\right] = \varphi\,[F_R, R_E]$$

where F_R is Froude number and R_E is Reynolds's number.

Problem 18.5 *Derive the dimensionless form of the discharge equation for a V-notch.*

Solution

The discharge Q through a V-notch depends on the available head (H), gravity (g), flow velocity (v), and the notch angle θ.

$$Q = \varphi\,(v, H, g, \theta) \qquad \ldots(a)$$

$$0 = \varphi\,(Q, v, H, g, \theta) \qquad \ldots(b)$$

Since there are 5 variables in all $n = 5$.

Now let us write the dimensions for each of the 5 variables.

$[Q = L^3 T^{-1}]$, $[v = LT^{-1}]$, $[H = L]$, $[g = LT^{-2}]$ $[\theta]$

Fundamental dimensions however are 2 (LT). Hence $(n - m) = (5 - 2) = 3$ = number of dimensionless terms or π terms including angle θ.

Equation (b) may re-written as,

$$\varphi_1\,(\pi_1, \pi_2, \pi_3) = 0$$

The next step involves selection of variables that are repeated as per the criteria already mentioned. Number of variables to be selected are 2 since the given variables are 2 dimensional consisting of only (LT). Since Q is a dependent variable it should be left out. Again the two variables must be so selected that each represents one property viz. geometric (length or diameter), flow (velocity or gravity) and fluid property (density or viscosity). Let us choose H and g as repeating variables whose dimensions L, LT^{-1} satisfy the condition that together they should contain all the two basic dimensions and yet not form a dimensionless group on their own.

$$\pi_1 = [L^0 T^0] = Q\, L^a g^b$$

$$[L^0 T^0] = [L^3 T^{-1}]\,[L]^a\,[LT^{-2}]^b$$

Now we can equate the powers of LT

$$\left.\begin{matrix} 3 + a + b = 0 \\ -1 - 2b = 0 \end{matrix}\right] \text{ which yields, } 2b = -1 \text{ and } a = -\frac{5}{2}$$

$$\pi_1 = [Q L^a g^b]$$

$$= Q\, L^{-\frac{5}{2}} g^{-\frac{1}{2}}$$

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$$\pi_1 = \left[\frac{Q}{\left(H^{\frac{5}{2}}\right)\left(\sqrt{g}\right)}\right] \quad \ldots(a)$$

Similarly,

$$\pi_2 = [L^0\ T^0] = v\ L^a\ g^b$$
$$[L^0\ T^0] = [L\ T^{-1}]\ [L]^a\ [LT^{-2}]^b$$

Now we can equate the powers of LT

$$\left.\begin{array}{l} 1+a+b=0 \\ -1-2b\ =0 \end{array}\right] \text{ which yields, } 2b = -1 \text{ and } a = -\frac{1}{2}$$

$$\pi_2 = [v\ L^a\ g^b]$$
$$= v\,L^{-\frac{1}{2}} g^{-\frac{1}{2}}$$

$$\pi_2 = \left[\frac{v}{\sqrt{[Hg]}}\right] \quad \ldots(b)$$

$$\pi_3 = \theta \quad \ldots(c)$$

Since $\varphi_1\ (\pi_1,\ \pi_2,\ \pi_3,) = 0,$

$$\varphi_1\left(\frac{Q}{H^{\frac{5}{2}}\sqrt{g}}\right), \left[\frac{v}{\sqrt{(H\ g)}}\right], \theta, = 0$$

$$\left(\frac{Q}{H^{\frac{5}{2}}\sqrt{g}}\right) = \varphi\left[\frac{v}{\sqrt{(H\ g)}}\right], \theta$$

Problem 18.6 *Derive an expression in the dimensionless form for pressure drop across circular orifice under turbulent flow conditions.*

Solution

The pressure drop across a circular orifice under turbulent flow conditions depends on the orifice diameter (D), the head causing the flow (H), the flow velocity (v), viscosity (μ), density (ρ) and the roughness factor (k). Thus,

$$\Delta p = \varphi\ (D, v, H, \rho, \mu, k) \quad \ldots(a)$$
$$0 = \varphi\ (\Delta p, D, v, H, \rho, \mu, k) \quad \ldots(b)$$

Since there are 7 variables in all $n = 7$.

Now let us write the dimensions for each of the 7 variables.

$[\Delta p = ML^{-1}\ T^{-1}]$, $[v = LT^{-1}]$, $[H = L]$, $[\rho = ML^{-3}]$ $[\mu = ML^{-1}\ T^{-1}]$ $[L]$

Fundamental dimensions however are 3. (MLT). Hence $(n - m) = (7 - 3) = 4$ = number of dimensionless terms or π terms.

Equation (b) may re-written as,

$$\varphi_1\ (\pi_1,\ \pi_2,\ \pi_3,\ \pi_4) = 0$$

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The next step involves selection of variables that are repeated as per the criteria already mentioned. Number of variables to be selected are 3 since the given variables are 3 dimensional consisting of (MLT). Since Δp is a dependent variable it should be left out. Again the three variables must be so selected that each represents one property viz, geometric (length or diameter), flow (velocity or gravity) and fluid property (density or viscosity). Let us choose D, v and ρ as repeating variables whose dimensions L, LT^{-1} ML^{-3} satisfy the condition that together they should contain all the three basic dimensions and yet not form a dimensionless group on their own.

$$\pi_1 = [M^0\ L^0\ T^0] = D^a, v^b, \rho^c, \Delta p$$
$$[M^0\ L^0\ T^0] = [L]^a\ [LT^{-1}]^b\ [M\ L^{-3}]^c\ [M\ L^{-1}\ T^{-2}]$$

Now we can equate the powers of MLT

$$\left.\begin{array}{l} c+1=0 \\ a+b-3c-1=0 \\ -b-2=0 \end{array}\right] \text{ which yields, } c=-1,\ b=-2 \text{ and } a=0$$

$$\pi_1 = D^a, v^b, \rho^c, \Delta p$$
$$= D^0, v^{-2}, \rho^{-1}, \Delta p$$

$$\pi_1 = \left[\frac{\Delta p}{v^2 \rho}\right] \quad \ldots(a)$$

Similarly, $\pi_2 = [M^0\ L^0\ T^0] = [D^a, v^b, \rho^c, L]$

$$[M^0\ L^0\ T^0] = [L]^a\ [LT^{-1}]^b\ [M\ L^{-3}]^c\ [L]$$

Now we can equate the powers of MLT

$$\left.\begin{array}{l} c = 0 \\ a + b - 3c + 1 = 0 \\ -b = 0 \end{array}\right] \text{ which yields, } c=0,\ b=-0 \text{ and } a=-1$$

$$\pi_2 = [D^{-1}, v^0, \rho^0, L] = \frac{L}{D}$$

$$\pi_2 = \left[\frac{L}{D}\right] \quad \ldots(b)$$

Similarly,

$$\pi_3 = [M^0\ L^0\ T^0] = [D^a, v^b, \rho^c, \mu]$$
$$[M^0\ L^0\ T^0] = [L]^a\ [LT^{-1}]^b\ [M\ L^{-3}]^c\ [ML^{-1}\ T^{-1}]$$

Now we can equate the powers of MLT

$$\left.\begin{array}{l} c+1=0 \\ a+b-3c-1=0 \\ -b-1=0 \end{array}\right] \text{ which yields, } c=-1,\ b=-1 \text{ and } a=-1$$

$$\pi_3 = [D^{-1}, v^{-1}, \rho^{-1}, \mu]$$

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$$\pi_3 = \left[\frac{\mu}{Dv\rho}\right] \quad \ldots(c)$$

Similarly, $\pi_4 = [M^0 L^0 T^0] = [D^a, v^b, \rho^c, k]$

$[M^0 L^0 T^0] = [L]^a [LT^{-1}]^b [M L^{-3}]^c [L]$

Now we can equate the powers of MLT

$$\left.\begin{array}{l} c = 0 \\ a + b - 3c + 1 = 0 \\ -b = 0 \end{array}\right] \text{which yields, } c = 0, b = -0 \text{ and } a = -1$$

$$\pi_4 = [D^a, v^b, \rho^c, k] = [D^{-1}, v^0, \rho^0, k]$$

$$\pi_4 = \left[\frac{k}{D}\right] \quad \ldots(d)$$

$$\varphi_1 (\pi_1, \pi_2, \pi_3, \pi_4) = 0$$

$$\varphi_1 \left[\left(\frac{\Delta p}{v^2\rho}\right), \left(\frac{L}{D}\right), \left(\frac{\mu}{Dv\rho}\right), \left(\frac{k}{D}\right)\right] = 0$$

$$\left(\frac{\Delta p}{v^2\rho}\right) = \varphi_1 \left[\left(\frac{L}{D}\right), \left(\frac{\mu}{Dv\rho}\right), \left(\frac{k}{D}\right)\right]$$

is the required expression.

Problem 18.7 *A uniform fluid stream of certain velocity is directed at right angles against a circular cylinder which is rotating on its geometrical axis with a certain angular velocity. Establish a dimensional relationship for the power required to rotate the cylinder.*

Solution

The power required to rotate the cylinder is depending upon the $\left(\frac{L}{D}\right)$ ratio, the fluid stream velocity (v), the angular velocity of the cylinder (ω), the kinematic viscosity of the fluid (υ) and the density of the fluid (ρ). Thus,

$$P = \varphi (L, D, v, \omega, \upsilon, \rho) \quad \ldots(a)$$

$$0 = \varphi (P, L, D, v, \omega, \upsilon, \rho) \quad \ldots(b)$$

Since there are 7 variables in all $n = 7$.

Now let us write the dimensions for each of the 7 variables.

$[P = ML^2 T^{-3}]$, $[L = L]$, $[D = L]$, $[v = LT^{-1}]$, $[\omega = T^{-1}]$, $[\upsilon = L^2 T^{-1}]$ $[\rho = ML^{-3}]$

Fundamental dimensions however are 3 (MLT). Hence $(n - m) = (7 - 3) = 4 =$ number of dimensionless terms or π terms.

Equation (b) may re-written as,

$$\varphi_1 (\pi_1, \pi_2, \pi_3, \pi_4) = 0$$

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The next step involves selection of variables that are repeated as per the criteria already mentioned. Number of variables to be selected are 3 since the given variables are 3 dimensional consisting of (MLT). Since P is a dependent variable it should be left out. Again the three variables must be so selected that each represents one property viz, geometric (length or diameter), flow (velocity or gravity) and fluid property (density or viscosity). Let us choose D, v and ρ as repeating variables whose dimensions L, LT^{-1} ML^{-3} satisfy the condition that together they should contain all the three basic dimensions and yet not form a dimensionless group on their own.

$$\pi_1 = [M^0\ L^0\ T^0] = D^a, v^b, \rho^c, P$$
$$[M^0\ L^0\ T^0] = [L]^a\ [LT^{-1}]^b\ [M\ L^{-3}]^c\ [M\ L^2\ T^{-3}]$$

Now we can equate the powers of MLT

$$\left.\begin{array}{l} c+1=0 \\ a+b-3c+2=0 \\ -b-3=0 \end{array}\right] \text{ which yields, } c=-1,\ b=-3 \text{ and } a=-2$$

$$\pi_1 = D^a, v^b, \rho^c, P = D^{-2}, v^{-3}, \rho^{-1}, P$$

$$\pi_1 = \left[\frac{P}{D^2\ v^3\ \rho}\right]$$

Similarly,

$$\pi_2 = [M^0\ L^0\ T^0] = D^a, v^b, \rho^c, \omega$$
$$[M^0\ L^0\ T^0] = [L]^a\ [LT^{-1}]^b\ [M\ L^{-3}]^c\ [T^{-1}]$$

Now we can equate the powers of MLT

$$\left.\begin{array}{l} c=0 \\ a+b-3c=0 \\ -b-1=0 \end{array}\right] \text{ which yields, } c=0,\ b=-1 \text{ and } a=1$$

$$\pi_2 = D^a, v^b, \rho^c, \omega = D, v^{-1}, \rho^0, \omega$$

$$\pi_2 = \left[\frac{\omega\ D}{v}\right]$$

Similarly,

$$\pi_3 = [M^0\ L^0\ T^0] = D^a, v^b, \rho^c, \upsilon$$
$$[M^0\ L^0\ T^0] = [L]^a\ [LT^{-1}]^b\ [M\ L^{-3}]^c\ [L^2\ T^{-1}]$$

Now we can equate the powers of MLT

$$\left.\begin{array}{l} c=0 \\ a+b-3c+2=0 \\ -b-1=0 \end{array}\right] \text{ which yields, } c=0,\ b=-1 \text{ and } a=-1$$

$$\pi_3 = D^a, v^b, \rho^c, \upsilon = D^{-1}, v^{-1}, \rho^0, \upsilon$$

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$$\pi_3 = \left[\frac{\upsilon}{v D}\right]$$

$$\pi_4 = \left[\frac{L}{D}\right]$$

From (b) $\varphi_1 (\pi_1, \pi_2, \pi_3, \pi_4) = 0$

$$\varphi_1 \left[\left(\frac{P}{D^2 v^3 \rho}\right), \left(\frac{\omega D}{v}\right), \left(\frac{\upsilon}{v D}\right), \left(\frac{L}{D}\right)\right] = 0$$

$$\left(\frac{P}{D^2 v^3 \rho}\right) = \varphi \left[\left(\frac{\omega D}{v}\right), \left(\frac{\upsilon}{v D}\right), \left(\frac{L}{D}\right)\right]$$

is the required expression.

Problem 18.8 *A rocket launched missile 1.5 m in diameter and 12 m long designed for an average in flight velocity of 15 m/s is to be tested in a wind tunnel using a* $\frac{1}{15}$ *scale model. Determine the average wind flow velocity on the scale model if all other conditions are similar.*

Solution

If we assume dynamic similarity exists the Reynolds number of both the scale model and the prototype must be equal.

Thus, $$\left[\frac{v L}{\upsilon}\right]_m = \left[\frac{v L}{\upsilon}\right]_P$$

$$v_m = v_p \left[\frac{L_P}{L_m}\right]\left[\frac{\upsilon_p}{\upsilon_m}\right] \qquad \text{but since } \upsilon_m = \upsilon_P$$

$$v = 15\ [15]\ [1] = 225 \text{ m/s}$$

Problem 18.9 *A model of a spill way is built to a scale of 1 : 100. If the discharge through the model spill way is 0.5 m^3/s, determine the discharge through the prototype.*

Solution

Since in spill ways gravity is the predominant force we use the Froude number to determine the discharge. From Eq. (18.8) we have,

$$\left[\frac{v}{\sqrt{gL}}\right]_m = \left[\frac{v}{\sqrt{gL}}\right]_P$$

$$\left[\frac{v_m}{\sqrt{gL_m}}\right] = \left[\frac{v_P}{\sqrt{gL_P}}\right]$$

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$$\frac{V_m}{V_P} = \left[\frac{\sqrt{L_m}}{\sqrt{L_P}}\right] = \sqrt{L_R}$$

Discharge ratio = Q_R = (velocity ratio) × (area ratio)

Discharge ratio = $Q_R = (v_R) \times (L_R)^2$

Thus $$\frac{Q_m}{Q_P} = (L_R)^{2.5}$$

$$\frac{0.5}{Q_P} = \left(\frac{1}{100}\right)^{2.5}$$

$$Q_p = 50{,}000 \text{ m}^3/\text{s}$$

Problem 18.10 *An airplane with a wing span of 2.5 m has to be designed for a speed of 300 km / hour in still air at 10° C. If a 1:20 scale model of the wing has to be tested in a wind tunnel under a pressure which is 15 times the atmospheric pressure, determine the velocity of air stream to be directed at the model to obtain conditions similar to actual flight conditions. Assume that all other factors are equal.*

Solution

If we assume dynamic similarity exists the Reynolds number of both the scale model and the prototype must be equal.

Thus, $$\left[\frac{v\ L}{\upsilon}\right]_m = \left[\frac{v\ L}{\upsilon}\right]_P$$

$$\left[\frac{v_m\ L_m}{\upsilon_m}\right] = \left[\frac{v_P\ L_P}{\upsilon_P}\right]$$

but since $$v_P = \frac{(300 \times 1000)}{3600} = 83.33 \text{ m/s}$$

$$\left[v_m \frac{\left(\frac{1}{20}\right)}{\upsilon_m}\right] = \left[(83.33) \times \frac{2.5}{\upsilon_P}\right]$$

Since viscosity is not affected by pressure difference between the prototype and the scale model,

$$\frac{v_P}{v_m} = \frac{[(83.33 \times 2.5) \times 20]}{v_m}$$

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$$\frac{v_P}{v_m} = \frac{[4166.5]}{v_m} = \frac{p_m}{p_P} = 15$$

Hence, $$\frac{[4166.5]}{v_m} = 15$$

$$v_m = 277.7 \text{ m/s} = \text{velocity of the air stream.}$$

Problem 18.11 *Hydraulic oil at the rate of 3 m^3/s is required to be conveyed through a circular conduit of diameter 1.5 m. If a scale model was constructed using a pipe of inside diameter = 15 cm, determine the volume flow rate through the model if the test fluid is water at 20° C. $\mu_{oil} = 1.07 \times 10^{-1}$ Pa.s and $\rho_{oil} = 906$ kg/m^3 .*

Solution

Volume flow rate through the model is $Q_m = A_m \times v_m$

$$Q_m = 0.7854\,(0.15)^2 \times v_m \quad \ldots(a)$$

If we assume dynamic similarity exists the Reynolds number of both the scale model and the prototype must be equal.

Thus, $$\left[\frac{v\,L}{\upsilon}\right]_m = \left[\frac{v\,L}{\upsilon}\right]_P$$

$$\left[\frac{v_m\,L_m}{\upsilon_m}\right] = \left[\frac{v_P\,L_P}{\upsilon_P}\right]$$

$$\left[\frac{v_m\,D_m}{\upsilon_m}\right] = \left[\frac{v_P\,D_P}{\upsilon_P}\right] \quad \ldots(b)$$

$$\upsilon_P = \frac{\mu_{oil}}{\rho_{oil}}$$

$$= \frac{(1.07 \times 10^{-1})}{(906)}$$

$$= 1.18 \times 10^{-4} \text{ m}^2/\text{s}$$

$$\upsilon_m = \frac{\mu_{water}}{\rho_{water}} = \frac{(1.02 \times 10^{-3})}{(998)}$$

$$= 1.02 \times 10^{-6} \text{ m}^2/\text{s}$$

$$v_p = \frac{Q_P}{A_P} = \frac{3}{0.7854\,(1.5)^2} = 1.69 \text{ m/s}$$

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$$\frac{v_m}{v_P} = \left[\frac{D_P}{D_m}\right] \times \left[\frac{\upsilon_m}{\upsilon_P}\right] \quad \text{...from (b)}$$

$$\frac{v_m}{(1.69)} = \left[\frac{1.5}{0.15}\right] \times \left[\frac{1.02 \times 10^{-6}}{1.18 \times 10^{-4}}\right]$$

$$v_m = 0.146 \text{ m/s}$$

Substituting in (a) $Q_m = 0.7854\,(0.15)^2 \times 0.146$

$$= 2.58 \times 10^{-3} \text{ m}^3/\text{s}$$

Problem 18.12 *A laboratory model of a ship has a submerged area = 5 m^2. If the scale ratio is 1:20, the total drag = 1500 N when cruising at 1.5 m/s determine the total drag on the prototype at its cruising speed. Neglect skin friction coefficient. Assume that the properties of the fluid are identical in both the cases.*

Solution

For water it is assumed $\rho = 1000 \text{ kg/m}^3$

Resistance to motion of a ship is the sum of the pressure drag, skin friction drag and the wave resistance. Consequently the forces involved are inertia, viscosity and gravity. While skin friction drag is computed based on dynamic similarities, wave resistance is based on Froude number. This requirement cannot be satisfied unless both the model and the prototype have the same dimensions. This problem is overcome by first recording the total drag on the model when cruising at the given velocity. The skin friction drag for the model is then computed using the standard formula. If the skin friction drag is now subtracted from the total drag, the remainder can be multiplied by the scale ratio using Froude modeling. This value plus the skin friction drag computed for the prototype together yields the total drag on the prototype.

The skin friction drag $[F_D]_m = \left[C_D \left(\frac{\rho V^2}{2}\right) A\right]$

If we neglect (C_D) then $[F_D]_m = \left[\left(\frac{\rho V^2}{2}\right) A\right]$

$$[F_D]_m = \left[\frac{(1000)\,(1.5)^2}{2} \times 5\right] = 1125 \text{ N}$$

Total drag on the model $[F_{DT}]_m = 1500 \text{ N}$

Wave resistance $[F_{DW}]_m = 1500 - 1125 = 375 \text{ N}$

We can now use Froude's model law to obtain the wave resistance of the prototype. The force ratio $[F_W]_R$ is given by,

$$\frac{[F_{DW}]_P}{[F_{DW}]_m} = [F_W]_R \quad \text{...(a)}$$

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$$[F_W]_R = \frac{\left[(\rho_p\ (v_p)^2\ )\ L_p^2\right]}{\left[(\rho_m\ (v_m)^2\ )\ L_m^2\right]}$$

$$= [((\rho_R\ (v_R)^2\ )\ L^2{}_R]$$

$[F_W]_R = [(\rho_R\ (v_R)^2\ )\ L^2{}_R]$. For this particular case since the properties of the fluid are identical for both the prototype and the model $\rho_R = 1$. Hence,

$$[F_W]_R = [(v_R)^2\ )\ L^2{}_R] \qquad \text{Since, } (v_R) = \sqrt{L_R}$$

$$[F_W]_R = [(\sqrt{L_R}\ )^2\ )\ L^2{}_R] = [L^3{}_R]$$

Substituting in Eq. (a)

$$\frac{[F_{DW}]_P}{[F_{DW}]_m} = [L^3{}_R]$$

$$[F_D]_m = [375] \times [(20)^3] = 3000 \text{ kN}$$

The skin friction drag $[F_D]_P = \left[C_D \left(\frac{\rho V_2}{2}\right) A\right]$

If we neglect (C_D) then $[F_D]_P = \left[\left(\frac{\rho V_2}{2}\right) A\right]$

$$[F_D]_P = \left[(1000)\left[\frac{1.5\left(\sqrt{20}\right)^2}{2}\right] \times 5(20)^2\right]$$

$$= 30000 \text{ kN}$$

Total drag on the prototype $[F_{DT}]_P = [F_D]_P + [F_D]_m$

$$= 30000 + 3000$$

$$= 33000 \text{ kN}$$

Problem 18.13 *The scale model of a water turbine while rotating at 600 rpm will discharge 0.15 m^3/s under ahead of 2 m. If the scale ratio is 1:25 and the diameter of the prototype is taken as 2.5 m determine the discharge, speed of rotation and the head under which the prototype should operate.*

Solution In geometrically similar rotating machinery as long as the vector diagrams of velocity of fluid entering or leaving the machinery are similar, for all practical purposes it is assumed that dynamic similarity exits. But they are best can be described as homologous units because the Reynolds numbers are never the same in model and the prototype which results in discrepancy to the extent of 2–3 percent in the end results.

Model laws followed in case of rotating hydraulic machinery are the same for both pumps and turbines.

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The basic identities are,

(1) $$\left[\frac{gH}{N^2 D^2}\right] = 1 \text{ or, } [H \approx N^2 D^2] \quad \text{...Head,}$$

(2) $$\left[\frac{Q}{N D^3}\right] = 1 \text{ or, } [Q \approx N D^3] \quad \text{...Discharge,}$$

(3) $$\left[\frac{P}{\rho N^3 D^5}\right] = 1 \text{ or, } [P \approx N^3 D^5] \quad \text{...Power}$$

In addition for homologous units,

The specific speeds (N_S) should be the same for geometrically similar hydraulic rotating machinery.

(4) $$N_S = \left[N \frac{\sqrt{Q}}{(gH)^{\frac{3}{4}}}\right]$$

(5) The size factor $$= \frac{D_P}{D_m} = \left[\frac{Q_P}{Q_m}\right] \times \left[\frac{N_P}{N_m}\right] \times \left[\frac{H_P}{H_m}\right]$$

From considerations of geometric similarity,

$$\left[\frac{D_P}{D_m}\right] = [\text{scale factor}] = \left(\frac{25}{1}\right)$$

$$\left[\frac{2.5}{D_m}\right] = \left(\frac{25}{1}\right)$$

$$[D_m] = 0.1 \text{ m}$$

Since $$\left[\frac{D_P}{D_m}\right] = \left[\frac{H_P}{H_m}\right]$$

$$\left[\frac{2.5}{0.1}\right] = \left[\frac{H_P}{2}\right]$$

$$[H_P] = 50 \text{ m}$$

From considerations of dynamic similarity,

$$\left[\frac{gH}{N^2 D^2}\right]_P = \left[\frac{gH}{N^2 D^2}\right]_m$$

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$$\left[\frac{H_P}{N_P^2\ D_P^2}\right] = \left[\frac{H_m}{N_m^2\ D_m^2}\right]$$

$$\frac{H_P}{H_m} = \frac{\left[N_P^2\ D_P^2\right]}{\left[N_m^2\ D_m^2\right]}$$

$$\frac{50}{2} = \frac{\left[N_P^2\ (2.5)^2\right]}{\left[(600)^2\ (0.1)^2\right]}$$

$$25 = \frac{\left[N_P^2\ (6.25)\right]}{3600}$$

$$N_P = 120$$

From considerations of discharge,

$$\left[\frac{Q}{N\,D^3}\right]_P = \left[\frac{Q}{N\,D^3}\right]_m$$

$$\left[\frac{Q_P}{N_P\ D_P^3}\right] = \left[\frac{Q_m}{N_m\ D_m^3}\right]$$

$$\left[\frac{Q_P}{(120)\ (2.5)^3}\right] = \left[\frac{0.15}{(600)\ (0.1)^3}\right]$$

$$\left[\frac{Q_P}{(120)\ (2.5)^3}\right] = 0.25$$

$$Q_P = 468.75\ \text{m}^3/\text{s}$$

Here the subscripts (P) and (m) relate to prototype and the scale model respectively.

QUESTIONS BANK

Section A

Review Questions

1. Write down the variables on which the pressure drop per unit length in turbulent flow through a horizontal pipe depends.
2. Discuss the purpose, advantages and disadvantages of dimensional analysis. What do you understand by the term dimensional homogeneity?
3. What is the limitation of Rayleigh's method of dimensional analysis? State

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the Buckingham's π theorem. Name any other method employed in dimensional analysis.

4. Write down the functional relationship in the non-dimensional form between the power required to drive a propeller and the other variables involved.
5. In terms of MLT determine the dimensions of Angular velocity, Power, and torque.
6. Derive the unit of mass under the FPS system.
7. Discuss the advantages and applications of model testing.
8. Name the three similarities that must exit between the model and the prototype. What is the significance of dimensionless number?
9. State the purpose of various model laws that are followed in model studies.
10. What do you understand by the term 'distorted model'? State their advantages and specific areas of application.

Section B

Multiple Choice Questions

Choose the right answer

1. The dimensions of Dynamic viscosity are
   (a) $ML^{-2}T^{-2}$ (b) $ML^{-1}T^{-1}$
   (c) $ML^{-1}T^{-2}$ (d) None of these
2. Identify the functional relationship for a V-Notch from among the choices given.
   (a) $F(Q,H,g,v \text{ and } \theta)$ (b) $F(v,D,r,m,H)$
   (c) $F\left(\frac{\Delta p}{L},v,D,\rho,\mu\right)$ (d) None of these
3. The choice of repeating variables in dimensional analysis is governed by
   (a) The need to exclude the dependent variable
   (b) The need to include a variable that contains geometric property
   (c) The need to include a variable that contains flow property.
   (d) All of these
4. Select a non-dimensional parameter from among the choice given.
   (a) Froude number (b) Weber number
   (c) Reynolds number (d) All of these
5. The non-dimensional parameters are always a ratio of inertia force to any one of these forces
   (a) Viscous force (b) Gravity force
   (c) Elastic force (d) All of these

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6. Froude number is

(a) $\left[\frac{v}{\sqrt{Lg}}\right]$ (b) $\left[\frac{v}{\sqrt{\frac{K}{r}}}\right]$

(c) $\left[\frac{v}{\sqrt{\frac{\sigma}{\rho L}}}\right]$ (d) None of these

7. For the same two fluids the pressure intensity ratio is

(a) $L_R{}^{\frac{7}{2}}$ (b) $L_R{}^2$

(c) L_R (d) None of these

8. Euler's model law is applicable where
   (a) the phenomenon of cavitations occurs
   (b) capillary raise in narrow passages
   (c) underwater testing of missiles
   (d) None of these
9. Limitations of hydraulic similitude are
   (a) Only qualitative results are possible
   (b) Model studies are expensive
   (c) The selection of scale ratio is not based on any criteria but is usually a matter of experience
   (d) All of these
10. The scale effects in models can be
   (a) positive only
   (b) negative only
   (c) both positive and negative
   (d) None of these

Section C

Problems

1. Given the functional relationship $f\left(Q, \frac{Dp}{L}, D, m\right)$ find the dimensionless form of the equation for discharge Q.

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2. The head loss due to friction (h_f) in a pipe depends on, pipe inside diameter (D), length (L), roughness magnitude (ε), flow velocity (v), gravity (g) and the viscosity (μ). Find the dimensionless form of the equation for h_f.

(*Ans.* $h_f = \varphi\left[(R_E), \left(\frac{\varepsilon}{D}\right)\left(\frac{L}{D}\right)\right]$)

3. The load (w) that can be supported by a journal bearing of length L, diameter (D) depends on the radial clearance (δ), eccentricity (e), oil film viscosity (μ) and rotational speed (N). Find the dimensionless form of the equation.

(*Ans.* $W\mu ND^2 = \varphi\left[\left(\frac{\delta}{D}\right), \left(\frac{e}{D}\right), \left(\frac{L}{D}\right)\right]$)

4. A blower is required to be installed atop a hill station where the density of air is 0.9 kg /m^3. Expected delivery from the blower is 3 m^3/s at a pressure differential of 250 N/m^2, while rotating at 1500 rpm. If a scale model is to be designed for test at sea level where the density is 1.2 kg/m^3, determine the discharge, and the RPM at which the model is required to operate. Assume geometric scale and ratio of viscosities as = unity.

(*Ans.* Q = 2.25 m^3/s and RPM = 1125)

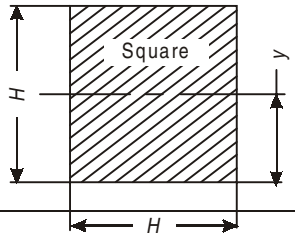
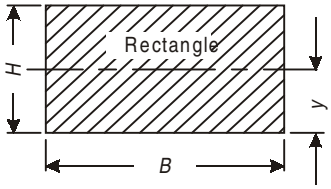
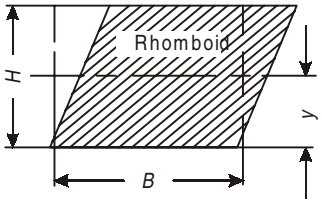
Answers to Multiple Choice Questions

1. (b)	**2.** (a)	**3.** (d)	**4.** (d)	**5.** (d)
6. (a)	**7.** (c)	**8.** (a)	**9.** (d)	**10.** (c)

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Appendices

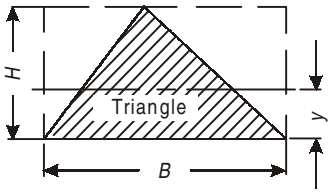
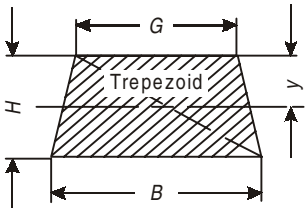
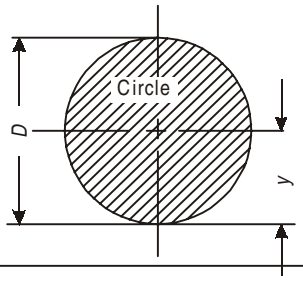
Appendix 1

Section	Area of section	Distance to centrodial axis-y	Moment of inertia about I_c
	H^2	$\frac{H}{2}$	$\frac{H^4}{12}$
	BH	$\frac{H}{2}$	$\frac{BH^3}{12}$
	BH	$\frac{H}{2}$	

Contd.

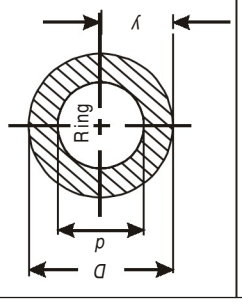
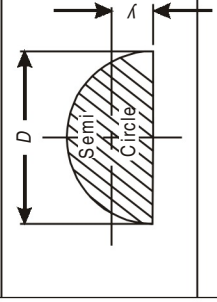
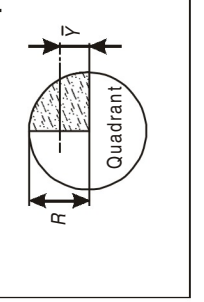
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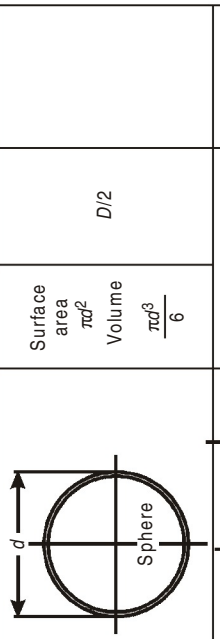
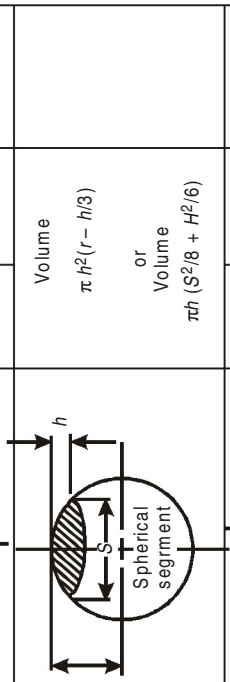
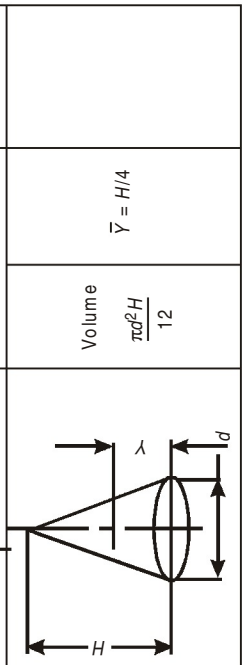
Contd.

Section	Area of section	Distance to centrodial axis-y	Moment of inertia about I_c
	$\frac{BH}{2}$	$\frac{H}{3}$	$\frac{BH^3}{36}$
	$\frac{H(G+B)}{2}$	$\frac{H(G+2B)}{3(G+B)}$	$\frac{H^3(G^2+4GB^2+B)}{36(G+B)}$
	$0.7854\ D^2$	$\frac{D}{2}$	$\frac{\pi D^4}{64}$

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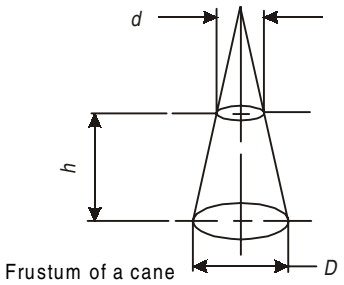
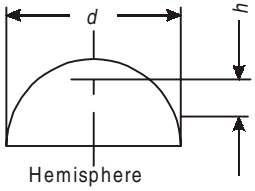
Appendix 1-A

Section	Area of section	Distance to centrodial axis-y	Moment of inertia about I_c
	$\frac{\pi (D^2 - d^2)}{4}$	$D/2$	$\frac{\pi (D^4 - d^4)}{64}$
	$\frac{\pi D^2}{8}$	$0.212D$	$(6.86 \times 10^{-3}) D^4$
	$\frac{\pi R^2}{4}$	$0.424R$	$(5.49 \times 10^{-2}) R^4$

Section	Area of section	Distance to centrodial axis-y	Moment of inertia about I_c
	Surface area πd^2 Volume $\frac{\pi d^3}{6}$	$D/2$	
	Volume $\pi h^2 (r - h/3)$ or Volume $\pi h (S^2/8 + H^2/6)$		
	Volume $\frac{\pi d^2 H}{12}$	$\bar{Y} = H/4$	

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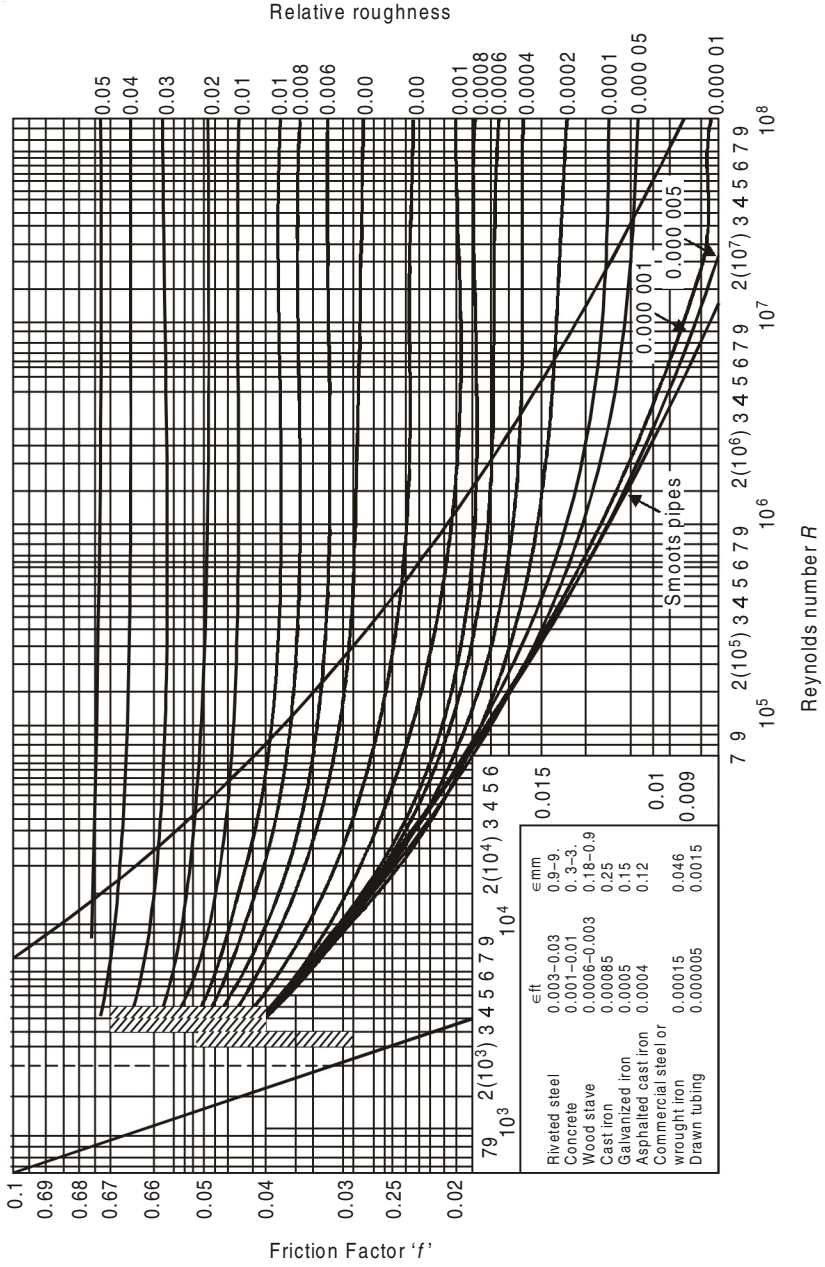
Appendix 1-B

Section	Area of section	Distance to centrodial axis-y	Moment of inertia about I_C
 Frustum of a cane	Volume $\left(\frac{\pi}{12}\right) h\,(D^2 + D) \times (d + d^2)$		
 Hemisphere	Volume $\frac{(\pi d^3)}{12}$	$Y = 3\,\frac{d}{16}$	

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Appendix 2

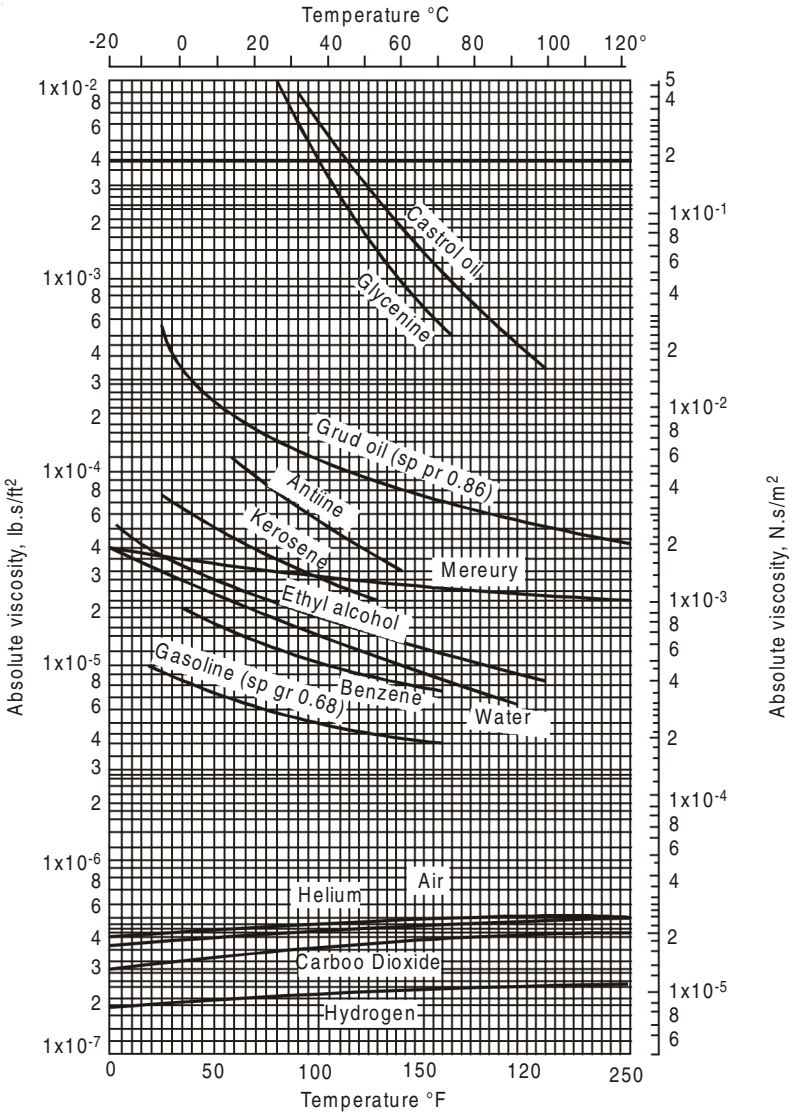
Moody's Diagram



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Appendix 3

Absolute viscosities of some common fluids



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APPENDIX 4

Properties of water

Temperature (°C)	Specific Weight γ (kN/m^3)	Density ρ (kg/m^3)	Dynamic Viscosity μ (Pa's) or (N-s/m^2)	Kinematic Viscosity υ (m^2/s)
0	9.81	1000	1.75×10^{-3}	1.75×10^{-6}
5	9.81	1000	1.52×10^{-3}	1.52×10^{-6}
10	9.81	1000	1.30×10^{-3}	1.30×10^{-6}
15	9.81	1000	1.15×10^{-3}	1.15×10^{-6}
20	9.79	998	1.02×10^{-3}	1.02×10^{-6}
25	9.78	997	8.91×10^{-4}	8.94×10^{-7}
30	9.77	996	8.00×10^{-4}	8.03×10^{-7}
35	9.75	994	7.18×10^{-4}	7.22×10^{-7}
40	9.73	992	6.51×10^{-4}	6.56×10^{-7}
45	9.71	990	5.94×10^{-4}	6.00×10^{-7}
50	9.69	988	5.41×10^{-4}	5.48×10^{-7}
55	9.67	986	4.98×10^{-4}	5.05×10^{-7}
60	9.65	984	4.60×10^{-4}	4.67×10^{-7}
65	9.62	981	4.31×10^{-4}	4.39×10^{-7}
70	9.62	978	4.02×10^{-4}	4.11×10^{-7}
75	9.56	975	3.73×10^{-4}	3.83×10^{-7}
80	9.53	971	3.50×10^{-4}	3.60×10^{-7}
85	9.50	968	3.30×10^{-4}	3.41×10^{-7}
90	9.47	965	3.11×10^{-4}	3.22×10^{-7}
95	9.44	962	2.92×10^{-4}	3.04×10^{-7}
100	9.40	958	2.82×10^{-4}	2.94×10^{-7}

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Appendix 5

Propeties of common liquids

	Specific Gravity sg	Specific Weight γ (kN/m³)	Density ρ (kg/m³)	Dynamic Viscosity μ (Pa's) or (N-s/m³)
Acetone	0.787	7.72	787	3.16×10^{-4}
Alcohol, ethye	0.787	7.72	787	1.00×10^{-3}
Alcohol, methyl	0.789	7.74	789	5.60×10^{-4}
Alcohol, Propyl	0.802	7.87	802	1.92×10^{-3}
Ammonia	0.826	8.10	826	—
Benzenu	0.876	8.59	876	6.03×10^{-4}
Carbon Tetrachloride	1.590	15.60	1.590	9.10×10^{-4}
Castor oil	0.960	9.42	960	6.51×10^{-1}
Ethylene	1.100	10.79	1.100	1.62×10^{-2}
Gasoline	0.68	6.67	680	2.87×10^{-4}
Glycerine	1.258	12.34	1.258	9.60×10^{-1}
Kerosene	0.823	8.07	823	1.64×10^{-3}
Linseed oil	0.930	9.12	930	3.31×10^{-2}
Mercury	13.54	132.8	13.540	1.53×10^{-3}
Propane	0.495	4.86	495	1.10×10^{-4}
Seawater	1.030	10.10	1.030	1.03×10^{-3}
Turpentine	0.870	8.53	770	1.37×10^{-3}
Fuel oil, medium	0.852	8.36	852	2.99×10^{-3}
Fuel oil, heavy	0.906	8.89	906	1.07×10^{-1}

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Appendix 6

Typical propeties of petroleum lubricating oils

Type	Specific Gravity sg	Kinematic Viscosity υ				Viscosity Index
		at 40°C (105°F)		at 100°C (212°F)		
		(m^2/s)	(ft^2/s)	(m^2/s)	(ft^2/s)	
Automotive hydraulic systems	0.887	3.99×10^{-5}	4.30×10^{-4}	7.29×10^{-6}	7.85×10^{-5}	141
Machine tool hydraulic systems	0.887	3.20×10^{-5}	3.44×10^{-4}	4.79×10^{-6}	5.16×10^{-5}	64
Light	0.895	6.70×10^{-5}	7.21×10^{-4}	7.29×10^{-6}	7.85×10^{-5}	66
Medium	0.901	1.96×10^{-4}	2.11×10^{-3}	1.40×10^{-6}	1.51×10^{-5}	70
Heavy	0.884	1.40×10^{-5}	1.51×10^{-4}	5.20×10^{-6}	5.60×10^{-5}	226
Low temperature machine tool lubricating oils						
Light	0.881	2.20×10^{-5}	2.37×10^{-4}	3.90×10^{-6}	4.20×10^{-5}	80
Medium	0.915	6.60×10^{-5}	7.10×10^{-4}	7.00×10^{-6}	7.53×10^{-5}	83
Heavy	0.890	2.00×10^{-4}	2.15×10^{-3}	1.55×10^{-5}	1.67×10^{-4}	80

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Appendix 7

Propeties of air

Temperature (°C)	Specific Weight γ (kN/m³)	Density ρ (kg/m³)	Dynamic Viscosity μ (Pa's) or (N-s/m²)	Kinematic Viscosity υ (m²/s)
–40	1.514	14.85	1.51×10^{-5}	9.98×10^{-6}
–30	1.452	14.24	1.56×10^{-5}	1.08×10^{-5}
–20	1.394	13.67	1.62×10^{-5}	1.16×10^{-5}
–10	1.341	12.15	1.67×10^{-5}	1.24×10^{-5}
0	1.292	12.67	1.72×10^{-5}	1.33×10^{-5}
10	1.247	12.23	1.77×10^{-5}	1.42×10^{-5}
20	1.204	11.81	1.81×10^{-5}	1.51×10^{-5}
30	1.164	11.42	1.86×10^{-5}	1.60×10^{-5}
40	1.127	11.05	1.91×10^{-5}	1.69×10^{-5}
50	1.092	10.71	1.95×10^{-5}	1.79×10^{-5}
60	1.060	10.39	1.99×10^{-5}	1.89×10^{-5}
70	1.029	10.09	2.04×10^{-5}	1.99×10^{-5}
80	0.9995	9.802	2.09×10^{-5}	2.09×10^{-5}
90	0.9720	9.532	2.13×10^{-5}	2.19×10^{-5}
100	0.9459	9.277	2.17×10^{-5}	2.30×10^{-5}
110	0.9213	9.034	2.22×10^{-5}	2.40×10^{-5}
120	0.8978	8.805	2.26×10^{-5}	2.51×10^{-5}

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Appendix 8

Properties of the atmosphere

Altitude (m)	SI Units		Density (kg/m³)
	Temperature (°C)	Pressure (kPa)	
0	15.00	101.3	1.225
200	13.70	98.6	1.202
400	12.40	96.6	1.179
600	11.10	94.6	1.156
800	9.80	92.1	1.134
1000	8.50	89.9	1.112
2000	2.00	79.5	1.007
3000	–4.49	70.1	0.9093
4000	–10.98	61.7	0.8194
5000	–17.47	54.0	0.7364
10000	–49.90	26.5	0.4135
15000	–5650	12.11	0.1948
20000	–56.50	5.53	0.0889
25000	–51.60	2.55	0.0401
3000	–46.64	1.20	0.0184

U.S. customary system units

Altitude (ft)	Temperature (°F)	Pressure (psi)	Density (slugs/ft³)
0	59.00	14.696	14.38×10^{-3}
500	57.22	14.433	2.34×10^{-3}
1000	55.43	14.173	2.25×10^{-3}
5000	41.17	12.227	2.05×10^{-3}
10000	23.34	10.106	1.76×10^{-3}
15000	5.51	8.293	1.50×10^{-3}
20000	–12.62	6.753	1.27×10^{-3}
30000	– 47.99	4.365	8.89×10^{-4}
40000	–69.70	2.720	5.85×10^{-4}
50000	–69.70	1.683	3.62×10^{-4}
60000	–69.70	1.040	2.24×10^{-4}
70000	–67.30	0.644	1.38×10^{-4}
80000	–61.81	0.400	8.45×10^{-5}
90000	–56.32	0.251	5.22×10^{-5}
100000	–50.84	0.158	3.25×10^{-5}

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Appendix 9

S.I Units of measurement

Length (L)	:	**Metre (m)** [kilometre (km) Centimeter (cm) Millimeter (mm)]
Area (A)	:	**Square meter (m^2)** [square cm (cm^2) square mm (mm^2)]
Volume (V)	:	**Cubic meter (m^3)** [liter (l), cm^3, mm^3] One liter = 0.001 m^3
Mass (m)	:	**Kilogram (kg)** [tonne (t) (one tonne = 1000 kg)]
Time (t)	:	**second (s)**
Rotational Speed (n)	:	**Revolutions per minute (RPM)** [Revolutions Per Second (RPS)]
Velocity (v)	:	**Metres/second (m/s)** [kilometres/hour (km/h)]
Volume flow Rate (Q)	:	**Cubic metres/second (m^3/s)** [liters/minute (L/min) 1 m^3/s = 60,000 L/min]
Force (F)	:	**Newtons (N)** [kg m/sce^2] (1 kgf = 9.81N)
Pressure (P)	:	**Pascal (Pa)** [N/m^2, bar (one bar = 10^5 Pa)]
Work/Energy (W)	:	**Joule (J)** [kilowatt hour (kWh) 1J = 1Nm = 1 kgm/sec^2]
Torque (M)	:	**Newton meter (Nm)** [Joule (J) 1 Nm = 1 J = 1 Ws]
Power (P)	:	**Watt (W)** [Joule/sec (J/s) 1 Watt = 1 J/s = 1 Nm/sec]
Dynamic Viscosity (μ)	:	**Pascal sec (Pas)** [Ns/m^2] (1 Ns/m^2 = 1000 mPas)
Kinematic Viscosity (υ)	:	**Metre square/second (m^2/s)** [Centistokes (cSt) 1 cSt = 10^{-6} m^2/s]
Temperature	:	**Kelvin (K)** [Degree Celsius (°C)]

Note : Highlighted units of measurement are the ones most commonly used.

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Exercises

The equation $p = \rho gh$ is not applicable to gases because the specific weight of gas changes with change in pressure.

At a height of 300 m above mean sea level the atmospheric pressure is 96.6 kilopascal as compared to 101.3 kPa at the surface.

Change in pressure is directly proportional to sp. wt of the liquid.

Gravity has no effect for laminar flow in a horizontal pipe. For vertical flow the gravitational effect is in the form of hydrastatic pressure.

Some times it is found convenient to calibrate gauges to read pressure in terms of mm of Hg. Instead of in Pa. The conversion is 1 mm of Hg. = 132.8 Pa.

772 mm Hg = **102.5 kPa** / 104 kPa / 104 kPa / none of these.

225 kPa (abs) = **124 kPa** / 123.5 kPa / 125 kPa / or none of these.

Standard atmosphere = **0.759 m** / 0.9 m / 1.2 m of Hg.

1.0 mm of Hg = 132.8 Pa. based on $p = \gamma h$ where $\gamma = 132.8$ kPa and $h = .001$m

Barometer reading is 75.5 cm of *hg*. Sp of *hg* = 13.6 convert the pressure in to Pa. (100,792) ans

What is gauge pressure? Pressure measured above atmospheric pressure 60,000 LPM = 1 m^3/sec

An error 1% in measuring the head over V-Notch results in an error of 2.5% in discharge. An error 1% in measuring the head over rectangular–Notch results in an error of 1.5% in discharge

Chapter 11. It is much faster to empty an overhead tank with a long vertical pipe than with an orifice. Head causing the flow is more The running cost of pumping the same fluid at the same flow rate through 200 mm diameter pipe instead of 300 mm diameter pipe is more by 7.6 times.

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