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FOUNDATIONS OF FLUID MECHANICS

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To My Teachers:

Theodore von Kármán

and

Stepan Prokofievitch Timoshenko

To Whom The Inspiration For This Book Is Largely Due

PREFACE

This book is intended primarily for engineering and applied science students as an introductory and/or a secondary course in fluid mechanics. However, it may also serve professional engineers who need a basic understanding of fluid mechanics for their current work, as well as preparation for advanced work. The book is further intended for those who have not had adequate foundations in fluid mechanics during their college work and who are independent and unassisted readers.

Fluid mechanics originally evolved from the combined results of the highly-developed classical hydrodynamics on one hand and the purely empirical hydraulics on the other. It was the Prandtl school that first developed the synthetic process between classical hydrodynamics and practical hydraulics. During the past two decades there has been a rapidly increasing improvement in modern technology and design. The trend is ever increasing. These advances demand a reassessment of current course content at college level, and fluid mechanics is no exception. We can no longer imagine that students can still be trained for the specific tasks they will have to perform in the course of their careers in engineering or applied science. But their training should have provided them with the ability to formulate their problems based on the fundamental laws of nature, to recognize the type of methods which are applicable, to read with understanding the relevant work and to come up with a usable answer. The need for a book on fluid mechanics which can realize this aim has long been felt.

In this textbook I have attempted to present, in a logical pattern of development, a unified comprehensive coverage of the basic foundations of the subject matter. It emphasizes the concepts and basic principles of fluid mechanics that are common to application in most branches of the engineering profession. The presentation demonstrates the universality of mathematical expressions portraying physical phenomena. It introduces a high level of general theoretical treatment of fundamentals, which are presented at an elementary level. A typical point showing the concept of generality is the treatment of the governing

equations for a viscous compressible flow; first the equations are derived, and then the general realtions are reduced in accordance with the limitations of the specific problem under investigation. In this way, the student gets a much clearer understanding and better appreciation of the underlying physical assumptions and limitations of the solution obtained when he sees what terms in the governing equations are neglected. Problems are formulated whenever possible, on the basis of fundamental physical principles rather than the utilization of arbitrary assumptions. Many illustrative examples are included to enhance the reader's thorough understanding of the subject, and develop his ability to analyze new and challenging problems in this field.

The order of the contents of this text is believed to present a rational classification of the topics treated. The introductory chapter begins with a discussion of the general aspects of fluid mechanics and the physical properties of fluids. The regimes of the mechanics of fluids are presented and categorized here so that the reader can grasp a general picture of fluid mechanics and its related fields before attempting to study any particular branch of the subject. It is vitally important that the reader recognizes the limitations in the application of the theory to any specific problem. Chapter 2 deals with fluid statics and relative motion of liquids. The fundamental differential equation of fluid statics is obtained in general vector form. The kinematical foundations of fluid motion are discussed in Chapter 3. It has been my observation that one of the fundamental sources of difficulty for students in fluid mechanics is the method of describing fluid motion. I have attempted to give a clear mathematical derivation and simple mathematical and numerical examples to illustrate the relation between the Lagrange method and the Euler method. The discussion of topics such as conservation of mass, circulation, and velocity potential are deferred to later chapters for obvious reasons.

Chapters 4 and 5 are devoted to the governing equations of fluid dynamics which form the backbone of the remaining chapters of the text. Chapter 4 gives the concepts and relationships of the stresses and rates of strain which are essential in the development of the governing equations of a viscous compressible fluid. Unless the materials presented in Chapter 4 are well understood, the derivation of the governing equations in Chapter 5 cannot be made meaningful. In today's environment we encounter many structures which have cylindrical and spherical configurations. The readers may find the comprehensive presentation of the basic equations of fluid dynamics in cylindrical and spherical coordinates in this chapter very helpful.

The problems of one- and multidimensional flows of an inviscid incompressible fluid are given in Chapters 6 and 7, respectively. To

illustrate the conciseness of the analysis in the presentation I shall use the two-dimensional source and sink flow as an example. The discussion begins with the physical definition of a source; the stream function and the velocity potential for source flow are obtained and discussed only after the flow satisfying the continuity equation and the condition of irrotationality are shown. I believe that only through this type of presentation can the student obtain a clear understanding of the basic physical problem which will affect his comprehension of future problems.

In Chapters 8, 9, and 10, a systematic development of the flow of a viscous incompressible fluid is presented. Chapter 8 begins with the discussion of the significance of the Reynolds number which was obtained from the Navier-Stokes equation (derived in Chapter 5) by means of the law of similarity. Topics such as measurement of viscosity and hydrodynamics of bearing lubrication are introduced only after the explanations of the significance of the Reynolds number, viscosity and parallel flow theory are made. The laminar boundary layer theory is presented in Chapter 9 which begins with the discussion of the properties of the Navier-Stokes equations and the boundary layer concept. Chapter 10 presents an introduction to turbulent flow in which the Reynolds stress has been clearly explained with an example of the rate of momentum transport of fluid masses. The analogy between the mean free path in kinetic theory of gases (given in Chapter 8) and the mixing length in turbulence is clearly shown.

The last two chapters are devoted to the flow of compressible fluids. Chapter 11 deals with an inviscid compressible flow. The discussion begins with the four controlling parameters in compressible flow which are obtained from the energy equation (derived in Chapter 5) by means of the law of similarity. Chapter 12 is concerned with the flow of a viscous fluid. The governing equations of viscous compressible fluids (derived in Chapter 5) have been clearly applied here. The topic on steady flow through a constant-area pipe with friction is included in this chapter. The presentation of this topic in this chapter is believed to be both logical and orderly.

It is my conviction that topics such as dimensions and unit systems and dimensional analysis should be presented in a unified fashion. If this chapter were placed at the beginning of the text it could easily overwhelm most students by the strange parameters or numbers which actually belong to the latter part of the text. This is the very reason that the chapter on dimensional method in fluid mechanics is presented in an appendix so that it can be referred to whenever required. Both British and metric units are discussed in detail here and the significant "numbers" of different regimes in the fluid mechanics are

obtained by dimensional analysis and are in complete agreement with results obtained from the similarity principle. A review of thermodynamic relations is also presented in an appendix to supplement the discussion of compressible flow theory given in Chapters 5, 11, and 12.

As the first course in solid mechanics is now being taught with vector methods and notations in many schools, these methods and notations therefore have been used consistently throughout the text. It is my belief that the vector analysis presentation gives a simple, clear, and suggestive exposition of the physical idea involved. A summary of vector analysis is given in Appendix A which may be helpful for those who have been little exposed to the subject as well as for those who wish to refer to vector identities and to the transformation formulae for orthogonal curvilinear coordinates. After a student has gained some basic knowledge of strength of material he should be able to follow the discussion in Chapter 4 without difficulty.

The material contained in this text can be covered in about six semester hours at junior-senior level or three semester hours in the first graduate course. A three-semester-hour terminal course could normally include most of the first four chapters plus portions of Chapters 5 through 7, with selected topics in Chapters 8 through 11. For those who are interested only in inviscid flow, Appendix D is specially prepared to replace Chapters 4 and 5. The instructor should select those topics in accordance with requirements of his class.

In preparing the manuscript, I have relied heavily on the works of authors who have written before me, particularly those authors whose writings are considered classics in this field. A list of selected references is given at the end of the text and I am indebted to many of these studies.

A number of my colleagues and friends have been patient enough to read portions of the manuscript and to offer valuable comments. In particular, I wish to take this opportunity to express my sincerest appreciation to Dr. J. O. Hinze of the Technological University, Delft, Dr. A. Miele of Rice University, Dr. M. Morduchow of the Polytechnic Institute of Brooklyn, Dr. P. H. Miller of the University of Missouri, Dr. R. W. Courter of the University of Wyoming, and Drs. L. R. Mack, D. G. Hull, J. W. Porter, and F. D. Masch of The University of Texas. I wish to heartily thank Drs. N. W. Newmark and W. J. Hall of the University of Illinois for their valuable comments and suggestions during the preparation of the manuscript. Mr. A. M. Bloom assisted the author with great devotion during the final preparation of the manuscript; his assistance in revising the manuscript and in proofreading was an essential help for the completion of the book. Finally, I greatly appreciate the efforts of Mrs. Georgia Courter, Mrs. Marie Kaak, and

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Austin, Texas

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1

INTRODUCTION

1.1. General Description of Fluid Mechanics

Fluid mechanics, one of the oldest branches of physics and the foundation for the understanding of many other aspects of the applied sciences and engineering, concerns itself with the investigation of the motion and equilibrium of fluids. It is a subject of widespread interest in almost all fields of engineering as well as in astrophysics, biology, biomedicine, meteorology, physical chemistry, plasma physics, and geophysics. Since the nineteenth century when the study of hydraulics as a science was associated with the growth of the fields of civil engineering and naval architecture, the scope of fluid mechanics has steadily broadened. The development of aeronautical, chemical, and mechanical engineering during the past few decades on the one hand and the exploration of space in the past few years on the other, have given added stimuli to the study of fluid mechanics so that it now ranks as one of the most important basic subjects in engineering science.

The frontier of fluid dynamic research has been extended into the exotic regimes of hypervelocity flight and flow of electrically conducting fluids. This has introduced new fields of interest such as hypersonic flow and magnetofluidynamics. In this connection it has become necessary to combine a knowledge of thermodynamics, mass transfer, heat transfer, electromagnetic theory, and fluid mechanics to fully understand the physical phenomena involved.

In spite of the dramatic recent advances in fluid mechanics, it is still possible to experience the application of fluid mechanics principles in daily life. The flight of birds in the air and the motion of fish in the water are governed by the laws of fluid mechanics. The baseball pitcher is dependent on the circulation principle to provide him with a baffling repertoire of pitches. The designs of airplanes and ships for air and ocean transportation are based on the theory of fluid mechanics. Even the natural phenomena such as destructive hurricanes and tornadoes may someday be explained by the principles of fluid mechanics. We live in an environment of air and of water to a great extent, so that almost everything we do is connected in someway to the science of fluid mechanics.

In order to proceed in a logical manner with the discussion of fluid properties, it is necessary to differentiate between a solid and a fluid. Matter exists in three states—solid, liquid, and gaseous. The latter two categories make up the fluid state. Later it will be shown that this division into solid and fluid states constitutes a natural grouping of matter from the standpoint of internal stresses and strains in elastic media. That is, the stress in a linear elastic solid is proportional to its strain (or deformation per unit length), while the stress in a fluid* is proportional to its time rate of strain. A simpler definition will suffice for the moment, however. When a force is applied to a solid, deformation will be produced in the body. If the force per unit area (stress) is less than the yield stress (within the proportional limit) of the material, the deformation disappears when the force is removed. Otherwise, it will acquire a permanent set, or even crack. If a shearing force (force component tangential to the surface) is applied to a fluid, however, the fluid will deform continuously regardless of the magnitude of the force. This continuous deformation under the action of a shearing stress is manifested in the tendency of fluids to flow rather than remain rigid as solids do. This behavior can be explained by the molecular properties of solids and fluids. Very strong intermolecular attractive forces exist in solids which give them the property of rigidity. These forces are weaker in liquids and extremely small in gases. This characteristic enables liquid molecules to move freely within a liquid mass while still maintaining a close proximity to one another. In addition, gas molecules have freedom to the extent that they completely fill any space allotted to them.

The study of all aspects of fluid behavior can be divided into three categories—statics, kinematics, and dynamics. In the first case the fluid elements are at rest with respect to one another and, thus, free of shearing stresses. The static pressure distributions in a fluid and on

*Newtonian fluid [see Eq. (1-6)].

bodies immersed in a fluid can be determined from a static analysis. The kinematics of fluids deals with a study of the translation, rotation, and rate of deformation of a fluid particle. This analysis is useful in determining methods of describing the motion of a particle and in analyzing flow pictures. Unfortunately, because a fluid represents distributed media, the interaction of fluid particles with one another precludes the determination of velocity and acceleration information from a kinematic study alone. It is necessary to perform a dynamic analysis of a fluid in motion to determine the effects of the fluid and its surroundings on the motion. The dynamic analysis involves the consideration of forces acting on the fluid particles in motion with respect to one another. Since relative motion of particles does exist, shearing forces are important in the analysis.

1.2. *Continuum Mechanics*

In its most fundamental form the description of the motion of a fluid involves a study of the behavior of all the discrete molecules which make up the fluid. Fortunately, in liquids, analysis by the molecular theory is not necessary for our purposes because the strong intermolecular cohesive forces compel the fluid to behave as a continuous mass of substance. In gases, since the molecular motion is varied and the quantity of molecules is large (e.g., for most gases the molecular density is 4.4×10^{20} molecules per in.^{3}), it is convenient to avoid the difficult molecular analysis of the fluid by considering the average effects of all the molecules within the gas. This can only be done with any degree of accuracy if the molecular density is high enough so that the gas can be treated as a continuous mass of fluid. Such a fluid model is called the continuum.

There are two factors which are important in determining the validity of the continuum model. The first is the distance between molecules. This distance is evidently not the same for all the molecules in the gas at any one time. Therefore, an average distance called the molecular mean free path is defined to give an indication of molecular density. The mean free path is a statistical average distance which molecules travel between collisions (the mean free path of atmospheric air is $2\text{–}3 \times 10^{-6}$ in). If the mean free path is very small in comparison to some characteristic length in the flow (less than one per cent), then the gas can be treated as a continuous medium. On the other hand, if it is large in comparison to some characteristic length, the gas can no longer be considered continuous and must be analyzed on the molecular scale. The other factor is the elapsed time between collisions. This time must

be sufficiently small so that the random statistical nature of the molecular activity is preserved.

The material in this book will deal primarily with continuous fluids. It will also be assumed that the elastic properties are the same at all points in the fluid and that they are identical in all directions from a specified point. These conditions satisfy the requirements for homogeneity and isotropy, respectively.

1.3. *Fluid Properties*

Certain characteristics of a continuous fluid are independent of the motion of the fluid. These characteristics are called the basic properties of the fluid. In order to adequately discuss these properties, a consistent set of standard units must be defined. In this text the following British system of units will be employed; however, all other consistent systems of units (Appendix C) are equally applicable to all the equations:

Mass	slug
Force	lbf
Length	ft
Time	sec

a. Pressure (p)

The pressure at a point in the fluid can be expressed mathematically as follows:

$$p = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \tag{1-1}$$

where ΔF is the incremental normal force exerted on the incremental area ΔA by the surrounding fluid particles. Actually in the physical case, $\Delta A \to \mathcal{E}^2$ where $\mathcal{E}^2$ is an infinitesimal area over which the medium can be considered a continuum. The pressure* represents the compressive stress in the fluid and has the same magnitude in all directions. The latter statement is demonstrated in the following manner: the hydrostatic pressures (all shearing stresses are zero) acting on a cube in a rectangular Cartesian coordinate system are p_x, p_y, and p_z as shown in Fig. 1.1. Now, if a plane is passed diagonally through the cube at an arbitrary angle so that it forms an area ΔA, the pressure acting on ΔA is also arbitrary and is denoted by p. Since the resulting tetrahedron is in equilibrium, the summation of forces** in any direction is zero. The projection of the arbitrary area ΔA on each of the three Cartesian

*A precise definition of pressure will be given in Chap. Four.

**The general case which includes the body force in the y-direction will be discussed in Chap. Two.

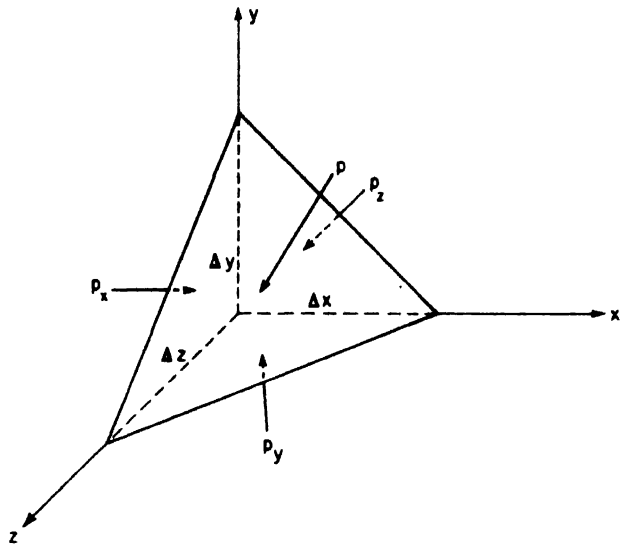


Figure 1.1

planes is given in terms of the cosines of the angles which the normal of plane ΔA makes with each of the three coordinate axes. That is:

$$\text{projection on } y-z \text{ plane} = \Delta A \cos\alpha = \Delta y\,\Delta z/2$$
$$\text{projection on } x-z \text{ plane} = \Delta A \cos\beta = \Delta x\,\Delta z/2$$
$$\text{projection on } x-y \text{ plane} = \Delta A \cos\gamma = \Delta x\,\Delta y/2$$

Hence,

$$\begin{aligned}\Sigma F_x = 0 &= p_x \frac{\Delta y\,\Delta z}{2} - p\,\Delta A\cos\alpha \\ \Sigma F_y = 0 &= p_y \frac{\Delta x\,\Delta z}{2} - p\,\Delta A\cos\beta \\ \Sigma F_z = 0 &= p_z \frac{\Delta x\,\Delta y}{2} - p\,\Delta A\cos\gamma\end{aligned} \tag{1-2}$$

or
$$p_x = p_y = p_z = p$$

Pressure has the units of force per unit area. In the system of units employed in this text, pressure is given in lbf/ft^2 (or lbf/in.^{2}).

b. Density (ρ) Specific Weight (γ), Specific Volume (v_{sp})

The density represents the mass of fluid contained within a unit volume. Density has the units of slugs/ft^3. Mathematically, the density at a point is given as

$$\rho = \lim_{\Delta\tau \to 0} \frac{\Delta m}{\Delta \tau} \tag{1-3}$$

where Δm is the incremental mass of the fluid surrounding the point and $\Delta\tau$ is the incremental volume surrounding the point. Again, in the physical case $\Delta\tau \to \mathcal{E}^3$ where $\mathcal{E}^3$ is the infinitesimal volume over which the substance can be considered a continuum.

Closely related to the density is the specific weight. It represents the weight of fluid per unit volume. The specific weight can be computed from the density by

$$\gamma = \rho g \tag{1-4}$$

where g is the gravitational acceleration (weight per unit mass). In Eq. (1-4), γ has the units of lbf/ft^3.

The specific volume of a fluid is the volume occupied by a unit mass. It is, therefore, given by

$$v_{sp} = \frac{1}{\rho} \tag{1-5}$$

Specific volume has the units of ft^3/slug.

c. Specific Gravity (S)

Of special significance in the study of liquids is the specific gravity. It is the ratio of the specific weight (or density) of a fluid at actual conditions to the specific weight (or density) of pure water at standard conditions (14.7 psi, 68°F).

d. Viscosity (μ)

The viscosity of a fluid is a very important property in the analysis of liquid behavior and fluid motion near a solid boundary. It is the result of intermolecular forces exerted as layers of fluid attempt to slide by one another. The shearing stress between layers of nonturbulent fluid moving in straight, parallel lines can be defined for the Newtonian fluid (see § 1.4g for definition of non-Newtonian fluid) as

$$\tau_{yx} \propto \frac{\partial u}{\partial y} \tag{1-6}$$

where τ_{yx} is the shearing stress on the surface (composed of the x-axis and a unit depth in the z-direction) with normal $\mathbf{n}$ in the y-direction and u is the velocity in the x-direction. That is, the shearing stress between fluid layers is proportional to the change in velocity per length across the layers (the transverse velocity gradient). The effect of viscosity on fluid motion is illustrated in Fig. 1. 2. This represents the variation of

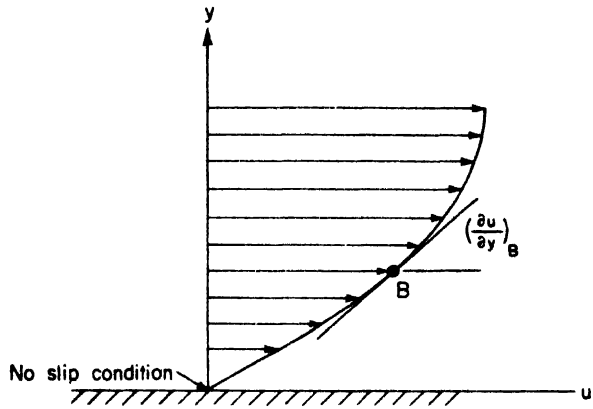


Figure 1.2

fluid velocity for motion near a solid boundary. Experiments have indicated that the velocity is zero at the boundary but increases considerably with distance away from the boundary. Based on additional experimental evidence, the relation between the shearing stress and the transverse velocity gradient is given by

$$\tau_{yx} = \mu \frac{\partial u}{\partial y} \tag{1-7}$$

The constant of proportionality μ in this equation is called the coefficient of viscosity or the dynamic viscosity. The law of shearing stress given by Eq. (1-7) is known as Newton's law of friction. The units of the viscosity coefficient are

$$[\mu] = \frac{[\tau_{yx}]}{\left[\frac{\partial u}{\partial y}\right]} = \frac{\text{lbf-sec}}{\text{ft}^2} \text{ or } \frac{\text{slugs}}{\text{ft-sec}}$$

It is apparent from the shape of the velocity profile in Fig. 1.2 that the shearing stress between the fluid and the wall is greater than the shearing stress between adjacent fluid layers. Later this will lead to the conclusion that, for fluids with small viscosity, the shearing stresses between fluid layers can be neglected except in the vicinity of solid boundaries.

For fluids which conform to Eq. (1-7), the coefficient of viscosity is a function of temperature primarily and pressure very slightly. In most cases the pressure dependence is negligible. An interesting factor in the variation of viscosity coefficient with temperature is that the viscosity of a liquid usually decreases with increasing temperature while the viscosity of a gas always increases with increasing temperature.

Kinematic Viscosity (ν). The ratio of the coefficient of viscosity to the density is called the kinematic viscosity, a quantity in which no force is involved. That is

$$\nu = \frac{\mu}{\rho} \tag{1-8}$$

The units of ν are ft^2/sec. At constant pressure, ν is primarily a function of temperature.

e. Temperature (T)

When two bodies of different heat content are brought into contact while isolated from all other bodies, there is a flow of heat until the bodies are in thermal equilibrium. In other words, when two bodies are in thermal equilibrium,* all changes in properties of the bodies cease. Under this condition they are said to have a property in common; this property is called temperature. Scales of temperature are then defined in terms of certain measurable physical properties such as the volumetric expansion of mercury. The magnitude of temperature can be related to the molecular activity resulting from heat transfer. As an example, the Fahrenheit temperature scale was set up in such a manner that the freezing point of water is designated by 32°F and the boiling point of water is at 212°F. The absolute minimum on any temperature scale is fixed by the condition at which molecular activity from heat transfer stops. In the Fahrenheit scale this is at −459.7°F. For convenience an absolute temperature scale associated with the Fahrenheit system was devised and named the Rankine scale. Thus, 0°R is equivalent to −459.7°F and $T(°R) = 459.7 + T(°F)$.

f. Thermal Conductivity (k)

The thermal conductivity of a substance has the same mathematical form as the viscosity coefficient but with different variables. That is, there is a law known as Fourier's heat-conduction law which states that the conductive heat flow per unit area (or heat flux) q_n is proportional to the temperature decrease per unit distance in a direction normal to the area through which the heat is flowing. Fourier's heat conduction law can be written:

$$q_n \propto -\frac{\partial T}{\partial n} \tag{1-9}$$

*This leads to the Zeroth Law of Thermodynamics. If two systems are equal in temperature to a third, they are equal in temperature to each other.

The constant of proportionality in this equation is called the thermal conductivity. The equation now takes the form

$$q_n = -k\frac{\partial T}{\partial n} \tag{1-10}$$

where the units for k are:

$$[k] = \frac{[q_n]}{[\partial T/\partial n]} = \frac{\text{Btu}}{\text{ft-hr-}^\circ\text{R}} \text{ or } \frac{\text{lbf}}{\text{sec-}^\circ\text{R}}$$

Note: 1 Btu (British thermal unit) = 778 ft-lbf.

g. Specific Heat

The inherent thermal properties of a flowing gas become important when the energies are considered. The specific heat is such a quantity. The specific heat is defined as the amount of heat required to raise the temperature of a unit mass of medium by one degree. The value of the specific heat depends on the type of process involved in raising the temperature of the unit mass. The two common processes used in evaluation of specific heat are the constant volume process and the constant pressure process. The specific heats of the processes above are designated by C_τ and C_p, respectively.

The definitions for these quantities are (see Appendix B)

$$C_\tau = \left(\frac{\partial E}{\partial T}\right)_\tau \tag{1-11}$$

where E is the internal energy per unit mass of the fluid and C_τ is a measure of the energy-carrying capacity of the gas molecules, and

$$C_p = \left(\frac{\partial H}{\partial T}\right)_p \tag{1-12}$$

where $H = E + pv_{sp}$ is the enthalpy or total heat content per unit mass of fluid and C_p is a measure of the ability of the gas to do external work in addition to internal energy (see Appendix B). The units of the specific heats can be deduced from the above mathematical definitions. That is

$$[C_\tau] = \frac{[E]}{[T]} = \frac{\text{Btu}}{\text{slug-}^\circ\text{R}} \text{ or } \frac{\text{ft-lbf}}{\text{slug}^\circ\text{R}}$$

$$[C_p] = \frac{[h]}{[T]} = \frac{\text{Btu}}{\text{slug-}^\circ\text{R}} \text{ or } \frac{\text{ft-lbf}}{\text{slug}^\circ\text{R}}$$

The ratio of specific heats given by

$$\kappa = \frac{C_p}{C_\tau} \tag{1-13}$$

appears often in the study of compressible fluids and has special significance. This ratio is a measure of the relative internal complexity of the molecules of the gas. From the kinetic theory of gases, it has been determined that the ratio of specific heats can be related to the degrees of freedom of the gas molecules by

$$\kappa = \frac{n+2}{n}$$

At normal temperatures, there are six possible degrees of freedom, three translational and three rotational. For nitrogen (a diatomic gas), $n = 5$ because one of the rotational degrees of freedom is small in comparison with the other two.

Thus

$$\kappa = \frac{5+2}{5} = \frac{7}{5} = 1.4 \quad \text{(nitrogen)}$$

A monatomic gas has 3 degrees of freedom in translation only. Hence,

$$\kappa = \frac{3+2}{3} = \frac{5}{3} = 1.67 \quad \text{(helium)}$$

The specific heats of a compressible gas are related to the gas constant, R. For a perfect gas (see Appendix B) this constant is found to be

$$R = C_p - C_v \tag{1-14}$$

Or, in terms of the universal gas constant, by

$$R = \frac{\mathscr{R}}{m} \tag{1-15}$$

where

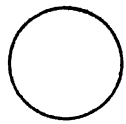
$$\mathscr{R} = 1545 \frac{\text{ft-lbf}}{\text{lbm-mole}^\circ\text{R}}, \; m \text{ is the molecular wt. of gas}$$

From Eq. (1–14), it can be seen that R has the same units as the specific heats.

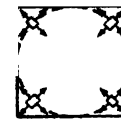
h. Surface Tension (σ_s)

The behavior of the free surface of a liquid or the interface between two immiscible liquids has long been an object of some interest and importance. The tendency for a soap film to form a surface of minimum area for its given volume (e. g., a sphere) and other similar phenomena has led to the common concept of surface tension, the apparent tangential force per unit length acting on a free liquid surface. Actually, a more accurate description of the phenomena is in terms of a free surface energy. This energy arises as a consequence of work being done on a molecule to bring it to a free surface or interface. This work is the result of action against an unbalance of molecular forces acting on

Figure 1.3



Circular Frame



Rectangular Frame

the surface molecule. An illustration will serve to clarify the concept. Consider the frames of Fig. 1.3. A soap film is transferred from the circular frame to the rectangular frame. In the process one strip of the film "stretches" to fill in corners of the rectangle. The film itself is inelastic so that in the "stretching" process any new film surface must be created by the projection of new molecules into the surface by the unbalance of molecular forces. The work done per unit area of surface formation, or the stretching force per unit length can be given by

$$E_{fs} = \frac{\text{work done}}{\text{unit area}}$$

$$\sigma_s = \frac{\text{stretching force}}{\text{unit length}}$$

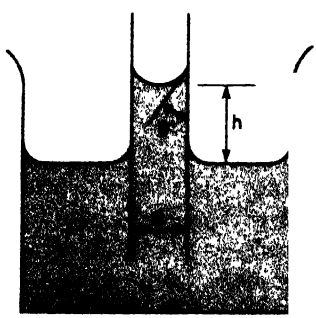
where E_{fs} is the free surface energy and σ_s the surface tension. The latter is numerically equivalent to the free surface energy per unit area.

Also associated with surface tension is the capillary deflection of liquids in small tubes. If a glass tube is inserted into a beaker of water (Fig. 1.4a), the water will eventually rise in the tube and display a concave spherical surface (called a meniscus). The deviation of the water level in the tube from that in the beaker can be given by

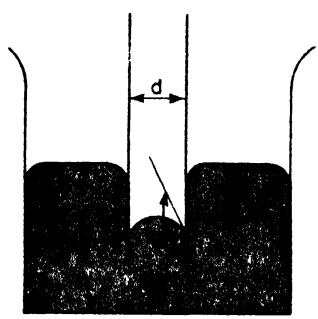
$$h \propto \frac{\sigma}{\gamma} \cos\theta \tag{1-16}$$

where θ is the angle between the tangent to the water surface and the surface of the glass.

Figure 1.4



Water



Mercury

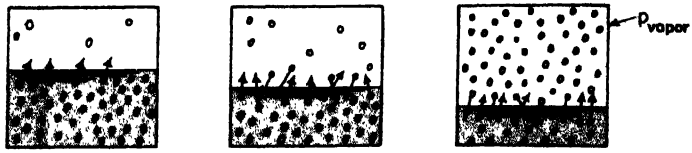


Figure 1.5

Water is a substance which "wets" the glass surface creating a concave meniscus and small contact angle. Mercury, a substance which does not wet the glass surface, has different surface properties. In fact, mercury in contact with a glass surface will display a convex meniscus. In addition if a capillary tube is inserted into a beaker of mercury, the mercury level in the tube will drop below that in the beaker. This is consistent with Eq. (1-16) since the angle of contact is now greater than 90° (Fig. 1.4b).

i. Vapor Pressure

All liquids have a tendency to evaporate when exposed to a gaseous atmosphere. The rate at which this evaporation occurs is dependent on the molecular energy of the liquid (that is, the type of material and its temperature) and the condition of the atmosphere adjoining the liquid. Consider the liquid contained in the sealed beaker of Fig. 1.5. Assume that a constant temperature is maintained within the beaker. Some of the liquid molecules have sufficient energy to break away from the liquid and enter the air in the vapor state (Fig. 1.5a). As time passes the air will contain vapor molecules to the extent that some of them will be forced to re-enter the liquid by the partial pressure of the air (Fig. 1.5b). Eventually an equilibrium condition will exist when the rate at which molecules are entering the liquid is equal to the rate at which they are leaving (Fig. 1.5c). In this condition the air above the liquid is saturated with liquid vapor molecules and the pressure on the liquid surface is called the vapor pressure. When a liquid has a very high vapor pressure it evaporates readily and is called a volatile liquid. The phenomenon of boiling of a liquid is closely related to the vapor pressure. In fact when the vapor pressure of a liquid slightly exceeds the pressure impressed on its surface, the liquid will boil. This means that boiling can be achieved either by raising the temperature of the liquid to raise its vapor pressure or by lowering the pressure of the surrounding gas below the liquid's vapor pressure.

The problem of cavitation encountered in hydromachinery and hydrofoil applications is a direct result of local pressures being less than or equivalent to the vapor pressure of the fluid. When this condition prevails, vaporous bubbles or cavities are formed in the flow. The unsteady nature of these cavities and their ultimate collapse with accom-

panying high pressures often lead to the pitting and erosion of machinery surfaces.

f. Bulk Modulus of Elasticity (K)

The bulk modulus of elasticity is defined as the ratio of the net increase of pressure on an element of fluid to the unit strain produced by the pressure change. That is

$$K = \lim_{\Delta v_{sp} \to 0} \frac{-\Delta p}{\Delta v_{sp}/v_{sp}} = \frac{dp}{(d\rho/\rho)} \tag{1-17}$$

Since $d\rho/\rho$ represents the relative change in density brought about by the pressure change dp, it is apparent that the bulk modulus is the inverse of the compressibility of the material at a given temperature. For example, the bulk moduli of elasticity for water and air are 300,000 psi and 15 psi, respectively, which means that air is about 20,000 times more compressible than water.

A tabulation of important properties for some common liquids and gases is presented in Table E-1.

1.4. *Regimes in the Mechanics of Fluids*

The various types of flow in fluid mechanics are distinguished according to the fluid properties which characterize the physical situation. These controlling factors are expressed in terms of certain measurable quantities such as the velocity of a body through the fluid, the density of the fluid, the viscosity of the fluid, etc. Furthermore, all of these factors can be put in nondimensional forms which are referred to as significant "numbers" (see Appendix C for a discussion of dimensional analysis). Thus, fluids of all kinds can be generally classified into the following categories:

a. Ideal Fluids (see Chaps. Six and Seven)

This branch of fluid dynamics is often called classical hydrodynamics. The fluid is assumed to be inviscid (zero viscosity) and incompressible (constant density), hence there is no tangential force between adjacent fluid layers. An extensive mathematical theory has been developed for the ideal fluid. Although the theory of ideal fluids fails to explain the phenomena of actual fluids, it gives reliable results in the calculation of lift, induced drag, and wave motion for gas flow at low velocity and for water.

b. Viscous Incompressible Fluids

The theory of viscous incompressible fluids in which the fluid den-

sity is assumed to be constant has widespread application to the flow of liquids (such as water) and the flow of air at low velocity. It explains the phenomena of the viscous forces, flow separation, and eddy flows. This branch of fluid mechanics can be subdivided into two classes:

(i) *Slightly Viscous Fluids* (See Chaps. Eight through Ten). These fluids exhibit only small shearing force when in motion except at very high speeds. The flow of slightly viscous fluids can be of two types: a smooth, orderly motion classified as laminar flow and a random, fluctuating motion classified as turbulent flow. The nature of a viscous flow can be characterized by a nondimensional parameter, the Reynolds number, $Re = uL/\nu$, which is defined as the ratio of the dynamic pressure ρu^2 and $\mu u/L$, a typical shearing stress. The flow of a slightly viscous fluid is characterized by a large Reynolds number with the value of this parameter being smaller for laminar flow than for turbulent flow. Light oil and water are examples of fluids with small viscosity; however, the flow of air at low speeds is also included here.

(ii) *Very Viscous Fluids* (see § 8. 7). The flow of these fluids is characterized by the presence of very large viscous forces and small inertia forces. The study of these fluids leads to applications of the theories of lubrication, visco-elasticity and plasticity. Typical fluids in this category are heavy oil and asphalt. The Reynolds number in this case is much smallar than in the previous case.

c. Gasdynamics—Theory of Compressible Flow (see Chaps. Eleven and Twelve)

The realm of gasdynamics deals with the dynamics and thermodynamics of the flow of a compressible fluid past a body. Gasdynamics can be further subdivided into the fields of study commonly referred to as subsonic ($M < 1$), transonic ($M \sim 1$), supersonic ($1 < M \leq 6$), and hypersonic ($6 < M \leq 40$) gasdynamics. The symbol M represents the Mach number which is the ratio of fluid velocity and fluid sonic velocity. The classification between supersonic and hypersonic flows is arbitrary because it depends on the shape of the body. Since an increase of velocity is accompanied by a change of density and temperature, gasdynamics is a combination of two sciences, fluid mechanics and thermodynamics. It is also called aerothermodynamics. In addition to Mach number, the controlling factors in this case are the Reynolds number, Prandtl number, $Pr = \mu C_p/k$, and the ratio of specific heats $\kappa = C_p/C_v$. The Prandtl number may be written as the ratio of kinematic viscosity and thermal diffusivity $\nu/(k/\rho C_p)$, and it is a measure of the relative importance of viscosity and heat conduction.

Regardless of speed ranges, the theory of gasdynamics can be divided into two parts, inviscid gasdynamics and viscous gasdynamics. The inviscid theory is important in the calculation of nozzle characteristics, shock waves, lift, and wave drag of a body, while the viscous theory is applicable to the calculation of the skin friction and heat transfer characteristics of a body moving through gas at very high speeds.

d. Rarefied Gasdynamics (Superaerodynamics)

This phase of fluid motion departs from the concept of continuum mechanics under which the previous groups have been defined. A dimensionless parameter, the Knudsen number, $Kn = \lambda/L$, aptly describes the degree of departure from continuum flow.

Here λ is the molecular mean free path and L is some characteristic length. Based on the Knudsen number, the following groups exist:

Continuum ($Kn \leq 0.01$). The equations of viscous compressible flow are applicable in this regime. Implicit in the use of these equations is the condition of zero velocity at a solid boundary (no-slip condition; see Fig. 1. 2).

Slip Flow ($0.01 < Kn \leq 0.10$). In most cases in this regime the continuum gasdynamics analysis is valid provided the slip boundary conditions are employed. These conditions provide for a finite velocity and a temperature jump at a solid boundary.

Transition Flow ($0.10 < Kn \leq 10$). The kinetic theory of gases must be employed to adequately describe this flow.

Free-Molecule Flow ($Kn > 10$). In this flow the molecules are so widely dispersed that molecular interaction can be neglected.

All of the superaerodynamic regimes are encountered at high altitude, the molecular density becoming less with increasing altitude.

e. Magnetofluidmechanics

The subject of magnetofluidmechanics is an extension not only of fluid mechanics, but also of four other classes of engineering science; namely, thermodynamics, mechanics, materials, and the electrical sciences. This discipline was initiated a relatively short time ago by astrophysicists and has since played an important role in engineering. Several alternative names which are widely used are; magnetofluidynamics, magnetohydrodynamics, magnetogasdynamics, and hydromagnetics.

Magnetofluidynamics is the study of the motion of an electrically conducting fluid in the presence of a magnetic field. The motion of the electrically conducting fluid in the magnetic field will induce electric currents in the fluid, thereby modifying the field. At the same time it will produce mechanical forces which also modify the motion.

The interaction between the field and the motion makes magnetofluidynamic analysis both difficult and interesting.

A gas at ordinary and moderately high temperatures is a nonconductor, but at very high temperatures (above 1.8×10^4 °R), thermal excitation takes place. This leads to dissociation and ionization. Ionized gas is often called a plasma which is an electrical conducting gas by virtue of its charged particles due to free electrons. Highly ionized gases are present in the sun and other stars.

The important parameters for magnetofluidynamics are about seventeen in number, too numerous to be mentioned here without going into the physics of the subject. Examples of engineering problems involving the flow of electrically conducting fluids are the re-entry of missiles and satellites, the plasma jet, controlled fusion research, and the magnetohydrodynamics generator.

f. Flow of Multicomponent Mixtures

This field is another extension of basic fluid mechanics. The flow of a single, homogeneous fluid which has been considered in basic fluid mechanics is enlarged upon here by the consideration of the flow of chemically reacting component mixtures. In addition to the momentum and energy transport which have been considered in fluid mechanics, mass transport must also be included in the present treatment. The property controlling mass transfer is the diffusion coefficient or mass diffusivity, D, which is analogous to the thermal diffusivity $k/\rho C_p$ and the kinematic viscosity (momentum diffusivity) ν, and has the same units, ft^2/sec. The dimensionless parameter in this case is called the Schmidt number, $Sc = \nu/D$. Areas associated with mixture flows are the so-called rate processes which include chemical kinetics, transport processes, distillation, absorption, and extraction.

g. Non-Newtonian Fluids

Not all fluids conform to Newtonian friction law of Eq. (1-7). Such "non-Newtonian" fluids demonstrate a nonlinear dependence of shearing stress on velocity gradient. Fluids belonging to this category are any number of thick gooey substances such as paste, printers' ink, and slurries.

There are a great number of empirical equations which have been proposed to express the steady-state relation between τ_{yx} and $\partial u/\partial y$. As an example, the following two-parameter equation was proposed by Ostwald-de Waele

$$\tau_{yx} = \alpha \left|\frac{du}{dy}\right|^{n-1} \frac{du}{dy} \tag{1-18}$$

For $n = 1$, this equation reduces to Newton's law of viscosity with $\alpha = \mu$ [see Eq. (1-7)].

The above classification is presented with the thought that the reader should grasp a general picture of the fluid mechanics and its related fields before attempting to study any particular branch of the subject. It is vitally important that the reader recognize the limitations in the application of the theory to any specific problem.

NOMENCLATURE FOR CHAPTER ONE

C_p	specific heat at constant pressure, Eq. (1-12)
C_τ	specific heat at constant volume, Eq. (1-11)
E	internal energy per unit mass, Eq. (1-11)
h	enthalpy per unit mass, Eq. (1-12)
g	gravitational acceleration (weight per unit mass), Eq. (1-4)
k	thermal conductivity, Eq. (1-10)
K	bulk modulus of elasticity, Eq. (1-17)
Kn	Knudsen number, § 1.4d
m	(1) mass, Eq. (1-3); (2) molecular wt of gas, Eq. (1-15)
M	Mach number, § 1.4c
n	(1) normal direction, Eq. (1-9) (2) number of degrees of freedom, § 1.3g
p	pressure, Eq. (1-1)
Pr	Prandtl number, § 1.4c
q_n	heat flux, Eq. (1-9)
$\mathscr{R}$	universal gas constant, Eq. (1-15)
Re	Reynolds number, § 1.4b(i)
S	specific gravity, § 1.3c
Sc	Schmidt number, § 1.4f
T	temperature, § 1.3e
u	velocity component in x-direction, Eq. (1-6)
v_{sp}	specific volume, Eq. (1-5)
x, y, z	Cartesian coordinates
α, β, γ	directional angles made by the normal to the surface from the origin, Eq. (1-2)
γ	specific weight, Eq. (1-4)
$\mathscr{E}$	an infinitesimal length, § 1.3a
κ	ratio of specific heats, Eq. (1-13)
λ	molecular mean free path, § 1.4d
μ	coefficient of viscosity, Eq. (1-7)
ν	kinematic viscosity, Eq. (1-8)
ρ	density, Eq. (1-8)
σ_s	surface tension, § 1.3h
τ_{yx}	shearing stress on the surface with normal in the y-direction, Eq. (1-7)

2

FLUID STATICS AND RELATIVE MOTION OF LIQUIDS

In §1.1 we have discussed that the study of all aspects of fluid behavior can be divided into three categories—statics, kinematics, and dynamics. In this chapter the science of fluid statics will be treated. If all particles* of the fluid are at rest with respect to a reference coordinate system, the fluid is considered to be static. On the other hand there are cases in which the fluid elements may be at rest with respect to each other or with respect to the boundaries enclosing them, but not with respect to a reference coordinate system. In these cases the fluid elements and the container enclosing the fluid may be moving with a uniform accelerated or decelerated motion, and the law of hydrostatics, taking into account the effects of uniform acceleration, can still be applied. Hence, when a fluid is either at rest or in relative equilibrium, the fluid elements are not subjected to any shearing stress.

The definition of the pressure at a point in a fluid is given in Eq. (1–1). In §1.3a we have shown that for a fluid system in equilibrium, the pressure at every point is independent of orientation, i. e.,

$$p_x = p_y = p_z = p \tag{1-2}$$

in rectangular Cartesian coordinates. Hence, the pressure of the fluid

*A particle is an infinitesimally small volume of the medium containing a large number of molecules.

is a scalar quantity. Equation (1–2) can be stated that at a point in a fluid the pressure is exerted equally in all directions. This is known as Pascal's principle. In deriving Eq. (1–2) we have neglected the external force* $(\gamma/6)\Delta x\,\Delta y\,\Delta z$ in the y-direction (Fig. 1.1).

If the gravitational force is added to Eq. (1–2), we have (see Fig. 1.1)

$$\sum F_x = 0 = p_x\left(\frac{\Delta y\,\Delta z}{2}\right) - p\Delta A\cos\alpha = (p_x - p)\frac{\Delta y\,\Delta z}{2}$$

$$\sum F_y = 0 = p_y\left(\frac{\Delta x\,\Delta z}{2}\right) - p\Delta A\cos\beta - \frac{\gamma}{6}(\Delta x\,\Delta y\,\Delta z)$$

$$= \left[(p_y - p) - \frac{\gamma}{3}\Delta y\right]\frac{\Delta x\,\Delta z}{2}$$

$$\sum F_z = 0 = p_z\left(\frac{\Delta x\,\Delta y}{2}\right) - p\Delta A\cos\gamma = (p_z - p)\frac{\Delta x\,\Delta y}{2}$$

As Δx, Δy, and Δz become infinitesimal, the term $\frac{1}{3}\gamma\Delta y$ in the above equation is a higher-order term which can be neglected in comparison with the other terms. Hence, for a fluid system in equilibrium a pressure at every point is represented by Eq. (1–2).

In §2.1 the fundamental differential equation of fluid statics is obtained in general vector form. Special forms of the equation for incompressible and compressible fluids, respectively, are given.

In §2.2 the application of the incompressible static pressure equation to methods of pressure measurement is discussed. Several types of pressure-measuring devices are explained and illustrated.

In §2.3 the application of the aerostatic equation in determining the variation of atmospheric properties in both the troposphere and stratosphere is discussed. The standard atmosphere is defined.

In §2.4 the hydrostatic equation is employed to determine the force exerted by a fluid on a submerged body. Both plane and curved surfaces are considered.

In §2.5 the principle of buoyancy is discussed. The Archimedes principle is explained.

In §2.6 the stability of bodies in a static fluid is discussed. Both the submerged body and the floating body are considered.

In §2.7 relative motion of liquids with uniform translational acceleration is discussed. The equation of the surface of liquids is obtained.

In §2.8 relative motion of liquids with uniform rotation is dis-

*External, body, volume, extraneous, and impressed forces are all equivalent in the scope of an ordinary fluid mechanics book. Volume or body forces usually include inertial forces, gravity forces, and electromagnetic forces. Such a force is proportional to the mass. External forces are gravity forces and electromagnetic forces which are more appropriate to use in the treatment of fluid statics.

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cussed. The equation of the surface of liquids in this case corresponds to a paraboloid of revolution.

A. FLUID STATICS

2.1. Fundamental Equation of Fluid Statics

Consider a fluid mass at rest which occupies a volume τ bounded by the surface A (Fig. 2.1). Let $d\tau$ be an element of volume and $\rho d\tau$

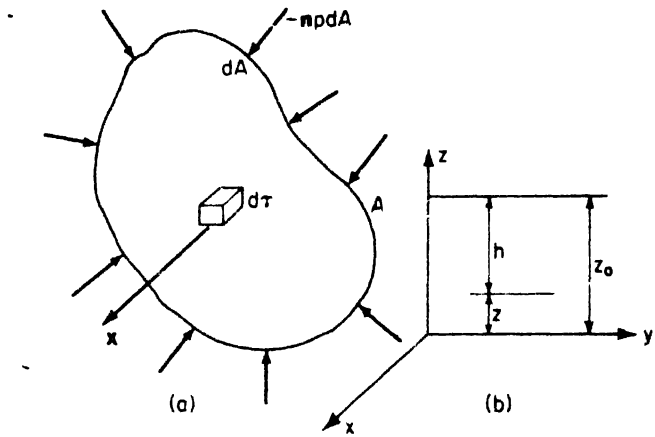


Figure 2.1

the elementary mass. The volume τ is in equilibrium under the action of the following forces:

(a) The resultant of the external forces is

$$\mathbf{F}_B = \iiint \mathbf{X}\rho \, d\tau \tag{2-1}$$

where $\mathbf{X}$ represents the external force per unit mass acting upon the elementary volume $d\tau$.

(b) The resultant of the surface forces is

$$\mathbf{F}_S = -\iint \mathbf{n}p \, dA \tag{2-2}$$

where $\mathbf{n}$ is the unit vector normal to the element of surface, taken positive when directed outwards, and the direction of $\mathbf{F}_S$ is acting inwards. In accordance with Gauss' theorem (Appendix A), Eq. (2-2) can be expressed as

$$\mathbf{F}_S = -\iint \mathbf{n}p \, dA = -\iiint \nabla p \, d\tau \tag{2-3}$$

For equilibrium, we have

$$\mathbf{F}_B + \mathbf{F}_S = \iiint (\mathbf{X}\rho - \nabla p)\, d\tau = 0 \tag{2-4}$$

Equation (2–4) is true for all τ no matter how small, and hence the fundamental equation of fluid statics is obtained,

$$-\nabla p = -\mathbf{X}\rho \tag{2-5}$$

It should be mentioned that the Euler equation [Eq. (5–20)] reduces to Eq. (2–5) when the inertial forces are considered to be negligible.

If gravity is the only external force acting on the fluid to be considered, the resulting scalar equations in rectangular Cartesian coordinates are obtained from Eq. (2–5) as

$$\frac{\partial p}{\partial x} = X_x \rho = 0 \tag{2-6a}$$

$$\frac{\partial p}{\partial y} = X_y \rho = 0 \tag{2-6b}$$

$$\frac{\partial p}{\partial z} = X_z \rho = -\rho g = -\gamma \tag{2-6c}$$

where X_z is the external force per unit mass in the positive direction of the z-axis and $g = -X_z$, the acceleration of gravity. Equations (2-6a) and (2-6b) indicate that the pressure is constant in planes normal to the z-axis. The variation of pressure in the z-direction is given by Eq. (2–6c) in the form

$$dp = -\rho g\, dz = -\gamma\, dz \tag{2-7}$$

In general, the fundamental equation of fluid statics, Eq. (2–7), cannot be integrated unless the nature of ρ and g are specified. For most practical applications the gravitational acceleration, g, can be considered constant. The determination of the pressure from Eq. (2–7) is therefore treated separately: the case for constant density (incompressible fluids), and the case for variable density (compressible fluids).

a. Constant Density Solution (Incompressible Fluids)

When the density is constant, Eq. (2–7) yields, after integrating,

$$p = -\rho g z + C = -\gamma z + C \tag{2-8}$$

where C is the constant of integration. If we choose the subscript 0 to represent conditions at the free surface of the fluid, we have

$$p - p_0 = \gamma(z_0 - z) = \gamma h \tag{2-9a}$$

or

$$z_0 = \left(\frac{p - p_0}{\gamma}\right) + z \tag{2-9b}$$

where h is the vertical distance below the free surface (Fig. 2.1b) corresponding to a pressure p, and z_0 the height of the free surface above an arbitrary datum. Equation (2–8) is usually referred to as the hydrostatic equation.

Thus the difference in pressure between two elevations in a static incompressible fluid can be expressed in terms of the height of the fluid between the elevations (the head of the fluid). This result is known as Torricelli's principle which is the basis for pressure-measuring devices such as manometers. Applications of Eq. (2–9) to various pressure-measuring devices will be discussed later in this chapter.

b. Variable Density Solution (Compressible Fluids)

For a perfect gas (Appendix B) the equation of state relating the density to the pressure and temperature is

$$\rho = \frac{p}{RT}$$

where R is the gas constant [see Eqs. (1–15) and (B–10b)]. With this expression for ρ, Eq. (2–7) can be written

$$\frac{dp}{p} = -\frac{g}{RT}\,dz \tag{2-10}$$

Thus, the pressure variation in a static compressible fluid can be determined from Eq. (2–10) if the temperature as a function of z is known. Equation (2–10), which may be called the "aerostatic equation", has an important application in the determination of the properties of the atmosphere. We shall discuss this type of application in §2.3.

2.2. *Application of the Hydrostatic Equation* [Eq. (2–8)]

The fundamental problem of fluid statics is the determination of the static pressure in a fluid. Various devices for measuring static pressure have been developed, based on the result given by Eq. (2–9). These devices are generally known as manometers. Before proceeding with a description of several types of manometers, it is advantageous to define the scales of pressure measurement. Pressure is usually expressed with reference to either absolute zero pressure (a complete vacuum) or local atmospheric pressure. However, any other arbitrary datum may also be used as reference. The absolute pressure is the

pressure expressed as a difference between its value and the absolute zero pressure. When a pressure is expressed as a difference between its value and the local atmospheric pressure, it is called a gage pressure. These two scales for pressure measurement can be expressed as follows:

$$p_{ab} = p - 0 = p \tag{2-11a}$$

$$p_{gage} = p - p_{atm} \tag{2-11b}$$

If the pressure p is less than the atmospheric pressure, the gage pressure has a negative value. This is called a vacuum pressure. The two scales for pressure measurement are illustrated in Fig. 2.2.

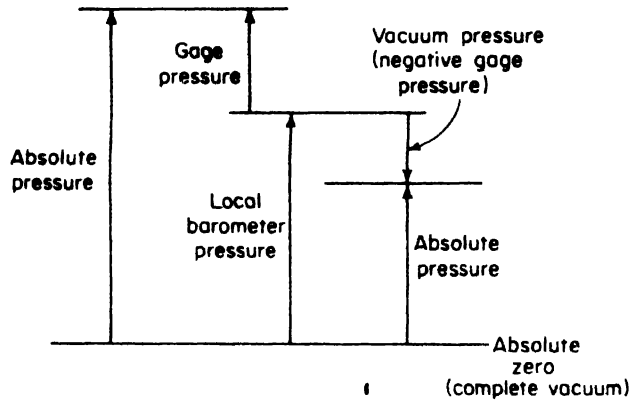


Figure 2.2

a. Simple Manometers

(i) *Barometer* (Fig. 2.3). The barometer is a device for measuring the local atmospheric pressure. It consists of an evacuated glass tube

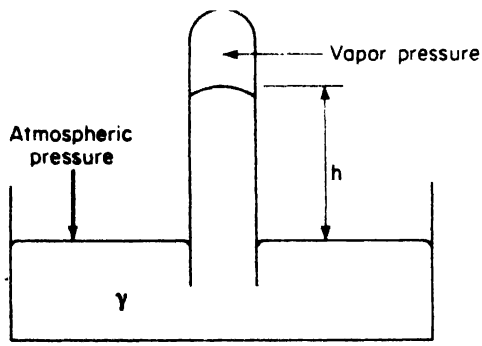


Figure 2.3

with open end submerged in a reservoir of fluid, usually mercury. The atmospheric pressure exerted on the surface of the mercury in the reservoir causes the mercury to rise up the tube against the pressure of the mercury vapor. Although this mercury vapor pressure is a function of temperature, it is exceedingly small at usual atmospheric temperatures and can be considered as vacuum. Hence the mercury barometer actually measures the difference in pressure between the atmosphere and almost vacuum which is the absolute pressure of the atmosphere. According to Eq. (2-9) the local atmospheric pressure is given as

$$p_{\text{atm}} = \gamma h \tag{2-12a}$$

where γ is the specific weight of the manometer fluid considered.

Under standard atmosphere [see Eq. (2-19)] the air has a specific weight of approximately 0.0765 lbf/ft^3 at a sea-level pressure of 14.7 lbf/in.^{2} and a temperature of 59° F. From Eq. (2-12a) the head h for different liquids corresponding to the standard atmosphere can be immediately determined. If the specific weight of water is taken as 62.4 lbf/ft^3, Eq. (2-12a) may be written in terms of the value for γ and the specific gravity S

$$p_{\text{atm}} = 62.4\,Sh \tag{2-12b}$$

Under standard atmosphere the head for water is

$$h = \frac{14.7 \times 144}{62.4} = 34 \text{ ft}$$

and for mercury ($S = 13.6$)

$$h = \frac{14.7 \times 144}{13.6 \times 62.4} = 2.5 \text{ ft or } 30 \text{ in.}$$

(ii) *Piezometer* (Fig. 2.4). The Piezometric tube is used in determining the static pressure head of flowing liquid at any section of a

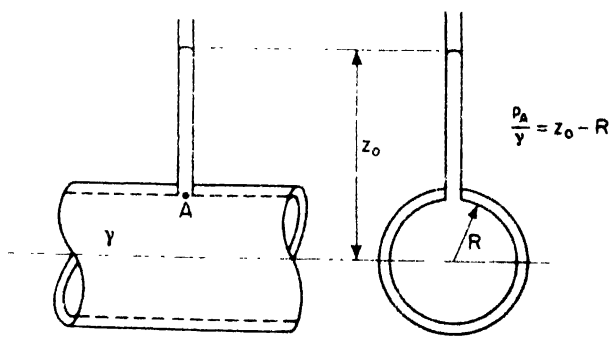


Figure 2.4

pipe. As seen in Fig 2.4. the Piezometer consists of a tube with the bottom opening mounted flush with the inside wall of the pipe and the top opening exposed to atmospheric pressure. The height h together with the known specific weight of the liquid determines the static pressure [Eq. (2–9b)] of the flowing liquid at any specified section. A Piezometer is frequently used for measuring moderate pressures in liquids.

b. Differential Manometers

One of the most commonly employed devices for measuring pressure is the differential manometer (Fig. 2.5). It consists of a continuous tube open at both ends and bent into a U shape. The tube is partially filled with a liquid so that when different pressures, p_A and p_B, are applied to the two ends of the tube, respectively, the fluid is deflected in proportion to the pressure difference. Thus, differential manometers measure the gage pressure.

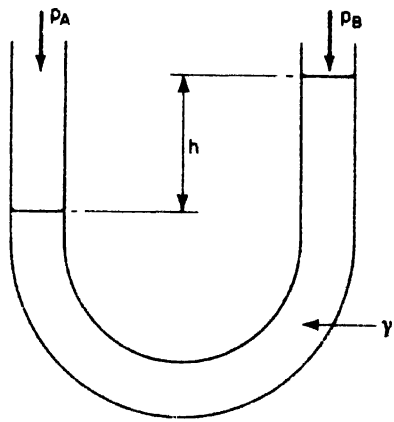


Figure 2.5

Consider the manometer in Fig. 2.6 in which the pressure at B is to be determined while the pressure at

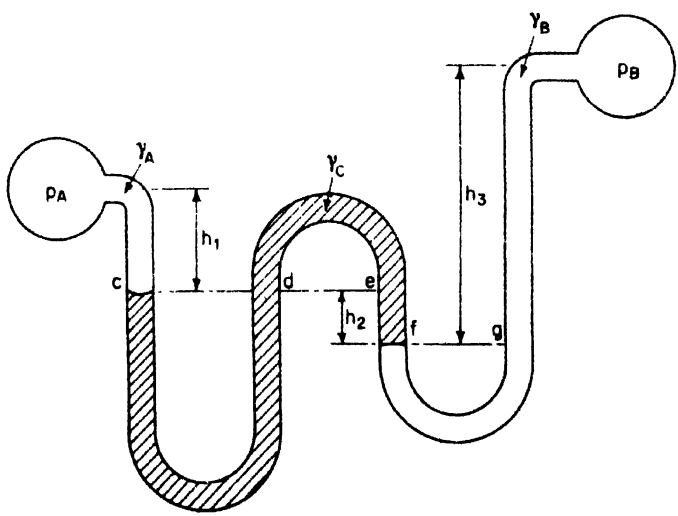


Figure 2.6

A is known. If the specific weight of the manometer liquid is γ_C and the specific weight of the fluid at A and B are γ_A and γ_B, respectively, the equation for pressure at A and B may be written [see Eq. (2–9)],

$$p_A = p_C - \gamma_A h_1 \tag{2–13a}$$

$$p_B = p_g - \gamma_B h_3 \tag{2–13b}$$

But the equation for pressure at f ($p_f = p_g$) is

$$p_g = p_f = p_c + \gamma_c h_2 = p_c + \gamma_c h_2 \tag{2–13c}$$

Combining Eqs. (2–13a)–(2–13c), we obtain the result for the pressure at B

$$p_B = p_A + \gamma_A h_1 + \gamma_C h_2 - \gamma_B h_3 \tag{2–13d}$$

The preceding example represents the solution of a typical problem of the differential manometer. When the reference pressure and the specific weights of the manometer fluids are known, the unknown pressure at the other end of a manometer can always be determined from the known quantities and the deflection of the fluid column.

For the determination of very small differences in pressure or precise determination of large pressure differences, the micromanometer may be used. The reader may refer to a text on fluid measurement for detailed discussions (e. g., Ref. 46).

Illustrative Example 2.1. (a) In Fig. 2.A, if the air pressure is 15 lb/in.^{2} abs, what is the absolute pressure in pounds per square inch at A?

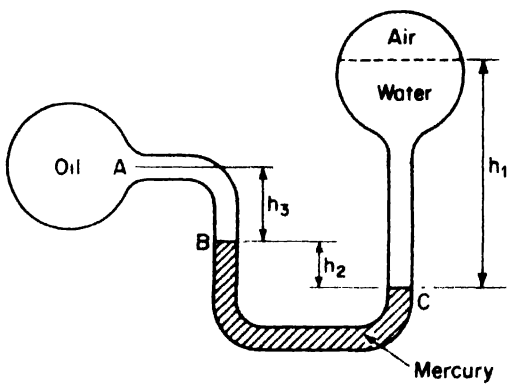


Figure 2.6

($h_1 = 10$ in., $h_2 = 3$ in., $h_3 = 6$ in.). (b) If the oil in the left-hand limb of Fig. 2.A is replaced by air, what is the pressure at A in order to maintain the same configuration as given in (a)?

Solution.

(a) $p_{air} + h_1\gamma_{H_2O} = h_3\gamma_{oil} + h_2\gamma_{Hg} + p_A$

$$p_A = 15 + \left[\frac{10}{12} - \frac{6}{12}(0.8) - \frac{3}{12}(13.6)\right]\left(\frac{62.4}{144}\right)$$

$$p_A = 15 + [0.833 - 0.4 - 3.4](0.434)$$

$$p_A = 15 - 1.286 = 13.71 \text{ lbf/in.}^2$$

(b) $$p_A = p_B = p_{air} + h_1\gamma_{H_2O} - h_2\gamma_{Hg}$$

$$p_A = 15 + \left[\frac{10}{12} - \frac{3}{12}(13.6)\right]\left(\frac{62.4}{144}\right)$$

$$p_A = 15 - 1.113 = 13.89 \text{ lbf/in.}^2$$

2.3. *Application of the Aerostatic Equation* —Atmospheric Equilibrium

The determination of the static properties of the atmosphere surrounding us is of considerable importance to aeronautical engineers and meteorologists. Since air is a compressible fluid, its density, in general, is a function of pressure and temperature. For a perfect gas such as air, the equation of state [Eq. (B–10b)] relates the density to the pressure and temperature. Hence, the pressure variation in the atmosphere can be determined by solving Eq. (2–10) provided the temperature and altitude relation is known.

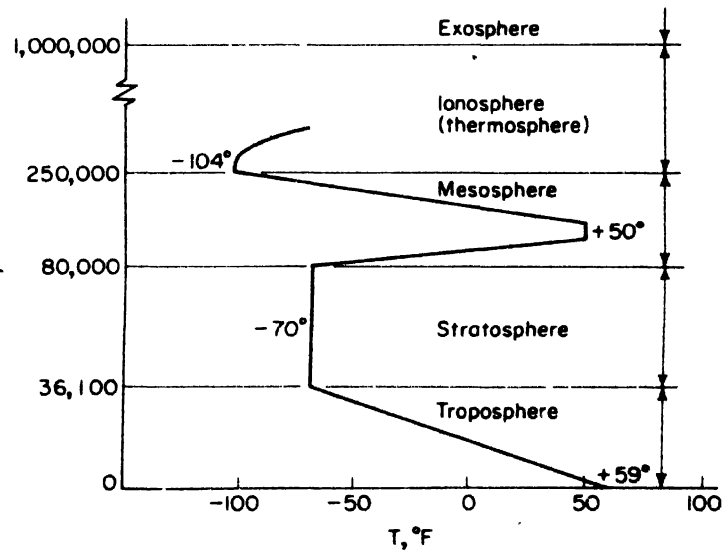


Figure 2.7

Experimental evidence of the temperature variation with altitude in different layers of the atmosphere is shown in Fig. 2.7. From the

standpoint of a single gas and continuum analysis the layers of primary importance are the troposphere and the stratosphere. Therefore, we shall investigate the atmospheric properties of these two layers by using the aerostatic equation [Eq. (2–10)].

a. The Troposphere

The temperature-altitude variation in the troposphere (the part of the atmosphere lying between the earth's surface and the stratosphere) has been found to be linear and may be written

$$T = T_0 - \alpha z \tag{2–14}$$

where $T_0 = 519°\text{R}$ is the absolute sea-level temperature and α is the constant lapse rate. In the "standard atmosphere" [see Eq. (2–19)] adopted for aeronautical calculations, the lapse rate is taken as $0.00365°\text{F/ft}$. Substituting Eq. (2–14) and its derivative into Eq. (2–10) we obtain

$$\frac{dp}{p} = \frac{g}{R\alpha}\frac{dT}{T} \tag{2–15}$$

Integration of Eq. (2–15) yields

$$\ln p = \frac{g}{R\alpha}\ln T + C \tag{2–16a}$$

The constant of integration C can be obtained at the sea-level condition. Hence, Eq. (2–16a) becomes

$$\ln\frac{p}{p_0} = \frac{g}{R\alpha}\ln\frac{T}{T_0} = \frac{g}{R\alpha}\ln\left(1 - \frac{\alpha z}{T_0}\right) \tag{2–16b}$$

from which the relation between pressure p and altitude z is obtained

$$\frac{p}{p_0} = \left(1 - \frac{\alpha z}{T_0}\right)^{g/R\alpha} \tag{2–16c}$$

The density-altitude relation can be derived from the equation of state for a perfect gas with the aid of Eq. (2–16c). The result is

$$\frac{\rho}{\rho_0} = \frac{\gamma}{\gamma_0} = \left(\frac{T}{T_0}\right)^{(g/R\alpha)-1} = \left(1 - \frac{\alpha z}{T_0}\right)^{(g/R\alpha)-1} \tag{2–17}$$

Combining Eqs. (2–16c) and (2–17) the relation between pressure and density is found:

$$\frac{\rho}{\rho_0} = \left(\frac{p}{p_0}\right)^{1-(R\alpha/g)} \tag{2–18}$$

At sea-level the international *standard atmosphere* has been chosen as

$$p_0 = 29.92\text{ in. (Hg)} = 2{,}116.2\text{ lbf/ft}^2 = 10{,}332\text{ kg/m}^2$$

$$T_0 = 59°F = 519°R = 15°C = 288°K$$
$$\gamma_0 = 0.07654\ \text{lbf/ft}^3 = 1.226\ \text{kg/m}^3$$
$$\rho_0 = 0.002378\ \text{slug/ft}^3 = 0.125\ \text{kg-sec}^2/\text{m}^4$$
$$\mu_0 = 3.719 \times 10^{-7}\ \text{lbf-sec/ft}^2 = 1.8158 \times 10^{-6}\ \text{kg-sec/m}^2$$
$$R = 1716.3\ \text{ft}^2/\text{sec}^2°F = 287.1\ \text{m}^2/\text{sec}^2°C$$
$$\alpha = 0.00365°F/\text{ft} = 0.0065°C/\text{m} \tag{2-19}$$

With the above values for α and R, Eq. (2–14) and Eqs. (2–16c)–(2–18) yield, respectively,

$$\frac{T}{T_0} = 1 - 7.03 \times 10^{-6}z$$
$$\frac{p}{p_0} = (1 - 7.03 \times 10^{-6}z)^{5.14}$$
$$\frac{\rho}{\rho_0} = (1 - 7.03 \times 10^{-6}z)^{4.14}$$
$$\frac{\rho}{\rho_0} = \left(\frac{p}{p_0}\right)^{0.8055} \tag{2-20}$$

The maximum value for z in Eq. (2–20) is about 36,000 feet.

b. The Stratosphere

The layer of air above the troposphere is called the stratosphere, and the temperature is more or less constant (around −70°F) in this layer. The distance of this layer is between 36,000 and 80,000 feet above sea-level. Now by letting the subscript 1 indicate the condition at the intersection between the stratosphere and the troposphere, Eq. (2–20) gives

$$\frac{T_1}{T_0} = 0.747$$
$$\frac{p_1}{p_0} = 0.224$$
$$\frac{\rho_1}{\rho_0} = 0.299 \tag{2-21}$$

For an isothermal process (constant temperature) Eq. (2–10) can be written

$$\frac{dp}{p} = -\frac{g}{R}\frac{dz}{T_1} \tag{2-22a}$$

and integrating,

standpoint of a single gas and continuum analysis the layers of primary importance are the troposphere and the stratosphere. Therefore, we shall investigate the atmospheric properties of these two layers by using the aerostatic equation [Eq. (2–10)].

a. The Troposphere

The temperature-altitude variation in the troposphere (the part of the atmosphere lying between the earth's surface and the stratosphere) has been found to be linear and may be written

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$$\frac{dp}{p} = \frac{g}{R\alpha}\frac{dT}{T} \tag{2–15}$$

Integration of Eq. (2–15) yields

$$\ln p = \frac{g}{R\alpha}\ln T + C \tag{2–16a}$$

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from which the relation between pressure p and altitude z is obtained

$$\frac{p}{p_0} = \left(1 - \frac{\alpha z}{T_0}\right)^{g/R\alpha} \tag{2–16c}$$

The density-altitude relation can be derived from the equation of state for a perfect gas with the aid of Eq. (2–16c). The result is

$$\frac{\rho}{\rho_0} = \frac{\gamma}{\gamma_0} = \left(\frac{T}{T_0}\right)^{(g/R\alpha)-1} = \left(1 - \frac{\alpha z}{T_0}\right)^{(g/R\alpha)-1} \tag{2–17}$$

Combining Eqs. (2–16c) and (2–17) the relation between pressure and density is found:

$$\frac{\rho}{\rho_0} = \left(\frac{p}{p_0}\right)^{1-(R\alpha/g)} \tag{2–18}$$

At sea-level the international *standard atmosphere* has been chosen as

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$$T_0 = 59°\text{F} = 519°\text{R} = 15°\text{C} = 288°\text{K}$$
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$$\rho_0 = 0.002378\ \text{slug/ft}^3 = 0.125\ \text{kg-sec}^2/\text{m}^4$$
$$\mu_0 = 3.719 \times 10^{-7}\ \text{lbf-sec/ft}^2 = 1.8158 \times 10^{-6}\ \text{kg-sec/m}^2$$
$$R = 1716.3\ \text{ft}^2/\text{sec}^2{}°\text{F} = 287.1\ \text{m}^2/\text{sec}^2{}°\text{C}$$
$$\alpha = 0.00365°\text{F/ft} = 0.0065°\text{C/m} \tag{2-19}$$

With the above values for α and R, Eq. (2-14) and Eqs. (2-16c)–(2-18) yield, respectively,

$$\frac{T}{T_0} = 1 - 7.03 \times 10^{-6}z$$
$$\frac{p}{p_0} = (1 - 7.03 \times 10^{-6}z)^{5.14}$$
$$\frac{\rho}{\rho_0} = (1 - 7.03 \times 10^{-6}z)^{4.14}$$
$$\frac{\rho}{\rho_0} = \left(\frac{p}{p_0}\right)^{0.8055} \tag{2-20}$$

The maximum value for z in Eq. (2-20) is about 36,000 feet.

b. The Stratosphere

The layer of air above the troposphere is called the stratosphere, and the temperature is more or less constant (around $-70°\text{F}$) in this layer. The distance of this layer is between 36,000 and 80,000 feet above sea-level. Now by letting the subscript 1 indicate the condition at the intersection between the stratosphere and the troposphere, Eq. (2-20) gives

$$\frac{T_1}{T_0} = 0.747$$
$$\frac{p_1}{p_0} = 0.224$$
$$\frac{\rho_1}{\rho_0} = 0.299 \tag{2-21}$$

For an isothermal process (constant temperature) Eq. (2-10) can be written

$$\frac{dp}{p} = -\frac{g}{R}\frac{dz}{T_1} \tag{2-22a}$$

and integrating,

$$\frac{p}{p_1} = e^{-(g/RT_1)(z-z_1)} \tag{2-22b}$$

The equation of state for a perfect gas in this case reduces to

$$\rho = \frac{p}{RT_1}$$

or

$$\frac{\rho}{\rho_1} = \frac{p}{p_1} = e^{-(g/RT_1)(z-z_1)} \tag{2-22c}$$

Equation (2–22c) represents the variation of pressure and density in a static compressible fluid with constant temperature.

One must be reminded that in the above discussions, both §2.3a and § 2.3 b, the gravitational acceleration was considered to be constant. Actually, it varies inversely with the square of the distance from the center of the earth. This correction appears to be insignificant (less than 0.5%) within the stratosphere.

2.4. *Forces on Submerged Surfaces*

a. Forces on Plane Surfaces

When a body is submerged in a liquid, a normal force is exerted on it as a consequence of the hydrostatic pressure acting on the surface of the body. The determination of the magnitude and the line of action of the resultant force is of practical interest to engineers. Consider an inclined flat surface A shown in Fig. 2.8 to be subjected to hydrostatic pressure on one side and atmospheric pressure on the other side. Let p denote the gage pressure on a differential area dA; the resultant force F_R on an area A is therefore

$$F_R = \int\int p\, dA \tag{2-23}$$

According to Eq. (2–9), Eq. (2–23) reduces to

$$F_R = \gamma \int\int h\, dA = \gamma \sin\theta \int\int y\, dA \tag{2-24}$$

where h is the vertical distance of an element dA from the free surface of the liquid and y is measured from the x-axis, the line of intersection between the extension of the inclined plane and the liquid surface (Fig. 2.8). According to the basic principle of mechanics, the ordinate of the centroid c of a plane surface is defined as

$$y_c = \frac{1}{A} \int\int y\, dA \tag{2-25}$$

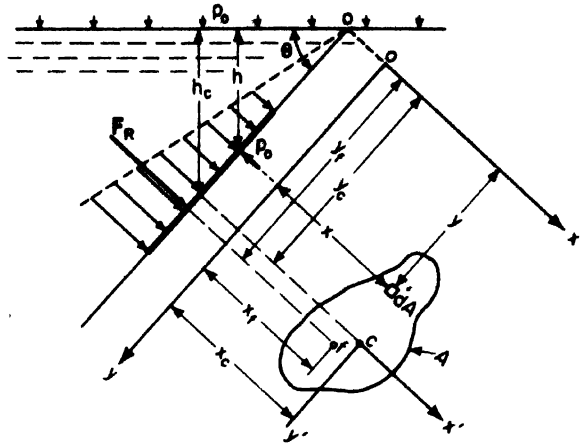


Figure 2.8

Thus, the combination of Eqs. (2–24) and (2–25) gives

$$F_R = (\gamma \sin \theta) y_c A = (\gamma h_c) A = p_c A \tag{2–26}$$

where h_c is the vertical distance of the centroid c below the free surface, and p_c is the average pressure acting at the centroid.

The point of action of the resultant force on the plane surface is called the center of pressure, f. Let y_f and x_f be the distances from the center of pressure to the x and y-axes, respectively. By equating the moment of the resultant force about the x-axis to the summation of the moments of these component forces, we have

$$y_f F_R = \int y \, dF = \iint yp \, dA = \gamma \sin \theta \iint y^2 \, dA \tag{2–27}$$

By solving for y_f and replacing F_R by Eq. (2–24),

$$y_f = \frac{\iint y^2 \, dA}{\iint y \, dA} \tag{2–28}$$

In the same manner the lateral location of the center of pressure can be obtained by taking the moment about the y-axis, hence

$$x_f F_R = \int x \, dF = \iint xp \, dA = \gamma \sin \theta \iint xy \, dA \tag{2–29}$$

from which

$$x_f = \frac{\iint xy \, dA}{\iint y \, dA} \tag{2–30}$$

By definition, the double integral in Eq. (2–27) is the moment of inertia, I_{xx}, of the plane surface about the x-axis and the double integral in Eq. (2–29) is the product of inertia, I_{xy}, of the plane surface about the x and y-axis. Furthermore, if the theorem of parallel axes is applied, then

$$I_{xx} = \int\int y^2\, dA = I_{x'x'} + Ay_c^2 \tag{2–31a}$$

$$I_{xy} = \int\int xy\, dA = I_{x'y'} + Ax_c y_c \tag{2–31b}$$

where $I_{x'x'}$ and $I_{x'y'}$ are the moment of inertia and the product of inertia of the area about the centroidal axes, i. e., x'-axis and x' and y'-axes, respectively. Substituting Eqs. (2–25) and (2–31) into Eqs. (2–28) and (2–30), respectively, the coordinates of the center of pressure of the plane surface are obtained.

$$y_f = \frac{I_{x'x'}}{Ay_c} + y_c \tag{2–32a}$$

$$x_f = \frac{I_{x'y'}}{Ay_c} + x_c \tag{2–32b}$$

When $x_f = x_c$ ($I_{x'y'} = 0$), the plane area is symmetrical about the y'-axis.

Illustrative Example 2.2. Determine the magnitude of the force exerted on the plane trapezoidal surface shown in Fig. 2.B and the location of the center of pressure. ($\gamma = 62.4$ lbf/ft^3 for water).

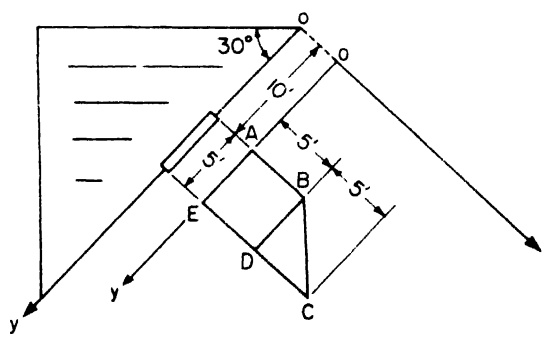


Figure 2.B

Solution. According to Eq. (2–25) the ordinate of the centroid of the surface *ABCDEA* is $y_c = (1/A)\int\int y\,dx\,dy$. The limits of x are $0 \le x \le 5$ for boundary *AB* and $5 \le x \le y - 5$ for boundary *BC*. (When $y = 10, x = 5$ and $y = 15, x = 10$). Hence,

$$y_c = \frac{1}{37.5}\int_{y=10}^{y=15} y\left[\int_0^5 dx + \int_5^{y-5} dx\right] dy$$

$$y_c = \frac{1}{37.5}\int_{y=10}^{y=15} \left[x\Big|_0^5 + x\Big|_5^{y-5}\right] y\,dy$$

$$y_c = \frac{1}{37.5}\int_{10}^{15} (y-5)y\,dy = \frac{1}{37.5}\left[\frac{1}{3}y^3 - \frac{5}{2}y^2\right]_{10}^{15}$$

$$y_c = \frac{1}{37.5}(479.17) = 12.78 \text{ ft}$$

Thus

$$F_R = \gamma \sin\theta \int\int y\,dA = 62.4\left(\frac{1}{2}\right)(479.17) = 14{,}950 \text{ lbf}$$

According to Eqs. (2–28) and (2–30) the coordinates of the center of pressure are:

$$y_f = \frac{\int\int y^2\,dA}{\int\int y\,dA} \quad \text{and} \quad x_f = \frac{\int\int xy\,dA}{\int\int y\,dA}$$

In a similar manner these integrals can be evaluated as follows

$$\int\int y^2\,dA = \int_{y=10}^{y=15} y^2\left[\int_0^5 dx + \int_5^{y-5} dx\right] dy$$

$$= \int_{10}^{15} (y-5)y^2\,dy = \left[\frac{1}{4}y^4 - \frac{5}{3}y^3\right]_{10}^{5}$$

$$= 6198 \text{ ft}^4$$

$$\int\int xy\,dA = \int_{y=10}^{y=15} \left[\int_0^5 x\,dx + \int_5^{y-5} x\,dx\right] y\,dy$$

$$= \int_{10}^{15} \left[\frac{x^2}{2}\Big|_0^5 + \frac{x^2}{2}\Big|_5^{y-5}\right] y\,dy$$

$$= \int_{10}^{15} \frac{1}{2}(y-5)^2 y\,dy = \left[\frac{1}{8}y^4 - \frac{5}{3}y^3 + \frac{25}{4}y^2\right]_{10}^{15}$$

$$= 1901 \text{ ft}^4$$

Thus

$$y_f = \frac{6198}{479.17} = 12.93 \text{ ft} \quad \text{from } x\text{-axis}$$

$$x_f = \frac{1901}{479.17} = 3.967 \text{ ft} \quad \text{from } AE$$

b. Forces on Curved Surfaces

Consider an arbitrary submerged curved surface shown in Fig. 2.9a. A rectangular coordinate system is introduced whose xy plane coincides with the free surface of the liquid and whose z-axis points below the

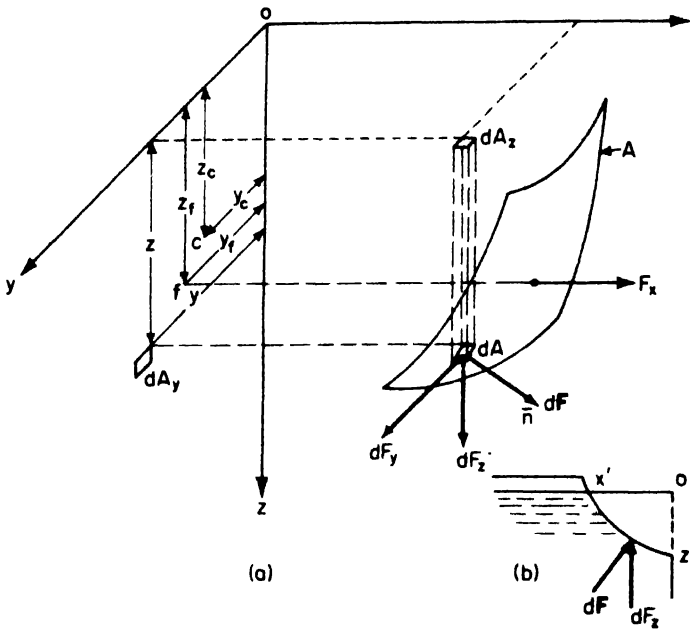


Figure 2.9

xy plane. At a distance z from the surface of the liquid an arbitrary surface element dA with normal $\mathbf{n}$ is subjected to a net force whose direction is coincident with the normal to dA. The force element is

$$d\mathbf{F} = p\,d\mathbf{A} = \gamma z \mathbf{n}\,dA \tag{2-33}$$

The components of the element of force in the x, y, and z-directions are

$$dF_x = \mathbf{i}\cdot d\mathbf{F} = l\gamma z\,dA \tag{2-34a}$$

$$dF_y = \mathbf{j}\cdot d\mathbf{F} = m\gamma z\,dA \tag{2-34b}$$

$$dF_z = \mathbf{k}\cdot d\mathbf{F} = n\gamma z\,dA \tag{2-34c}$$

where l, m, and n are the direction cosines of $\mathbf{n}$ (Appendix A). The components of the surface element projected on the yz, xz, and xy plane are, respectively,

$$dA_x = (\mathbf{n}\cdot\mathbf{i})\,dA = l\,dA \tag{2-35a}$$

$$dA_y = (\mathbf{n}\cdot\mathbf{j})\,dA = m\,dA \tag{2-35b}$$

$$dA_z = (\mathbf{n}\cdot\mathbf{k})\,dA = n\,dA \tag{2-35c}$$

Substituting Eq. (2-35) into Eq. (2-34), we have

$$dF_x = \gamma z\,dA_x \tag{2-36a}$$

$$dF_y = \gamma z\,dA_y \tag{2-36b}$$

$$dF_z = \gamma z\,dA_z \tag{2-36c}$$

Integrating Eq. (2-36), the total force in each of the directions of the three coordinate axes is obtained:

$$F_x = \gamma \int\int z\,dA_x \tag{2-37a}$$

$$F_y = \gamma \int\int z\,dA_y \tag{2-37b}$$

$$F_z = \gamma \int\int z\,dA_z \tag{2-37c}$$

Now let z_c be the distance of the centroid of A_x (projected area onto the yz plane) from the y-axis, then following the method discussed in §2.4a

$$z_c A_x = \int\int z\,dA_x \tag{2-38}$$

and therefore

$$F_x = \gamma z_c A_x \tag{2-39}$$

If z_f and y_f are taken to be the coordinates of the point of action of F_x on the yz plane, we have, by taking moments

$$z_f F_x = \int z\,dF_x \tag{2-40a}$$

and

$$y_f F_x = \int y\,dF_x \tag{2-40b}$$

With the aid of Eqs. (2-36) and (2-39), the coordinates of the center of pressure of area A_x are obtained:

$$z_f = \frac{1}{A_x z_c}\int\int z^2\,dA_x = \frac{I_{yy}}{A_x z_c} \tag{2-41a}$$

$$y_f = \frac{1}{A_x z_c}\int\int yz\,dA_x = \frac{I_{yz}}{A_x z_c} \tag{2-41b}$$

where I_{yy} is the moment of inertia of area A_x about the y-axis and I_{yz} the product of inertia with respect to axes y and z.

In a similar manner the total force in the y-direction is obtained

$$F_y = \gamma z_c A_y \tag{2-42}$$

where A_y is the projection of area A onto the xz plane. If z_e and x_e are taken to be the coordinates of the point of action of F_y on the xz plane, we have

$$z_e F_y = \int z \, dF_y \tag{2-43a}$$

and

$$x_e F_y = \int x \, dF_y \tag{2-43b}$$

With the aid of Eqs. (2-36) and (2-42), the coordinates of the center of pressure of area A_y are obtained:

$$z_e = \frac{1}{A_y z_c} \int\int z^2 \, dA_y = \frac{I_{xx}}{A_y z_c} \tag{2-44a}$$

$$x_e = \frac{1}{A_y z_c} \int\int xz \, dA_y = \frac{I_{xz}}{A_y z_c} \tag{2-44b}$$

where I_{xx} is the moment of inertia of area A_y about the x-axis and I_{xz} the product of inertia with respect to axes x and z.

If we compare the results given in Eqs. (2-39) and (2-41) with that of Eqs. (2-26), (2-28), and (2-30), we see that the force on a curved surface in any horizontal direction is equal to the force on the projected surface which is perpendicular to the force.

From Eq. (2-36) the vertical component of the force on surface element dA is

$$dF_z = \gamma z \, dA_z \tag{2-36c}$$

This is equivalent to the weight of the prism of liquid lying on element dA (Fig. 2.9a). Therefore the total vertical component of the force on area A is

$$F_z = \gamma \int\int z \, dA_z = \gamma \tau \tag{2-45}$$

where τ is the volume of the body of liquid between the plane of the free surface and the submerged curved surface. Equation (2-45) is equivalent to the weight of the body of liquid resting on area A. The line of action of F_z is in the vertical direction passing through the center of gravity of the volume τ.

If a curved surface is floating on the surface of the liquid as shown in Fig. 2.9b, the vertical component of the force will be upward. Thus, the vertical force component on surface $x'z'$ in Fig. 2.9b is the weight of the hypothetical volume of liquid lying on $x'z'$ [Eq. (2-45)], acting upward. The line of action is also in the vertical direction passing through the centroid of this hypothetical volume of liquid.

Illustrative Example 2.3. Figure 2.C shows the parabolic section of a dam of width 10 ft. Determine the magnitude, direction, and location of the resultant water pressure force acting on the dam.

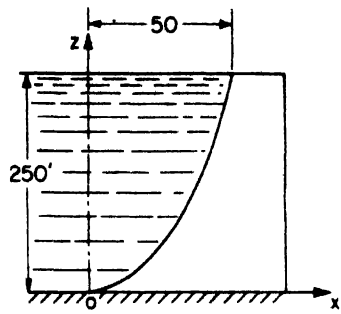


Figure 2.C

Solution. The equation of the parabolic profile is $10z = x^2$. According to Eq. (2-37a) the horizontal component of the force is

$$F_x = \gamma \int\int (250 - z)\, dA_x$$

$$= \gamma \int_0^{250} \int_0^{10} (250 - z)\, dy\, dz$$

$$F_x = (10)(62.4)\left[250z - \frac{1}{2}z^2\right]_0^{250} = 19{,}500{,}000 \text{ lbf}$$

The vertical component of the force [Eq. (2-37c)] is

$$F_z = \gamma \int\int (250 - z)\, dA_z = \gamma \int_0^{50} \int_0^{10} (250 - z)\, dy\, dx$$

$$F_z = 10\gamma \int_0^{50} \left(250 - \frac{x^2}{10}\right) dx$$

$$F_z = (10)(62.4)\left[250x - \frac{x^3}{30}\right]_0^{50} = 5{,}200{,}000 \text{ lbf}$$

The resultant force is

$$F = \sqrt{F_x^2 + F_z^2} = 20{,}181{,}500 \text{ lbf}$$

The inclination angle to the vertical direction is

$$\theta = \tan^{-1}\left(\frac{19{,}500{,}000}{5{,}200{,}000}\right) = 75°04'$$

The location of the resultant force is

$$z_f F_x = \int z\, dF_x = \gamma \int_0^{250} \int_0^{10} (250 - z)z\, dy\, dz$$

$$= (10)(62.4)\left[250\frac{z^2}{2} - \frac{z^3}{3}\right]_0^{250}$$

from which

$$z_f = \frac{1{,}625{,}000{,}000}{19{,}500{,}000} = 83.3 \text{ ft} \quad \text{from} \quad x\text{-axis}$$

and

$$x_f F_z = \int x\, dF_z = \gamma \int_0^{50} \int_0^{10} \left(250 - \frac{x^2}{10}\right) x\, dy\, dx$$

$$x_f F_z = (10)(62.4)\left[250\frac{x^2}{2} - \frac{x^4}{40}\right]_0^{50}$$

from which

$$x_f = \frac{97{,}500{,}000}{5{,}200{,}000} = 18.75 \text{ ft} \quad \text{from } z\text{-axis}$$

2.5. *Hydrostatic Lift — Buoyancy*

In the preceding section we have evidenced a vertical component of force, acting upward on a curved surface when it is floating on the surface of the liquid. This hydrostatic lift is called the buoyant force, and the phenomenon is called buoyancy. Now, examine the case of a body when immersed in a liquid. Consider a solid body of arbitrary shape completely submerged in a homogeneous liquid as shown in Fig. 2.10. According to §2.4b, it is evident that resultant horizontal forces acting on areas A_x and A_y are zero.

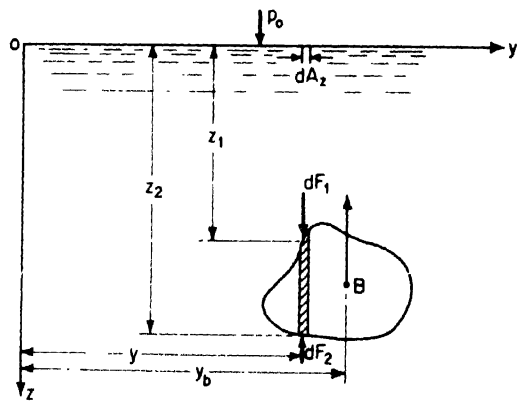


Figure 2.10

The vertical forces acting on the two ends of a vertical prism of cross section dA_z (Fig. 2.10) are, respectively,

$$dF_1 = (p_o + p_1)\,dA_z \tag{2-46a}$$

and

$$dF_2 = (p_o + p_2)\,dA_z \tag{2-46b}$$

The difference between the upward and downward forces is the buoyant force dF_B on the vertical prism, i. e.,

$$dF_B = dF_2 - dF_1 = (p_2 - p_1)\,dA_z = \gamma(z_2 - z_1)\,dA_z \tag{2-47}$$

But $(z_2 - z_1)\,dA_z = d\tau$, the volume of the vertical prism. The buoyant force F_B on the entire submerged body is obtained by integrating Eq. (2–47). Thus

$$F_B = \int\int \gamma(z_2 - z_1)\,dA_z = \int\int\int \gamma\,d\tau = \gamma\tau \tag{2–48}$$

where τ is the total volume of the submerged body. Equation (2–48) simply equates the buoyant force with the weight of the liquid displaced by the submerged body of volume τ. Hence Eq. (2–48) represents the familiar Archimedes' principle which states: The buoyant force on a submerged body is equal to the weight of liquid it has displaced, and acts through the centroid of the displaced volume.

The line of action of the buoyant force, which passes through the center of buoyancy, B, can be located by taking the moment of the force with respect to the z-axis. Thus

$$y_b F_B = \int y\,dF_B \tag{2–49}$$

Substituting Eqs. (2–47) and (2–48) into Eq. (2–49), the y-coordinate of the center of buoyancy is obtained:

$$y_b = \frac{1}{\tau}\int\int\int y\,d\tau \tag{2–50}$$

which is also the centroid of the displaced volume.

The buoyant force of a partially immersed body, according to Archimedes' principle, is equal to the weight of the displaced liquid. It can easily be seen that the buoyant force must just balance the weight of the body when the floating body is statically in equilibrium. This principle will be illustrated further in the problem section.

2.6. *Stability of Submerged and Floating Bodies*

a. Submerged Bodies

A body is said to be in stable equilibrium if it is insensitive to slight disturbances trying to unbalance it. Otherwise, it is in unstable equilibrium. When a body is submerged in a liquid the equilibrium requires that the weight of the body acting through its center of gravity, c, is colinear with an equal hydrostatic lift acting through the center of buoyancy B. In general, the location of point c does not have to coincide with point B if the body is not homogeneous.

A submerged body is considered stable if its center of gravity, c, is located below the center of buoyancy. This can be seen from Fig. 2.11b that if the body is tilted slightly the buoyant force produces a

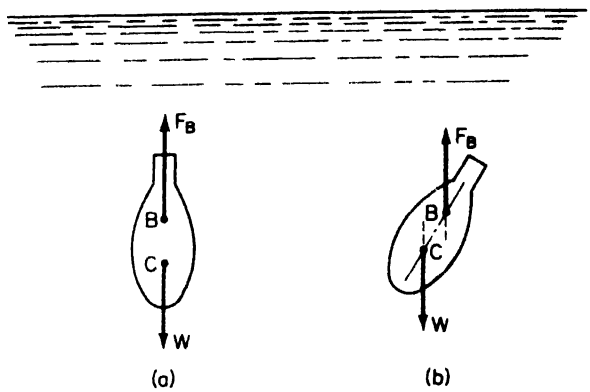


Figure 2.11

restoring moment trying to return the body to its original position (Fig. 2.11a). On the other hand, if point c is above point B (Fig. 2.12a), any disturbance from the equilibrium position will create a moment which will rotate the body away from its original position (Fig. 2.12b). If the weight and buoyant force act through the same point, the body will always assume the position in which it is placed; hence it is in neutral equilibrium.

b. Floating Bodies

A floating body such as a ship is stable if its center of gravity is below the center of buoyancy. However, it may still be stable even if its center of gravity lies above the center of buoyancy. Let us now examine the condition of the degree of stability possessed by the

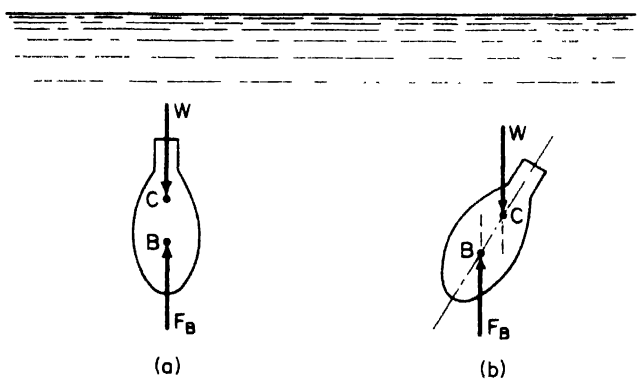


Figure 2.12

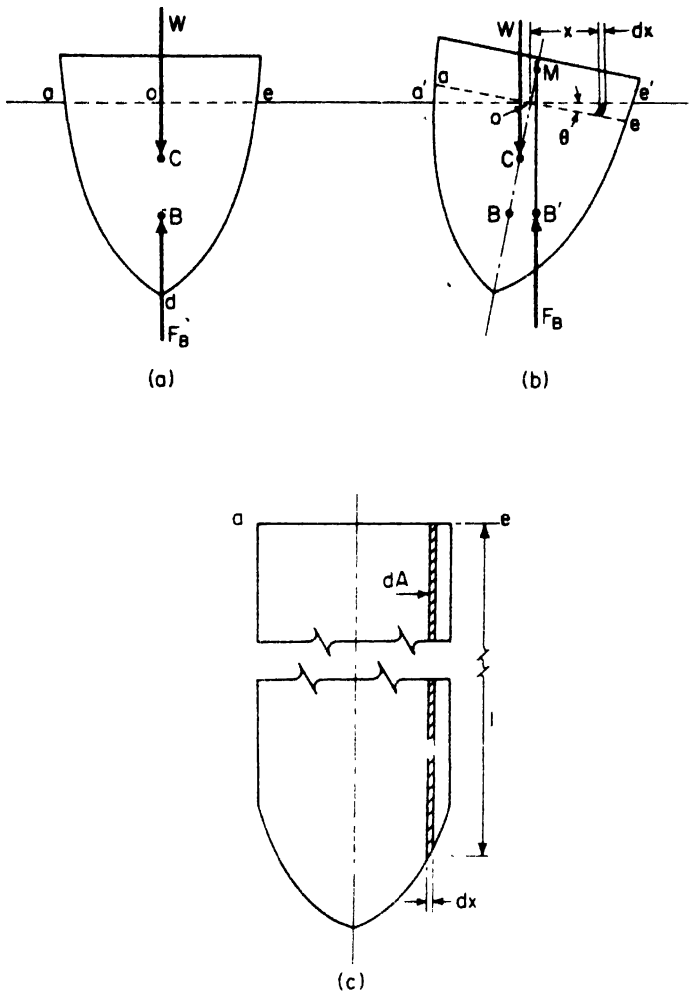


Figure 2.13

system. Consider a floating body as shown in Fig. 2.13a where the center of gravity *c* lies vertically above the center of buoyancy *B*. The water plane *ae* is defined as the intersection of the free surface of the liquid with the body. The area in the body defined by this plane as shown in Fig. 2.13c is the water plane area. The intersection of the axis passing through points *c* and *B* with the water plane is denoted by *o*.

When the body is rotated about an axis through *o* and perpendicular to the plane of the paper from its equilibrium position, the point *B* shifts to a new location, *B'* (Fig. 2.13b). This shift of the center of

buoyancy to point B' is due to the fact that the volume of displacement τ assumes a different shape when the body is in a tilted position. In Fig. 2.13b the metacenter M is defined as the point of intersection of the vertical line through B' and the extension of $\overline{BC}$. It is a measure of the stability of a floating body. It can be seen from Fig. 2.13b that if M lies above c, the forces F_B and W produce a restoring moment trying to return the body to its original position. On the other hand, if M is found to be below c, the moment produced by F_B and W would overturn the body.

Referring to Fig. 2.13b it is seen that after the body is tilted through a small angle θ, the buoyant force through B' is

$$F_B = \gamma(\text{vol } a'de') \tag{2-51a}$$

which is equal to

$$\gamma(\text{vol } ade) + \gamma(\text{vol } oee') - \gamma(\text{vol } oa'a) \tag{2-51b}$$

Since the total displacement volume τ remains unchanged during the rolling of the body, the weights of the wedges oee' and $oa'a$ are equal and their moment is a couple. It is this couple which must be balanced by the restoring moment produced by the shift of the total buoyant force F_B. The couple due to the wedges (Figs. 2.13b and 2.13c) is calculated as follows:

$$\iint_{oa'a} x\gamma(x \sin\theta)l\,dx + \iint_{oee'} x\gamma(x \sin\theta)l\,dx = \gamma\theta \iint x^2\,dA \tag{2-52}$$

where $l\,dx = dA$ and θ is assumed small. Equating Eq. (2-52) to the moment produced by F_B due to the shift of the center of buoyancy, we have

$$F_B(\overline{MB})\theta = \gamma\theta \iint x^2\,dA \tag{2-53}$$

The integral on the right-hand side of Eq. (2-53) is the moment of inertia I of the water plane area of the body about the axis of turning. Therefore

$$\overline{MB} = \frac{\gamma I}{F_B} = \frac{I}{\tau} \tag{2-54}$$

and the metacentric height is

$$\overline{MC} = \overline{MB} - \overline{CB} = \left(\frac{I}{\tau} - \overline{CB}\right) \tag{2-55}$$

Since the restoring moment is

$$F_B(\overline{MC})\theta = \gamma\tau\left(\frac{I}{\tau} - \overline{CB}\right)\theta \tag{2-56}$$

the body would be unstable if $(I/\tau - \overline{CB}) < 0$.

Illustrative Example 2.4. A rectangular barge (see Fig. 2.13) is 50 ft long, 20 ft wide and 10 ft deep. It has a draft of 7 ft when fully loaded. The center of gravity of the barge is on the axis of symmetry at the water surface. Determine the stability condition of the barge and the metacentric height.

Solution. The water displaced by the submerged part of the barge is

$$\tau = (50)(20)(7) = 7{,}000 \text{ ft}^3$$

The moment of inertia of the waterplane area of the body about the axis of turning is

$$I = \frac{(50)(20)^3}{12} = 33{,}330 \text{ ft}^4$$

Hence,

$$\frac{I}{\tau} = \frac{33{,}330}{7{,}000} = 4.76 \text{ ft}$$

The distance between the center of gravity and the center of buoyancy is

$$\overline{CB} = 7.0 - 3.5 = 3.5 \text{ ft}$$

Therefore, the metacentric height is [Eq. (2–55)]

$$\overline{MC} = \frac{I}{\tau} - \overline{CB} = 1.26 \text{ ft}$$

above the center of gravity. Since

$$\frac{I}{\tau} - \overline{CB} = 1.26 > 0$$

the floating barge is stable.

B. RELATIVE MOTION OF LIQUIDS

2.7. Uniform Translational Acceleration

As previously mentioned that in the absence of relative motion between fluid elements or between fluid elements and boundaries the fluid masses are not subjected to any shearing stress. This is true not only when the fluid is at rest, but also when it is subjected to a uniformly accelerated motion after the relative equilibrium state is established. In the latter case the fluid behaves as if it were in rigid body motion, i. e., in static equilibrium. In Fig. 2.14 is shown a container of liquid undergoing uniformly translational acceleration a_x along the

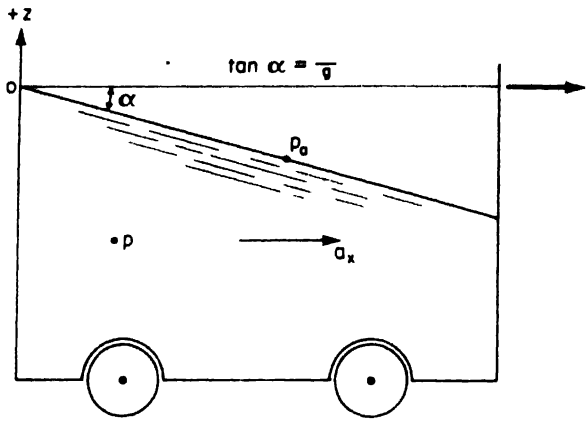


Figure 2.14

x-direction. According to D'Alembert's principle in dynamics, a dynamic system can be treated as a static system provided the effects of acceleration are accounted for by the addition of a fictitious "inertia force." If we add this inertia force per unit volume to Eq. (2-6a) it yields

$$-\frac{\partial p}{\partial x} = a_x \rho \tag{2-57}$$

However, the summation of forces in the y-direction and z-direction are the same as that of Eqs. (2-6b) and (2-6c), i. e.,

$$-\frac{\partial p}{\partial y} = 0 \tag{2-6b}$$

$$-\frac{\partial p}{\partial z} = \rho g \tag{2-6c}$$

Integrating Eqs. (2-57), (2-6b), and (2-6c) yields, respectively,

$$p = -\rho a_x x + f_1(y, z) \tag{2-58a}$$

$$p = f_2(x, z) \tag{2-58b}$$

$$p = -\rho g z + f_3(x, y) \tag{2-58c}$$

where f_1, f_2, and f_3 are arbitrary functions of integration. Since Eq. (2-58b) indicates that p is a function of x and z only, Eqs. (2-58a) and (2-58c) give

$$f_1(z) = -\rho g z + C$$

and

$$f_3(x) = -\rho a_x x + C$$

Finally we have

$$p = -\rho a_x x - \rho g z + C \tag{2-59a}$$

where C is an arbitrary constant. Equation (2–59a) evidently satisfies the differential equations for p [Eqs. (2–57), (2–6b), and (2–6c)]. Since the free surface is an isobaric (constant pressure) surface, and the pressure on the surface of the liquid is equal to the atmospheric pressure p_a, the constant C can be evaluated at $x = 0$, $z = 0$ (Fig. 2.14), i. e.,

$$C = p_a$$

Thus

$$p - p_a = -\rho a_x x - \rho g z \tag{2-59b}$$

To determine the shape of the surface of the fluid under consideration, we need to put $p = p_a$ in Eq. (2–59b) and solve for z.

$$z = -\frac{a_x}{g} x \tag{2-60}$$

which is the equation of the surface of the liquid. Equation (2–60) indicates that the liquid surface is a plane with a negative slope equal to $-a_x/g$. The analysis above can be easily extended to include the case with uniform acceleration in the direction along the z-axis.

2.8. Uniform Rotation

If a body of liquid is rotated about a vertical axis at a uniform angular velocity, it is subjected to a centrifugal force, pointed away

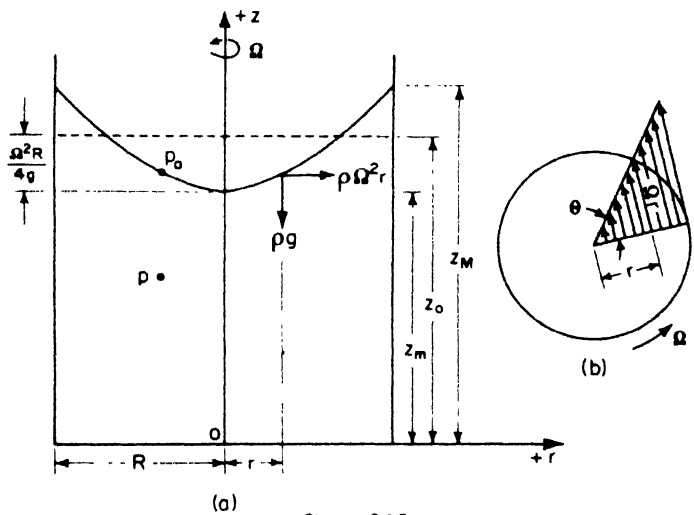


Figure 2.15

from the vertical axis, which will alter the originally horizontial position of the free surface of the liquid. In Fig. 2.15a is shown a cylindrical container of liquid which is rotating at a uniform angular velocity Ω. The free surface has now risen at the cylindrical walls and is depressed at the center. By D'Alembert's principle we can add the element of centrifugal force per unit volume (inertia *force* per unit volume) in the radial direction of the equilibrium equations, and Eq. (2-5) can be expressed in cylindrical coordinates (Figs. 2.15a and 2.15b) as follows:

$$-\frac{\partial p}{\partial r} = -\rho\Omega^2 r \tag{2-61}$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} = 0 \tag{2-62}$$

$$-\frac{\partial p}{\partial z} = \rho g \tag{2-63}$$

For constant ρ and Ω, the integration of Eqs. (2-61), (2-62), and (2-63) gives, respectively,

$$p = \tfrac{1}{2}\rho\Omega^2 r^2 + f_1(\theta, z) \tag{2-64}$$

$$p = f_2(r, z) \tag{2-65}$$

$$p = -\rho g z + f_3(r, \theta) \tag{2-66}$$

The arbitrary functions of integration f_1 and f_2 are determined from Eqs. (2-64), (2-65), and (2-66) in the form

$$f_1(z) = -\rho g z + C$$

$$f_2(r) = \tfrac{1}{2}\rho\Omega^2 r^2 + C$$

Thus,

$$p = \tfrac{1}{2}\rho\Omega^2 r^2 - \rho g z + C \tag{2-67a}$$

where C is an arbitrary constant. It can be easily shown that Eq. (2-67a) satisfies all the three differential equations for p [Eqs. (2-61), (2-62), and (2-63)]. As mentioned previously, once rotational equilibrium is reached, the free surface of the liquid is depressed at the center of rotation z_m (Fig. 2.15a). The constant C which is evaluated at $r = 0$, $z = z_m$ yields

$$C = \rho g z_m + p_a$$

and Eq. (2-67a) becomes

$$p - p_a = \tfrac{1}{2}\rho\Omega^2 r^2 - \rho g(z - z_m) \tag{2-67b}$$

where p_a is the pressure at the free surface (isobaric surface).

The equation of the surface of liquid under consideration can be determined by putting $p = p_a$ in Eq. (2-67b):

$$z = \frac{\Omega^2}{2g} r^2 + z_m \tag{2-68}$$

Equation (2–68) represents the curve of a parabola with vertical axis whose vertex is at z_m. The isobaric surface therefore corresponds to a paraboloid of revolution. If z_0 is denoted as the height of the level of the liquid at rest and z_M the maximum height of the liquid at the container wall during the rotation, the value of z_m can be determined by equating the original and final volume of the liquid

$$\pi R^2 z_0 = \pi R^2 z_M - \tfrac{1}{2}\pi R^2 (z_M - z_m)$$

or

$$z_0 = \tfrac{1}{2}(z_M + z_m) \tag{2-69}$$

But Eq. (2–68) gives

$$z_M = \frac{\Omega^2}{2g} R^2 + z_m \tag{2-70a}$$

Thus,

$$z_m = -\frac{\Omega^2 R^2}{4g} + z_0 \tag{2-70b}$$

This is the height of the depression at the center of the free surface of the liquid measured from the r-axis.

Illustrative Example 2.5. If the mercury U-tube shown in Fig. 2.D is rotated about a vertical axis through o, calculate the speed when the difference of level between the tubes is 8 in. What is the original height of the mercury column?

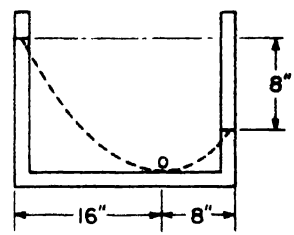


Figure 2.D

Solution. According to Eqs. (2–68) and (2–70a), we have, respectively,

$$z = \frac{\Omega^2 r^2}{2g} + z_m$$

$$z_M = \frac{\Omega^2 R^2}{2g} + z_m$$

Taking the difference of the two expressions,

$$z_M - z = \frac{\Omega^2}{2g}(R^2 - r^2)$$

Substituting $z_M - z = 8$ in., $R = 16$ in., and $r = 8$ in., we have

$$8 = \frac{\Omega^2}{(64.4)(12)}[(16)^2 - (8)^2]$$

Hence the rotating speed is

$$\Omega = \sqrt{32.2} = 5.675 \text{ rad/sec}$$

or

$$\Omega = \frac{(5.675)(60)}{2\pi} = 54.2 \text{ rpm}$$

The original height of the mercury column is

$$z_0 = \frac{1}{2}(z_M + z_m) = \frac{1}{4}\frac{\Omega^2 R^2}{g}$$

$$z_0 = \frac{(5.675)^2(16)^2}{(4)(32.2)(12)} = 5.33 \text{ in.}$$

NOMENCLATURE FOR CHAPTER TWO

a_t	uniform translational acceleration, Eq. (2-57)
A	area, §2.1
c	centroid of a plane surface, Eq. (2-25)
$\mathbf{F}$	resultant force, Eqs. (2-1), (2-2), (2-4), and (2-33)
F_B	buoyant force, Eq. (2-48)
f	center of pressure, Eq. (2-27)
F	magnitude of resultant force
g	gravitational acceleration
h	vertical distance below the free surface corresponding to a pressure p, Fig. 2.16
$I_{xx}, I_{\eta\eta}$	moment of inertia, Eqs. (2-31a) and (2-41a)
I_{xy}, I_{yz}, I_{xz}	product of inertia, Eqs. (2-31b), (2-41b), and (2-44b)
l, m, n	direction cosines of n, Eq. (2-34)
$\overline{MC}$	metacentric height, Eq. (2-55)
$\mathbf{n}$	unit vector normal to an arbitrary surface, Eq. (2-33) and Fig. 2.9
p	pressure, Eq. (2-8)
p_{abs}	absolute pressure, Eq. (2-11a) and Fig. 2.2
p_{gage}	gage pressure, Eq. (2-11b) and Fig. 2.2
p_{atm}	atmospheric pressure, Eq. (2-12a) and Fig. 2.2
r, θ, z	cylindrical coordinates, Fig. 2.15
R	gas constant, Eq. (2-19)
T	temperature, Eq. (2-10)
x, y, z	Cartesian coordinates, Fig. 2.1b
$\mathbf{X}$	external force per unit mass, Eq. (2-1)
z	altitude, §2.3
z_0	height of free surface above an arbitrary datum, Eq. (2-9b) and Fig. 2.1b
α	constant lapse rate, Eq. (2-14)
γ	specific weight, Eqs. (2-6c) and (2-45)
μ	coefficient of viscosity, Eq. (2-19)
ρ	density, Eq. (2-1)

τ volume, Eqs. (2-1) and (2-45)
Ω uniform angular velocity, Eq. (2-61)
$(\ )_1$ condition at the intersection between stratosphere and the troposphere, §2.3b
$(\ )_c$ quantities referring to centroid c
$(\ )_f$ quantities referring to center of pressure f
$(\ )_0$ quantities referring to sea-level condition, §2.3a

PROBLEMS

2.1. In the figure a cylindrical element of fluid with cross-sectional area dA and height dz is shown. Derive the equilibrium equation in the z-direction.

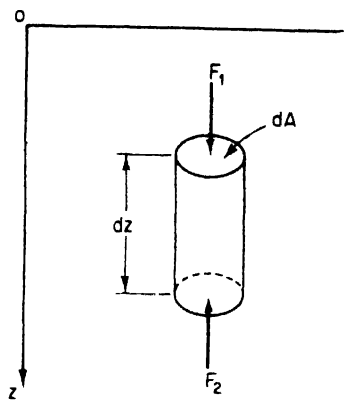


Figure P.2.1

2.2. If the vapor pressure in a barometer tube (Fig. 2.3) is assumed to be 0.1 lbf/in.^{2} and the level of the mercury, h, is 29.8 in., what is the atmospheric pressure in lbf/in.^{2} and in kg/m^2? The specific gravity of mercury is 13.6.

2.3. The specific gravity of gasoline is 0.673. If the gasoline rises to a height z_0 (Fig. 2.4) above the center of the pipe, what is the gage pressure in lbf/in.^{2} at point A and at an angle of 45° from point A?

2.4. In Fig. 2.6 the center portion of the differential monometer is filled with oil of specific gravity γ_c of 0.8, and the remainder of the tube with water of specific gravity of 1.0. Find the pressure difference $p_B - p_A$ in lbf/in.^{2} if $h_3 = 9$ in., $h_2 = 3$ in., and $h_1 = 6$ in.

2.5. If water is replaced by air for the left-hand limb in Prob. 2.4, find the pressure difference $p_B - p_A$ in lbf/in.^{2}

2.6. What pressure at A is required in order to make $h_2 = 0$ in Fig. 2.A? The air pressure above the water is assumed to be atmospheric.

2.7. If the pressure and density of the atmosphere vary polytropically (see Appendix B) in the form $p/p_0 = (\rho/\rho_0)^n$ where n is the polytropic expansion exponent and the subscript o refers to the sea-level condition, verify Eq. (2-20) for $n = 1.241$.

2.8. Plot the density ratio ρ/ρ_0 versus the altitude in both the troposphere and the stratosphere.

2.9. Calculate the temperature-lapse rate, in both °F/ft and °C/m in the troposphere if the pressure-density relation is assumed to be isentropic (see Appendix B), i. e., $p/p_0 = (\rho/\rho_0)^k$ where $k = 1.405$ for air.

2.10. If the temperature-lapse rate is given as 0.0065°C/m, plot the pressure in mm Hg. versus temperature in °C in the troposphere. The *standard atmosphere* at sea-level is given in Eq. (2-19).

2.11. A triangle gate having a hinged horizontal base 6 ft long and an altitude of 3 ft is inclined 60° from the vertical with the vertex pointing upward. The base of the gate is 10 ft below the water surface. What normal force must be applied at the vertex of the gate to open it?

2.12. Verify the results of the illustrative example 2.2 by using Eqs. (2-26), (2-32a), and (2-32b). The values of $y_c, x_c, I_{x'x'}$ and $I_{x'y}$ for the area *ABCDEA* can be obtained without performing the integration.

2.13. Determine the total force of the water pressure and the location of the center of pressure on a circular gate inclined 45° with the water surface. The gate is 4 ft in diameter and its center is 6 ft below the water surface.

2.14. Find the specific gravity of the liquid on the left-hand side of the gate so that the liquid pressure is sufficient to keep the gate in an equilibrium position (see Fig. P.2.14).

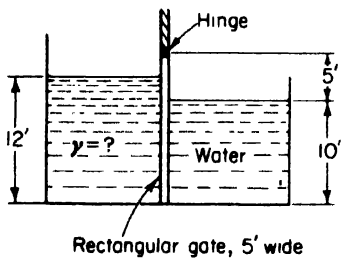


Figure P.2.14

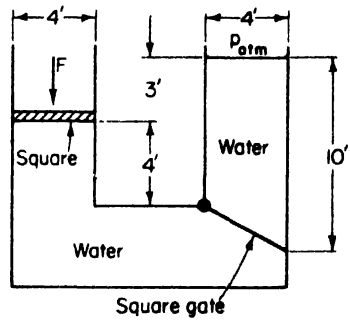


Figure P.2.15

2.15. Calculate the magnitude of the force F shown in Fig. P.2.15 required to keep the square gate in equilibrium.

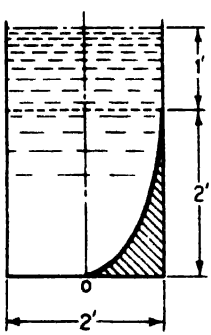


Figure P.2.17

2.16. Verify the results of the illustrative example 2.3 using Eqs. (2-39) and (2-45) for F_x and F_z, respectively.

2.17. Determined the forces on the walls of the parabolic channel of width 1 ft. (see Fig. P.2.17). Find the location of the center of pressure.

2.18. If the quarter circular gate of width 10 ft is hinged at the bottom (see Fig. P.2.18), find the force F_H necessary to hold the gate in place.

2.19. If water stands at a depth of 3 ft on the right-hand side of the circular gate of width 10 ft, find the force F_H necessary to hold the gate in place (see Fig. P.2.19).

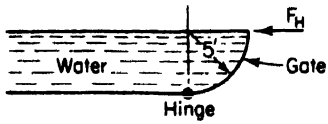


Figure P.2.18

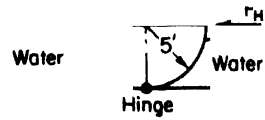


Figure P.2.19

2.20. Calculate the moment required to keep the gate of width 10 ft (see Fig. P.2.20) in equilibrium.

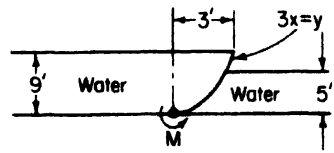


Figure P.2.20

2.21. A right circular cone 5 ft in diameter, 5 ft in height, and weighing 100 lbf, floats in water with its vertex up. An anchor weighing 100 lbf/ft^3 is attached to the lower end of the cone. Determine the total weight of the anchor if the base of the cone is submerged 4 ft below the water surface.

2.22. A log 10 in. in diameter and 20 ft long has a specific gravity of 0.60. One end of the log is hinged to a wall 3 ft above the surface of the water; the other end floats in the water. Find the length of log submerged in the water and the reaction at the hinge.

2.23. If the log given in Prob. 2.22 is hinged to a wall 10 ft below the surface of the water, find the length of log submerged in the water and the reaction at the hinge.

2.24. A barge 60 ft long and 16 ft high has a draft of 10 ft. If the center of gravity is on the axis of symmetry, 8 ft above the bottom, what is the minimum width of the barge for stable equilibrium?

2.25. Determine the maximum length of a log which has a diameter of 5 ft so that the log, in vertical position, will float in stable equilibrium on the surface of water. The specific gravity of the log is 0.60.

2.26. A rectangular scow weighing 50 tons floats in water. It is 50 ft long, 20 ft wide, and 10 ft high. If the center of gravity of the scow is taken at its geometrical center, find the metacentric height.

2.27. A rectangular scow 30 ft long, 15 ft wide, and 10 ft high has a draft of 6 ft. What load must be moved transversely from the center to a point 3 ft from the edge in order that the scow may list through an angle of 10°?

2.28. A car has a rectangular cross section, 15 ft long, 5 ft wide, and 6 ft high. If the car is $\frac{2}{3}$ full of oil (sp gr 0.8), what is the maximum acceleration it can undergo without spilling any oil? What force is required to produce the maximum acceleration

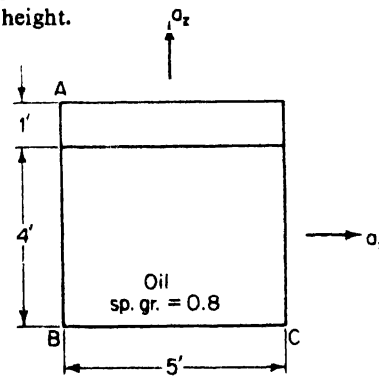


Figure P.2.29

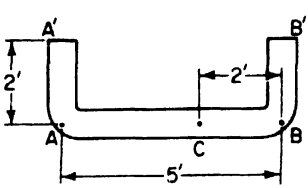


Figure P.2.30

(the weight of the car is assumed to be negligible)?

2.29. In the figure (P.2.29) $a_x = 20$ ft/sec^2 and $a_z = 8.05$ ft/sec^2. Find the pressure at A, B, and C.

2.30. The U-tube in the figure (P.2.30) is one-half filled with water. When accelerated to the right 16.1 ft/sec^2, determine the pressure at A and B.

2.31. An open cylindrical vessel, 3 ft in diameter, 5 ft high, and one-half filled with water, rotates about its vertical axis with a constant speed of 100 rpm. Determine (a) the depth of the water at both the center and the wall of the vessel; (b) the total pressure on the cylindrical walls; (c) the total pressure on the bottom of the vessel.

2.32. An open cylindrical vessel 4 ft in diameter contains oil (sp gr 0.8) 5 ft deep when at rest. If it is rotated about its vertical axis at a speed 250 rpm., determine the smallest depth the vessel can have without spilling water over the sides.

2.33. If the U-tube shown in Prob. 2.30 is rotated about a vertical axis through BB' with a speed of 50 rpm, determine the height of the column AA' and BB'. The tube is one-half filled with water when at rest.

2.34. If the U-tube shown in Prob. 2.30 is rotated about a vertical axis through C at 60 rpm, determine the gage pressure at A and B. The tube is one-half filled with mercury when at rest.

2.35. A conical vessel with vertical axis has an altitude of 4 ft and is filled with water. Its base, 3 ft in diameter, is horizontal and uppermost. If the vessel is rotated about its axis with a speed of 100 rpm, how much water will remain in it?

3

KINEMATICS OF FLUIDS

Fluid kinematics is that branch of fluid mechanics which deals with the motion of fluid particles without consideration of the character of the particles or the influence of forces upon their motion. The types of motion from the kinematic point of view are translation, rate of deformation, and rotation of the fluid particles. Each type of motion may occur singly, or in any combination. Since the results of kinematic studies of the fluid motion are the groundwork on which the dynamical results (study of forces responsible for the motion of fluids) are constructed, it is desirable to acquire a basic understanding of this branch of mechanics.

In §3.1 the methods of describing fluid motion are discussed in detail. Both the Lagrangian and Eulerian methods of description are mathematically formulated and explained.

In §3.2 a general mathematical treatment of the kinematics of translation, rotation, and rate of deformation is presented. A detailed discussion of rotation and vorticity is presented later in this chapter. The fundamentals of rates of deformation are not discussed here since a knowledge of the stress and rate of strain relations is essential for their complete description. In addition the concepts of conservation of mass (continuity equation) and circulation are treated later in Chaps. Five and Seven, respectively, because certain physical concepts which are necessary for their description have not yet been discussed.

In §3.3 the equations of streamlines, path lines, and streak lines are given and a graphical presentation is used to illustrate the physical phenomena of these lines.

In §3.4 the material derivative is defined, and the acceleration vector in Cartesian, cylindrical, and spherical coordinates is presented.

In §3.5 a detailed discussion of rotation and vorticity is presented and the equations of vortex lines are derived.

3.1. *Methods of Describing Fluid Motion*

In order to obtain a complete picture of the flow of the fluid, one must determine the position (spatial coordinates x, y, z) of every particle of a fluid for every instant of time. The velocity and acceleration of a fluid particle can then be found from its change in position as time goes on. We can study the motion of fluid particles by two methods. It is possible either to describe the motion of each particle of fixed identity for all time, or to determine the motion of a fluid at every point and at every time without considering the whereabouts of indi-

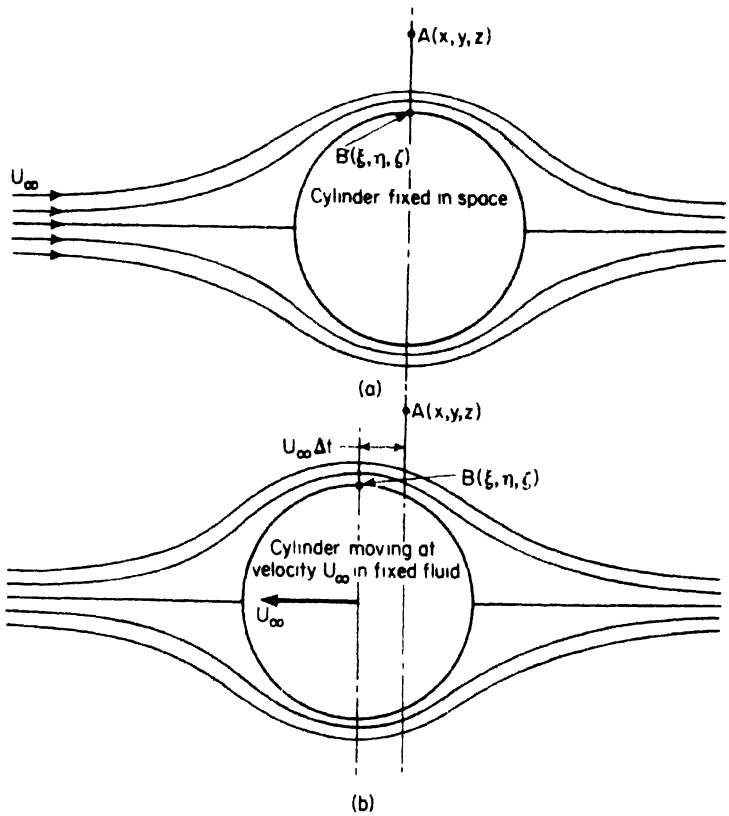


Figure 3.1

vidual particles. The former method of representing the fluid motion is called the Lagrangian method (actually attributed to Euler, but used extensively by Lagrange) while the latter method is the Eulerian method. In most engineering applications, since a knowledge of the behavior of individual particles is not necessary, the method of Euler is generally used.

Before proceeding with a description of these methods, it is desirable to discuss briefly the concept of time dependence. In general, fluid properties and flow characteristics of individual particles will be different in the field of flow (the entire region filled with fluid). If the velocities at different points in space are independent of time, the flow field is called steady. A time-dependent flow, on the other hand, is called unsteady flow. An unsteady flow in one space reference can often be transferred to another space reference where the flow becomes steady. As an example of this, consider the flow pattern (Fig. 3.1a) which results from the flow of a uniform stream past an infinitely long circular cylinder at rest. An observer at A (coordinates x, y, z) would consider the motion of the fluid to be steady as the velocity at any point in space remains invariant with time. On the other hand, when the cylinder moves with a uniform velocity in a fluid at rest (Fig. 3.1b), the flow becomes unsteady with respect to the observer at the same point A. In this case the velocity at any point in space with respect to A does not remain constant with time as the cylinder moves past it. If the observer stays at point B (coordinates ξ, η, ζ) attached to the circular cylinder, he would see the same flow pattern at all times as in the case shown in Fig. 3.1a. Hence the flow, with respect to point B attached to the cylinder is steady.

a. Lagrangian Method

In the Lagrangian representation a rectangular system of coordinates is normally used. A certain particle of the fluid is given a fixed identity by specifying the initial position $\boldsymbol{\xi}$ at a given time $t = t_0$. At a later time, $t = t$, the same particle is at position $\mathbf{r}$. Then the position of the particle is completely specified if the position vector $\mathbf{r}$ or the three corresponding coordinates x, y, z are given as a function of time t and the initial position $\boldsymbol{\xi}$.

$$\mathbf{r} = \mathbf{F}(\boldsymbol{\xi}, t) \tag{3-1a}$$

where $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \qquad \boldsymbol{\xi} = \mathbf{i}x_0 + \mathbf{j}y_0 + \mathbf{k}z_0$

or

$$\begin{aligned} x &= F_1(x_0, y_0, z_0, t) \\ y &= F_2(x_0, y_0, z_0, t) \\ z &= F_3(x_0, y_0, z_0, t) \end{aligned} \tag{3-1b}$$

The initial coordinates $\hat{\xi}$ of a particle will be referred to as the material coordinates of the particle and may be used conveniently to identify the particle itself, i.e., the particle $\hat{\xi}$. The spatial coordinates r of the particle may be referred to as its position.

The velocity and acceleration of a fluid particle $\hat{\xi}$ can be obtained from the material deriatives of the position [Eq. (3–1b)] of the particle, thus for the velocity,

$$u = \left(\frac{dx}{dt}\right)_{\hat{\xi}} = \left(\frac{\partial x}{\partial t}\right)_{\hat{\xi}}$$

$$v = \left(\frac{\partial y}{\partial t}\right)_{\hat{\xi}} \tag{3-2}$$

$$w = \left(\frac{\partial z}{\partial t}\right)_{\hat{\xi}}$$

and for the acceleration,

$$a_x = \left(\frac{\partial^2 x}{\partial t^2}\right)_{\hat{\xi}}$$

$$a_y = \left(\frac{\partial^2 y}{\partial t^2}\right)_{\hat{\xi}} \tag{3-3}$$

$$a_z = \left(\frac{\partial^2 z}{\partial t^2}\right)_{\hat{\xi}}$$

The subscript $\hat{\xi}$ means that the differentiation must be carried out with $\hat{\xi}$ constant.

It is apparent that this method has many favorable aspects. The motion and trajectory of each and every particle of fluid is known, so that at any time it is possible to trace the history of each fluid particle. In addition, by virtue of the fact that particles are initially identified and traced through their motion, conservation of mass is inherent.

However, this seemingly attractive method of analysis has one serious drawback. The fundamental equations of motion in Lagrangian form are simple in appearance but very difficult to solve because of their nonlinear nature. This difficulty is compounded by the fact that the use of approximations and the assumption of steady flow does not appreciably simplify the equations. Therefore, the abundance of information (much of which is not necessary in the solution of many practical problems) available with the Lagrangian technique is, indeed, difficult to obtain. In fact, most practical applications of the method are limited to one-dimensional flow. Thus, it is desirable to investigate another means of describing fluid motion.

b. Eulerian Method

In this method the individual fluid particles are not identified. Instead, a fixed position in space is chosen, and the velocity of particles

at this position as a function of time is sought. Mathematically, the velocity of particles at any point in the space can be written

$$\mathbf{q} = \mathbf{f}(\mathbf{r}, t) \tag{3-4a}$$

where

$$\mathbf{q} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$$

or

$$\begin{aligned} u &= f_1(x, y, z, t) \\ v &= f_2(x, y, z, t) \\ w &= f_3(x, y, z, t) \end{aligned} \tag{3-4b}$$

The relationship between the Eulerian and Lagrangian methods can now be shown. According to the Lagrangian method, we have a set of Eqs. (3-2) for each particle which can be combined with Eq. (3-4b) as follows:

$$\begin{aligned} \frac{dx}{dt} &= u(x, y, z, t) \\ \frac{dy}{dt} &= v(x, y, z, t) \\ \frac{dz}{dt} &= w(x, y, z, t) \end{aligned} \tag{3-5}$$

The integration of Eqs. (3-5) leads to three constants of integration which can be considered as initial coordinates x_0, y_0, z_0 of the fluid particle. Hence the solutions of Eq. (3-5) give the equations of Lagrange:

$$\begin{aligned} x &= F_1(x_0, y_0, z_0, t) \\ y &= F_2(x_0, y_0, z_0, t) \\ z &= F_3(x_0, y_0, z_0, t) \end{aligned} \tag{3-1b}$$

It is evident that in principle the Lagrangian method of description can always be derived by Eqs. (3-2) from the Eulerian method. Actually the solutions of the set of three simultaneous differential Eqs. (3-5) are generally very difficult. The following simple example serves to illustrate the concepts.

Illustrative Example 3.1. Assuming that the velocity components for a two-dimensional flow system can be given in the Eulerian system by

$$\begin{aligned} u &= A(x + y) + Ct \\ v &= B(x + y) + Et \end{aligned} \tag{a}$$

Find the displacement of a fluid particle in the Lagrangian system.

Solution. In terms of the displacements x and y, the velocities may also be represented by

$$u = \frac{dx}{dt} \qquad v = \frac{dy}{dt} \tag{b}$$

Equating Eqs. (a) and (b) leads to

$$\frac{dx}{dt} = A(x + y) + Ct \qquad \frac{dy}{dt} = B(x + y) + Et \tag{c}$$

The simultaneous solution of these two differential equations can be carried out by the operator method as follows: Eqs. (c) may be written

$$\frac{dx}{dt} - Ax - Ay = Ct \qquad \frac{dy}{dt} - Bx - By = Et$$

Therefore, if we let $D(\ ) = d/dt(\ )$, these equations become

$$(D - A)x - Ay = Ct \tag{d}$$

$$-Bx + (D - B)y = Et \tag{e}$$

Eliminating x from these equations leads to

$$-ABy + (D - B)(D - A)y = E(D - A)t + BCt$$

This equation can now be rearranged and solved in the following way:

$$D[D - (A + B)]y = E - (EA - BC)t$$

Thus,

$$y = C_1 + C_2 e^{(A+B)t} - \left[\frac{B(E + C)}{(A + B)^2}\right]t - \left[\frac{BC - AE}{2(A + B)}\right]t^2 \tag{g}$$

The solution for x, obtained by substitution of Eq. (g) into Eq. (e), is

$$x = -C_1 + \frac{A}{B}C_2 e^{(A+B)t} - \left[\frac{A(E + C)}{(A + B)^2}\right]t + \left[\frac{BC - AE}{2(A + B)}\right]t^2 - \frac{(E + C)}{(A + B)^2} \tag{h}$$

The constants C_1 and C_2 are now found from the initial conditions which are $x = x_0, y = y_0$ at $t = t_0 = 0$. These constants are

$$C_1 = -\frac{B}{A + B}\left[x_0 - \frac{A}{B}y_0 + \frac{E + C}{(A + B)^2}\right]$$

$$C_2 = \frac{B}{A + B}\left[x_0 + y_0 + \frac{E + C}{(A + B)^2}\right]$$

Therefore, the solutions given by Eq. (g) and Eq. (h) can be written in the form

$$x = F_1(x_0, y_0, t)$$
$$y = F_2(x_0, y_0, t)$$

The Lagrange system is thus functions of the initial positions, x_0 and y_0, and the time, t.

Illustrative Example 3.2. If $A = 2$, $B = 1$, $C = 3$, $E = \frac{1}{2}$ as given in illustrative example 3.1, show graphically the displacement of a fluid particle in the Lagrange system in an interval of time $t_0 = 0 \leq t \leq 1$ sec.

Solution. For $A = 2, B = 1, C = 3, E = \frac{1}{2}$, we have from Eqs. (g) and (h), respectively,

$$y = -\frac{1}{3}x_0 + \frac{2}{3}y_0 - \frac{7}{54} + \left(\frac{x_0}{3} + \frac{y_0}{3} + \frac{7}{54}\right)e^{3t} - \frac{7}{18}t - \frac{1}{3}t^2$$

$$x = \frac{1}{3}x_0 - \frac{2}{3}y_0 - \frac{7}{27} + \left(\frac{2}{3}x_0 + \frac{2}{3}y_0 + \frac{7}{27}\right)e^{3t} - \frac{7}{9}t + \frac{1}{3}t^2$$

For $x_0 = 1$ and $y_0 = 1$ the graphical representation of the displacement of a fluid particle in the Lagrange system in an interval of time $0 \leq t \leq 1$ is shown in Fig. 3.A.

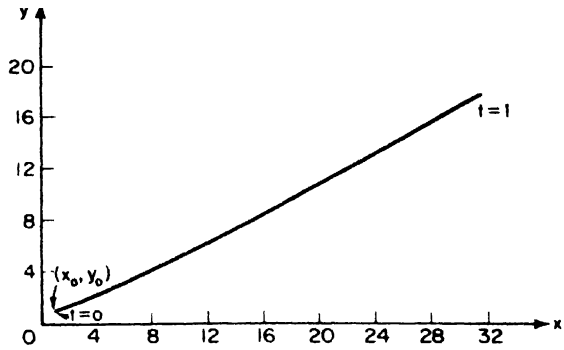


Figure 3.A

The Eulerian Method is particularly attractive in the solution of problems in multidimensional steady flow. This is because the nonlinear differential equations of Euler (§5.2) can often be linearized when appropriate approximations are made. This simplification cannot be made when corresponding Lagrangian equations are applied. The Eulerian coordinate representation will be used to describe fluid properties and flow characteristics in the subsequent chapters.

3.2. Translation, Rotation, and Rate of Deformation

In general, the movement of a fluid element may consist of a translation, a rotation, and a rate of deformation (Fig. 3.2).

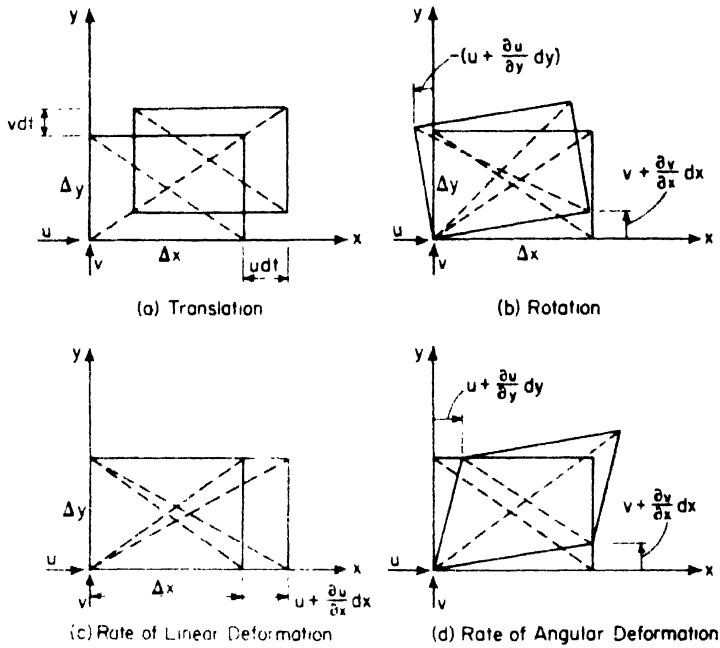


Figure 3.2

Consider the movement of a fluid (Fig. 3.3). The velocity at point $P(x, y, z)$ is $\mathbf{q}$ and at point $P_1(x_1, y_1, z_1)$, a small distance $d\mathbf{r}$ from P, is $\mathbf{q}_1$. The velocity vector $\mathbf{q}_1$ can be written

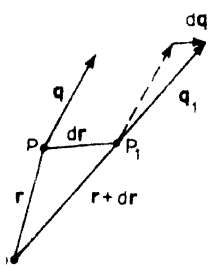


Figure 3.3

$$\begin{aligned}\mathbf{q}_1 &= i u_1 + j v_1 + k w_1 = \mathbf{q} + d\mathbf{q} \\ &= iu + jv + kw + \frac{\partial \mathbf{q}}{\partial x}dx + \frac{\partial \mathbf{q}}{\partial y}dy + \frac{\partial \mathbf{q}}{\partial z}dz \\ &= \mathbf{i}\left[u + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\right] \\ &\quad + \mathbf{j}\left[v + \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz\right] \\ &\quad + \mathbf{k}\left[w + \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz\right]\end{aligned} \tag{3-6}$$

Since the vorticity $\boldsymbol{\Omega}$ is defined by

$$\boldsymbol{\Omega} = \nabla \times \mathbf{q}$$
$$= \mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{3-7}$$

Eq (3-6) can be rearranged as

$$\begin{aligned}
\mathbf{q}_1 = \mathbf{i}\Big\{u &+ \left[\frac{\partial u}{\partial x}\,dx + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dy + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)dz\right] \\
&+ \frac{1}{2}\left[\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dz - \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dy\right]\Big\} \\
&+ \mathbf{j}\Big\{v + \left[\frac{\partial v}{\partial y}\,dy + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dx + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)dz\right] \\
&+ \frac{1}{2}\left[\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx - \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)dz\right]\Big\} \\
&+ \mathbf{k}\Big\{w + \left[\frac{\partial w}{\partial z}\,dz + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)dx + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)dy\right] \\
&+ \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)dy - \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dx\right]\Big\} \\
&= \mathbf{q} + \frac{1}{2}\boldsymbol{\Omega} \times d\mathbf{r} + \mathbf{D}
\end{aligned} \tag{3-8}$$

where

$$\begin{aligned}
\mathbf{D} = \mathbf{i}&\left[\frac{\partial u}{\partial x}\,dx + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dy + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)dz\right] \\
&+ \mathbf{j}\left[\frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dx + \frac{\partial v}{\partial y}\,dy + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)dz\right] \\
&+ \mathbf{k}\left[\frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)dx + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)dy + \frac{\partial w}{\partial z}\,dz\right]
\end{aligned}$$

Equation (3-8) represents the most general form of the movement of a fluid element. The first term $\mathbf{q}$ represents the translational velocity vector which indicates linear motion of all parts of the fluid element without change of shape. Hence, all of the velocity gradients are zero (i.e., $\partial u/\partial x = \partial u/\partial y = \partial u/\partial z = \cdots = \partial w/\partial z = 0$). The second term, $\frac{1}{2}\boldsymbol{\Omega} \times d\mathbf{r}$ represents a rigid rotation of the fluid element which will be discussed further. The third term represents the rate of strain term (or rate of deformation tensor) which will be discussed in greater detail in Chap. Four.

Illustrative Example 3.3. Given a velocity field with components

$$\begin{aligned}
u(x, y, z) &= cx + 2\omega_0 y + u_0 \\
v(x, y, z) &= cy + v_0 \\
w(x, y, z) &= -2cz + w_0
\end{aligned} \tag{a}$$

where c, ω_0, u_0, v_0, and w_0 are constants. If Eq. (a) gives the velocity components at a point (x, y, z), determine the velocity components at a neighboring point (x_1, y_1, z_1) and determine the different types of motion which are involved.

Solution. By Eq. (3–8)

$$\mathbf{q}_1 = \mathbf{q} + \frac{1}{2}\boldsymbol{\Omega} \times d\mathbf{r} + \mathbf{D}$$

(i) Translational Velocity

$$\begin{aligned} \mathbf{q} &= u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \\ \mathbf{q} &= (cx + 2\omega_0 y + u_0)\mathbf{i} + (cy + v_0)\mathbf{j} + (w_0 - 2cz)\mathbf{k} \end{aligned} \tag{b}$$

(ii) Rotational Velocity

$$\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\Omega} = \frac{1}{2}\nabla \times \mathbf{q} = \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \frac{1}{2}\left[\mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]$$

we note that $\partial u/\partial z = \partial v/\partial x = \partial v/\partial z = \partial w/\partial x = \partial w/\partial y = 0$.
Thus, $\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\Omega} = \frac{1}{2}[0 - (2\omega_0)]\mathbf{k} = -\omega_0\mathbf{k}$
This represents a constant angular velocity about the z-axis.

$$\frac{1}{2}\boldsymbol{\Omega} \times d\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -\omega_0 \\ dx & dy & dz \end{vmatrix} = \mathbf{i}(0 + \omega_0\, dy) + \mathbf{j}(-\omega_0\, dx - 0) + \mathbf{k}(0)$$

$$= (\omega_0\, dy)\mathbf{i} - (\omega_0\, dx)\mathbf{j} \tag{c}$$

(iii) Rate of Strain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = c; \qquad \frac{\partial w}{\partial z} = -2c, \qquad \text{i. e., uniform rate of normal strain}$$

$$\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \omega_0, \qquad \text{uniform rate of shearing strain}$$

$$\frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = 0$$

$$\frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0$$

Thus,

$$\mathbf{D} = \mathbf{i}(c\, dx + \omega_0\, dy) + \mathbf{j}(\omega_0\, dx + c\, dy) + \mathbf{k}(-2c\, dz) \tag{d}$$

Thus, the motion at the point (x_1, y_1, z_1) has three contributions:

(i) Translational velocity given by Eq. (b),
(ii) Rotational velocity given by Eq. (c), and
(iii) Rate of strain velocity given by Eq. (d).

3.3. *Streamlines, Path Lines, and Streak Lines*

a. *Streamlines*

In the discussion of the Eulerian method of description, we have defined the velocity field by the velocity vector given in Eq. (3–4a). This means that the velocity as a function of time is given in direction and magnitude at every point in the space considered. If for a fixed instant of time a space curve is drawn so that it is tangent everywhere to a velocity vector, then this curve is called a streamline. It is seen that the method of Euler gives a series of instantaneous streamlines of the state of motion, and the flow pattern changes with each instant of time. But for steady flow the orientation of the streamlines will be fixed (Fig. 3.1a). From the above definition of a streamline ($\mathbf{q} \parallel d\mathbf{r}$) we can write

$$\mathbf{q} \times d\mathbf{r} = 0 \tag{3–9}$$

at a given time. The differential equations of the streamlines, as derived from Eq. (3–9), are:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{3–10}$$

Illustrative Example 3.4. The velocity vector $\mathbf{q}$ is given by

$$\mathbf{q} = \mathbf{i}x - \mathbf{j}y \tag{a}$$

Determine the equation of the streamlines.

Solution. According to Eq. (3–9) we can write

$$\mathbf{q} \times d\mathbf{r} = (\mathbf{i}x - \mathbf{j}y) \times (\mathbf{i}\,dx + \mathbf{j}\,dy) = 0 \tag{b}$$

Simplifying,

$$(x\,dy + y\,dx)\mathbf{k} = 0$$

or

$$\frac{dx}{x} = -\frac{dy}{y} \tag{c}$$

This equation is identical to Eq. (3–10) in which $u = x$ and $v = -y$. Integrating Eq. (c) we obtain the equation for the streamlines,

$$xy = C \tag{d}$$

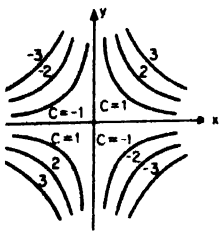


Figure 3.B

where C is an arbitrary constant.

The graphical representation of Eq. (d) is given in Fig. 3.B. It is seen that it represents streamlines of a steady two-dimensional flow at a 90° corner. The velocity vector $\mathbf{q} = \mathbf{i}x - \mathbf{j}y$ is everywhere tangent to the streamline $xy = C$.

b. Path Lines

As mentioned in §3.1, the method of Lagrange gives information about the paths of the different fluid particles as a function of the time. In other words, a path line is the trajectory of a fluid particle of fixed identity which is determined from Eqs. (3–5)

$$\frac{dx}{dt} = u(x, y, z, t)$$

$$\frac{dy}{dt} = v(x, y, z, t) \tag{3–5}$$

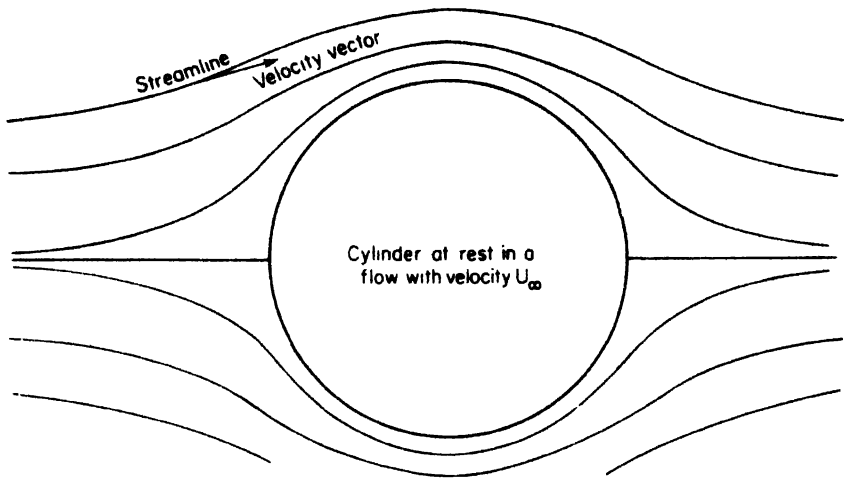
$$\frac{dz}{dt} = w(x, y, z, t)$$

with initial conditions: $x(t_0) = x_0$, $y(t_0) = y_0$, and $z(t_0) = z_0$. For a steady flow, the velocity $\mathbf{q}(\mathbf{r})$ is independent of time; and Eq. (3–5) becomes identical to Eqs. (3-10). It is evident that the path lines are identical with the streamlines in a steady flow.

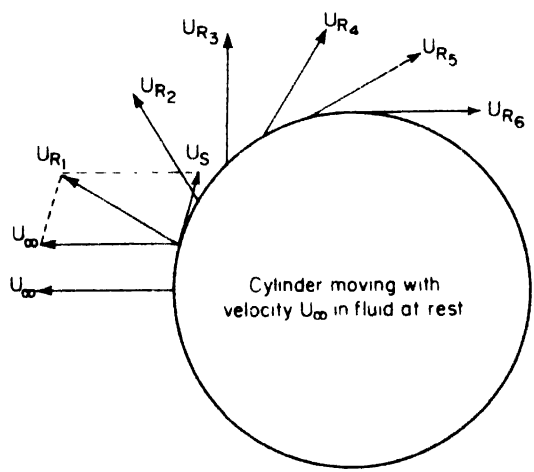
As an illustration of path lines, let us refer to Fig. 3.4. In Fig. 3.4a a solid circular cylinder is placed in a fixed position in a moving fluid. This is a steady-state flow pattern as explained previously. Since by definition, the velocity is tangential to the streamlines the velocity on the surface of the cylinder is everywhere tangent to the cylinder. Figure 3.4b shows the unsteady motion created by a cylinder moving with a uniform velocity U_∞ in a fluid at rest. To an observer away from the cylinder the resultant velocity on the cylinder in this case is the vector sum of the tangential velocity on the surface and the velocity of the cylinder. Hence, the instantaneous streamlines are the lines which coincide with the resultant velocity vectors, U_{Ri}.

Let us consider a particle A on the surface of the cylinder which is initially, $t = t_0$, located at x_0 as shown in Fig. 3.4c. The cylinder is assumed to be moving with a uniform velocity U_∞ in the x-direction. Initially this particle is on streamline 1, and it will move along streamline 1 with resultant velocity U_{R_1} until it reaches point B after a time Δt has elapsed. During this time the cylinder together with the stream-

line pattern has also traveled a distance, $U_\infty \Delta t$. The streamlines 1, 2, 3, etc., are all separated by an amount $U_\infty \Delta t$ in the direction of motion. As the particle reaches B, it will no longer follow streamline 1 but will be moving along streamline 2. At an increment of time Δt later, the particle will reach C and hence change its course along the streamline 3. If we limit Δt to an infinitesimal quantity, the path of the particle will be the path line. It is interesting to note that each

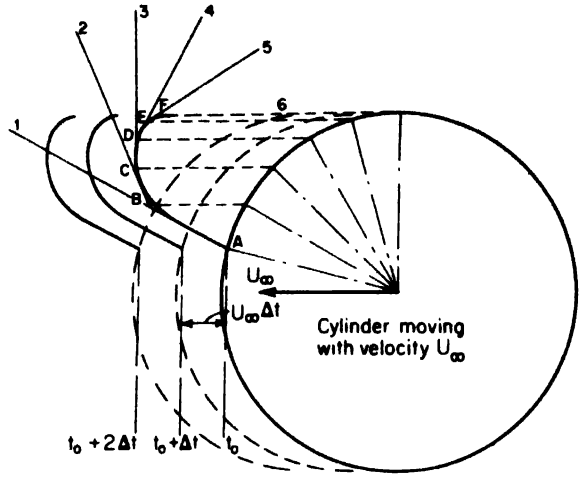


(a) Streamlines, Steady Flow

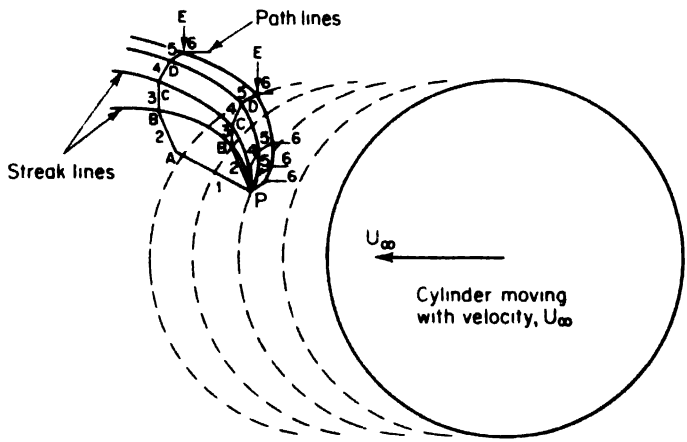


(b) Streamlines, Unsteady Flow

Figure 3.4



(c) Path Lines, Unsteady Flow



(d) Streak Lines, Unsteady Flow

Figure 3.4

element of the particle path becomes smaller than the preceding one, so that eventually the particle will come to rest.

c. *Streak Lines*

A streak line is the locus of points at a given time, t_1, that connects the temporary location of all the particles that have passed

through a fixed point in the flow field. This line is of particular interest in experimental flow visualization. If dye is injected into a liquid at a fixed point in the flow field, then at a later time t_1 the dye will indicate the end points of path lines of particles which have passed through the injection point.

The equation of the streak line at time t can be derived by the Lagrangian method. If a fluid particle (x_0, y_0, z_0) passes a fixed point (x_1, y_1, z_1) in the course of time, the Lagrangian method of description gives the equation

$$\begin{aligned} f_1(x_0, y_0, z_0, t) &= x_1 \\ f_2(x_0, y_0, z_0, t) &= y_1 \\ f_3(x_0, y_0, z_0, t) &= z_1 \end{aligned} \tag{3-11}$$

or solving for x_0, y_0, z_0

$$\begin{aligned} x_0 &= G_1(x_1, y_1, z_1, t) \\ y_0 &= G_2(x_1, y_1, z_1, t) \\ z_0 &= G_3(x_1, y_1, z_1, t) \end{aligned} \tag{3-12}$$

If the positions (x, y, z) of the particles which have passed the fixed point $(x_1, y_1. z_1)$ are determined, then a streak line can be drawn through these points. The equation of the streak line at time t is given by

$$\begin{aligned} x &= F_1(x_0, y_0, z_0, t) \\ y &= F_2(x_0, y_0, z_0, t) \\ z &= F_3(x_0, y_0, z_0, t) \end{aligned} \tag{3-13}$$

Upon the substitution of Eq. (3-12) into Eq. (3-13) one obtains

$$\begin{aligned} x &= F_1[G_1, G_2, G_3, t] \\ y &= F_2[G_1, G_2, G_3, t] \\ z &= F_3[G_1, G_2, G_3, t] \end{aligned} \tag{3-14}$$

or

$$\mathbf{r} = \mathbf{F}[\mathbf{G}(\mathbf{r}_1, t), t]$$

The following simple example may serve as an illustration of solving Eqs. (3-14).

Let

$$u = \frac{x}{t} \qquad v = y \qquad w = 0 \tag{a}$$

The equations of path lines are

$$u = \frac{dx}{dt} = \frac{x}{t} \qquad v = \frac{dy}{dt} = y \qquad w = \frac{dz}{dt} = 0 \tag{b}$$

The solutions of Eqs. (b) give

$$x = x_0 \frac{t}{t_0} \qquad y = y_0 e^{t-t_0} \qquad z = z_0 \tag{c}$$

The particle at (x_1, y_1) at all times can be obtained from the inverse relation of Eq. (c)

$$x_0 = \frac{x_1}{s} t_0 \qquad y_0 = y_1 e^{-s+t_0} \qquad z_0 = z_1 \tag{d}$$

where the parameter s lies in the interval $t_0 \leq s \leq t$. Substituting Eqs. (d) into Eqs. (c), we obtain the coordinates of the streak line

$$x = x_1 \frac{t}{s} \qquad y = y_1 e^{t-s} \qquad z = z_1 \tag{e}$$

These results are plotted in Fig. 3.C.

It is seen from the above example that for a steady flow, streak

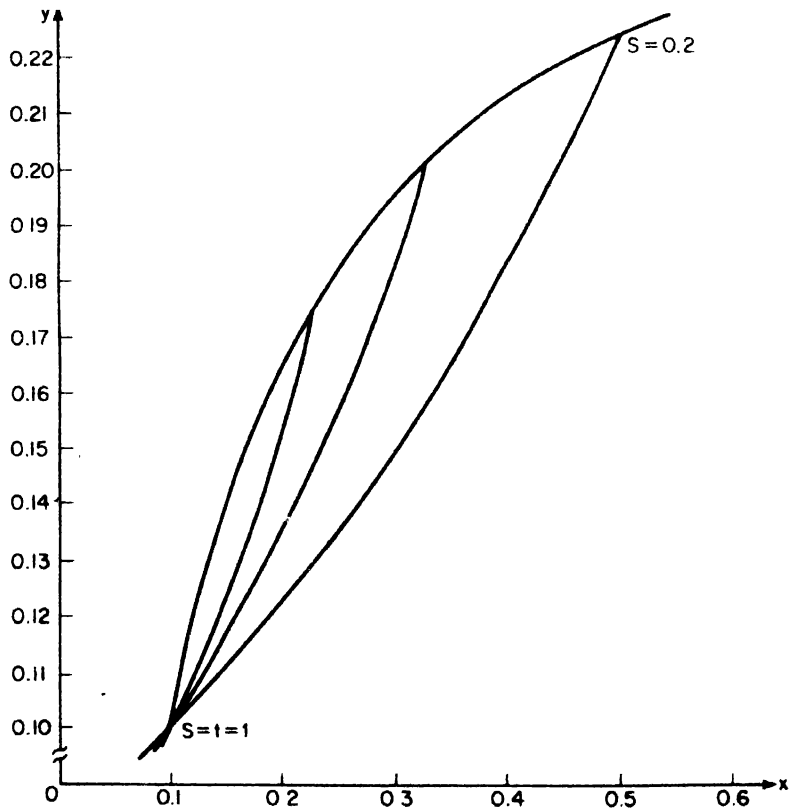


Figure 3.C

lines are identical to path lines, and hence they coincide with streamlines.

In Fig. 3.4d is shown a graphical construction of streak lines. The physical situation is identical to the case presented in Fig. 3.3c in connection with the discussion of path lines. Let the point P be arbitrarily selected as a fixed point where dye is injected. In this case the streamlines 1, 2, 3, etc., are so located that the horizontal spacing between them through P is equal. At an instant of time t_0 while the cylinder is moving with a uniform velocity U_∞ in the x-direction, a dyed particle will begin to move along streamline 1 with velocity U_{R_1} towards A. After a small increment of time Δt, streamline 2 will be at P and another particle will begin to move from P along the direction of streamline 2 with velocity U_{R_2}. Shortly afterward, streamline 2 will catch up with the first particle at A and the particle will change its course from streamline 1 to streamline 2 with corresponding velocity U_{R_2}. At the time $t + 2\Delta t$, streamline 3 will move to P and a third particle will start to move along the direction of streamline 3. At a later time, streamline 3 will meet the second particle at B' and then the first particle at B and both particles will begin to move along the direction of streamline 3. The process goes on with time as long as the motion continues. The line which joins the points of the path lines that have passed through P for a given time represents the streak line originating at P.

3.4. *The Material Derivative and Acceleration*

The velocity of particles at any point in the space can be written in Eulerian form

$$\mathbf{q} = \mathbf{f}(\mathbf{r}, t) \tag{3-4a}$$

The velocity in the x-direction at time t can be reduced from Eq. (3-4a) in the form

$$u = f_1(x, y, z, t) \tag{3-4b}$$

At an infinitesimal time interval Δt later, this particle will move to the position $x + \Delta x$, $y + \Delta y$, $z + \Delta z$ with an x-component of velocity $u + \Delta u$. The increments Δx, Δy, and Δz are

$$\Delta x = u\Delta t \qquad \Delta y = v\Delta t \qquad \Delta z = w\Delta t$$

Hence we have

$$\begin{aligned} u + \Delta u &= f_1(x + u\Delta t,\ y + v\Delta t,\ z + w\Delta t,\ t + \Delta t) \\ &= f_1(x, y, z, t) + \left(u\frac{\partial f_1}{\partial x}\Delta t + v\frac{\partial f_1}{\partial y}\Delta t + w\frac{\partial f_1}{\partial z}\Delta t + \frac{\partial f_1}{\partial t}\Delta t\right) \end{aligned}$$

$$+ \text{ higher order terms in } \Delta t \tag{3-15}$$

Taking the limiting form of Eq. (3-15) one obtains

$$\lim_{\Delta t \to 0} \frac{\Delta u}{\Delta t} = \frac{Du}{Dt} = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)u \tag{3-16}$$

The total differential with respect to time

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{3-17}$$

is known as the material derivative (also substantial derivative or particle derivative) with respect to time. The first term in Eq. (3-17) which is associated with time variation at a fixed position is called the local derivative and the last three terms, associated with the change of a quantity due to movement of the particle is called convective derivative. The symbol d/dt is also used for the material derivative.

The acceleration of a fluid particle of fixed identity can be expressed by the material derivative of the velocity vector $\mathbf{q}$, i.e.,

$$\mathbf{a} = \frac{D\mathbf{q}}{Dt} = \frac{\partial \mathbf{q}}{\partial t} + u\frac{\partial \mathbf{q}}{\partial x} + v\frac{\partial \mathbf{q}}{\partial y} + w\frac{\partial \mathbf{q}}{\partial z} = \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q} \tag{3-18}$$

In Cartesian coordinates, the acceleration components are

$$\begin{aligned} a_x &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ a_y &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ a_z &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned} \tag{3-19}$$

In cylindrical coordinates (r, θ, z) with velocity components v_r, v_θ, v_z, the components of acceleration are

$$\begin{aligned} a_r &= \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} \\ a_\theta &= \frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \\ a_z &= \frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z} \end{aligned} \tag{3-20}$$

In spherical coordinates $(\omega, \varphi, \theta)$ with velocity components v_θ, v_ω, v_φ, the components of acceleration are

$$\begin{aligned} a_\theta &= \frac{\partial v_\theta}{\partial t} + v_\omega\frac{\partial v_\theta}{\partial \omega} + \frac{v_\theta}{\omega}\frac{\partial v_\theta}{\partial \varphi} + \frac{v_\theta}{\omega \sin\varphi}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_\omega}{\omega} + \frac{v_\theta v_\varphi \cot\varphi}{\omega} \\ a_\omega &= \frac{\partial v_\omega}{\partial t} + v_\omega\frac{\partial v_\omega}{\partial \omega} + \frac{v_\varphi}{\omega}\frac{\partial v_\omega}{\partial \varphi} + \frac{v_\theta}{\omega \sin\varphi}\frac{\partial v_\omega}{\partial \theta} - \frac{v_\varphi^2 + v_\theta^2}{\omega} \end{aligned} \tag{3-21}$$

$$a_\varphi = \frac{\partial v_\varphi}{\partial t} + v_\omega \frac{\partial v_\varphi}{\partial \omega} + \frac{v_\varphi}{\omega}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_\theta}{\omega \sin \varphi}\frac{\partial v_\varphi}{\partial \theta} + \frac{v_\omega v_\varphi}{\omega} - \frac{v_\theta^2 \cot \varphi}{\omega}$$

The derivation of acceleration components in cylindrical and spherical coordinates can be found in Chap. Five.

Illustrative Example 3.6. Given the velocity field $\mathbf{q} = \mathbf{i}Ax^2y + \mathbf{j}By^2zt + \mathbf{k}czt^2$, determine the acceleration of a fluid particle of fixed identity.

Solution. (a) Using Eq. (3-18) we have

$$\mathbf{a} = \frac{\partial \mathbf{q}}{\partial t} + u\frac{\partial \mathbf{q}}{\partial x} + v\frac{\partial \mathbf{q}}{\partial y} + w\frac{\partial \mathbf{q}}{\partial z}$$

$$= \mathbf{j}(By^2z) + \mathbf{k}(2czt) + (Ax^2y)[\mathbf{i}(2Axy)] + (By^2zt)[\mathbf{i}(Ax^2) + \mathbf{j}(2Byzt)]$$

$$+(czt^2)[\mathbf{j}(By^2t) + \mathbf{k}(ct^2)]$$

$$= \mathbf{i}A(2Ax^3y^2 + Bx^2y^2zt) + \mathbf{j}B(y^2z + 2By^3z^2t^2 + cy^2zt^3) + \mathbf{k}c(2zt + czt^4)$$

The components of the acceleration are,

$$a_x = A(2Ax^3y^2 + Bx^2y^2zt)$$

$$a_y = B(y^2z + 2By^3z^2t^2 + cy^2zt^3)$$

$$a_z = c(2zt + czt^4)$$

(b) Using Eq. (3-19)

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$= (Ax^2y)(2Axy) + (By^2zt)(Ax^2)$$

$$= A(2Ax^3y^2 + Bx^2y^2zt)$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$= By^2z + (By^2zt)(2Byzt) + (czt^2)(By^2t)$$

$$= B(y^2z + 2By^3z^2t^2 + cy^2zt^3)$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$= 2czt + (czt^2)(ct^2)$$

$$= c(2zt + czt^4)$$

3.5. *Vorticity*

The vorticity is actually an antisymmetric tensor (Ref. 32) and its three distinct elements transform like the components of a vector in Cartesian coordinates. This is the reason that vorticity components can

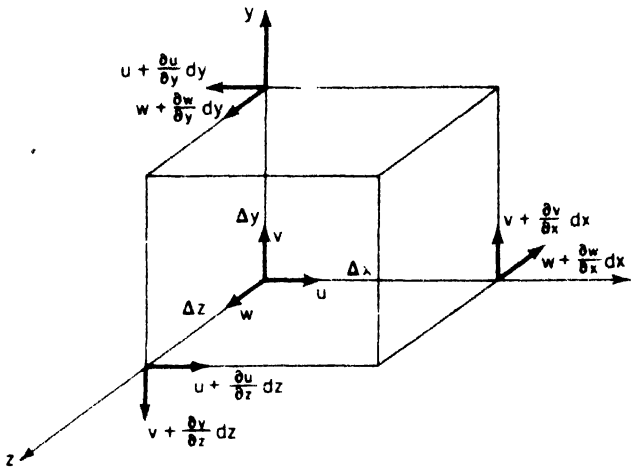


Figure 3.5

be treated as vectors. Referring to Fig. 3.5, we can obtain the angular velocities of the segments Δy and Δz about the x-axis as follows:

$$\text{Angular velocity of segment } \Delta y = \frac{w + (\partial w/\partial y)\,\Delta y - w}{\Delta y} = \frac{\partial w}{\partial y}$$

$$\text{Angular velocity of segment } \Delta z = -\frac{v + (\partial v/\partial z)\,\Delta z - v}{\Delta z} = -\frac{\partial v}{\partial z}$$

Therefore, the average angular velocity component in the x-direction is

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \tag{3-22}$$

The average angular velocity components in the y and z-directions may be obtained in a similar manner as follows:

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \tag{3-23}$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{3-24}$$

Finally the angular velocity vector $\boldsymbol{\omega}$ of a fluid element in terms of the velocity field is

$$\boldsymbol{\omega} = \mathbf{i}\omega_x + \mathbf{j}\omega_y + \mathbf{k}\omega_z = \frac{1}{2}\left[\mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] \tag{3-25}$$

Comparing Eq. (3-25) and Eq. (3-7), we have the following relation connecting the vorticity vector and the angular velocity vector

$$\Omega = 2\omega \tag{3-26}$$

If a line is drawn in the fluid so that the tangent to it at each point is in the direction of the vorticity vector Ω at that point, the line is called a vortex line. If $d\mathbf{l}$ is in the direction of the vorticity vector then

$$\Omega \times d\mathbf{l} = 0$$

or

$$\mathbf{i}(\Omega_y dz - \Omega_z dy) + \mathbf{j}(\Omega_z dx - \Omega_x dz) + \mathbf{k}(\Omega_x dy - \Omega_y dx) = 0 \tag{3-27}$$

From the above relations, equations of vortex lines are obtained:

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z} \tag{3-28}$$

a. Vorticity in Polar Coordinates

In the two-dimensional polar coordinate system (Fig. 3.6) the angular velocity of segment Δr is

$$\frac{v_\theta + (\partial v_\theta/\partial r)\,\Delta r - v_\theta}{\Delta r} = \frac{\partial v_\theta}{\partial r}$$

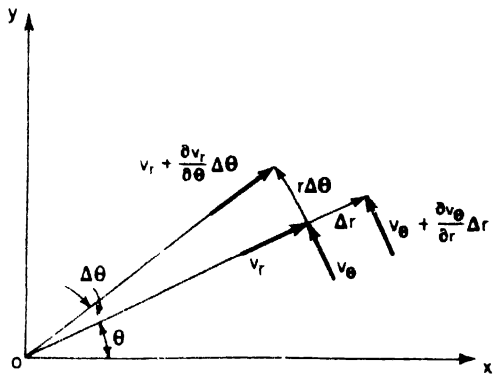


Figure 3.6

Also, the angular velocity of segment $r\Delta\theta$ is

$$-\frac{v_r + (\partial v_r/\partial\theta)\,\Delta\theta - v_r}{r\Delta\theta} = -\frac{1}{r}\frac{\partial v_r}{\partial\theta}$$

The additional term arising from the angular velocity about the center, o, is

$$\frac{v_\theta}{r}$$

Hence, the vorticity in polar coordinates is

$$\Omega_z = 2\omega_z = \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} \tag{3-29}$$

b. Vorticity in Orthogonal Curvilinear Coordinates

The general form of the vorticity components can be written in terms of the orthogonal curvilinear coordinates (see Appendix A) as

$$\mathbf{i}_1\Omega_1 = \frac{\mathbf{i}_1 h_1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_2}(h_3 q_3) - \frac{\partial}{\partial x_3}(h_2 q_2)\right] \tag{3-30a}$$

$$\mathbf{i}_2\Omega_2 = \frac{\mathbf{i}_2 h_2}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_3}(h_1 q_1) - \frac{\partial}{\partial x_1}(h_3 q_3)\right] \tag{3-30b}$$

$$\mathbf{i}_3\Omega_3 = \frac{\mathbf{i}_3 h_3}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 q_2) - \frac{\partial}{\partial x_2}(h_1 q_1)\right] \tag{3-30c}$$

(i) *Cylindrical Coordinates.* For the case of cylindrical coordinates we have

$$dx_1 = dr \qquad h_1 = 1 \qquad q_1 = v_r$$
$$dx_2 = d\theta \qquad h_2 = r \qquad q_2 = v_\theta$$
$$dx_3 = dz \qquad h_3 = 1 \qquad q_3 = v_z$$

Thus Eqs. (3-30) become

$$\Omega_z = \frac{1}{r}\left[\frac{\partial}{\partial r}(rv_\theta) - \frac{\partial}{\partial \theta}(v_r)\right]$$

$$\Omega_r = \frac{1}{r}\left[\frac{\partial}{\partial \theta}(v_z) - \frac{\partial}{\partial z}(rv_\theta)\right]$$

$$\Omega_\theta = \frac{\partial}{\partial z}(v_r) - \frac{\partial}{\partial r}(v_z)$$

or

$$\Omega_z = \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} \tag{3-31a}$$

$$\Omega_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \tag{3-31b}$$

$$\Omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \tag{3-31c}$$

(ii) *Spherical Coordinates.* In this case the orthogonal coordinates are:

$$dx_1 = d\omega \qquad h_1 = 1 \qquad q_1 = v_\omega$$
$$dx_2 = d\varphi \qquad h_2 = \omega \qquad q_2 = v_\varphi$$

$$dx_3 = d\theta \qquad h_3 = \omega \sin \varphi \qquad q_3 = v_\theta$$

Equations (3–30) therefore become:

$$\Omega_\omega = \frac{1}{\omega^2 \sin \varphi}\left[\frac{\partial}{\partial \varphi}(\omega v_\theta \sin \varphi) - \frac{\partial}{\partial \theta}(\omega v_\varphi)\right]$$

$$\Omega_\varphi = \frac{1}{\omega \sin \varphi}\left[\frac{\partial}{\partial \theta}(v_\omega) - \frac{\partial}{\partial \omega}(\omega v_\theta \sin \varphi)\right]$$

$$\Omega_\theta = \frac{1}{\omega}\left[\frac{\partial}{\partial \omega}(\omega v_\theta) - \frac{\partial}{\partial \varphi}(v_\omega)\right]$$

or

$$\Omega_\omega = \frac{1}{\omega}\frac{\partial v_\theta}{\partial \varphi} - \frac{1}{\omega \sin \varphi}\frac{\partial v_\varphi}{\partial \theta} + \frac{v_\theta}{\omega}\cot \varphi \tag{3–32a}$$

$$\Omega_\varphi = \frac{1}{\omega \sin \varphi}\frac{\partial v_\omega}{\partial \theta} - \frac{\partial v_\theta}{\partial \omega} - \frac{v_\theta}{\omega} \tag{3–32b}$$

$$\Omega_\theta = \frac{\partial v_\varphi}{\partial \omega} + \frac{v_\varphi}{\omega} - \frac{1}{\omega}\frac{\partial v_\omega}{\partial \varphi} \tag{3–32c}$$

Illustrative Example 3.7. The velocity vector in the flow field is given by

$$\mathbf{q} = \mathbf{i}(Az - By) + \mathbf{j}(Bx - Cz) + \mathbf{k}(Cy - Ax)$$

where A, B, and C are nonzero constants. Determine the equations of the vortex lines.

Solution. According to Eq. (3–27) we have

$$\mathbf{i}(\Omega_y dz - \Omega_z dy) + \mathbf{j}(\Omega_z dx - \Omega_x dz) + \mathbf{k}(\Omega_x dy - \Omega_y dx) = 0 \tag{3–27}$$

and from Eqs. (3–22), (3–23), and (3–24), we obtain

$$\Omega_x = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = C + C = 2C$$

$$\Omega_y = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = A + A = 2A$$

$$\Omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = B + B = 2B$$

With these values for Ω_x, Ω_y, and Ω_z Eq. (3–27) reduces to

$$\text{(i)} \quad 2A\,dz - 2B\,dy = 0 \quad \text{or} \quad z - \frac{B}{A}y = C_1 \tag{a}$$

$$\text{(ii)} \quad 2B\,dx - 2C\,dz = 0 \quad \text{or} \quad x - \frac{C}{B}z = C_2 \tag{b}$$

$$\text{(iii)} \quad 2C\,dy - 2A\,dx = 0 \quad \text{or} \quad y - \frac{A}{C}x = C_3 \tag{c}$$

Equations (a), (b), and (c) are the equations of vortex lines. They are straight lines in this case.

NOMENCLATURE FOR CHAPTER THREE

a_x, a_y, a_z	acceleration components in Cartesian coordinates, Eq. (3-3)
a_r, a_θ, a_z	acceleration components in cylindrical coordinates, Eq. (3-20)
$a_\omega, a_\varphi, a_\theta$	acceleration components in spherical coordinates, Eq. (3-21)
$\mathbf{q}$	velocity vector, Eqs. (3-4a) and (3-6)
$\mathbf{r}$	position vector, Eq. (3-1a)
t	time
u, v, w	velocity components in Cartesian coordinates, Eq. (3-2)
v_r, v_θ, v_z	velocity components in cylindrical coodinates, Eq. (3-20)
$v_\omega, v_\varphi, v_\theta$	velocity components in spherical coordinates, Eq. (3-21)
x, y, z	Cartesian coordinates
r, θ, z	cylindrical coordinates, Eq. (3-20)
ω, φ, θ	spherical coordinates, Eq. (3-21)
$\boldsymbol{\omega}$	angular velocity vector, Eq. (3-25)
$\boldsymbol{\Omega}$	vorticity vector, Eq. (3-7)
$\Omega_x, \Omega_y, \Omega_z$	vorticity components in Cartesian coordinates, Eq. (3-27)
$\Omega_r, \Omega_\theta, \Omega_z$	vorticity components in cylindrical coordinates, Eq. (3-31)
$\Omega_\omega, \Omega_\varphi, \Omega_\theta$	vorticity components in spherical coordinates, Eq. (3-32)

PROBLEMS

3.1. The velocity at a point in a fluid for a one-dimensional flow may be given in the Eulerian coordinates by $u = Ax + Bt$. Show that $x = f(x_0, t)$ in the Lagrange coordinates can be obtained from the Eulerian system. The initial position of the fluid particle is designated by x_0 and the initial time $t_0 = 0$ may be assumed.

3.2. For a two-dimensional flow the velocities at a point in a fluid may be expressed in the Eulerian coordinates by $u = x + y + 2t$ and $v = 2y + t$.

(a) Determine the Lagrange coordinates as functions of the initial positions x_0 and y_0 and the time t.

(b) Assuming $x_0 = 1$, $y_0 = 1$, and $t_0 = 0$, plot the displacement of a fluid particle in the Lagrangian coordinates during the interval $t = 0$ to $t = 1$.

3.3. The velocities at a point in a fluid in the Eulerian system are given by

$$u = x + y + z + t$$

$$v = 2(x + y + z) + t$$

$$w = 3(x + y + z) + t$$

Show that the displacements of a fluid particle in the Lagrangian system are

$$x = \tfrac{5}{6}x_0 - \tfrac{1}{6}y_0 - \tfrac{1}{6}z_0 + \tfrac{1}{6}(x_0 + y_0 + z_0 + \tfrac{1}{12})e^{6t} - \tfrac{1}{12}t - \tfrac{1}{4}t^2 - \tfrac{1}{72}$$

$$y = -\tfrac{1}{3}x_0 + \tfrac{2}{3}y_0 - \tfrac{1}{3}z_0 + \tfrac{1}{3}(x_0 + y_0 + z_0 + \tfrac{1}{12})e^{6t} - \tfrac{1}{6}t - \tfrac{1}{36}$$

$$z = -\tfrac{1}{2}x_0 - \tfrac{1}{2}y_0 + \tfrac{1}{2}z_0 + \tfrac{1}{2}(x_0 + y_0 + z_0 + \tfrac{1}{12})e^{6t} - \tfrac{1}{4}t - \tfrac{1}{4}t^2 - \tfrac{1}{24}$$

where x_0, y_0, and z_0 are the initial position in the Lagrangian coordinates when $t = t_0 = 0$.

3.4. Find the equation of the streamlines for the flow $\mathbf{q} = -\mathbf{i}(3y^2) - \mathbf{j}(6x)$. Plot the streamline passing the point (1, 1).

3.5. The velocity components in a two-dimensional flow field for an incompressible fluid are given by $u = e^x \cosh(y)$ and $v = -e^x \sinh(y)$. Determine the equation of the streamlines for this flow.

3.6. The velocity components in a three-dimensional flow field for an incompressible fluid are given by $u = 2x$, $v = -y$, and $w = -z$. Determine the equations of the streamline passing the point (1, 1, 1).

3.7. In Fig. 3.4b is shown the unsteady-state streamlines of a cylinder moving with a uniform velocity U_∞ in the fluid at rest. Calculate and draw the unsteady-state streamlines at an interval of 10° ($\theta = 10°$). The tangential component of velocity on the surface of the cylinder in a uniform stream, U_∞, is given by $v_\theta = -2U_\infty \sin\theta$.

3.8. In Fig. 3.4c is shown the path line of a particle A on the surface of the cylinder moving with a uniform velocity U_∞ in the fluid at rest. Using the results calculated in Prob. 3.7, construct the path line of the particle A which is initially located on the surface of the cylinder at $\theta = 170°$, i.e., 10° with the direction of U_∞.

3.9. In Fig. 3.4d is shown streak lines passing through the point P for a given time. Using the results obtained in Probs. 3.7 and 3.8, construct the streak lines in the vicinity of the cylinder.

3.10. Determine the acceleration of a fluid particle from the following flow field

$$\mathbf{q} = \mathbf{i}(Axy^2t) + \mathbf{j}(Bx^2yt) + \mathbf{k}(cxyz)$$

3.11. In cylindrical coordinates (r, θ, z) the velocity components of a flow are given as $v_r = Ar^2z\cos\theta$, $v_\theta = Brz\sin\theta$, $v_z = cz^2t$. Determine the components of the acceleration of a fluid particle.

3.12. In spherical coordinates $(\omega, \varphi, \theta)$, the velocity components are given as $v_\omega = (\omega^2/t^2)\sin\theta$, $v_\theta = (\omega/t)\sin\varphi\cos\theta$, $v_\varphi = (\omega/t)\cot\varphi\csc\theta$.

Determine the components of acceleration of a fluid particle. What are the acceleration components of a fluid particle at position, $\omega = 10$ ft, $\theta = 30°$, $\varphi = 60°$, and at time $t = 1$ sec?

3.13. Find the vorticity of the fluid motion for the given velocity components.

(a) $u = A(x + y), \qquad v = -A(x + y)$

(b) $u = 2Axz, \qquad v = A(c^2 + x^2 - z^2)$

(c) $u = Ay^2 + By + C, \qquad v = 0$

3.14. Find the vorticity in polar coordinates for the following velocity components

(a) $v_r = r \sin \theta, \qquad v_\theta = 2r \cos \theta$

(b) $v_r = \frac{A}{r} \cos \theta, \qquad v_\theta = 0$

(c) $v_r = \frac{A}{r}, \qquad v_\theta = 0$

(d) $v_r = \left(1 - \frac{A}{r^2}\right) \cos \theta, \qquad v_\theta = -\left(1 + \frac{A}{r^2}\right) \sin \theta - \frac{B}{r}$

3.15. Find the vorticity in the spherical coordinates for the velocity components,

$$v_\omega = \left(1 - \frac{A}{\omega^3}\right) \cos \varphi$$

$$v_\varphi = -\left(1 + \frac{A}{2\omega^3}\right) \sin \varphi$$

$$v_\theta = 0$$

4

GENERAL THEORY OF STRESS AND RATE OF STRAIN

In Chap. One the regimes of fluid mechanics were briefly discussed. For a single continuous fluid the most complete description of such a flow is the theory of a viscous compressible flow. The mathematical equations governing the motion of a viscous compressible fluid are based on the three conservation laws: conservation of mass, conservation of momentum, and conservation of energy of the fluid. Because of the presence of viscosity in the fluid, the complicated viscous forces due to shearing stresses and the dissipation energy due to work done by the shearing stresses must be taken into consideration in the respective governing equations. For this reason the concepts and relationships of the stresses and rates of strain are essential in the development of the governing equations of a viscous compressible fluid.

In §4.1 the nature of stresses is discussed in detail. The nine components of stress are derived from the respective resultant stress on each of the surfaces of a rectangular parallelepiped. From the condition of equilibrium of the elementary volume the nine components of stress are reduced to six components σ_{xx}, σ_{yy}, σ_{zz}, σ_{xy}, σ_{yz}, and σ_{zx}.

In §4.2 the transformation of stress-components is discussed. The invariance of the stress to the choice of coordinate systems is shown.

Hence, the stress is a tensor (tensor of order 2) quantity.

In §4.3 the nature of strains is discussed and the difference between the strain in elasticity and the rate of strain in fluid mechanics is also emphasized. Finally, the six components of the rates of strain ε_{xx}, ε_{yy}, ε_{zz}, ε_{xy}, ε_{yz}, and ε_{zx} are derived.

In §4.4 the transformation of the rate of strain components is discussed. The invariance of the rate of strain to the choice of coordinate systems clearly indicates that the rate of strain is also a tensor of order 2.

In §4.5 the relation between stress and rate of strain is established. The formulation of the relation above is based on their tensor character and some reasonable and logical assumptions. The stress and rate of strain relation makes possible the expression of surface forces in terms of velocity gradients in the governing equations of a viscous compressible fluid.

Vector methods and notation are used in this chapter. The tensor quantity is also mentioned and identified. The reader may refer to Appendix A for vector analysis and notation and Ref. 32 for the tensor concept.

4.1. *Nature of Stresses*

The forces acting on an element of a continuous medium in general are of two types, namely, external or body forces and internal or surface forces (also called contact forces). Gravitational and electromagnetic forces are examples of body forces which are distributed over the volume of the medium. Body forces are usually expressed per unit mass of the medium acted upon. Surface forces, on the other hand, are regarded as acting on an element of volume through its bounding surface.

Consider a surface ΔA of a fluid in contact with a solid body. Then forces are exerted between the parts of the solid and the fluid close to ΔA. But according to Newton's third law, the forces exerted on the fluid across a portion of ΔA are equal and opposite to the force (assuming its inertia and body forces are vanishingly small) exerted on the body across the portion of ΔA. The set of forces on the surface of the body transmitted from the surroundings is equivalent to a resultant force F and a couple M (Fig. 4.1). Since we define a stress as the limit of the ratio of the force to the area on which it acts when the area shrinks to a point (within the limitations of the continuum), it follows that the ratio of the couple (with vanishingly small moment arm) to the area must ultimately vanish. We can therefore specify the intensity

of the action at any point on the surface by the limit of the ratio of force to area. This is called the stress at the point which may, in general, be in any direction. For a fluid at rest, it is normal to the surface and is in the nature of a pressure. When fluids are in motion, however, there are also shearing stresses in addition to normal stress.

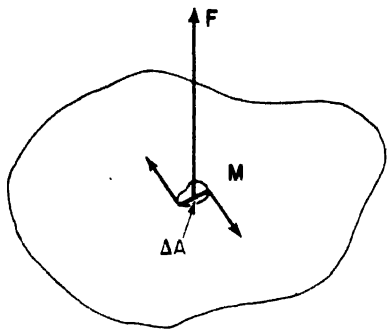


Figure 4.1

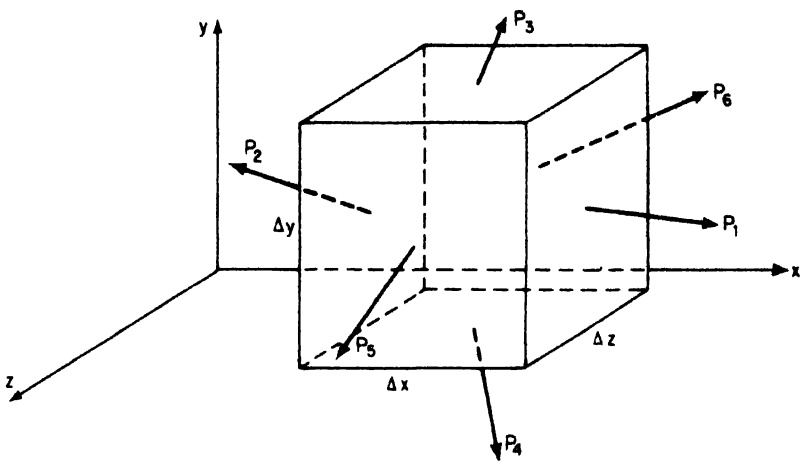


Figure 4.2

Let us consider a rectangular parallelepiped, Fig. 4.2, of dimensions Δx, Δy, and Δz having its center at x, y, z, and take the mean stresses on the surfaces to be the stresses at the middle of the surface concerned. The middle points of the two surfaces perpendicular to the x-axis have

coordinates $(x \pm \Delta x/2, y, z)$, and the middle points of the other surfaces are at $(x, y \pm \Delta y/2, z)$ and $(x, y, z \pm \Delta z/2)$.

In the study of fluid motion we are only interested in the distorting forces which cause the fluid to undergo deformation and, therefore, the mean force at (x, y, z) can be taken to be zero, i. e., $\mathbf{P}(x, y, z) = 0$. Also the effects of body forces (proportional to the cube of an infinitesimal length) on distortion are of smaller order than those of surface forces and may be neglected. Then since $\mathbf{P}_1$ is the resultant stress acting at $(x + \Delta x/2, y, z)$, an expansion in Taylor's series in powers of $\Delta x/2$, yields

$$\mathbf{P}_1 = \frac{\partial \mathbf{P}}{\partial x}\left(\frac{\Delta x}{2}\right) + \frac{1}{2}\frac{\partial^2 \mathbf{P}}{\partial x^2}\left(\frac{\Delta x}{2}\right)^2 + \cdots$$

Similarly, since $\mathbf{P}_2$ acts at $(x - \Delta x/2, y, z)$,

$$\mathbf{P}_2 = -\frac{\partial \mathbf{P}}{\partial x}\left(\frac{\Delta x}{2}\right) + \frac{1}{2}\frac{\partial^2 \mathbf{P}}{\partial x^2}\left(\frac{\Delta x}{2}\right)^2 + \cdots$$

If $\Delta x \to \epsilon$ where ϵ approaches a vanishingly small linear dimension, we may neglect powers of Δx beyond the first. Hence

$$\mathbf{P}_1 = -\mathbf{P}_2 = \frac{\partial \mathbf{P}}{\partial x}\left(\frac{\Delta x}{2}\right)$$

In a like manner the resultant stresses acting at $(x, y \pm \Delta y/2, z)$ and $(x, y, z \pm \Delta z/2)$ are, respectively,

$$\mathbf{P}_3 = -\mathbf{P}_4 = \frac{\partial \mathbf{P}}{\partial y}\left(\frac{\Delta y}{2}\right)$$

$$\mathbf{P}_5 = -\mathbf{P}_6 = \frac{\partial \mathbf{P}}{\partial z}\left(\frac{\Delta z}{2}\right)$$

The homogeneous state of stress is hence given by $\mathbf{P}_1$, $\mathbf{P}_3$, and $\mathbf{P}_5$ which have fundamentally the nature of a rate of change of a vector in a given direction, i. e., a force with second direction normal to the surface on which it acts. Hence stress is a tensor of order 2 (see Appendix A) which is completely specified by its magnitude and two directions.

If each of the $\mathbf{P}_1$, $\mathbf{P}_3$, and $\mathbf{P}_5$ is resolved into three components along the axes of x, y, and z, we have

$$\mathbf{P}_1 = -\mathbf{P}_2 = \mathbf{i}_x\sigma_{xx} + \mathbf{i}_y\sigma_{xy} + \mathbf{i}_z\sigma_{xz} = \mathbf{P}_x \tag{4-1a}$$

$$\mathbf{P}_3 = -\mathbf{P}_4 = \mathbf{i}_x\sigma_{yx} + \mathbf{i}_y\sigma_{yy} + \mathbf{i}_z\sigma_{yz} = \mathbf{P}_y \tag{4-1b}$$

$$\mathbf{P}_5 = -\mathbf{P}_6 = \mathbf{i}_x\sigma_{zx} + \mathbf{i}_y\sigma_{zy} + \mathbf{i}_z\sigma_{zz} = \mathbf{P}_z. \tag{4-1c}$$

where $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ are the unit vectors in the x, y, and z-directions, respectively. The symbol for both normal stresses and shearing stresses is denoted by σ with two subscripts. The first subscript indicates the axis to which the surface is perpendicular, and the second indicates

the direction to which the stress is parallel. For example, σ_{xx} is the value of the normal stress acting on a plane whose normal is parallel to the x-axis and the stress itself is also in the x-direction; σ_{xy} is the shearing stress acting on the same plane with σ_{xx} while the stress itself is parallel to the y-axis. The notation for designating stresses is shown by Fig. 4.3.

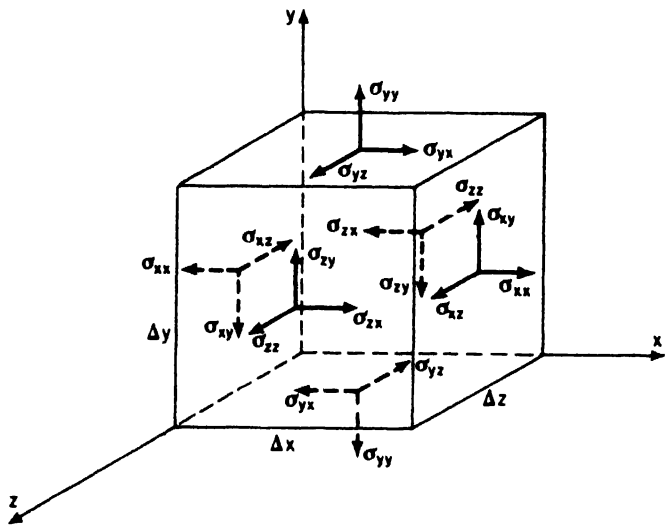


Figure 4.3

The six shearing stresses shown in Fig. 4.3 are not, however, independent. This can be proved by taking the moment about any axis through the center of the rectangular parallelepiped and establishing the condition of equilibrium of the elementary volume. Since the inertia forces of the fluid give a moment of higher order and are thus negligible here, the equations of moments about axes through the center of the rectangular parallelepiped give

$$\sigma_{yz} = \sigma_{zy} \qquad \sigma_{zx} = \sigma_{xz} \qquad \sigma_{xy} = \sigma_{yx} \tag{4-2}$$

Thus, six, rather than nine, quantities, $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}$ suffice to determine the state of stress at a point.

Illustrative Example 4.1. Verify the equality of the shearing stresses given in Eq. (4-2).

Solution. Consider an element of fluid as shown in Fig. 4.A. Let us take the moment about the x-axis through the center of the element, O. The torque produced by the normal forces $\sigma_{yy}\,dx\,dz$ and $\sigma_{zz}\,dx\,dy$ (not shown) are zero and by the shearing forces are:

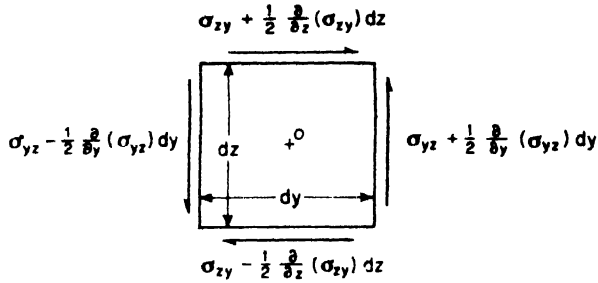


Figure 4.A

$$\left[\sigma_{yz} + \frac{1}{2}\frac{\partial}{\partial y}(\sigma_{yz})\,dy\right] dx\,dz\left(\frac{1}{2}dy\right) + \left[\sigma_{yz} - \frac{1}{2}\frac{\partial}{\partial y}(\sigma_{yz})\,dy\right] dx\,dz\left(\frac{1}{2}dy\right)$$
$$- \left[\sigma_{zy} + \frac{1}{2}\frac{\partial}{\partial z}(\sigma_{zy})\,dz\right] dx\,dy\left(\frac{1}{2}dz\right) - \left[\sigma_{zy} - \frac{1}{2}\frac{\partial}{\partial z}(\sigma_{zy})\,dz\right] dx\,dy\left(\frac{1}{2}dz\right)$$
$$= (\sigma_{yz} - \sigma_{zy})\,dx\,dy\,dz \qquad \text{(a)}$$

According to Newton's law, the torque must be equal to the angular momentum of a rotating fluid element about the axis of rotation (x-axis through the center O). Hence,

$$(\sigma_{yz} - \sigma_{zy})\,dx\,dy\,dz = \left[(\rho\,dx\,dy\,dz)\frac{(dy)^2}{12} + (\rho\,dx\,dy\,dz)\frac{(dz)^2}{12}\right]\frac{d\omega_x}{dt} \qquad \text{(b)}$$

But dy and dz are infinitesimal quantities and $d\omega_x/dt$ cannot become infinite. Equation (b) reduces to

$$\sigma_{yz} = \sigma_{zy} \qquad \text{(c)}$$

Similarly, we can easily verify the equality of the shearing stresses

$$\sigma_{zx} = \sigma_{xz} \qquad \text{and} \qquad \sigma_{xy} = \sigma_{yx} \qquad \text{(d)}$$

4.2. Transformation of Stress-Components

We shall now proceed to show that the stresses at a given point across any plane can be calculated from equilibrium conditions when the values of the six components of stress at the point are given. Let us simplify our calculation by analyzing the case of a two-dimensional stress. Consider two sets of orthogonal axes x, y and x', y' that are rotated with respect to each other. The direction cosines between the two sets of axes are given by

$$\cos x'x = \cos\theta = l, \qquad \cos x'y = \sin\theta = m$$
$$\cos y'x = \sin\theta = m, \qquad \cos y'y = \cos\theta = l$$

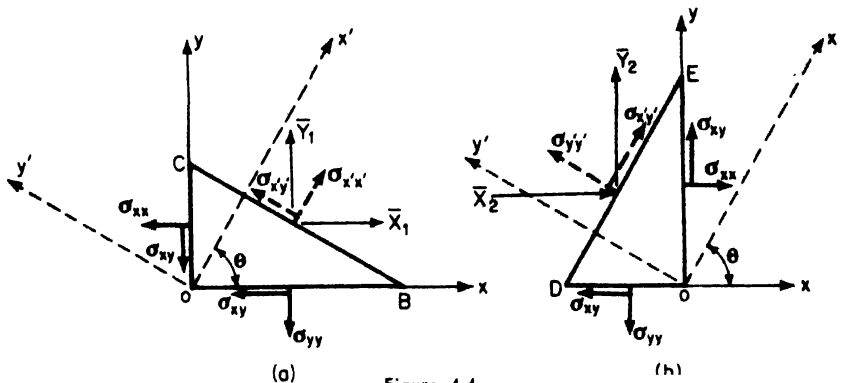


Figure 4.4

Consider the equilibrium of forces acting on an elementary wedge of unit width (Fig. 4.4a) whose inclined plane BC is perpendicular to the x'-axis. Let σ_{xx}, σ_{xy} and σ_{yy}, σ_{xy} be the stress-components acting on the planes OC and OB, respectively, and $\bar{X}_1$ and $\bar{Y}_1$ the x and y-components of stress on the plane BC. If A denotes the area of the plane BC of the element, the areas of OB and OC are Am and Al, respectively. The condition of equilibrium of forces in the x-direction, after neglecting the higher order terms of inertia and body forces, gives

$$\bar{X}_1 A = \sigma_{xx} Al + \sigma_{xy} Am$$

or

$$\bar{X}_1 = l\sigma_{xx} + m\sigma_{xy} \tag{4-3}$$

Similarly, the condition of equilibrium of forces in the y-direction gives

$$\bar{Y}_1 = l\sigma_{xy} + m\sigma_{yy} \tag{4-4}$$

The conditions of equilibrium of forces on the plane DE (Fig. 4.4b) corresponding to Eqs. (4–3) and (4–4) are

$$\bar{X}_2 = l\sigma_{xy} - m\sigma_{xx} \tag{4-3a}$$

$$\bar{Y}_2 = l\sigma_{yy} - m\sigma_{xy} \tag{4-4a}$$

Now let the stress-components of the new system of coordinate axes x' and y' be $\sigma_{x'x'}$, $\sigma_{y'y'}$, and $\sigma_{x'y'}$. The transformation equations for $\sigma_{x'x'}$, $\sigma_{y'y'}$, and $\sigma_{x'y'}$ can be obtained from Eqs. (4–3), (4–4), (4–3a), and (4–4a) with the aid of Figs. 4.4a and 4.4b as follows:

$$\sigma_{x'x'} = l\bar{X}_1 + m\bar{Y}_1 = l^2\sigma_{xx} + m^2\sigma_{yy} + 2lm\sigma_{xy} \tag{4-5a}$$

$$\sigma_{y'y'} = l\bar{Y}_2 - m\bar{X}_2 = m^2\sigma_{xx} + l^2\sigma_{yy} - 2lm\sigma_{xy} \tag{4-5b}$$

$$\sigma_{x'y'} = l\bar{Y}_1 - m\bar{X}_1 = lm(\sigma_{yy} - \sigma_{xx}) + (l^2 - m^2)\sigma_{xy} \tag{4-5c}$$

Equations (4–5) are the equations for the transformation of stress in the two-dimensional case for a rotation of the coordinate system through an angle θ. In terms of an angle θ, i. e., $\cos x'x = \cos\theta = l$ and $m = \sin\theta$, Eqs. (4–5) can be written in the following forms:

$$\sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta \tag{4–6}$$

$$\sigma_{y'y'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta - \sigma_{xy}\sin 2\theta \tag{4–7}$$

$$\sigma_{x'y'} = -\frac{\sigma_{xx} - \sigma_{yy}}{2}\sin 2\theta + \sigma_{xy}\cos 2\theta \tag{4–8}$$

With reference to Appendix A we can state that any of the physical quantities that are considered here may be identified as one of the following quantities:

(a) A scalar is a quantity characterized only by magnitude for a complete description.

(b) A vector is a quantity which has magnitude and one direction for a complete description.

(c) A tensor of order 2 is a quantity which is completely specified by magnitude and two directions.

In the last section we have shown that stress has the nature of rate of change of a vector in a given direction; hence, it is a tensor (tensor of order 2) quantity. Mathematically a vector is defined as a quantity whose components transform like the coordinates of a Cartesian coordinate system, and it has only one invariant as shown in Fig. 4.7 and Eq. (4–27). In the same way a Cartesian tensor is defined as a quantity whose components transform in a certain definite way when a coordinate system is subject to a linear transformation, i. e., to a rotation. For a tensor we shall find at least two invariants [see Eqs. (4–9) and (4–10)] for a two-dimensional case.

The first invariant of the stress tensor to the choice of coordinate systems can be easily obtained by adding the first two equations of Eqs. (4–5) and using the relation $l^2 + m^2 = 1$ as follows:

$$\sigma_{x'x'} + \sigma_{y'y'} = \sigma_{xx} + \sigma_{yy} = \sigma_{11} + \sigma_{22} \tag{4–9}$$

where σ_{11} and σ_{22} are the principal stresses. The second invariant of stress tensor is obtained from Eqs. (4–5) in the following form:

$$\sigma_{x'x'}\sigma_{y'y'} - \sigma_{x'y'}^2 = [(l^2\sigma_{xx} + m^2\sigma_{yy} + 2lm\sigma_{xy})(m^2\sigma_{xx} + l^2\sigma_{yy} - 2lm\sigma_{xy})] - [lm(\sigma_{yy} - \sigma_{xx}) + (l^2 - m^2)\sigma_{xy}]^2$$

or simplifying and using the relation $l^2 + m^2 = 1$,

$$\sigma_{x'x'}\sigma_{y'y'} - \sigma_{x'y'}^2 = \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2 = \sigma_{11}\sigma_{22} \tag{4–10}$$

Eqs. (4–9) and (4–10) constitute the two invariants for two-dimensional stress-components.

In terms of the general three-dimensional stress-components (Fig. 4.5) the equilibrium equations corresponding to Eqs. (4–3) and (4–4) are

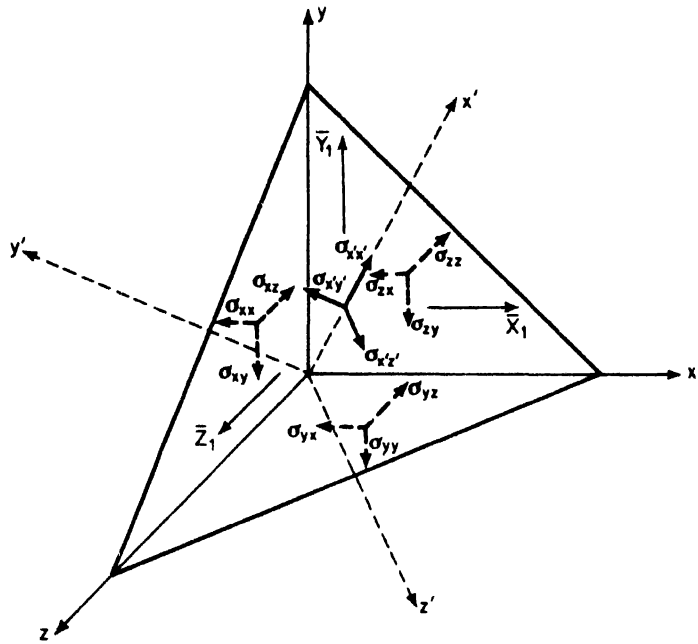


Figure 4.5

$$\bar{X}_1 = l_1\sigma_{xx} + m_1\sigma_{xy} + n_1\sigma_{zx} \tag{4–11a}$$

$$\bar{Y}_1 = l_1\sigma_{xy} + m_1\sigma_{yy} + n_1\sigma_{yz} \tag{4–11b}$$

$$\bar{Z}_1 = l_1\sigma_{zx} + m_1\sigma_{yz} + n_1\sigma_{zz} \tag{4–11c}$$

where $\bar{X}_1$, $\bar{Y}_1$, and $\bar{Z}_1$ are the x, y, and z-components of stress on the plane perpendicular to the new x'-axis (Fig. 4.5) and $l_1 = \cos x'x$, $m_1 = \cos x'y$, $n_1 = \cos x'z$. Similarly the x, y, and z-components of stress on planes perpendicular to the y'-axis, z'-axis, respectively, are

$$\bar{X}_2 = l_2\sigma_{xx} + m_2\sigma_{xy} + n_2\sigma_{zx} \tag{4–12a}$$

$$\bar{Y}_2 = l_2\sigma_{xy} + m_2\sigma_{yy} + n_2\sigma_{yz} \tag{4–12b}$$

$$\bar{Z}_2 = l_2\sigma_{zx} + m_2\sigma_{yz} + n_2\sigma_{zz} \tag{4–12c}$$

and

$$\bar{X}_3 = l_3\sigma_{xx} + m_3\sigma_{xy} + n_3\sigma_{zx} \tag{4–13a}$$

$$\bar{Y}_3 = l_3\sigma_{xy} + m_3\sigma_{yy} + n_3\sigma_{yz} \tag{4-13b}$$

$$\bar{Z}_3 = l_3\sigma_{xz} + m_3\sigma_{yz} + n_3\sigma_{zz} \tag{4-13c}$$

where $l_2 = \cos y'x$, $m_2 = \cos y'y$, $n_2 = \cos y'z$, $l_3 = \cos z'x$, $m_3 = \cos z'y$, and $n_3 = \cos z'z$. The direction cosines between the new coordinate axes x', y', z' and the original coordinate axes x, y, z can be conveniently tabulated as follows:

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

Consider the plane normal to the x'-axis with normal stress $\sigma_{x'x'}$ and shearing stresses $\sigma_{x'y'}$ and $\sigma_{z'x'}$ acting on it. Using Eq. (4–11) and the notations for the direction cosines we find

$$\begin{aligned}\sigma_{x'x'} &= \bar{X}_1 l_1 + \bar{Y}_1 m_1 + \bar{Z}_1 n_1 \\ &= l_1^2\sigma_{xx} + m_1^2\sigma_{yy} + n_1^2\sigma_{zz} + 2l_1m_1\sigma_{xy} + 2m_1n_1\sigma_{yz} + 2n_1l_1\sigma_{zx}\end{aligned} \tag{4-14a}$$

$$\begin{aligned}\sigma_{x'y'} &= \bar{X}_1 l_2 + \bar{Y}_1 m_2 + \bar{Z}_1 n_2 \\ &= l_1l_2\sigma_{xx} + m_1m_2\sigma_{yy} + n_1n_2\sigma_{zz} + (l_1m_2 + m_1l_2)\sigma_{xy} \\ &\quad + (m_1n_2 + n_1m_2)\sigma_{yz} + (l_1n_2 + n_1l_2)\sigma_{zx}\end{aligned} \tag{4-14b}$$

$$\begin{aligned}\sigma_{z'x'} &= \bar{X}_1 l_3 + \bar{Y}_1 m_3 + \bar{Z}_1 n_3 \\ &= l_1l_3\sigma_{xx} + m_1m_3\sigma_{yy} + n_1n_3\sigma_{zz} + (l_1m_3 + m_1l_3)\sigma_{xy} \\ &\quad + (m_1n_3 + n_1m_3)\sigma_{yz} + (l_1n_3 + n_1l_3)\sigma_{zx}\end{aligned} \tag{4-14c}$$

In a like manner we can find the normal and shearing stresses acting on the planes perpendicular to the y'-axis and z'-axis, respectively,

$$\sigma_{y'y'} = l_2^2\sigma_{xx} + m_2^2\sigma_{yy} + n_2^2\sigma_{zz} + 2l_2m_2\sigma_{xy} + 2m_2n_2\sigma_{yz} + 2n_2l_2\sigma_{zx} \tag{4-15a}$$

$$\sigma_{z'z'} = l_3^2\sigma_{xx} + m_3^2\sigma_{yy} + n_3^2\sigma_{zz} + 2l_3m_3\sigma_{xy} + 2m_3n_3\sigma_{yz} + 2n_3l_3\sigma_{zx} \tag{4-15b}$$

$$\begin{aligned}\sigma_{y'z'} &= l_2l_3\sigma_{xx} + m_2m_3\sigma_{yy} + n_2n_3\sigma_{zz} + (l_2m_3 + m_2l_3)\sigma_{xy} \\ &\quad + (m_2n_3 + m_3n_2)\sigma_{yz} + (n_2l_3 + l_2n_3)\sigma_{zx}\end{aligned} \tag{4-15c}$$

Equations (4–14) and (4–15) are the equations for the transformation of three-dimensional stress-components of two Cartesian coordinate systems.

Since the transformation of coordinates is orthogonal, the direction cosines are related by equations of the types

$$l_1^2 + l_2^2 + l_3^2 = 1 \tag{4-16a}$$

$$l_1^2 + m_1^2 + n_1^2 = 1 \tag{4-16b}$$

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \tag{4-16c}$$

$$l_1 m_1 + l_2 m_2 + l_3 m_3 = 0 \tag{4-16d}$$

From Eqs. (4–14) and (4–15) and the orthogonality relations in (4–16), the invariants for three-dimensional stress-components are found as follows:

$$\sigma_{x'x'} + \sigma_{y'y'} + \sigma_{z'z'} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \tag{4-17}$$

$$\begin{aligned}\sigma_{x'x'}\sigma_{y'y'} + \sigma_{y'y'}\sigma_{z'z'} + \sigma_{z'z'}\sigma_{x'x'} - \sigma_{x'y'}^2 - \sigma_{y'z'}^2 - \sigma_{z'x'}^2 \\ = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \sigma_{xy}^2 - \sigma_{yz}^2 - \sigma_{zx}^2\end{aligned} \tag{4-18}$$

$$\begin{aligned}\sigma_{x'x'}\sigma_{y'y'}\sigma_{z'z'} + 2\sigma_{x'y'}\sigma_{y'z'}\sigma_{z'x'} - \sigma_{x'x'}\sigma_{y'z'}^2 - \sigma_{y'y'}\sigma_{z'x'}^2 - \sigma_{z'z'}\sigma_{y'z'}^2 \\ = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{yz}^2\end{aligned} \tag{4-19}$$

Illustrative Example 4.2. Verify the following invariant for two dimensional stress-components:

$$(\sigma_{x'x'} - \sigma_{y'y'})^2 + 4\sigma_{x'y}^2 = (\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy} \tag{a}$$

Solution. Taking the difference of Eqs. (4–6) and (4–7) we obtain

$$\sigma_{x'x'} - \sigma_{y'y'} = (\sigma_{xx} - \sigma_{yy})\cos 2\theta + 2\sigma_{xy}\sin 2\theta \tag{b}$$

Squaring,

$$\begin{aligned}(\sigma_{x'x'} - \sigma_{y'y'})^2 = (\sigma_{xx} - \sigma_{yy})^2 \cos^2 2\theta + 4\sigma_{xy}^2 \sin^2 2\theta \\ + 2(\sigma_{xx} - \sigma_{yy})\sigma_{xy} \sin 2\theta \cos 2\theta\end{aligned} \tag{c}$$

From Eq. (4–8) we have

$$\begin{aligned}4\sigma_{x'y'}^2 = (\sigma_{xx} - \sigma_{yy})^2 \sin^2 2\theta + 4\sigma_{xy}^2 \cos^2 2\theta \\ - 2(\sigma_{xx} - \sigma_{yy})\sigma_{xy} \sin 2\theta \cos 2\theta\end{aligned} \tag{d}$$

Combining Eqs. (c) and (d), there results

$$(\sigma_{x'x'} - \sigma_{y'y'})^2 + 4\sigma_{x'y'}^2 = (\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2 \tag{e}$$

The physical meaning of Eq. (e) can be interpreted as follows: Let θ be the angle of an arbitrary coordinate system relative to the principal axis. Relative to these coordinates, the principal stresses, σ_{11} and σ_{22}, are:

$$\sigma_{xx} = \sigma_{11} \qquad \sigma_{yy} = \sigma_{22} \qquad \sigma_{xy} = 0 \tag{f}$$

Substituting Eq. (f) into Eq. (e) we have

$$(\sigma_{x'x'} - \sigma_{y'y'})^2 + 4\sigma_{x'y'}^2 = (\sigma_{11} - \sigma_{22})^2 \tag{g}$$

But the combination of Eqs. (f) and (c) results in

$$(\sigma_{x'x'} - \sigma_{y'y'})^2 = (\sigma_{11} - \sigma_{22})^2 \cos^2 2\theta \tag{h}$$

Hence, Eq. (g) can be written as

$$4\sigma_{x'y'}^2 = (\sigma_{11} - \sigma_{22})^2 \sin^2 2\theta \tag{i}$$

or

$$4(\sigma_{x'y'})_{\max}^2 = (\sigma_{11} - \sigma_{22})^2 \tag{j}$$

It is seen that the invariant for stress-components given in Eq. (a) has a physical meaning, being four times the square of the maximum shearing stress, i. e.,

$$\begin{aligned}(\sigma_{x'x'} - \sigma_{y'y'})^2 + 4\sigma_{x'y'}^2 &= (\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy} \\ &= (\sigma_{11} - \sigma_{22})^2 = 4(\sigma_{x'y'})_{\max}^2\end{aligned} \tag{k}$$

4.3. *Nature of Strains*

Strain is a nondimensional deformation which measures the change of relative positions of the parts of a body under any cause. In general, in a body under the influence of external forces, the displacements may consist of a translation, a rotation, and a distortion. Since a translation and a rotation represent rigid-body displacements which do not induce any strain in the body, they will not be considered here. Strain is classified into two kinds, viz., the normal strain and the shearing

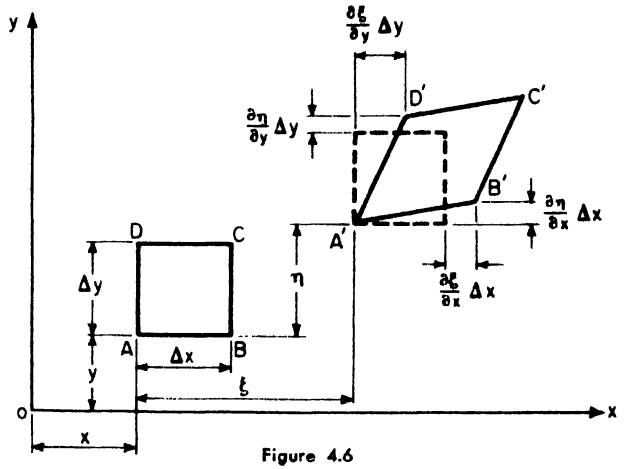


Figure 4.6

strain. The ratio of the change in length to the original length of a straight line element is defined as a normal (or direct) strain. On the other hand, the shearing strain is expressed in terms of the change in angle between two linear elements from the unstrained state to the strained state (see Fig. 4.6).

We shall begin our discussion of strain with the two-dimensional case. Consider a small rectangular element $ABCD$ with coordinates of the point A at (x, y) and with sides Δx and Δy in the unstrained position (Fig. 4.6). After strain, the element is displaced to the position $A'B'C'D'$ with coordinates of the point A' (displaced from A) at $(x + \xi, y + \eta)$. The coordinates of the point B' which is displaced from B are $x + \xi + \Delta x + (\partial\xi/\partial x)\,\Delta x$, $y + \eta + (\partial\eta/\partial x)\Delta x$ and D' displaced from D are $x + \xi + (\partial\xi/\partial y)\Delta y$, $y + \eta + \Delta y + (\partial\eta/\partial y)\,\Delta y$. The projected lengths of $A'B'$ are therefore $\Delta x + (\partial\xi/\partial x)\,\Delta x$ in the x-direction and $(\partial\eta/\partial x)\,\Delta x$ in the y-direction. The square of the length $A'B'$ is

$$(A'B')^2 = \left(\Delta x + \frac{\partial \xi}{\partial x}\Delta x\right)^2 + \left(\frac{\partial \eta}{\partial x}\Delta x\right)^2$$

The direct strain-component in the x-direction $(\varepsilon_{xx})_s$ is defined as

$$(\varepsilon_{xx})_s = \frac{A'B' - AB}{AB} \tag{4-20}$$

where the subscript ( )$_s$ indicates that the quantity applies to the solid case. From Eq. (4–20) the length $A'B'$ can be written as

$$A'B' = [1 + (\varepsilon_{xx})_s]AB = [1 + (\varepsilon_{xx})_s]\Delta x \tag{4-21}$$

Substituting the expression above into the equation for $(A'B')^2$ and dropping higher-order terms, we obtain

$$(\varepsilon_{xx})_s = \frac{\partial \xi}{\partial x}$$

Similarly, the normal strain-component in the y-direction is obtained from the length $C'D'$ which is

$$(\varepsilon_{yy})_s = \frac{\partial \eta}{\partial y}$$

The shearing strain $(\gamma_{xy})_s$ at the point A is the change of the angle between the length AB and AD. From the projected lengths of $A'B'$ and $C'D'$ we find that the right angle DAB between the length AB and AD is diminished by the angle

$$(\gamma_{xy})_s = \frac{(\partial\eta/\partial x)\,\Delta x}{\Delta x(1 + \partial\xi/\partial x)} + \frac{(\partial\xi/\partial y)\,\Delta y}{\Delta y(1 + \partial\eta/\partial y)}$$

After neglecting higher-order terms, we have

$$(\gamma_{xy})_s = \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y} \tag{4-22}$$

or

$$(\varepsilon_{xy})_s = \frac{(\gamma_{xy})_s}{2} = \frac{1}{2}\left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\right)$$

It is important to note that the strains have the nature of a change in displacement in a given unit length in a given direction. Hence, strain is a tensor of order 2.

In elasticity we are interested in actual displacement in a unit length, i. e., $\Delta\xi$, $\Delta\eta$. In a fluid there is, by definition, no resistance to deformation but only to the time rate of deformation. Hence in fluid dynamics the rate of change of strain with time (rate of strain) will be used in place of strain in elasticity. The rates of strain in fluid dynamics corresponding to Eqs. (4–20), (4–21), and (4–22) are

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial}{\partial t}\left(\frac{\partial \xi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial t}\right) = \frac{\partial u}{\partial x} \\ \varepsilon_{yy} &= \frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial y}\right) = \frac{\partial v}{\partial y} \\ \frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\right) &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \end{aligned} \tag{4–23}$$

where u and v are the fluid velocities in the x and y directions respectively.

The rates of strain components in the general three-dimensional case can be obtained in a similar manner. They are

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{yx} \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \gamma_{zy} \\ \gamma_{zx} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \gamma_{xz} \end{aligned} \tag{4–24}$$

where w is the velocity component in the z-direction. The six quantities above are called the rates of strain components.

4.4. Transformation of the Rates of Strain

Now we proceed to calculate the values of the rates of strain in a new coordinate system as functions of the original rates of strain. In order to facilitate the understanding of the transformation we will first develop the two-dimensional case and then later extend it to three dimensions. Let the new coordinate system (x', y') be obtained from the original coordinate system (x, y) by a rotation through an angle θ (Fig. 4.7). Then if the velocity components in the new coordinate system (x', y') be u', v', we have

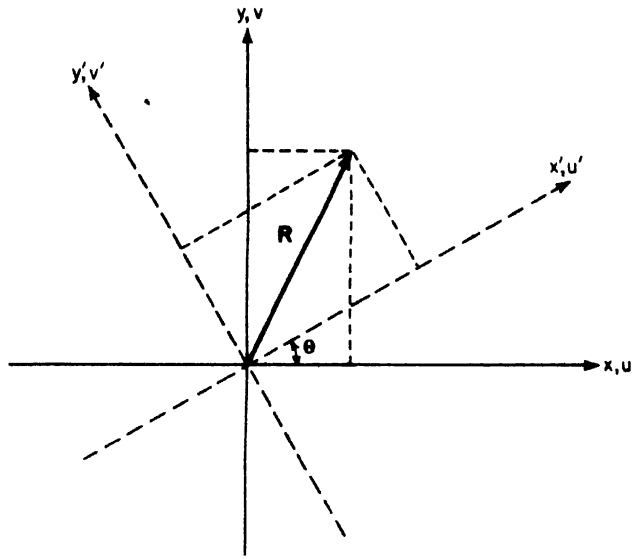


Figure 4.7

$$x' = lx + my$$
$$y' = -mx + ly \tag{4-25a}$$

$$x = lx' - my'$$
$$y = mx' + ly' \tag{4-25b}$$

and

$$u' = lu + mv$$
$$v' = -mu + lv \tag{4-26}$$

From Eq. (4–25) we find the only invariant of a vector **R** is

$$\mathbf{R}\cdot\mathbf{R} = x^2 + y^2 = x'^2 + y'^2 \tag{4-27}$$

With Eqs. (4–25) and (4–26) we can calculate the rates of strain in the new coordinate system (x', y') as follows:

$$\begin{aligned}\varepsilon_{x'x'} = \frac{\partial u'}{\partial x'} &= \frac{\partial u'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial x'} \\ &= \left(l\frac{\partial u}{\partial x} + m\frac{\partial v}{\partial x}\right)l + \left(l\frac{\partial u}{\partial y} + m\frac{\partial v}{\partial y}\right)m \\ &= l^2\varepsilon_{xx} + m^2\varepsilon_{yy} + lm\gamma_{xy} \end{aligned} \tag{4-28a}$$

$$\varepsilon_{y'y'} = \frac{\partial v'}{\partial y'} = m^2\varepsilon_{xx} + l^2\varepsilon_{yy} - lm\gamma_{xy} \tag{4-28b}$$

$$\gamma_{x'y'} = \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} = 2lm(\varepsilon_{yy} - \varepsilon_{xx}) + (l^2 - m^2)\gamma_{xy} \qquad (4\text{-}28c)$$

where the direction cosines l and m were given in §4.2. In terms of the angle θ the rates of strain can be written as

$$\varepsilon_{x'x'} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} + \frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta \qquad (4\text{-}29)$$

$$\varepsilon_{y'y'} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} - \frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta \qquad (4\text{-}30)$$

$$\frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta \qquad (4\text{-}31)$$

Equations (4–29), (4–30), and (4–31) are the transformation formulae for the rates of strain. By comparing these equations with Eqs. (4–6), (4–7), and (4–8), it is seen that the rates of strain tensor ε_{xx}, ε_{yy}, and $\gamma_{xy}/2$ transform exactly the same as the stress tensor σ_{xx}, σ_{yy}, and σ_{xy}.

Since the rate of strain is also a tensor of order 2, there must exist at least two invariants of the rate of strain to the choice of coordinate systems. These can be found from Eqs. (4–28) as follows:

$$\varepsilon_{x'x'} + \varepsilon_{y'y'} = \varepsilon_{xx} + \varepsilon_{yy} \qquad (4\text{-}32)$$

$$\begin{aligned}\varepsilon_{x'x'}\varepsilon_{y'y'} - \frac{\gamma_{x'y'}^2}{4} &= (l^2\varepsilon_{xx} + m^2\varepsilon_{yy} + lm\gamma_{xy})(m^2\varepsilon_{xx} + l^2\varepsilon_{yy} - lm\gamma_{xy}) \\ &\quad - \frac{1}{4}[2lm(\varepsilon_{yy} - \varepsilon_{xx}) + (l^2 - m^2)\gamma_{xy}]^2 \\ &= \varepsilon_{xx}\varepsilon_{yy}(l^4 + 2l^2m^2 + m^4) - \frac{\gamma_{xy}^2}{4}(l^4 + m^4 + 2l^2m^2) \\ &= \varepsilon_{xx}\varepsilon_{yy} - \frac{\gamma_{xy}^2}{4} \qquad (4\text{-}33)\end{aligned}$$

Equation (4–32) shows that the divergence of the velocity vector at a given point is independent of the orientation of the coordinate axes, and Eq. (4–33) is related to the dissipation function which will be discussed later.

Now let us consider the general case of the rates of strain in three dimensions. Referring to the table of direction cosines in §4.2, the relations between coordinates and velocity components in the two systems are

$$\begin{aligned} x' &= l_1x + m_1y + n_1z \\ y' &= l_2x + m_2y + n_2z \\ z' &= l_3x + m_3y + n_3z \\ x &= l_1x' + l_2y' + l_3z' \\ y &= m_1x' + m_2y' + m_3z' \\ z &= n_1x' + n_2y' + n_3z' \end{aligned} \qquad (4\text{-}34)$$

and

$$\begin{aligned} u' &= l_1 u + m_1 v + n_1 w \\ v' &= l_2 u + m_2 v + n_2 w \\ w' &= l_3 u + m_3 v + n_3 w \end{aligned} \tag{4-35}$$

The equations above can be used to calculate the rates of strain in the new coordinate system (x', y', z'). They are

$$\begin{aligned} \varepsilon_{x'x'} &= \frac{\partial u'}{\partial x'} = \frac{\partial u'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial u'}{\partial z}\frac{\partial z}{\partial x'} \\ &= \left(l_1\frac{\partial u}{\partial x} + m_1\frac{\partial v}{\partial x} + n_1\frac{\partial w}{\partial x}\right)l_1 + \left(l_1\frac{\partial u}{\partial y} + m_1\frac{\partial v}{\partial y} + n_1\frac{\partial w}{\partial y}\right)m_1 \\ &\quad + \left(l_1\frac{\partial u}{\partial z} + m_1\frac{\partial v}{\partial z} + n_1\frac{\partial w}{\partial z}\right)n_1 \\ &= l_1^2\varepsilon_{xx} + m_1^2\varepsilon_{yy} + n_1^2\varepsilon_{zz} + l_1 m_1\gamma_{xy} + m_1 n_1\gamma_{yz} + n_1 l_1\gamma_{zx} \end{aligned} \tag{4-36a}$$

$$\varepsilon_{y'y'} = \frac{\partial v'}{\partial y'} = l_2^2\varepsilon_{xx} + m_2^2\varepsilon_{yy} + n_2^2\varepsilon_{zz} + l_2 m_2\gamma_{xy} + m_2 n_2\gamma_{yz} + n_2 l_2\gamma_{zx} \tag{4-36b}$$

$$\varepsilon_{z'z'} = \frac{\partial w'}{\partial z'} = l_3^2\varepsilon_{xx} + m_3^2\varepsilon_{yy} + n_3^2\varepsilon_{zz} + l_3 m_3\gamma_{xy} + m_3 n_3\gamma_{yz} + n_3 l_3\gamma_{zx} \tag{4-36c}$$

$$\begin{aligned} \gamma_{x'y'} &= \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} = 2l_1 l_2\varepsilon_{xx} + 2m_1 m_2\varepsilon_{yy} + 2n_1 n_2\varepsilon_{zz} \\ &\quad + (l_1 m_2 + m_1 l_2)\gamma_{xy} + (m_1 n_2 + n_1 m_2)\gamma_{yz} + (n_1 l_2 + l_1 n_2)\gamma_{zx} \end{aligned} \tag{4-36d}$$

$$\begin{aligned} \gamma_{y'z'} &= \frac{\partial w'}{\partial y'} + \frac{\partial v'}{\partial z'} = 2l_2 l_3\varepsilon_{xx} + 2m_2 m_3\varepsilon_{yy} + 2n_2 n_3\varepsilon_{zz} \\ &\quad + (l_2 m_3 + m_2 l_3)\gamma_{xy} + (m_2 n_3 + n_2 m_3)\gamma_{yz} + (n_2 l_3 + l_2 n_3)\gamma_{zx} \end{aligned} \tag{4-36e}$$

$$\begin{aligned} \gamma_{z'x'} &= \frac{\partial u'}{\partial z'} + \frac{\partial w'}{\partial x'} = 2l_3 l_1\varepsilon_{xx} + 2m_3 m_1\varepsilon_{yy} + 2n_3 n_1\varepsilon_{zz} \\ &\quad + (l_3 m_1 + m_3 l_1)\gamma_{xy} + (m_3 n_1 + n_3 m_1)\gamma_{yz} + (n_3 l_1 + l_3 n_1)\gamma_{zx} \end{aligned} \tag{4-36f}$$

Equations (4–36) are the transformation formulae for the rates of strain in the three-dimensional case. From Eqs. (4–36), (4–14), and (4–15) we can conclude that the rates of strain follow the same rules of transformation as the stress tensor.

With the aid of the orthogonality relation shown in (4–16), the

invariants for the rates of strain in the three-dimensional case are found as follows:

$$\varepsilon_{x'x'} + \varepsilon_{y'y'} + \varepsilon_{z'z'} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \tag{4-37}$$

$$\varepsilon_{x'x'}\varepsilon_{y'y'} + \varepsilon_{y'y'}\varepsilon_{z'z'} + \varepsilon_{z'z'}\varepsilon_{x'x'} - \tfrac{1}{4}(\gamma_{x'y'}^2 + \gamma_{y'z'}^2 + \gamma_{z'x'}^2)$$
$$= \varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{yy}\varepsilon_{zz} + \varepsilon_{zz}\varepsilon_{xx} - \tfrac{1}{4}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2) \tag{4-38}$$

$$\varepsilon_{x'x'}\varepsilon_{y'y'}\varepsilon_{z'z'} + \tfrac{1}{4}(\gamma_{x'y'}\gamma_{y'z'}\gamma_{z'x'} - \varepsilon_{x'x'}\gamma_{y'z'}^2 - \varepsilon_{y'y'}\gamma_{z'x'}^2 - \varepsilon_{z'z'}\gamma_{x'y'}^2)$$
$$= \varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz} + \tfrac{1}{4}(\gamma_{xy}\gamma_{yz}\gamma_{zx} - \varepsilon_{xx}\gamma_{yz}^2 - \varepsilon_{yy}\gamma_{zx}^2 - \varepsilon_{zz}\gamma_{xy}^2) \tag{4-39}$$

Illustrative Example 4.3. Let the new coordinate system (x', y') be obtained from the original coordinate system (x, y) by a rotation through an angle of 45°. Verify the invariants of the rates of strain [Eqs. (4–32) and (4–33)] for a rectilinear flow with a linear velocity profile, i. e.,

$$u = ay, \qquad v = 0 \tag{a}$$

Solution. The rates of strain with respect to the original coordinate system (x, y) are,

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = a \qquad \varepsilon_{xx} = \frac{\partial u}{\partial x} = 0 \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} = 0 \tag{b}$$

With respect to the new coordinate system, we have [see Eqs. (4–25) and (4–26)]

$$\begin{aligned}
u' &= lu + mv = \frac{1}{\sqrt{2}}(u + v) = \frac{1}{\sqrt{2}}ay \\
v' &= -mu + lv = -\frac{1}{\sqrt{2}}(u - v) = -\frac{1}{\sqrt{2}}ay \\
x' &= lx + my = \frac{1}{\sqrt{2}}(x + y) \\
y' &= -mx + ly = -\frac{1}{\sqrt{2}}(x - y) \\
x &= lx' - my' = \frac{1}{\sqrt{2}}(x' - y') \\
y &= mx' + ly' = \frac{1}{\sqrt{2}}(x' + y')
\end{aligned} \tag{c}$$

The rates of strain with respect to the new coordinate system are therefore,

$$\begin{aligned}
\gamma_{x'y'} &= \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} = \frac{\partial v'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial u'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial y'} \\
\gamma_{x'y'} &= \frac{1}{2}(-a) + \frac{1}{2}(a) = 0 \\
\varepsilon_{x'x'} &= \frac{\partial u'}{\partial x'} = \frac{\partial u'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial x'} = \frac{a}{2}
\end{aligned} \tag{d}$$

$$\varepsilon_{y'y'} = \frac{\partial v'}{\partial y'} = \frac{\partial v'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial y'} = -\frac{a}{2}$$

Hence, the invariants of the rates of strain are obtained from Eqs. (b) and (d) as follows:

$$\varepsilon_{x'x'} + \varepsilon_{y'y'} = \frac{a}{2} - \frac{a}{2} = 0 = \varepsilon_{xx} + \varepsilon_{yy}$$

$$\varepsilon_{x'x'}\varepsilon_{y'y'} - \frac{1}{4}\gamma_{x'y'}^2 = -\frac{a^2}{4} = \varepsilon_{xx}\varepsilon_{yy} - \frac{1}{4}\gamma_{xy}^2$$

4.5 Relation Between Stress and Rate of Strain

In elasticity the relationship between the stress and strain of a solid body within the elastic limit is governed by Hooke's law. The generalized Hooke's law states that each of the six stress-components may be expressed as a linear function of the six components of strain, and vice versa. The validity of Hooke's law has been verified by experiments for continuous, homogeneous, and isotropic materials. In a fluid the physical law connecting the stress and rate of strain can also be made by the following simple and reasonable assumptions:

(a) The stress-components may be expressed as a linear function of the rates of strain components.

(b) The relations between stress-components and rates of strain components must be invariant to a coordinate transformation consisting of either a rotation or a mirror reflection of axes.

(c) The stress-components must reduce to the hydrostatic pressure p when all the gradients of velocities are zero.

The first assumption is comparable to the small deflection theory in elasticity where only small strains and stresses are concerned. This is simply the first approximation to the more complete expressions for the stress and strain relations. The Newtonian friction law for rectilinear flow is a special case of this linear stress and rate of strain law. The second assumption requires the law connecting stress and rate of strain to be a real physical law which cannot depend upon a choice of the coordinate system. The third assumption is introduced so that the continuity condition is fulfilled with the hydrostatic case.

With the assumptions above we can proceed to determine the relations between stress and rate of strain in the two-dimensional case and then the extension to the three-dimensional form can be easily made. With the assumption (a) the stress-rate of strain relation can be expressed in the form

$$\sigma_{xx} = A_1\varepsilon_{xx} + B_1\varepsilon_{yy} + C_1\gamma_{xy} + D_1 \qquad (4\text{–}40a)$$

$$\sigma_{yy} = A_2\varepsilon_{xx} + B_2\varepsilon_{yy} + C_2\gamma_{xy} + D_2 \qquad (4\text{–}40b)$$

$$\sigma_{xy} = A_3\varepsilon_{xx} + B_3\varepsilon_{yy} + C_3\gamma_{xy} + D_3 \tag{4-40c}$$

where the A's, B's, C's, and D's are constants to be determined. The assumption (b) requires that the stress-rate of strain relation remains unaltered with respect to a new coordinate system, i. e.,

$$\sigma_{x'x'} = A_1\varepsilon_{x'x'} + B_1\varepsilon_{y'y'} + C_1\gamma_{x'y'} + D_1 \tag{4-41a}$$

$$\sigma_{y'y'} = A_2\varepsilon_{x'x'} + B_2\varepsilon_{y'y'} + C_2\gamma_{x'y'} + D_2 \tag{4-41b}$$

$$\sigma_{z'z'} = A_3\varepsilon_{x'x'} + B_3\varepsilon_{y'y'} + C_3\gamma_{x'y'} + D_3 \tag{4-41c}$$

Substituting Eqs. (4-40) into Eq. (4-6) we obtain

$$\begin{aligned}\sigma_{x'x'} = &\left[\frac{A_1}{2}(1 + \cos 2\theta) + \frac{A_2}{2}(1 - \cos 2\theta) + A_3 \sin 2\theta\right]\varepsilon_{xx} \\ &+ \left[\frac{B_1}{2}(1 + \cos 2\theta) + \frac{B_2}{2}(1 - \cos 2\theta) + B_3 \sin 2\theta\right]\varepsilon_{yy} \\ &+ \left[\frac{C_1}{2}(1 + \cos 2\theta) + \frac{C_2}{2}(1 - \cos 2\theta) + C_3 \sin 2\theta\right]\gamma_{xy} \\ &+ \left[\frac{D_1}{2}(1 + \cos 2\theta) + \frac{D_2}{2}(1 - \cos 2\theta) + D_3 \sin 2\theta\right]\end{aligned} \tag{4-42}$$

Combining Eqs. (4-29), (4-30), and (4-31) with Eq. (4-41a), we have

$$\begin{aligned}\sigma_{x'x'} = &\left[\frac{A_1}{2}(1 + \cos 2\theta) + \frac{B_1}{2}(1 - \cos 2\theta) - C_1 \sin 2\theta\right]\varepsilon_{xx} \\ &+ \left[\frac{A_1}{2}(1 - \cos 2\theta) + \frac{B_1}{2}(1 + \cos 2\theta) + C_1 \sin 2\theta\right]\varepsilon_{yy} \\ &+ \left[\frac{A_1}{2}\sin 2\theta - \frac{B_1}{2}\sin 2\theta + C_1 \cos 2\theta\right]\gamma_{xy} + D_1\end{aligned} \tag{4-43}$$

Since Eqs. (4-42) and (4-43) are identical, the coefficients of ε_{xx}, ε_{yy}, γ_{xy}, and the constant in the two equations must be identical for all θ. Hence the nine constants in Eq. (4-40) are reduced to four.

$$\begin{aligned} &A_1 = B_2 = A, && B_1 = A_2 = B \\ &C_1 = -A_3 = B_3 = -C_2 = C, && D_1 = D_2 = D \\ &C_3 = \frac{A_1 - A_2}{2} = \frac{A - B}{2}, && D_3 = 0\end{aligned} \tag{4-44}$$

It should be noted that the corresponding transformation applied to $\sigma_{y'y'}$ and $\sigma_{x'y'}$ gives nothing in addition to the relation in Eq. (4-44). Hence Eq. (4-40) can be rewritten as

$$\sigma_{xx} = A\varepsilon_{xx} + B\varepsilon_{yy} + C\gamma_{xy} + D \tag{4-45a}$$

$$\sigma_{yy} = B\varepsilon_{xx} + A\varepsilon_{yy} - C\gamma_{xy} + D \tag{4-45b}$$

$$\sigma_{xy} = -C(\varepsilon_{xx} - \varepsilon_{yy}) + \frac{A - B}{2}\gamma_{xy} \tag{4-45c}$$

Now let us consider the new coordinate system (x_1, y_1) which is related to the original coordinate system (x, y) by

$$x_1 = -x \qquad \text{and} \qquad y_1 = y \tag{4-46}$$

Thus the new coordinate system is a mirror reflection of the original system with respect to the y-axis. With reference to the new coordinate system, the velocity components are

$$u_1 = -u \qquad \text{and} \qquad v_1 = v \tag{4-47}$$

and the rates of strain and stresses are, respectively,

$$\begin{aligned} \varepsilon_{x_1x_1} &= \frac{\partial u_1}{\partial x_1} = \frac{\partial u}{\partial x} = \varepsilon_{xx}, \qquad \varepsilon_{y_1y_1} = \frac{\partial v_1}{\partial y_1} = \frac{\partial v}{\partial y} = \varepsilon_{yy} \\ \gamma_{x_1y_1} &= \frac{\partial v_1}{\partial x_1} + \frac{\partial u_1}{\partial y_1} = -\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = -\gamma_{xy} \end{aligned} \tag{4-48}$$

and

$$\sigma_{x_1x_1} = \sigma_{xx}, \qquad \sigma_{y_1y_1} = \sigma_{yy}, \qquad \sigma_{x_1y_1} = -\sigma_{xy} \tag{4-49}$$

Substituting Eqs. (4-48) and (4-49) into Eq. (4-45) we obtain

$$\sigma_{x_1x_1} = A\varepsilon_{x_1x_1} + B\varepsilon_{y_1y_1} - C\gamma_{x_1y_1} + D \tag{4-50a}$$

$$\sigma_{y_1y_1} = B\varepsilon_{x_1x_1} + A\varepsilon_{y_1y_1} + C\gamma_{x_1y_1} + D \tag{4-50b}$$

$$\sigma_{x_1y_1} = C(\varepsilon_{x_1x_1} - \varepsilon_{y_1y_1}) + \frac{A - B}{2}\gamma_{x_1y_1} \tag{4-50c}$$

But according to the assumption (b), the stress-rate of strain relation should have the same form in Eqs. (4-45) and (4-50) independent of the coordinate system. Hence

$$C = 0$$

According to the assumption (c), Eq. (4-50a) gives

$$D = -p \tag{4-51}$$

The constant $(A - B)/2$ in the last equation of Eq. (4-50) is the proportionality constant connecting the shearing stress and rate of shearing strain which is generally denoted by the dynamic coefficient of viscosity μ. Finally the relations between stress and rate of strain in the two-dimensional case given in Eqs. (4-45) are reduced to

$$\sigma_{xx} = 2\mu\varepsilon_{xx} + B\nabla\cdot\mathbf{q} - p \tag{4-52a}$$

$$\sigma_{yy} = 2\mu\varepsilon_{yy} + B\nabla\cdot\mathbf{q} - p \tag{4-52b}$$

$$\sigma_{xy} = \mu\gamma_{xy} = 2\mu\varepsilon_{xy} \tag{4-52c}$$

where $B = -\frac{2}{3}\mu$ will be discussed in the three-dimensional case.

The stress-rate of strain relations given in Eqs. (4-52) can be extended to three-dimensional flows. They are

$$\sigma_{xx} = 2\mu\varepsilon_{xx} + B\nabla\cdot\mathbf{q} - p \tag{4-53a}$$

$$\sigma_{yy} = 2\mu\varepsilon_{yy} + B\nabla\cdot\mathbf{q} - p \tag{4-53b}$$

$$\sigma_{zz} = 2\mu\varepsilon_{zz} + B\nabla\cdot\mathbf{q} - p \tag{4-53c}$$

$$\sigma_{xy} = 2\mu\varepsilon_{xy} = \sigma_{yx} \tag{4-53d}$$

$$\sigma_{yz} = 2\mu\varepsilon_{yz} = \sigma_{zy} \tag{4-53e}$$

$$\sigma_{zx} = 2\mu\varepsilon_{zx} = \sigma_{xz} \tag{4-53f}$$

The sum of the three normal stresses is

$$\sigma_{xx} + \sigma_{yy} + \sigma_{zz} + 3p = (2\mu + 3B)\nabla\cdot\mathbf{q} \tag{4-54}$$

For an incompressible fluid, $\nabla\cdot\mathbf{q} = 0$, the mean normal stress is equal to the negative hydrostatic pressure, i.e.,

$$\frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} = -p$$

Since the left-hand side of Eq. (4-54) cannot become negative, the term $2\mu + 3B$ must be positive for the compressible fluid, i.e.,

$$B = -\tfrac{2}{3}(\mu - \mu_1)$$

where μ_1 is referred to as the second viscosity coefficient which is significant only when there is exchange of molecular energy between translational degrees of freedom and vibrational and rotational degrees of freedom. For a perfect gas, to which the present approach is confined, the second viscosity coefficient μ_1 has been verified to be negligibly small. Hence Eqs. (4-9) and (4-17) can be interpreted as follows:

$$\sigma_{x'x'} + \sigma_{y'y'} = \sigma_{xx} + \sigma_{yy} = -2p \tag{4-9a}$$

$$\sigma_{x'x'} + \sigma_{y'y'} + \sigma_{z'z'} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = -3p \tag{4-17a}$$

NOMENCLATURE FOR CHAPTER FOUR

l, m, n	direction cosines, §4.2 and Eq. (4-11a)
p	hydrostatic pressure, Eq. (4-9)
u, v, w	velocity components in Cartesian coordinates
x, y, z	Cartesian coordinates
$(\gamma_{xy})_s$	shearing strain-component, Eq. (4-22)
$\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$	rate of shearing strain in Cartesian coordinates, Eq. (4-24)
$(\varepsilon_{xx})_s$	direct strain-component in the x-direction, §4.3 and Eq. (4-20)
$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$	rate of linear strain in Cartesian coordinates, Eq. (4-24)
θ	angle of rotation of the coordinates, Eq. (4-6)
μ	coefficient of viscosity, Eq. (4-52a)

μ_1	second viscosity coefficient, §4.5
σ_{11}, σ_{22}	principal stresses, Eq. (4-9)
$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$	normal stresses in Cartesian coordinates, Eq. (4-1)
$\sigma_{xy}, \sigma_{yz}, \sigma_{zx}$	shearing stresses in Cartesian coordinates, Eq. (4-1)

PROBLEMS

4.1. Verify the equality of the following shearing stresses:
(a) $\sigma_{xy} = \sigma_{yx}$
(b) $\sigma_{zx} = \sigma_{xz}$

4.2. Verify Eq. (4-10).

4.3. Show that the principal stresses may be expressed as

$$\sigma_{11} = \sigma_{\max} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$$

$$\sigma_{22} = \sigma_{\min} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$$

4.4. Verify Eq. (4-17).

4.5. Verify Eq. (4-18).

4.6. Verify Eq. (4-19).

4.7. Show that the direct strain-component in the y-direction is given by (Fig. 4.6).

$$(\varepsilon_{yy})_s = \frac{\partial \eta}{\partial y}$$

4.8. Show that the following velocity components represent a rigid-body motion

$$u = a + by - cz$$

$$v = d - bx + ez$$

$$w = f + cx - ey$$

where a, b, c, d, e, and f are arbitrary constants.

4.9. Show that the vorticity is an invariant, i. e.,

$$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'}$$

4.10. If a new coordinate system (x', y') is obtained from the original coordinate system (x, y) by a rotation through an angle of 30°, verify the invariants of the rates of strain [Eqs. (4-32) and (4-33)] for the flow:

$$u = ay, \qquad v = 0$$

4.11. Verify Eq. (4-37).

4.12. Verify Eq. (4-38).

4.13. Verify Eq. (4-39).

4.14. Show that Eq. (4-24) implies that the rates of strain tensor is symmetric.

4.15. Determine the rates of strain and explain the nature of the rates of strain for the following velocity components:

(a) $u = cx, \quad v = cy, \quad w = -2cz$

(b) $u = c, \quad v = w = 0$

(c) $u = 2cy, \quad v = w = 0$

(d) $u = u(x, y), \quad v = v(x, y), \quad w = 0$

where c is an arbitrary constant.

4.16. Show that the linear stress-rate of strain relation given by Eq. (4-53) implies that stress tensor is symmetric.

4.17. Determine the components of stress from the results obtained in Prob. 4.15.

5

FUNDAMENTAL EQUATIONS OF THE FLOW OF VISCOUS COMPRESSIBLE FLUIDS

It was mentioned in the preceding chapter that the most complete description of a single continuous fluid is the theory of a viscous compressible flow. In order to get a complete and fundamental understanding of the fluid dynamic problems, it is believed that the most general governing equations of continuity, momentum, and energy should first be derived and explained. The complete solution for the governing equations of a viscous compressible flow is still not possible because of insurmountable mathematical difficulties. However, many fluid mechanics problems can be described from the governing equations which are subject to great simplification in special cases. In this way one gets a much clearer understanding and better appreciation of the underlying physical assumptions and limitations of the solution obtained when he sees what terms in the governing equations are neglected.

In the study of a viscous compressible flow, it is necessary to obtain the three components of velocity and the state of the fluid, p, ρ, and T, as a function of both space and time. These six unknowns can be determined in principle from the six governing equations. The derivation of these equations is based consistently on vector methods and notation because the equations apply to Cartesian coordinates as well as to

cylindrical and spherical coordinates.

In § 5.1 the continuity equation based on the conservation of mass is derived.

In § 5.2 the equations of motion based on the conservation of momentum are derived for a viscous compressible fluid. The stresses and velocity gradients relations given in Chap. Four are fully used. These equations are simplified to the cases of a viscous compressible fluid with constant viscosity, viscous incompressible fluid, and nonviscous fluid for both compressible and incompressible flows.

In § 5.3 the energy equation based on the conservation of energy is derived. The energy added both by heat conduction and work done for stresses is shown. The energy equation for an incompressible fluid and a nonviscous fluid are also given.

In § 5.4 the equation of state for a perfect gas is discussed, especially as to its limitations. The equation of state for a real gas is also mentioned.

In § 5.5 the governing equations for a viscous compressible fluid in cylindrical coordinates are derived. The stresses and velocity gradients relations are also discussed.

In § 5.6 the complete derivation of the governing equations for a viscous compressible fluid in spherical coordinates is presented.

5.1. *The Equation of Continuity—Conservation of Mass*

The law of conservation of mass states that mass can be neither created nor destroyed. Referring to Fig. 5.1, moving surface A encloses

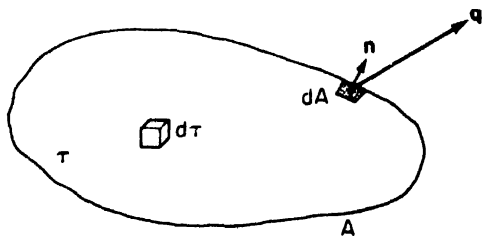


Figure 5.1

a mass of fluid within volume τ lying entirely in the fluid. If $\rho\Delta\tau$ is an element of mass, then the mathematical expression of the law of conservation of mass can be written

$$\frac{D}{Dt}(\rho\Delta\tau) = 0 \tag{5-1}$$

This is one form of the equation of continuity. Equation (5–1) can also be expressed in the form

$$\frac{D\rho/Dt}{\rho} + \frac{D(\Delta\tau)/Dt}{\Delta\tau} = 0 \tag{5–2}$$

from which it can be concluded that Eq. (5–2) is always satisfied in such a way that a decrease in volume is accompanied by an increase in density and vice versa. If we consider the total mass of the fluid enclosed in the surface A, Eq. (5–1) becomes

$$\lim_{\Delta\tau\to 0} \sum \left[\frac{1}{\rho}\frac{D\rho}{Dt}\Delta\tau + \frac{D(\Delta\tau)}{Dt}\right] = 0 \tag{5–3}$$

The first term in Eq. (5–3) can be rewritten in the integral form

$$\iiint \frac{1}{\rho}\left[\frac{\partial\rho}{\partial t} + \mathbf{q}\cdot\nabla\rho\right] d\tau \tag{5–4}$$

The second term in Eq. (5–3) represents the rate of change of volume which is equivalent to (see Appendix A)

$$\lim_{\Delta\tau\to 0} \sum \frac{D}{Dt}(\Delta\tau) = \iiint \nabla\cdot\mathbf{q}\, d\tau \tag{5–5}$$

Combining Eqs. (5–4) and (5–5), we find

$$\iiint \frac{1}{\rho}\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{q})\right] d\tau = 0 \tag{5–6}$$

from which the equation of continuity is obtained:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{q}) = 0 \tag{5–7}$$

Equation (5–7) may be written in a different form:

$$\frac{\partial\rho}{\partial t} + \mathbf{q}\cdot(\nabla\rho) + \rho\nabla\cdot\mathbf{q} = 0$$

or

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{q} = 0 \tag{5–8}$$

Hence $\nabla\cdot\mathbf{q}$ in compressible fluids is equivalent to the change of density when the fluid is compressed. In Cartesian coordiates Eq. (5–7) can be written

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{5–9a}$$

Equations (5–7) or (5–9a) may be interpreted as follows: For any closed surface drawn in the fluid, the increase in the mass of fluid within the surface in any time interval must be equal to the excess of mass that flows into the volume through the surface over the mass that flows

out. In the case of an incompressible fluid, Eq. (5-8) reduces to

$$\nabla \cdot \mathbf{q} = 0 \tag{5-9b}$$

Illustrative Example 5.1. Consider a two-dimensional incompressible steady flow field with velocity components in rectangular coordinates given by

$$u(x, y) = \frac{k(x^2 - y^2)}{(x^2 + y^2)^2} \tag{a}$$

$$v(x, y) = \frac{2kxy}{(x^2 + y^2)^2} \tag{b}$$

with k an arbitrary nonzero constant. Is the equation of continuity satisfied?

Solution. For a two-dimensional incompressible flow, the continuity equation, Eq. (5-9b), becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

The derivatives of Eqs. (a) and (b) with respect to x and y, respectively, yield

$$\frac{\partial u}{\partial x} = k(x^2 - y^2)\frac{(-2)(2x)}{(x^2 + y^2)^3} + \frac{k(2x)}{(x^2 + y^2)^2} = -4kx\frac{(x^2 - y^2)}{(x^2 + y^2)^3} + \frac{2kx}{(x^2 + y^2)^2} \tag{c}$$

$$\frac{\partial v}{\partial y} = 2kxy\frac{(-2)(2y)}{(x^2 + y^2)^3} + \frac{2kx}{(x^2 + y^2)^2} \tag{d}$$

Combining Eqs. (c) and (d), we obtain

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{-4kx^3 + 4kxy^2 + 2kx^3 + 2kxy^2}{(x^2 + y^2)^3} + \frac{-8kxy^2 + 2kx^3 + 2kxy^2}{(x^2 + y^2)^3}$$
$$= 0$$

Hence, the equation of continuity is satisfied.

5.2. *Equations of Motion (Navier-Stokes Equations)—Conservation of Momentum*

Consider a fluid mass in motion which at time t occupies a volume τ bounded by the surface A. Let $d\tau$ be an element of volume and $\rho\, d\tau$ the elementary mass moving with velocity $\mathbf{q}$ (see Fig. 5.2). Then $-\rho\, d\tau(D\mathbf{q}/Dt)$ is the force of inertia (D'Alembert's principle)[1] acting upon the elementary mass. The resultant of the inertia forces or the time rate of change of linear momentum is

[1] See Chap. Two.

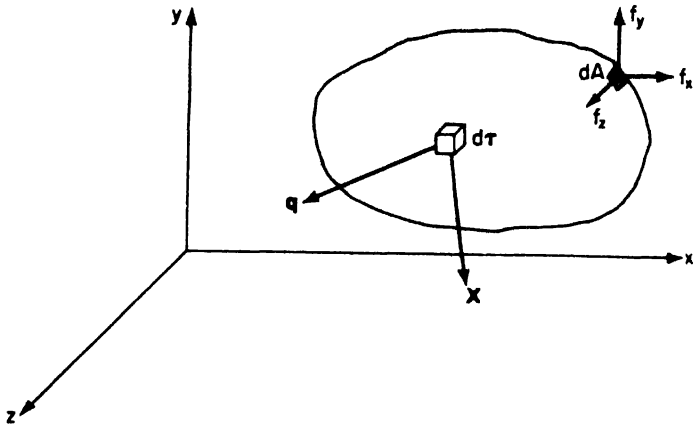


Figure 5.2

$$\mathbf{F}_I = -\int\int\int \frac{D\mathbf{q}}{Dt}\rho\, d\tau \tag{5-10}$$

If the body force per unit mass acting upon the elementary volume is denoted by X, then $\mathbf{X}\rho\, d\tau$ will be the elementary body force. The resultant of the body forces is

$$\mathbf{F}_B = \int\int\int \mathbf{X}\rho\, d\tau \tag{5-11}$$

The elementary surface force acting on an element of surface dA is expressed by the vector

$$\mathbf{f} = \mathbf{i}_x f_x + \mathbf{i}_y f_y + \mathbf{i}_z f_z = \mathbf{i}_x(\mathbf{P}_x \cdot d\mathbf{A}) + \mathbf{i}_y(\mathbf{P}_y \cdot d\mathbf{A}) + \mathbf{i}_z(\mathbf{P}_z \cdot d\mathbf{A}) \tag{5-12a}$$

where $d\mathbf{A}$ is the element of the vectoral area and $\mathbf{P}_x$, $\mathbf{P}_y$, and $\mathbf{P}_z$ are given in Eq. (4-1). The resultant of the surface forces is

$$\mathbf{F}_s = \mathbf{i}_x \int\int \mathbf{P}_x \cdot d\mathbf{A} + \mathbf{i}_y \int\int \mathbf{P}_y \cdot d\mathbf{A} + \mathbf{i}_z \int\int \mathbf{P}_z \cdot d\mathbf{A} \tag{5-12b}$$

In accordance with Gauss's divergence theorem, Eq. (5-12b) becomes

$$\mathbf{F}_s = \mathbf{i}_x \int\int\int \nabla \cdot \mathbf{P}_x d\tau + \mathbf{i}_y \int\int\int \nabla \cdot \mathbf{P}_y d\tau + \mathbf{i}_z \int\int\int \nabla \cdot \mathbf{P}_z d\tau \tag{5-12c}$$

By Newton's law of motion, the total force acting on a fluid mass enclosed in an arbitrary volume fixed in space is equal to the time rate of change of linear momentum. This is the law of conservation of momentum. Equating the resultants of the body and surface forces in Eqs. (5-11) and (5-12c), respectively, with the resultant of the inertia forces in Eq. (5-10), we find

$$\iiint \left\{-\rho\frac{D\mathbf{q}}{Dt} + \rho\mathbf{X} + [\mathbf{i}_x(\nabla\cdot\mathbf{P}_x) + \mathbf{i}_y(\nabla\cdot\mathbf{P}_y) + \mathbf{i}_z(\nabla\cdot\mathbf{P}_z)]\right\} d\tau = 0 \qquad (5\text{-}13)$$

Equation (5–13) is true for all τ no matter how small, hence the general equation of motion is obtained,

$$\rho\frac{D\mathbf{q}}{Dt} = \rho\mathbf{X} + \mathbf{i}_x(\nabla\cdot\mathbf{P}_x) + \mathbf{i}_y(\nabla\cdot\mathbf{P}_y) + \mathbf{i}_z(\nabla\cdot\mathbf{P}_z) \qquad (5\text{-}14)$$

Substituting the stress-components in Eq. (4–1) into Eq. (5–14) for $\mathbf{P}_x$, $\mathbf{P}_y$, and $\mathbf{P}_z$, the equations of motion in Cartesian coordinates are obtained as follows:

$$\rho\frac{Du}{Dt} = \rho X_x + \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z} \qquad (5\text{-}15a)$$

$$\rho\frac{Dv}{Dt} = \rho X_y + \frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{zy}}{\partial z} \qquad (5\text{-}15b)$$

$$\rho\frac{Dw}{Dt} = \rho X_z + \frac{\partial\sigma_{zx}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} \qquad (5\text{-}15c)$$

where X_x, X_y, and X_z are the components of body force per unit mass along the x, y, and z-directions, respectively. The operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{q}\cdot\nabla = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \qquad (5\text{-}16)$$

is sometimes called the Eulerian derivative which implies the rate of change of a certain fluid particle due to both nonstationary effects and convective effects in the direction of particle motion [see Eq. (3–17)]. Equations (5–15a), (5–15b), and (5–15c) may be expressed in terms of the velocity derivatives by using Eqs. (4–53) for the stress-components. The three equations are

$$\rho\frac{Du}{Dt} = \rho\left(\frac{\partial u}{\partial t} + \mathbf{q}\cdot\nabla u\right) = \rho X_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left\{\mu\left[2\frac{\partial u}{\partial x} - \frac{2}{3}(\nabla\cdot\mathbf{q})\right]\right\} + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] \qquad (5\text{-}17a)$$

$$\rho\frac{Dv}{Dt} = \rho\left(\frac{\partial v}{\partial t} + \mathbf{q}\cdot\nabla v\right) = \rho X_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left\{\mu\left[2\frac{\partial v}{\partial y} - \frac{2}{3}(\nabla\cdot\mathbf{q})\right]\right\} + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \qquad (5\text{-}17b)$$

$$\rho\frac{Dw}{Dt} = \rho\left(\frac{\partial w}{\partial t} + \mathbf{q}\cdot\nabla w\right) = \rho X_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left\{\mu\left[2\frac{\partial w}{\partial z} - \frac{2}{3}(\nabla\cdot\mathbf{q})\right]\right\} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] \qquad (5\text{-}17c)$$

The three equations above are known as the Navier-Stokes equations

of motion of a viscous compressible fluid. The equations of motion for flows with certain constant physical properties of the fluid may be reduced from Eqs. (5-17) as follows:

(i) *Viscous Compressible Fluid with Constant Viscosity.* If the coefficient of viscosity, μ, is constant, Eqs. (5-17) may be reduced to an equation in vector form

$$\rho\left(\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q}\cdot\nabla\mathbf{q}\right) = \rho\mathbf{X} - \nabla p + \mu\nabla^2\mathbf{q} + \frac{\mu}{3}\nabla(\nabla\cdot\mathbf{q}) \tag{5-18}$$

(ii) *Viscous Incompressible Fluid.* For incompressible fluids where both the density ρ and viscosity coefficient μ are constant, Eq. (5-18) reduces to

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q}\cdot\nabla\mathbf{q} = \mathbf{X} - \frac{\nabla p}{\rho} + \nu\nabla^2\mathbf{q} \tag{5-19}$$

where $\nu = \mu/\rho$ is called the kinematic viscosity and $\nabla\cdot\mathbf{q} = 0$ for an incompressible fluid.

(iii) *Nonviscous or Inviscid Fluid.* For a fluid which can exert no shearing stress at all, Eq. (5-18) reduces to the following simple form,

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q} = \mathbf{X} - \frac{\nabla p}{\rho} \tag{5-20}$$

Equation (5-20), usually called Euler's equation, is valid for both incompressible flows and compressible flows. For incompressible flows the mass density ρ is a constant while for the cases of subsonic, transonic, supersonic, and hypersonic flows, the mass density in general is a function of both pressure and temperature.

Illustrative Example 5.2. Consider an incompressible steady flow with constant viscosity. The velocity components are given by

$$u(y) = y\frac{U}{h} + \frac{h^2}{2\mu}\left(-\frac{dp}{dx}\right)\frac{y}{h}\left(1 - \frac{y}{h}\right) \tag{a}$$

$$v = w = 0 \tag{b}$$

If the body force is neglected, does $u(y)$ satisfy the equation of motion? h, U, and dp/dx are constants, and $p = p(x)$.

Solution. For the given assumptions, the equation of motion can be written as [see Eq. (5-19)]

$$(\mathbf{q}\cdot\nabla)\mathbf{q} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\mathbf{q} \tag{c}$$

In Cartesian coordinates we have

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$
$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \qquad \text{(d)}$$
$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Substituting Eq. (b) into Eq. (d) yields

$$\frac{1}{\rho}\frac{dp}{dx} = \frac{\mu}{\rho}\frac{d^2u}{dy^2} \qquad \text{(e)}$$

Combining Eqs. (a) and (e), we obtain

$$\frac{dp}{dx} = \mu\frac{d}{dy}\left[\frac{U}{h} + \frac{h^2}{2\mu}\left(-\frac{dp}{dx}\right)\left(\frac{1}{h} - \frac{2y}{h^2}\right)\right]$$

Simplifying,

$$\frac{dp}{dx} = \mu\left[\left(\frac{h^2}{2\mu}\right)\left(-\frac{dp}{dx}\right)\left(\frac{-2}{h^2}\right)\right] = \frac{dp}{dx}$$

Therefore, the equation of motion is satisfied.

5.3. *The Energy Equation—Conservation of Energy*

According to the First Law of Thermodynamics (see Appendix B), the energy added to a closed system increases the internal energy per unit mass of the fluid. The energy is added both by heat and by work done on the fluid. Let Q be the heat added per unit mass of gas and E the internal energy per unit mass of gas. The rate of work done by the stresses on a unit volume of the gas is

$$\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\gamma_{xy} + \sigma_{yz}\gamma_{yz} + \sigma_{zx}\gamma_{zx} \qquad (5\text{-}21)$$

The mathematical formulation of the First Law of Thermodynamics in terms of time variation of energy may be written

$$\rho\frac{dQ}{dt} + (\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\gamma_{xy} + \sigma_{yz}\gamma_{yz} + \sigma_{zx}\gamma_{zx}) = \rho\frac{DE}{Dt} \qquad (5\text{-}22)$$

where Q is a function of the time only. Substituting the expressions for stresses and rates of strain from Eqs. (4-53) and (4-24) into Eq. (5-21), we have

$$\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\gamma_{xy} + \sigma_{yz}\gamma_{yz} + \sigma_{zx}\gamma_{zx}$$
$$= -p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$+\mu\left\{-\frac{2}{3}(\nabla\cdot\mathbf{q})^2+2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]\right.$$
$$\left.+\left[\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2\right]\right\}$$
$$=-p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\Phi \tag{5-23}$$

where Φ is called the dissipation function and represents the time rate at which energy is being dissipated per unit volume through the action of viscosity. As mentioned in § 4.4, the dissipation function Φ which is a form of the strain energy can also be obtained from the invariants of the rate of strain. From Eqs. (4–37) and (4–38) we may write

$$\Phi=\frac{\alpha}{2}(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})^2+\frac{\beta}{2}[-4(\varepsilon_{xx}\varepsilon_{yy}+\varepsilon_{yy}\varepsilon_{zz}+\varepsilon_{zz}\varepsilon_{xx})$$
$$+(\gamma_{xy}^2+\gamma_{yz}^2+\gamma_{zx}^2)]$$
$$=\frac{\alpha}{2}(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})^2-\beta(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})^2$$
$$+\beta(\varepsilon_{xx}^2+\varepsilon_{yy}^2+\varepsilon_{zz}^2)+\frac{\beta}{2}(\gamma_{xy}^2+\gamma_{yz}^2+\gamma_{zx}^2)$$
$$=\frac{1}{2}(\alpha-2\beta)(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})^2+\beta(\varepsilon_{xx}^2+\varepsilon_{yy}^2+\varepsilon_{zz}^2)$$
$$+\frac{\beta}{2}(\gamma_{xy}^2+\gamma_{yz}^2+\gamma_{zx}^2) \tag{5-23a}$$

where α and β are arbitrary constants. If the velocity gradients given in Eq. (4–24) are applied and $(\alpha-2\beta)/2=-\frac{2}{3}\mu$, $\beta=2\mu$ are chosen, Eq. (5–23a) becomes

$$-\frac{2}{3}\mu(\nabla\cdot\mathbf{q})^2+2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]$$
$$+\mu\left[\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2\right]=\Phi \tag{5-23b}$$

Equation (5–23b) is the dissipation function which is identical to the Φ function in Eq. (5–23). In terms of the dissipation function, Eq. (5–22) can be rewritten

$$\rho\frac{dQ}{dt}+\Phi=\rho\frac{DE}{Dt}+p\nabla\cdot\mathbf{q} \tag{5-24}$$

From Eq. (5–8) we have

$$\frac{p}{\rho}\nabla\cdot\mathbf{q}=-\frac{p}{\rho^2}\frac{D\rho}{Dt}=\frac{D}{Dt}\left(\frac{p}{\rho}\right)-\frac{1}{\rho}\frac{Dp}{Dt} \tag{5-25}$$

By substituting Eq. (5–25) into Eq. (5–24), the desired energy equation becomes

$$\rho\frac{dQ}{dt} + \Phi = \rho\frac{Dh}{Dt} - \frac{Dp}{Dt} \tag{5–26}$$

where $h = E + p/\rho$ is the enthalpy of the gas per unit mass and Q is to be specified.

Now let us consider the mathematical formulation of the law of heat conduction. According to the Fourier heat-conduction law, the heat flow across an element of surface in Δt seconds is proportional to the greatest rate of decrease of the temperature T, that is,

$$-k\Delta A\Delta t\frac{\partial T}{\partial n} \tag{5–27}$$

where k is a constant of proportionality known as the thermal conductivity of a substance in Btu per unit time per unit length per degree R. The negative sign in Eq. (5–27) indicates that heat flows from points of higher temperature to those of lower temperature.

The total amount of heat flowing out in Δt seconds from an arbitrary volume τ bounded by a closed surface A is

$$-\Delta t\int\int k\frac{\partial T}{\partial n}dA = -\Delta t\int\int k(\nabla T)\cdot d\mathbf{A} \tag{5–28}$$

Applying the divergence theorem to the right-hand member of Eq. (5–28) yields

$$-\Delta t\int\int k(\nabla T)\cdot d\mathbf{A} = -\Delta t\int\int\int \nabla\cdot[k(\nabla T)]\,d\tau \tag{5–29}$$

Hence the heat added to the system per unit volume, $\rho\Delta Q$, becomes

$$\rho\Delta Q = \Delta t\,\nabla\cdot[k(\nabla T)]$$

or, in the limit as $\Delta t \to 0$

$$\rho\frac{dQ}{dt} = \nabla\cdot[k(\nabla T)] \tag{5–30}$$

In Cartesian coordinates, Eq. (5–30) becomes

$$\rho\frac{dQ}{dt} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) \tag{5–31}$$

Finally, we have the thermal energy equation when direct heating from chemical reaction and radiation heating is absent.

$$\nabla\cdot(k\nabla T) + \Phi = \rho\frac{Dh}{Dt} - \frac{Dp}{Dt} \tag{5–32}$$

In Cartesian coordinates, Eq. (5–32) has the form

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\Phi=\rho\frac{\partial(C_pT)}{\partial t}+\rho u\frac{\partial(C_pT)}{\partial x}+\rho v\frac{\partial(C_pT)}{\partial y}$$
$$+\rho w\frac{\partial(C_pT)}{\partial z}-\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right) \tag{5-33}$$

where $h = C_pT$

For gases with ordinary densities it has been shown by the kinetic theory of gases and by experiments that both μ and k are functions of the temperature only. The energy equation for special cases can be derived from Eq. (5-32) as follows:

(i) *Viscous Incompressible Fluid.* When the fluid is considered as an incompressible viscous fluid (k = constant and μ = constant), the energy equation [Eq. (5-32)] becomes

$$k\nabla^2T+\Phi_{(i)}=\rho C_v\frac{\partial T}{\partial t}+\rho C_v\mathbf{q}\cdot(\nabla T) \tag{5-34}$$

where
$$\Phi_{(i)}=2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]$$
$$+\mu\left[\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2\right]$$

and
$$C_v\frac{DT}{Dt}=\frac{D}{Dt}(C_p-R)T=\frac{D}{Dt}\left(C_pT-\frac{p}{\rho}\right)=\frac{Dh}{Dt}-\frac{1}{\rho}\frac{Dp}{Dt}+\frac{p}{\rho^2}\frac{D\rho}{Dt}$$
$$=\frac{Dh}{Dt}-\frac{1}{\rho}\frac{Dp}{Dt}$$

(ii) *Nonviscous Fluid.* The energy equation for a nonviscous fluid can be obtained by putting $\Phi = 0$ in Eq. (5-32), viz.,

$$\nabla\cdot(k\nabla T)=\rho\frac{Dh}{Dt}-\frac{Dp}{Dt} \tag{5-35}$$

For an adiabatic flow, i.e., $dQ = 0$, Eq. (5-35) becomes

$$\rho\frac{Dh}{Dt}-\frac{Dp}{Dt}=0 \tag{5-36}$$

Illustrative Example 5.3. Consider the case of simple Couette flow (see § 8.3 and § 12.1) with the velocity and temperature distributions as follows:

$$u=\frac{Uy}{h},\qquad v=0.\qquad p=\text{constant} \tag{a}$$

$$\frac{T-T_w}{T_\infty-T_w}=\frac{y}{h}+\frac{\mu U^2}{2k(T_\infty-T_w)}\left(\frac{y}{h}\right)\left(1-\frac{y}{h}\right) \tag{b}$$

where T_w and T_∞ are temperatures (constant in value) of the stationary and moving plates, respectively, and μ, h, and k are constants. Verify that Eqs. (a) and (b) are the solution of the energy equation [Eq. (5–34)].

Solution. For a two-dimensional incompressible steady flow the energy equation [Eq. (5–34)] may be written

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \Phi_{(i)} \tag{c}$$

where

$$\Phi_{(i)} = 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 = \mu\frac{U^2}{h^2} \tag{d}$$

From Eq. (b) we obtain

$$\begin{aligned} &\frac{\partial T}{\partial x} = 0, \qquad \frac{\partial^2 T}{\partial x^2} = 0 \\ &\frac{\partial T}{\partial y} = \frac{T_\infty - T_w}{h} + \frac{\mu U^2}{2k}\left(\frac{1}{h} - \frac{2y}{h^2}\right) \\ &\frac{\partial^2 T}{\partial y^2} = -\frac{\mu U^2}{kh^2} \end{aligned} \tag{e}$$

Substituting Eqs. (d) and (e) into Eq. (c), we have

$$0 = k\left(-\frac{\mu U^2}{kh^2}\right) + \mu\frac{U^2}{h^2} = 0$$

Hence, Eqs. (a) and (b) are solutions of the energy equation, Eq. (5–34).

5.4. The Equation of State—Perfect Gases

The equation of state of a substance is a relation between its pressure, specific volume, and temperature. For every homogeneous substance, solid, liquid, or gas, there exists an equation of state. The exact form of the function connecting these three scalar quantities is, in general, extremely complicated. It is often necessary to express this complicated relation in graphical form.

Experimental data of the thermal properties of gases show that for low pressure the equation of state approaches the same form for all gases, namely (see Appendix B)

$$\frac{p}{\rho} = RT \tag{5–37}$$

where R is the gas constant which, in the system adopted in the present work, is for air (see Table C–2 for other unit systems).

$$R = 1715 \frac{\text{ft}^2}{\text{sec}^2 \cdot \text{R}} \tag{5-38}$$

A gas (air, oxygen, or helium) which obeys the law of the equation of state [Eq. (5-37)] is called a perfect gas.

If intermolecular action among molecules of a gas is neglected, the distance between molecules being assumed to be much greater than the size of molecules and the volume of the molecules being considered to be negligible relative to the total apparent volume of the gas, the equation of state for a perfect gas [Eq. (5-37)] can also be derived from the principles of the kinetic theory of gases.

For real gases (ammonia, freon) where the size of molecules relative to the intermolecular distance may no longer be negligible at a certain pressure, and therefore, neither the forces of intermolecular attraction nor the volume of the molecule can be neglected, the equation of state for a perfect gas [Eq. (5-37)] is no longer satisfied. Among the equations proposed for real gases, the Van der Waals equation is the one most frequently applied. The Van der Waals equation is

$$p = \rho RT\left(\frac{1}{1 - \beta\rho} + \frac{\alpha\rho}{RT}\right) \tag{5-39}$$

where α and β are constants for a given gas. Equation (5-39) reduces to Eq. (5-37) when $\alpha = \beta = 0$.

5.5. Fundamental Equations in Cylindrical Coordinates

a. The Equation of Continuity

With reference to Eq. (5-7) and Appendix A we can readily write the equation of continuity in cylindrical coordinates (r, θ, z) as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} + \frac{\rho v_r}{r} = 0 \tag{5-40}$$

In the case of an incompressible fluid, Eq. (5-40) becomes

$$\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} + \frac{\rho v_r}{r} = 0 \tag{5-41}$$

b. The Equations of Motion

In cylindrical coordinates (r, θ, z) shown in Fig. 5.3 we must re-examine the terms in Eq. (5-13) which involve the inertia forces and the surface forces. Referring to Fig. 5.4, the components of the acceleration vector after neglecting higher-order terms can be expressed as follows:

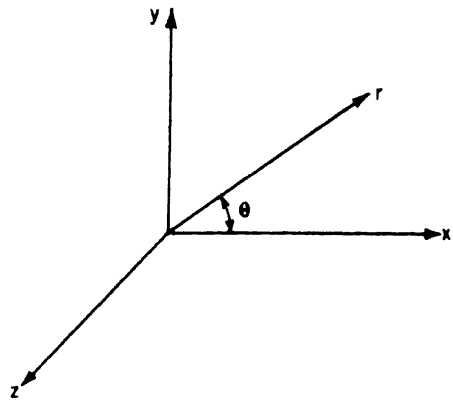


Figure 5.3

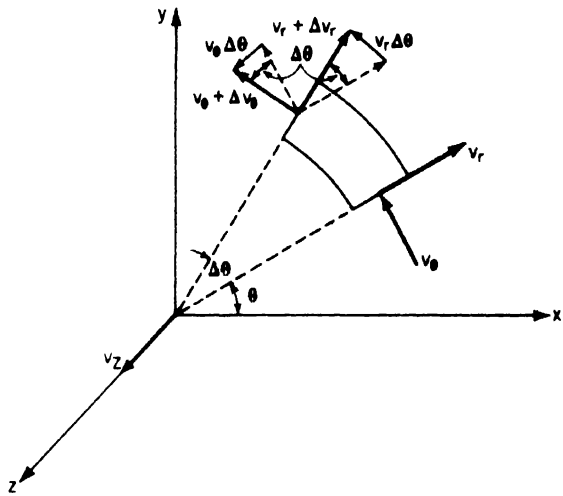


Figure 5.4

$$\frac{D\mathbf{q}}{Dt} = \mathbf{i}_r\left(\frac{Dv_r}{Dt} - v_\theta\frac{d\theta}{dt}\right) + \mathbf{i}_\theta\left(\frac{Dv_\theta}{Dt} + v_r\frac{d\theta}{dt}\right) + \mathbf{i}_z\frac{Dv_z}{Dt} \quad (5\text{–}42)$$

where v_r, v_θ, and v_z are the velocity components in the r, θ, and z-directions, respectively, and $\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_z$ are the unit vectors in the direction indicated by the respective subscripts. Substituting

$$v_\theta = r\frac{d\theta}{dt}$$

into Eq. (5–42), we have the resultant of the inertia forces in cylindrical coordinates

$$\iiint \rho \frac{Dq}{Dt} d\tau = \iiint \left[\mathbf{i}_r \left(\frac{Dv_r}{Dt} - \frac{v_\theta^2}{r} \right) + \mathbf{i}_\theta \left(\frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r} \right) + \mathbf{i}_z \frac{Dv_z}{Dt} \right] \rho \, d\tau \quad (5\text{–}43)$$

In order to write the resultant of surface forces in cylindrical coordinates, let us consider an element of a cylindrical volume with surface forces acting on respective surfaces (Fig. 5.5a). In the r-θ

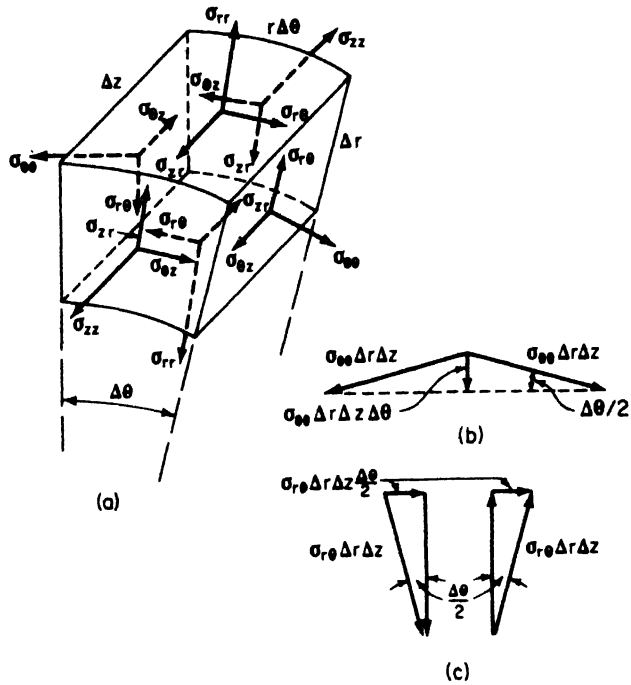


Figure 5.5

plane (Fig. 5.5b) we see that the normal forces $\sigma_{\theta\theta} \Delta r \Delta z$ contribute the components of force $-\sigma_{\theta\theta} \, \Delta r \Delta z \Delta\theta$ in the r-direction and the shearing forces $\sigma_{r\theta} \Delta r \Delta z$ give a component of force $\sigma_{r\theta} \Delta r \Delta z \Delta\theta$ in the θ-direction (Fig. 5.5c). Hence corresponding to Eq. (5–12b) in Cartesian coordinates, the resultant of surface forces in cylindrical coordinates is

$$\mathbf{F}_s = \mathbf{i}_r \left(\iint \mathbf{P}_r \cdot d\mathbf{A} - \iiint \frac{\sigma_{\theta\theta}}{r} r \, dr \, dz \, d\theta \right) + \mathbf{i}_\theta \left(\iint \mathbf{P}_\theta \cdot d\mathbf{A} + \iiint \frac{\sigma_{r\theta}}{r} r \, dr \, dz \, d\theta \right)$$
$$+ \mathbf{i}_z \left(\iint \mathbf{P}_z \cdot d\mathbf{A} \right) \quad (5\text{–}44)$$

In accordance with Gauss's divergence theorem, Eq. (5–44) takes the form

$$\mathbf{F}_s = \mathbf{i}_r \iiint \left(\nabla \cdot \mathbf{P}_r - \frac{\sigma_{\theta\theta}}{r}\right) d\tau + \mathbf{i}_\theta \iiint \left(\nabla \cdot \mathbf{P}_\theta + \frac{\sigma_{r\theta}}{r}\right) d\tau + \mathbf{i}_z \iiint \left(\nabla \cdot \mathbf{P}_z\right) d\tau \tag{5-45}$$

where

$$\begin{aligned} \mathbf{P}_r &= \mathbf{i}_r \sigma_{rr} + \mathbf{i}_\theta \sigma_{r\theta} + \mathbf{i}_z \sigma_{zr} \\ \mathbf{P}_\theta &= \mathbf{i}_r \sigma_{r\theta} + \mathbf{i}_\theta \sigma_{\theta\theta} + \mathbf{i}_z \sigma_{\theta z} \\ \mathbf{P}_z &= \mathbf{i}_r \sigma_{zr} + \mathbf{i}_\theta \sigma_{\theta z} + \mathbf{i}_z \sigma_{zz} \end{aligned} \tag{5-46}$$

If the resultant of the body forces is denoted by

$$\mathbf{F}_B = \iiint (\mathbf{i}_r X_r + \mathbf{i}_\theta X_\theta + \mathbf{i}_z X_z) d\tau \tag{5-47}$$

the equations of motion in cylindrical coordinates are obtained from Eqs. (5-43), (5-45), (5-46), and (5-47) as follows:

$$\begin{aligned} \rho\left(\frac{Dv_r}{Dt} - \frac{v_\theta^2}{r}\right) &= \rho X_r + \frac{1}{r}\left[\frac{\partial(r\sigma_{rr})}{\partial r} + \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial(r\sigma_{zr})}{\partial z}\right] - \frac{\sigma_{\theta\theta}}{r} \\ &= \rho X_r + \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} \\ \rho\left(\frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r}\right) &= \rho X_\theta + \frac{1}{r}\left[\frac{\partial(r\sigma_{r\theta})}{\partial r} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial(r\sigma_{\theta z})}{\partial z}\right] + \frac{\sigma_{r\theta}}{r} \\ &= \rho X_\theta + \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} \\ \rho\left(\frac{Dv_z}{Dt}\right) &= \rho X_z + \frac{1}{r}\left[\frac{\partial(r\sigma_{zr})}{\partial r} + \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial(r\sigma_{zz})}{\partial z}\right] \\ &= \rho X_z + \frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{zr}}{r} \end{aligned}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z} \tag{5-48}$$

The relations between the stresses and the rates of strain derived in § 4.5 were in Cartesian coordinates. Since these relations should not depend on any particular coordinate system, they can be written in cylindrical coordinates as follows:

$$\begin{aligned} \sigma_{rr} &= -p + 2\mu\varepsilon_{rr} - \tfrac{2}{3}\mu\nabla \cdot \mathbf{q} \\ \sigma_{\theta\theta} &= -p + 2\mu\varepsilon_{\theta\theta} - \tfrac{2}{3}\mu\nabla \cdot \mathbf{q} \\ \sigma_{zz} &= -p + 2\mu\varepsilon_{zz} - \tfrac{2}{3}\mu\nabla \cdot \mathbf{q} \end{aligned} \tag{5-49}$$

and

$$\begin{aligned} \sigma_{r\theta} &= \mu\gamma_{r\theta} \\ \sigma_{\theta z} &= \mu\gamma_{\theta z} \\ \sigma_{zr} &= \mu\gamma_{zr} \end{aligned} \tag{5-50}$$

In Appendix A the components of the rates of strain are given in terms of the velocity gradients in general orthogonal coordinates [Eqs. (A-52)-(A-57)]. In cylindrical coordinates, they are

$$= \frac{\partial v_r}{\partial r}, \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}, \qquad \varepsilon_{zz} = \frac{\partial v_z}{\partial z}$$

$$\gamma_{r\theta} = \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

$$\gamma_{\theta z} = \frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \tag{5-51}$$

$$\gamma_{zr} = \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}$$

The above relations can also be derived in the manner employed in the case of Cartesian coordinates. In Fig. 5.6 the solid area represents the original element in the r-θ plane and the dotted area, the element being deformed. In this case the rate of shearing strain is

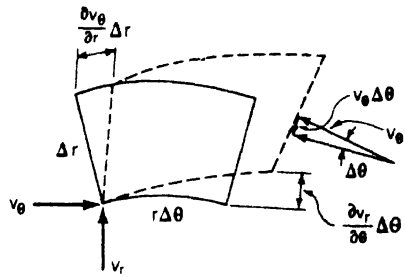


Figure 5.6

$$\gamma_{r\theta} = \frac{(\partial v_r/\partial\theta)\Delta\theta}{r\Delta\theta} + \frac{(\partial v_\theta/\partial r)\Delta r}{\Delta r} \qquad \frac{v_\theta \Delta\theta}{r\Delta\theta} = \frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} \qquad \frac{v_\theta}{r} \tag{5-51a}$$

The first two terms in Eq. (5-51a) are similar to the expression derived in Cartesian coordinates; however, the third term in the equation above is due to the contribution of v_θ to the velocity deformation in the r-direction. The other five components of the rates of strain in Eq. (5-51) can be intrepreted similarly.

Now, if we introduce the relations of stresses and rates of strain in Eqs. (5-49) and (5-50) and the rates of strain and velocity gradients relations in Eq. (5-51) into Eq. (5-48), the equations of motion of viscous compressible fluids in cylindrical coodinates are obtained:

$$\rho\left(\frac{Dv_r}{Dt} \quad \frac{v_\theta^2}{r}\right) = \rho X_r \quad \frac{\partial p}{\partial r} \quad \frac{\partial}{\partial r}\mu\left(2\frac{\partial v_r}{\partial r} \quad \frac{2}{3}\nabla\cdot\mathbf{q}\right.$$

$$+ \frac{\partial}{\partial\theta}\left[\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{\partial v_\theta}{\partial r}\right.\right. \qquad \left.\left.\frac{\partial}{\partial z}\mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)\right]\right.$$

$$+ \frac{2\mu}{r}\left(\frac{\partial v_r}{\partial r} - \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} - \frac{v_r}{r}\right) \tag{5-52a}$$

$$\rho\left(\frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r}\right) = \rho X_\theta - \frac{1}{r}\frac{\partial p}{\partial\theta} + \frac{1}{r}\frac{\partial}{\partial\theta}\left[\mu\left(\frac{2}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{2v_r}{r} - \frac{2}{3}\nabla\cdot\mathbf{q}\right)\right]$$

$$+ \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)\Big] + \frac{\partial}{\partial z}\left[\mu\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta} + \frac{\partial v_\theta}{\partial z}\right)\right]$$

$$+\frac{2\mu}{r}\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right) \tag{5-52b}$$

$$\rho\left(\frac{Dv_z}{Dt}\right)=\rho X_z-\frac{\partial p}{\partial z}+\frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial v_z}{\partial z}-\frac{2}{3}\nabla\cdot\mathbf{q}\right)\right]+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\mu\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta}+\frac{\partial v_\theta}{\partial z}\right)\right]$$
$$+\frac{\partial}{\partial r}\left[\mu\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)\right]+\frac{\mu}{r}\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right) \tag{5-52c}$$

where

$$\nabla\cdot\mathbf{q}=\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_z}{\partial z}+\frac{v_r}{r}$$

For certain special flows, Eqs. (5-52) reduce as follows:

(i) *Viscous Compressible Fluid with Constant Viscosity.* If the coefficient of viscosity is constant, Eqs. (5-52) become

$$\rho\left(\frac{Dv_r}{Dt}-\frac{v_\theta^2}{r}\right)=\rho X_r-\frac{\partial p}{\partial r}+\mu\left[\nabla^2 v_r-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{1}{3}\frac{\partial}{\partial r}(\nabla\cdot\mathbf{q})\right] \tag{5-53a}$$

$$\rho\left(\frac{Dv_\theta}{Dt}+\frac{v_r v_\theta}{r}\right)=\rho X_\theta-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left[\nabla^2 v_\theta+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2}+\frac{1}{3}\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla\cdot\mathbf{q})\right] \tag{5-53b}$$

$$\rho\frac{Dv_z}{Dt}=\rho X_z-\frac{\partial p}{\partial z}+\mu\left[\nabla^2 v_z+\frac{1}{3}\frac{\partial}{\partial z}(\nabla\cdot\mathbf{q})\right] \tag{5-53c}$$

where

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}$$

(ii) *Viscous Incompressible Flow.* When the density and coefficient of viscosity are both constant, Eqs. (5-52) become

$$\rho\left(\frac{Dv_r}{Dt}-\frac{v_\theta^2}{r}\right)=\rho X_r-\frac{\partial p}{\partial r}+\mu\left(\nabla^2 v_r-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}\right) \tag{5-54a}$$

$$\rho\left(\frac{Dv_\theta}{Dt}+\frac{v_r v_\theta}{r}\right)=\rho X_\theta-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left(\nabla^2 v_\theta+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r}\right) \tag{5-54b}$$

$$\rho\frac{Dv_z}{Dt}=\rho X_z-\frac{\partial p}{\partial z}+\mu\nabla^2 v_z \tag{5-54c}$$

(iii) *Nonviscous Fluid.* For a fluid which can exert no shearing stress, Eqs. (5-52) become

$$\rho\left(\frac{Dv_r}{Dt}-\frac{v_\theta^2}{r}\right)=\rho X_r-\frac{\partial p}{\partial r} \tag{5-55a}$$

$$\rho\left(\frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r}\right) = \rho X_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} \tag{5-55b}$$

$$\rho\frac{Dv_z}{Dt} = \rho X_z - \frac{\partial p}{\partial z} \tag{5-55c}$$

c. The Energy Equation

In cylindrical coordinates, the energy equation [Eq. (5-32)] becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \Phi = \rho\frac{Dh}{Dt} - \frac{Dp}{Dt} \tag{5-56}$$

where

$$\nabla T = \mathbf{i}_r\frac{\partial T}{\partial r} + \mathbf{i}_\theta\frac{1}{r}\frac{\partial T}{\partial \theta} + \mathbf{i}_z\frac{\partial T}{\partial z}$$

$$\nabla\cdot(k\nabla T) = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{k}{r}\frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(rk\frac{\partial T}{\partial z}\right)\right]$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

$$\Phi = 2\mu\left(\varepsilon_{rr}^2 + \varepsilon_{\theta\theta}^2 + \varepsilon_{zz}^2 + \frac{\gamma_{r\theta}^2}{2} + \frac{\gamma_{\theta z}^2}{2} + \frac{\gamma_{zr}^2}{2}\right) - \frac{2}{3}\mu(\nabla\cdot\mathbf{q})^2$$

$$= 2\mu\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)^2\right.$$

$$\left. + \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)^2 - \frac{1}{3}(\nabla\cdot\mathbf{q})^2\right]$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + v_z\frac{\partial}{\partial z}$$

In some special cases of fluid flow where certain physical properties are constant, Eq. (5-56) can be reduced to the following forms:

(i) *Viscous Incompressible Fluid.* For an incompressible fluid where k, ρ, and μ are constant, Eq. (5-56) becomes

$$k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} + \Phi_{(i)} = \rho C_\tau\frac{\partial T}{\partial t} + \rho C_\tau\left(v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z\frac{\partial T}{\partial z}\right) \tag{5-57}$$

where

$$\Phi_{(i)} = 2\mu\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)^2\right.$$

$$\left. + \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)^2\right]$$

(ii) *Nonviscous Compressible Fluid.* When shearing stresses are absent from the fluid, Eq. (5-56) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) = \rho\frac{Dh}{Dt} - \frac{Dp}{Dt} \tag{5-58}$$

For an adiabatic flow, i.e., $dQ = 0$, Eq. (5-58) reduces to

$$\rho\frac{Dh}{Dt} - \frac{Dp}{Dt} = 0 \tag{5-59}$$

Illustrative Example 5.4. If the velocity components in cylindrical coordinates are given by

$$\begin{aligned} v_r &= c_1\left(\frac{r_0^2}{r^2} - 1\right)\cos\theta \\ v_\theta &= c_1\left(\frac{r_0^2}{r^2} + 1\right)\sin\theta - \frac{c_2}{r} \\ v_z &= 0 \end{aligned} \tag{a}$$

where $0 < r_0 \leq r$ and r_0, c_1, and c_2 are arbitrary constants, verify that the equation of continuity [Eq. (5-40)] is satisfied.

Solution. According to Eq. (5-40), we have

$$\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r} = 0 \tag{b}$$

Upon the substitution of the proper derivatives of the velocity components of Eq. (a) into Eq. (b), the result becomes

$$-\frac{2c_1r_0^2}{r^3}\cos\theta + \frac{c_1r_0^2}{r^3} + \frac{c_1}{r}\cos\theta + 0 + \left(\frac{c_1r_0^2}{r^3} - \frac{c_1}{r}\right)\cos\theta = 0 \tag{c}$$

Hence, Eq. (b) is satisfied by given velocity components.

Illustrative Example 5.5. Consider a two-dimensional viscous incompressible steady flow with velocity components

$$\begin{aligned} v_r &= v_\theta = 0 \\ v_z &= \frac{1}{4\mu}\left(\frac{dp}{dz}\right)r^2 + A\ln r + B \\ p &= p(z) \end{aligned} \tag{a}$$

where A, B, and μ are constants and $0 < r_0 \leq r$. Is the equation of motion with negligible body force satisfied?

Solution. According to Eqs. (5-54) and (5-48) we have

$$\begin{aligned} \rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) &= \rho X_r - \frac{\partial p}{\partial r} \\ + \mu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right) & \end{aligned} \tag{b}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = \rho X_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r}\right) \quad \text{(c)}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = \rho X_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right) \quad \text{(d)}$$

Under the present assumptions it is easy to see that Eqs. (b) and (c) are identically equal to zero, and Eq. (d) reduces to

$$\frac{dp}{dz} = \mu\left(\frac{d^2 v_z}{dr^2} + \frac{1}{r}\frac{dv_z}{dr}\right) = \mu\left(\frac{1}{2\mu}\frac{dp}{dz} - \frac{A}{r^2} + \frac{1}{r}\frac{1}{2\mu}\frac{dp}{dz}r + \frac{A}{r^2}\right) = \frac{dp}{dz}$$

Therefore, the equation of motion is satisfied.

5.6. *Fundamental Equations in Spherical Coordinates*

a. The Equation of Continuity

Using the formula of $\nabla\cdot(\rho\mathbf{q})$ in general orthogonal curvilinear coordinates (Appendix A), the continuity equation in spherical coordinates $(\omega, \varphi, \theta)$ becomes

$$\frac{\partial \rho}{\partial t} + \frac{1}{\omega^2}\frac{\partial(\omega^2 \rho v_\omega)}{\partial \omega} + \frac{1}{\omega \sin\varphi}\frac{\partial(\rho v_\varphi \sin\varphi)}{\partial \varphi} + \frac{1}{\omega \sin\varphi}\frac{\partial(\rho v_\theta)}{\partial \theta} = 0 \qquad (5\text{-}60)$$

For an incompressible fluid, Eq. (5-60) reduces to

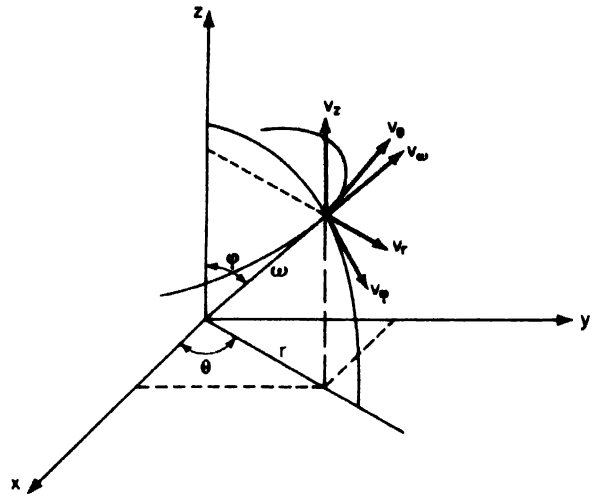
$$\frac{1}{\omega^2}\frac{\partial(\omega^2 v_\omega)}{\partial \omega} + \frac{1}{\sin\varphi}\frac{\partial(v_\varphi \sin\varphi)}{\partial \varphi} + \frac{1}{\sin\varphi}\frac{\partial v_\theta}{\partial \theta} = 0 \qquad (5\text{-}61)$$

b. The Equations of Motion

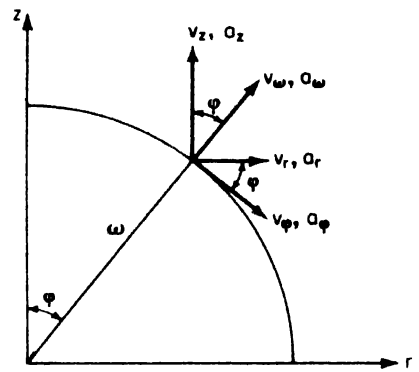
The acceleration terms in the equations of motion in spherical coordinates $(\omega, \varphi, \theta)$ shown in Fig. 5.7 can be calculated with the aid of the velocity and acceleration components previously found in cylindrical coordinates (r, θ, z). The relations between cylindrical coordinates and spherical coordinates are as follows:

$$r = \omega \sin\varphi, \qquad \theta = \theta, \qquad z = \omega \cos\varphi \qquad (5\text{-}62)$$

The components of velocity and acceleration in cylindrical coordinates given in Eq. (5-42) in terms of spherical coordinates are



(a)



(b)

Figure 5.7

$$
\begin{aligned}
v_r &= \frac{dr}{dt} = \omega \cos\varphi \frac{d\varphi}{dt} + \sin\varphi \frac{d\omega}{dt} \\
v_\theta &= r\frac{d\theta}{dt} = \omega \sin\varphi \frac{d\theta}{dt} \\
v_z &= \frac{dz}{dt} = -\omega \sin\varphi \frac{d\varphi}{dt} + \cos\varphi \frac{d\omega}{dt} \\
a_r &= \frac{Dv_r}{Dt} - \frac{v_\theta^2}{l} = \frac{Dv_r}{Dt} - \frac{v_\theta^2}{\omega \sin\varphi}
\end{aligned}
\tag{5-63}
$$

$$a_\theta = \frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r} = \frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{\omega \sin \varphi}$$

$$a_z = \frac{Dv_z}{Dt}$$

The components of velocity and acceleration in spherical coordinates can be obtained with the aid of Eq. (5–63) by the following procedures:

$$v_\theta = \omega \sin \varphi \frac{d\theta}{dt}$$

$$v_\omega = v_r \sin \varphi + v_z \cos \varphi = \omega \cos \varphi \sin \varphi \frac{d\varphi}{dt} + \sin^2 \varphi \frac{d\omega}{dt}$$

$$v_\varphi = -v_z \sin \varphi + v_r \cos \varphi = \omega \frac{d\varphi}{dt}$$

$$a_\theta = \frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{\omega \sin \varphi} = \frac{Dv_\theta}{Dt} + \left(\omega \cos \varphi \frac{d\varphi}{dt} + \sin \varphi \frac{d\omega}{dt}\right) \frac{v_\theta}{\omega \sin \varphi}$$

$$= \frac{Dv_\theta}{Dt} + v_\theta \cot \theta \frac{d\varphi}{dt} + \frac{v_\theta}{\omega} \frac{d\omega}{dt}$$

$$= \frac{Dv_\theta}{Dt} + \frac{v_\theta v_\omega}{\omega} + \frac{v_\theta v_\varphi}{\omega} \cot \varphi$$

$$a_\omega = a_r \sin \varphi + a_z \cos \varphi = \left(\frac{Dv_r}{Dt} - \frac{v_\theta^2}{\omega \sin \varphi}\right) \sin \varphi + \frac{Dv_z}{Dt} \cos \varphi$$

$$= \left[\cos \varphi \frac{d\varphi}{dt} \frac{d\omega}{dt} - \omega \sin \varphi \left(\frac{d\varphi}{dt}\right)^2 + \omega \cos \varphi \frac{d^2\varphi}{dt^2} + \cos \varphi \frac{d\varphi}{dt} \frac{d\omega}{dt} + \sin \varphi \frac{d^2\omega}{dt^2}\right] \sin \varphi - \frac{v_\theta^2}{\omega} + \left[-\sin \varphi \frac{d\omega}{dt} \frac{d\varphi}{dt} - \omega \cos \varphi \left(\frac{d\varphi}{dt}\right)^2 - \omega \sin \varphi \frac{d^2\varphi}{dt^2} + \cos \varphi \frac{d^2\omega}{dt^2} - \sin \varphi \frac{d\omega}{dt} \frac{d\varphi}{dt}\right] \cos \varphi$$

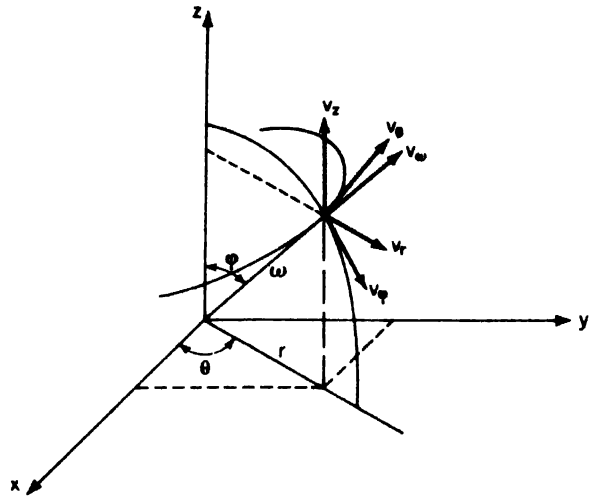
$$= \frac{Dv_\omega}{Dt} - \frac{v_\varphi^2 + v_\theta^2}{\omega}$$

$$a_\varphi = -a_z \sin \varphi + a_r \cos \varphi = -\frac{Dv_z}{Dt} \sin \varphi + \left(\frac{Dv_r}{Dt} - \frac{v_\theta^2}{\omega \sin \varphi}\right) \cos \varphi$$

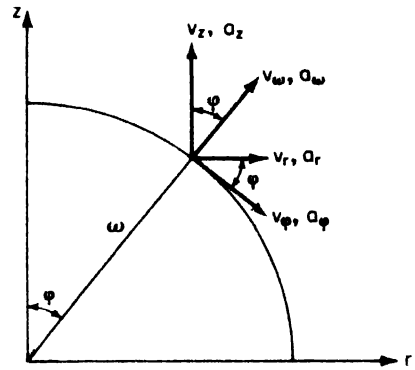
$$= -\sin \varphi \frac{D}{Dt}\left(-\omega \sin \varphi \frac{d\varphi}{dt} + \cos \varphi \frac{d\omega}{dt}\right) - \frac{v_\theta^2}{\omega} \cot \varphi + \cos \varphi \frac{D}{Dt}\left(\omega \cos \varphi \frac{d\varphi}{dt} + \sin \varphi \frac{d\omega}{dt}\right) = \frac{Dv_\varphi}{Dt} + \frac{v_\omega v_\varphi}{\omega} - \frac{v_\theta^2 \cot \varphi}{\omega}$$

(5–64)

Hence the resultant of the inertia forces in spherical coordinates can be written



(a)



(b)

Figure 5.7

$$v_r = \frac{dr}{dt} = \omega \cos \varphi \frac{d\varphi}{dt} + \sin \varphi \frac{d\omega}{dt}$$

$$v_\theta = r\frac{d\theta}{dt} = \omega \sin \varphi \frac{d\theta}{dt}$$

$$v_z = \frac{dz}{dt} = -\omega \sin \varphi \frac{d\varphi}{dt} + \cos \varphi \frac{d\omega}{dt}$$

$$a_r = \frac{Dv_r}{Dt} - \frac{v_\theta^2}{r} = \frac{Dv_r}{Dt} - \frac{v_\theta^2}{\omega \sin \varphi} \qquad (5\text{–}63)$$

$$a_\theta = \frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r} = \frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{\omega \sin \varphi}$$

$$a_z = \frac{Dv_z}{Dt}$$

The components of velocity and acceleration in spherical coordinates can be obtained with the aid of Eq. (5–63) by the following procedures:

$$v_\theta = \omega \sin \varphi \frac{d\theta}{dt}$$

$$v_\omega = v_r \sin \varphi + v_z \cos \varphi = \omega \cos \varphi \sin \varphi \frac{d\varphi}{dt} + \sin^2 \varphi \frac{d\omega}{dt}$$

$$v_\varphi = -v_z \sin \varphi + v_r \cos \varphi = \omega \frac{d\varphi}{dt}$$

$$a_\theta = \frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{\omega \sin \varphi} = \frac{Dv_\theta}{Dt} + \left(\omega \cos \varphi \frac{d\varphi}{dt} + \sin \varphi \frac{d\omega}{dt}\right) \frac{v_\theta}{\omega \sin \varphi}$$

$$= \frac{Dv_\theta}{Dt} + v_\theta \cot \theta \frac{d\varphi}{dt} + \frac{v_\theta}{\omega} \frac{d\omega}{dt}$$

$$= \frac{Dv_\theta}{Dt} + \frac{v_\theta v_\omega}{\omega} + \frac{v_\theta v_\varphi}{\omega} \cot \varphi$$

$$a_\omega = a_r \sin \varphi + a_z \cos \varphi = \left(\frac{Dv_r}{Dt} - \frac{v_\theta^2}{\omega \sin \varphi}\right) \sin \varphi + \frac{Dv_z}{Dt} \cos \varphi$$

$$= \left[\cos \varphi \frac{d\varphi}{dt} \frac{d\omega}{dt} - \omega \sin \varphi \left(\frac{d\varphi}{dt}\right)^2 + \omega \cos \varphi \frac{d^2\varphi}{dt^2} + \cos \varphi \frac{d\varphi}{dt} \frac{d\omega}{dt} + \sin \varphi \frac{d^2\omega}{dt^2}\right] \sin \varphi - \frac{v_\theta^2}{\omega} + \left[-\sin \varphi \frac{d\omega}{dt} \frac{d\varphi}{dt} - \omega \cos \varphi \left(\frac{d\varphi}{dt}\right)^2 - \omega \sin \varphi \frac{d^2\varphi}{dt^2} + \cos \varphi \frac{d^2\omega}{dt^2} - \sin \varphi \frac{d\omega}{dt} \frac{d\varphi}{dt}\right] \cos \varphi$$

$$= \frac{Dv_\omega}{Dt} - \frac{v_\varphi^2 + v_\theta^2}{\omega}$$

$$a_\varphi = -a_z \sin \varphi + a_r \cos \varphi = -\frac{Dv_z}{Dt} \sin \varphi + \left(\frac{Dv_r}{Dt} - \frac{v_\theta^2}{\omega \sin \varphi}\right) \cos \varphi$$

$$= -\sin \varphi \frac{D}{Dt}\left(-\omega \sin \varphi \frac{d\varphi}{dt} + \cos \varphi \frac{d\omega}{dt}\right) - \frac{v_\theta^2}{\omega} \cot \varphi + \cos \varphi \frac{D}{Dt}\left(\omega \cos \varphi \frac{d\varphi}{dt} + \sin \varphi \frac{d\omega}{dt}\right) = \frac{Dv_\varphi}{Dt} + \frac{v_\omega v_\varphi}{\omega} - \frac{v_\theta^2 \cot \varphi}{\omega}$$

(5–64)

Hence the resultant of the inertia forces in spherical coordinates can be written

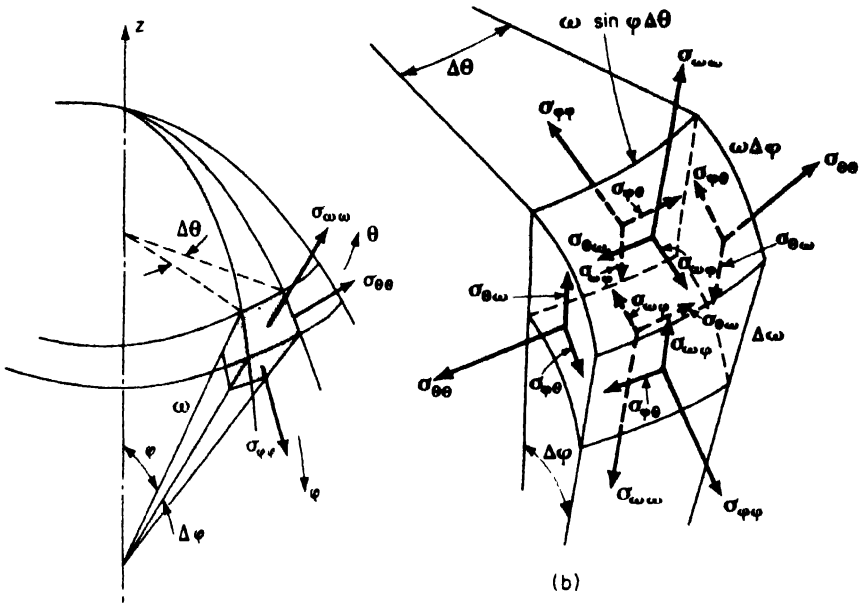


Figure 5.8

$$\mathbf{F}_I = -\int\int\int \rho \frac{D\mathbf{q}}{Dt} d\tau = -\int\int\int \left| \mathbf{i}_\omega \left(\frac{Dv_\omega}{Dt} - \frac{v_\varphi^2 + v_\theta^2}{\omega} \right) \right.$$
$$\left. + \mathbf{i}_\varphi \left(\frac{Dv_\varphi}{Dt} + \frac{v_\omega v_\varphi}{\omega} - \frac{v_\theta^2 \cot \varphi}{\omega} \right) + \mathbf{i}_\theta \left(\frac{Dv_\theta}{Dt} + \frac{v_\theta v_\omega}{\omega} + \frac{v_\theta v_\varphi}{\omega} \cot \varphi \right) \right| \rho \, d\tau \tag{5-65}$$

where $\mathbf{i}_\omega$, $\mathbf{i}_\varphi$, and $\mathbf{i}_\theta$ are the unit vectors in the direction indicated by the respective subscript.

The components of the surface forces in spherical coordinates (ω, φ, θ) are shown in Fig. 5.8. Let us consider the components of surface force acting in the ω-direction. Besides the three parallel forces $\sigma_{\omega\omega} \sin \varphi \, \Delta\varphi \, \Delta\theta$, $\sigma_{\theta\omega} \omega \, \Delta\varphi \, \Delta\omega$, and $\sigma_{\omega\varphi} \omega \sin \varphi \, \Delta\theta \, \Delta\omega$ here, there are also components of force due to the nonparallel forces $\sigma_{\theta\theta} \omega \, \Delta\varphi \, \Delta\omega$ and $\sigma_{\varphi\varphi} \omega \sin \varphi \, \Delta\theta \, \Delta\omega$. In order to determine the resultant of surface forces, the following detailed analysis of force components is believed to be very helpful.

(i) *Components of Nonparallel Forces in the ω-Direction.* In Fig. 5.9a it is seen that a component of force $\sigma_{\theta\theta} \omega \, \Delta\varphi \, \Delta\omega \, \Delta\theta$ in the ω-φ plane is contributed by the surface forces $\sigma_{\theta\theta} \omega \, \Delta\varphi \, \Delta\omega$. The force $\sigma_{\theta\theta} \omega \, \Delta\varphi \, \Delta\omega \, \Delta\theta$ in turn has a component of force $-\sigma_{\theta\theta} \omega \sin \varphi \, \Delta\omega \, \Delta\varphi \, \Delta\theta$ in the ω-direction (Fig. 5.9b). There is also the following component of force: $-\sigma_{\varphi\varphi} \omega \sin \varphi \, \Delta\theta \, \Delta\omega \, \Delta\varphi$

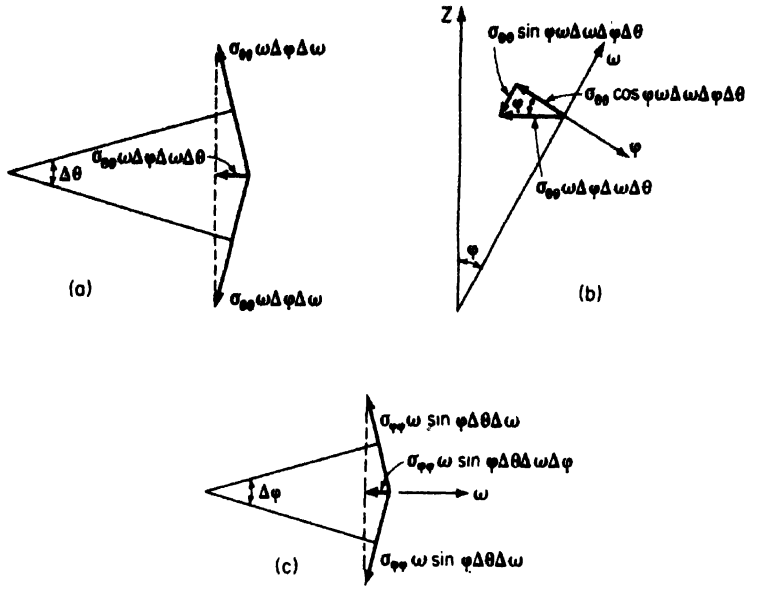


Figure 5.9

resulting from the forces $\sigma_{\varphi\varphi}\omega\sin\varphi\,\Delta\theta\,\Delta\omega$ which can be seen in Fig. 5.9c. Finally, the resultant of surface forces in the ω-direction due to nonparallel surface forces is

$$-\int\int\int\left(\frac{\sigma_{\theta\theta}}{\omega}+\frac{\sigma_{\varphi\varphi}}{\omega}\right)d\tau \tag{5-66}$$

where

$$d\tau=\omega^2\sin\varphi\Delta\omega\Delta\varphi\Delta\theta$$

(ii) *Components of Nonparallel Forces in the φ-Direction.* In Fig. 5.9b

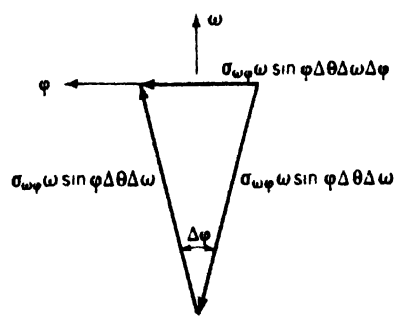


Figure 5.10

there is also a component of force $-\sigma_{\theta\theta}\omega \cos\varphi\Delta\omega\Delta\varphi\Delta\theta$ in the φ-direction due to the surface forces $\sigma_{\theta\theta}\omega\Delta\varphi\Delta\omega$. The other component of force in the φ-direction is $\sigma_{\omega\varphi}\omega \sin\varphi\Delta\theta\Delta\omega\Delta\varphi$. This is the result of the surface forces $\sigma_{\omega\varphi}\omega \sin\varphi\Delta\theta\Delta\omega$ (Fig. 5.10). The resultant of surface forces in the φ-direction due to nonparallel surface forces is

$$\iiint \left(\frac{\sigma_{\omega\varphi} - \sigma_{\theta\theta}\cot\varphi}{\omega}\right) d\tau \tag{5-67}$$

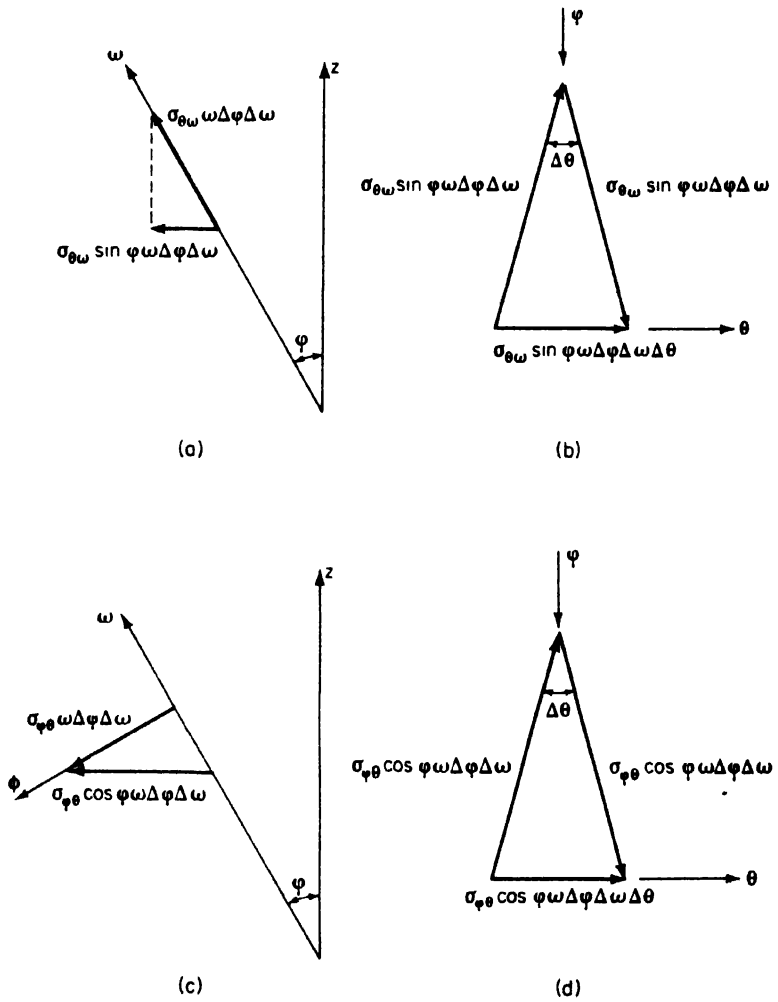


Figure 5.11

(iii) *Components of Nonparallel Forces in the θ-Direction.* The surface force $\sigma_{\theta\omega}\omega\Delta\varphi\Delta\omega$ produces a component of force $\sigma_{\theta\omega}\omega \sin\varphi\Delta\varphi\Delta\omega$ in the ω-φ plane. A pair of the forces $\sigma_{\theta\omega}\omega \sin\varphi\Delta\varphi\Delta\omega$ in turn gives a component of force $\sigma_{\theta\omega}\omega \sin\varphi\Delta\varphi\Delta\omega\Delta\theta$ in the θ-direction. This is illustrated in Figs. 5.11a and 5.11b. Again in the ω-φ plane, we have a surface force $\sigma_{\varphi\theta}\omega\Delta\varphi\Delta\omega$ which produces a component of force $\sigma_{\varphi\theta}\omega \cos\varphi\Delta\varphi\Delta\omega$. The force $\sigma_{\varphi\theta}\omega \cos\varphi\Delta\varphi\Delta\omega$ gives a component of force $\sigma_{\varphi\theta}\omega \cos\varphi\Delta\varphi\Delta\omega\Delta\theta$ in the θ-direction (Figs. 5.11c and 5.11d). The resultant of surface forces in the θ-direction due to nonparallel forces is

$$\iiint \left(\frac{\sigma_{\theta\omega}}{\omega} + \frac{\sigma_{\varphi\theta}\cot\varphi}{\omega}\right)d\tau \tag{5-68}$$

Hence the resultant of surface forces in spherical coordinates can be written

$$\begin{aligned}\mathbf{F}_s = {} & \mathbf{i}_\omega\left(\iint \mathbf{P}_\omega\cdot d\mathbf{A} - \iiint \frac{\sigma_{\varphi\varphi} + \sigma_{\theta\theta}}{\omega}d\tau\right) \\ & + \mathbf{i}_\varphi\left(\iint \mathbf{P}_\varphi\cdot d\mathbf{A} + \iiint \frac{\sigma_{\omega\varphi} - \sigma_{\theta\theta}\cot\varphi}{\omega}d\tau\right) \\ & + \mathbf{i}_\theta\left(\iint \mathbf{P}_\theta\cdot d\mathbf{A} + \iiint \frac{\sigma_{\theta\omega} + \sigma_{\varphi\theta}\cot\varphi}{\omega}d\tau\right)\end{aligned} \tag{5-69}$$

With the aid of Gauss's divergence theorem, Eq. (5-69) becomes

$$\begin{aligned}\mathbf{F}_s = {} & \mathbf{i}_\omega\iiint\left(\nabla\cdot\mathbf{P}_\omega - \frac{\sigma_{\varphi\varphi} + \sigma_{\theta\theta}}{\omega}\right)d\tau \\ & + \mathbf{i}_\varphi\iiint\left(\nabla\cdot\mathbf{P}_\varphi + \frac{\sigma_{\omega\varphi} - \sigma_{\theta\theta}\cot\varphi}{\omega}\right)d\tau \\ & + \mathbf{i}_\theta\iiint\left(\nabla\cdot\mathbf{P}_\theta + \frac{\sigma_{\theta\omega} + \sigma_{\varphi\theta}\cot\varphi}{\omega}\right)d\tau\end{aligned} \tag{5-70}$$

where

$$\begin{aligned}\mathbf{P}_\omega &= \mathbf{i}_\omega\sigma_{\omega\omega} + \mathbf{i}_\varphi\sigma_{\omega\varphi} + \mathbf{i}_\theta\sigma_{\theta\omega} \\ \mathbf{P}_\varphi &= \mathbf{i}_\omega\sigma_{\omega\varphi} + \mathbf{i}_\varphi\sigma_{\varphi\varphi} + \mathbf{i}_\theta\sigma_{\varphi\theta} \\ \mathbf{P}_\theta &= \mathbf{i}_\omega\sigma_{\theta\omega} + \mathbf{i}_\varphi\sigma_{\varphi\theta} + \mathbf{i}_\theta\sigma_{\theta\theta}\end{aligned} \tag{5-71}$$

If the resultant of the body forces is given by

$$\mathbf{F}_B = \iiint(\mathbf{i}_\omega X_\omega + \mathbf{i}_\varphi X_\varphi + \mathbf{i}_\theta X_\theta)d\tau \tag{5-72}$$

the equations of motion in spherical coordinates are obtained from Eqs. (5-69), (5-70), (5-71), and (5-72) as follows:

$$\begin{aligned}\rho\left(\frac{Dv_\omega}{Dt} - \frac{v_\varphi^2 + v_\theta^2}{\omega}\right) = {} & \rho X_\omega + \frac{1}{\omega\sin\varphi}\left[\frac{\sin\varphi}{\omega}\frac{\partial(\omega^2\sigma_{\omega\omega})}{\partial\omega}\right. \\ & \left. + \frac{\partial(\sigma_{\omega\varphi}\sin\varphi)}{\partial\varphi} + \frac{\partial\sigma_{\theta\omega}}{\partial\theta}\right] - \frac{\sigma_{\varphi\varphi} + \sigma_{\theta\theta}}{\omega}\end{aligned} \tag{5-73a}$$

$$\rho\left(\frac{Dv_\varphi}{Dt}+\frac{v_\omega v_\varphi}{\omega}-\frac{v_\theta^2\cot\varphi}{\omega}\right)=\rho X_\varphi+\frac{1}{\omega\sin\varphi}\left[\frac{\sin\varphi}{\omega}\frac{\partial(\omega^2\sigma_{\omega\varphi})}{\partial\omega}+\frac{\partial(\sigma_{\varphi\varphi}\sin\varphi)}{\partial\varphi}+\frac{\partial\sigma_{\varphi\theta}}{\partial\theta}\right]+\frac{\sigma_{\omega\varphi}}{\omega}-\frac{\sigma_{\theta\theta}\cot\varphi}{\omega} \tag{5-73b}$$

$$\rho\left(\frac{Dv_\theta}{Dt}+\frac{v_\theta v_\omega}{\omega}+\frac{v_\theta v_\varphi\cot\varphi}{\omega}\right)=\rho X_\theta+\frac{1}{\omega\sin\varphi}\left[\frac{\sin\varphi}{\omega}\frac{\partial(\omega^2\sigma_{\theta\omega})}{\partial\omega}+\frac{\partial(\sigma_{\varphi\theta}\sin\varphi)}{\partial\varphi}+\frac{\partial\sigma_{\theta\theta}}{\partial\theta}\right]+\frac{\sigma_{\theta\omega}}{\omega}+\frac{\sigma_{\varphi\theta}\cot\varphi}{\omega} \tag{5-73c}$$

where

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+v_\omega\frac{\partial}{\partial\omega}+\frac{v_\varphi}{\omega}\frac{\partial}{\partial\varphi}+\frac{v_\theta}{\omega\sin\varphi}\frac{\partial}{\partial\theta}$$

$$\nabla\cdot\mathbf{P}_\omega=\frac{1}{\omega^2}\frac{\partial(\omega^2\sigma_{\omega\omega})}{\partial\omega}+\frac{1}{\omega\sin\varphi}\frac{\partial(\sin\varphi\,\sigma_{\omega\varphi})}{\partial\varphi}+\frac{1}{\omega\sin\varphi}\frac{\partial\sigma_{\theta\omega}}{\partial\theta}$$

$$\nabla\cdot\mathbf{P}_\varphi=\frac{1}{\omega^2}\frac{\partial(\omega^2\sigma_{\omega\varphi})}{\partial\omega}+\frac{1}{\omega\sin\varphi}\frac{\partial(\sin\varphi\,\sigma_{\varphi\varphi})}{\partial\varphi}+\frac{1}{\omega\sin\varphi}\frac{\partial\sigma_{\varphi\theta}}{\partial\theta}$$

$$\nabla\cdot\mathbf{P}_\theta=\frac{1}{\omega^2}\frac{\partial(\omega^2\sigma_{\theta\omega})}{\partial\omega}+\frac{1}{\omega\sin\varphi}\frac{\partial(\sin\varphi\,\sigma_{\varphi\theta})}{\partial\varphi}+\frac{1}{\omega\sin\varphi}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}$$

The relations between the stresses and the rates of strain in spherical coordinates are

$$\begin{aligned}\sigma_{\omega\omega}&=-p+2\mu\varepsilon_{\omega\omega}-\tfrac{2}{3}\mu\nabla\cdot\mathbf{q}\\ \sigma_{\varphi\varphi}&=-p+2\mu\varepsilon_{\varphi\varphi}-\tfrac{2}{3}\mu\nabla\cdot\mathbf{q}\\ \sigma_{\theta\theta}&=-p+2\mu\varepsilon_{\theta\theta}-\tfrac{2}{3}\mu\nabla\cdot\mathbf{q}\end{aligned} \tag{5-74a}$$

and

$$\begin{aligned}\sigma_{\omega\varphi}&=\mu\gamma_{\omega\varphi}\\ \sigma_{\varphi\theta}&=\mu\gamma_{\varphi\theta}\\ \sigma_{\theta\omega}&=\mu\gamma_{\theta\omega}\end{aligned} \tag{5-74b}$$

The components of the rates of strain in terms of velocity gradients in spherical coordinates can be obtained from their relations in the general orthogonal coordinates (see Appendix A, § A.5). They are

$$\begin{aligned}\varepsilon_{\omega\omega}&=\frac{\partial v_\omega}{\partial\omega}\\ \varepsilon_{\varphi\varphi}&=\frac{1}{\omega}\frac{\partial v_\varphi}{\partial\varphi}+\frac{v_\omega}{\omega}\\ \varepsilon_{\theta\theta}&=\frac{1}{\omega\sin\varphi}\frac{\partial v_\theta}{\partial\theta}+\frac{v_\omega}{\omega}+\frac{v_\varphi\cot\varphi}{\omega}\\ \gamma_{\omega\varphi}&=\omega\frac{\partial}{\partial\omega}\left(\frac{v_\varphi}{\omega}\right)+\frac{1}{\omega}\frac{\partial v_\omega}{\partial\varphi}\end{aligned} \tag{5-75}$$

$$\gamma_{\varphi\theta} = \frac{\sin\varphi}{\omega}\frac{\partial}{\partial\varphi}\left(\frac{v_\theta}{\sin\varphi}\right) + \frac{1}{\omega\sin\varphi}\frac{\partial v_\varphi}{\partial\theta}$$

$$\gamma_{\theta\omega} = \frac{1}{\omega\sin\varphi}\frac{\partial v_\omega}{\partial\theta} + \omega\frac{\partial}{\partial\omega}\left(\frac{v_\theta}{\omega}\right)$$

The relations above can also be derived in the manner employed in the case of Cartesian coordinates. It can easily be seen that the derivation of $\gamma_{\omega\varphi}$ exactly corresponds to $\gamma_{r\theta}$ in cylindrical coordinates which is given in Eq. (5-51a).

If we introduce the relations of stress and rates of strain in Eqs. (5-74a) and (5-74b) and the rates of strain and velocity gradients relations of Eqs. (5-75) into Eqs. (5-73), the equations of motion of viscous compressible fluids in spherical coordinates are obtained:

$$\rho\left(\frac{Dv_\omega}{Dt} - \frac{v_\varphi^2 + v_\theta^2}{\omega}\right) = \rho X_\omega - \frac{\partial p}{\partial\omega} + \frac{\partial}{\partial\omega}\left[\mu\left(2\frac{\partial v_\omega}{\partial\omega} - \frac{2}{3}\nabla\cdot\mathbf{q}\right)\right]$$
$$+ \frac{1}{\omega}\frac{\partial}{\partial\varphi}\left\{\mu\left[\omega\frac{\partial}{\partial\omega}\left(\frac{v_\varphi}{\omega}\right) + \frac{1}{\omega}\frac{\partial v_\omega}{\partial\varphi}\right]\right\} + \frac{1}{\omega\sin\varphi}\frac{\partial}{\partial\theta}\left\{\mu\left[\frac{1}{\omega\sin\varphi}\frac{\partial v_\omega}{\partial\theta}\right.\right.$$
$$\left.\left. + \omega\frac{\partial}{\partial\omega}\left(\frac{v_\theta}{\omega}\right)\right]\right\} + \frac{\mu}{\omega}\left[4\frac{\partial v_\omega}{\partial\omega} - \frac{2}{\omega}\frac{\partial v_\varphi}{\partial\varphi} - \frac{4v_\omega}{\omega} - \frac{2}{\omega\sin\varphi}\frac{\partial v_\theta}{\partial\theta} - \frac{2v_\varphi\cot\varphi}{\omega}\right.$$
$$\left. + \omega\cot\varphi\frac{\partial}{\partial\omega}\left(\frac{v_\varphi}{\omega}\right) + \frac{\cot\varphi}{\omega}\frac{\partial v_\omega}{\partial\varphi}\right] \qquad (5\text{-}76a)$$

$$\rho\left(\frac{Dv_\varphi}{Dt} + \frac{v_\omega v_\varphi}{\omega} - \frac{v_\theta^2\cot\varphi}{\omega}\right) = \rho X_\varphi - \frac{1}{\omega}\frac{\partial p}{\partial\varphi} + \frac{1}{\omega}\frac{\partial}{\partial\varphi}\left[\mu\left(\frac{2}{\omega}\frac{\partial v_\varphi}{\partial\varphi} + \frac{2v_\omega}{\omega} - \frac{2}{3}\nabla\cdot\mathbf{q}\right)\right]$$
$$+ \frac{1}{\omega\sin\varphi}\frac{\partial}{\partial\theta}\left\{\mu\left[\frac{\sin\varphi}{\omega}\frac{\partial}{\partial\varphi}\left(\frac{v_\theta}{\sin\varphi}\right) + \frac{1}{\omega\sin\varphi}\frac{\partial v_\varphi}{\partial\theta}\right]\right\} + \frac{\partial}{\partial\omega}\left\{\mu\left[\omega\frac{\partial}{\partial\omega}\left(\frac{v_\varphi}{\omega}\right)\right.\right.$$
$$\left.\left. + \frac{1}{\omega}\frac{\partial v_\omega}{\partial\varphi}\right]\right\} + \frac{\mu}{\omega}\left\{2\left(\frac{1}{\omega}\frac{\partial v_\varphi}{\partial\varphi} - \frac{1}{\omega\sin\varphi}\frac{\partial v_\theta}{\partial\theta} - \frac{v_\varphi\cot\varphi}{\omega}\right)\cot\varphi\right.$$
$$\left. + 3\left[\omega\frac{\partial}{\partial\omega}\left(\frac{v_\varphi}{\omega} + \frac{1}{\omega}\frac{\partial v_\omega}{\partial\varphi}\right)\right]\right\} \qquad (5\text{-}76b)$$

$$\rho\left(\frac{Dv_\theta}{Dt} + \frac{v_\theta v_\omega}{\omega} + \frac{v_\varphi v_\theta}{\omega}\cot\varphi\right) = \rho X_\theta - \frac{1}{\omega\sin\varphi}\frac{\partial p}{\partial\theta} + \frac{1}{\omega\sin\varphi}\frac{\partial}{\partial\theta}\left[\mu\left(\frac{2}{\omega\sin\varphi}\frac{\partial v_\theta}{\partial\theta}\right.\right.$$
$$\left.\left. + \frac{2v_\omega}{\omega} + \frac{2v_\varphi\cot\varphi}{\omega} - \frac{2}{3}\nabla\cdot\mathbf{q}\right)\right] + \frac{\partial}{\partial\omega}\left\{\mu\left[\frac{1}{\omega\sin\varphi}\frac{\partial v_\omega}{\partial\theta} + \omega\frac{\partial}{\partial\omega}\left(\frac{v_\theta}{\omega}\right)\right]\right\}$$
$$+ \frac{1}{\omega}\frac{\partial}{\partial\varphi}\left\{\mu\left[\frac{\sin\varphi}{\omega}\frac{\partial}{\partial\varphi}\left(\frac{v_\theta}{\sin\varphi}\right) + \frac{1}{\omega\sin\varphi}\frac{\partial v_\varphi}{\partial\theta}\right]\right\} + \frac{\mu}{r}\left\{3\left[\frac{1}{\omega\sin\varphi}\frac{\partial v_\omega}{\partial\theta}\right.\right.$$
$$\left.\left. + \omega\frac{\partial}{\partial\omega}\left(\frac{v_\theta}{\omega}\right)\right] + 2\cot\varphi\left[\frac{\sin\varphi}{\omega}\frac{\partial}{\partial\varphi}\left(\frac{v_\theta}{\sin\varphi}\right) + \frac{1}{\omega\sin\varphi}\frac{\partial v_\varphi}{\partial\theta}\right]\right\} \qquad (5\text{-}76c)$$

where

$$\nabla\cdot\mathbf{q} = \frac{1}{\omega}\frac{\partial}{\partial\omega}(\omega^2 v_\omega) + \frac{1}{\omega\sin\varphi}\frac{\partial(v_\varphi\sin\varphi)}{\partial\varphi} + \frac{1}{\omega\sin\varphi}\frac{\partial v_\theta}{\partial\theta}$$

These general equations can be reduced to simpler forms in the following manner when certain physical properties are constant:

(i) *Viscous Compressible Fluid with Constant Viscosity.* If the viscosity of the fluid is constant, Eqs. (5-76) can be reduced to the following forms:

$$\rho\left(\frac{Dv_\omega}{Dt} - \frac{v_\varphi^2 + v_\theta^2}{\omega}\right) = \rho X_\omega - \frac{\partial p}{\partial \omega} + \mu\left\{\nabla^2 v_\omega - \frac{2v_\omega}{\omega^2} - \frac{2}{\omega^2}\frac{\partial v_\varphi}{\partial \varphi} - \frac{2v_\varphi \cot \varphi}{\omega^2}\right.$$
$$- \frac{2}{\omega^2 \sin \varphi}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{3}\left[\frac{\partial^2 v_\omega}{\partial \omega^2} + \frac{2}{\omega}\left(\frac{\partial v_\omega}{\partial \omega} - \frac{v_\omega}{\omega}\right) + \frac{1}{\omega}\frac{\partial^2 v_\varphi}{\partial \omega\, \partial \varphi}\right.$$
$$\left.\left. - \frac{1}{\omega^2}\left(\frac{\partial v_\varphi}{\partial \varphi} + v_\varphi \cot \varphi\right) + \frac{\cot \varphi}{\omega}\frac{\partial v_\varphi}{\partial \omega} + \frac{1}{\omega \sin \varphi}\left(\frac{\partial^2 v_\theta}{\partial \omega\, \partial \theta} - \frac{1}{\omega}\frac{\partial v_\theta}{\partial \theta}\right)\right]\right\} \tag{5-77a}$$

$$\rho\left(\frac{Dv_\varphi}{Dt} + \frac{v_\omega v_\varphi}{\omega} - \frac{v_\theta^2 \cot \varphi}{\omega^2}\right) = \rho X_\varphi - \frac{1}{\omega}\frac{\partial p}{\partial \varphi} + \mu\left\{\nabla^2 v_\varphi + \frac{2}{\omega^2}\frac{\partial v_\omega}{\partial \varphi} - \frac{v_\varphi}{\omega^2 \sin^2 \varphi}\right.$$
$$- \frac{2 \cos \varphi}{\omega^2 \sin^2 \varphi}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{3}\left[\frac{1}{\omega^2}\frac{\partial^2 v_\varphi}{\partial \varphi^2} + \frac{1}{\omega}\left(\frac{\partial^2 v_\omega}{\partial \varphi\, \partial \omega} + \frac{2}{\omega}\frac{\partial v_\omega}{\partial \varphi} + \frac{\cot \varphi}{\omega}\frac{\partial v_\varphi}{\partial \varphi}\right)\right.$$
$$\left.\left. + \frac{1}{\omega^2 \sin \varphi}\left(\frac{\partial^2 v_\theta}{\partial \varphi\, \partial \theta} - \cot \varphi \frac{\partial v_\theta}{\partial \theta} - v_\varphi\right)\right]\right\} \tag{5-77b}$$

$$\rho\left(\frac{Dv_\theta}{Dt} + \frac{v_\theta v_\omega}{\omega} + \frac{v_\varphi v_\theta \cot \varphi}{\omega}\right) = \rho X_\theta - \frac{1}{\omega \sin \varphi}\frac{\partial p}{\partial \theta} + \mu\left\{\nabla^2 v_\theta - \frac{v_\theta}{\omega^2 \sin^2 \theta}\right.$$
$$+ \frac{2}{\omega^2 \sin \varphi}\frac{\partial v_\omega}{\partial \theta} + \frac{2 \cos \varphi}{\omega^2 \sin^2 \varphi}\frac{\partial v_\varphi}{\partial \theta} + \frac{1}{3}\frac{1}{\omega \sin \varphi}\left[\frac{1}{\omega \sin \varphi}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\omega}{\partial \theta\, \partial \omega}\right.$$
$$\left.\left. + \frac{2}{\omega}\frac{\partial v_\omega}{\partial \theta} + \frac{1}{\omega}\frac{\partial^2 v_\varphi}{\partial \theta\, \partial \varphi} + \frac{\cos \varphi}{\omega \sin \varphi}\frac{\partial v_\varphi}{\partial \theta}\right]\right\} \tag{5-77c}$$

(ii) *Viscous Incompressible Fluid.* For an incompressible fluid where $\rho =$ constant, $\mu =$ constant, and $\nabla \cdot \mathbf{q} = 0$, Eqs. (5-76) reduce to

$$\left(\frac{Dv_\omega}{Dt} - \frac{v_\varphi^2 + v_\theta^2}{\omega}\right) = X_\omega - \frac{1}{\rho}\frac{\partial p}{\partial \omega} + \nu\left(\nabla^2 v_\omega - \frac{2v_\omega}{\omega^2} - \frac{2}{\omega^2}\frac{\partial v_\varphi}{\partial \varphi} - \frac{2v_\varphi \cot \varphi}{\omega^2}\right.$$
$$\left. - \frac{2}{\omega^2 \sin \varphi}\frac{\partial v_\theta}{\partial \theta}\right) \tag{5-78a}$$

$$\left(\frac{Dv_\varphi}{Dt} + \frac{v_\omega v_\varphi}{\omega} - \frac{v_\theta^2 \cot \varphi}{\omega}\right) = X_\varphi - \frac{1}{\rho\omega}\frac{\partial p}{\partial \varphi}$$
$$+ \nu\left(\nabla^2 v_\varphi + \frac{2}{\omega^2}\frac{\partial v_\omega}{\partial \varphi} - \frac{v_\varphi}{\omega^2 \sin^2 \varphi} - \frac{2 \cos \varphi}{\omega^2 \sin^2 \varphi}\frac{\partial v_\theta}{\partial \theta}\right) \tag{5-78b}$$

$$\left(\frac{Dv_\theta}{Dt} + \frac{v_\theta v_\omega}{\omega} + \frac{v_\varphi v_\theta \cot \varphi}{\omega}\right) = X_\theta - \frac{1}{\rho\omega \sin \varphi}\frac{\partial p}{\partial \theta}$$
$$+ \nu\left(\nabla^2 v_\theta - \frac{v_\theta}{\omega^2 \sin^2 \varphi} + \frac{2}{\omega^2 \sin \varphi}\frac{\partial v_\omega}{\partial \theta} + \frac{2 \cos \varphi}{\omega^2 \sin^2 \varphi}\frac{\partial v_\varphi}{\partial \theta}\right) \tag{5-78c}$$

where

$$\nabla^2 = \frac{1}{\omega^2}\frac{\partial}{\partial \omega}\left(\omega^2 \frac{\partial}{\partial \omega}\right) + \frac{1}{\omega^2 \sin \varphi}\frac{\partial}{\partial \varphi}\left(\sin \varphi \frac{\partial}{\partial \varphi}\right) + \frac{1}{\omega^2 \sin^2 \varphi}\frac{\partial^2}{\partial \theta^2}$$

(iii) *Nonviscous Fluid.* For a nonviscous fluid the equations of motion can be obtained from Eqs. (5-76) simply by letting $\mu = 0$, that is,

$$\rho\left(\frac{Dv_\omega}{Dt} - \frac{v_\varphi^2 + v_\theta^2}{\omega}\right) = \rho X_\omega - \frac{\partial p}{\partial \omega} \tag{5-79a}$$

$$\rho\left(\frac{Dv_\varphi}{Dt} + \frac{v_\omega v_\varphi}{\omega} - \frac{v_\theta^2 \cot \varphi}{\omega}\right) = \rho X_\varphi - \frac{1}{\omega}\frac{\partial p}{\partial \varphi} \tag{5-79b}$$

$$\rho\left(\frac{Dv_\theta}{Dt} + \frac{v_\theta v_\omega}{\omega} + \frac{v_\varphi v_\theta \cot \varphi}{\omega}\right) = \rho X_\theta - \frac{1}{\omega \sin \varphi}\frac{\partial p}{\partial \theta} \tag{5-79c}$$

c. The Energy Equation

In spherical coordinates, the energy equation [Eq. (5-32)] becomes

$$\frac{1}{\omega^2}\frac{\partial}{\partial \omega}\left(\omega^2 k \frac{\partial T}{\partial \omega}\right) + \frac{1}{\omega^2 \sin \varphi}\frac{\partial}{\partial \varphi}\left(k \frac{\partial T}{\partial \varphi}\sin \varphi\right) + \frac{1}{\omega^2 \sin^2 \varphi}\frac{\partial}{\partial \theta}\left(k \frac{\partial T}{\partial \theta}\right) + \Phi = \rho \frac{Dh}{Dt} - \frac{Dp}{Dt} \tag{5-80}$$

where

$$\nabla T = \mathbf{i}_\omega \frac{\partial T}{\partial \omega} + \mathbf{i}_\varphi \frac{1}{\omega}\frac{\partial T}{\partial \varphi} + \mathbf{i}_\theta \frac{1}{\omega \sin \varphi}\frac{\partial T}{\partial \theta}$$

$$\nabla \cdot (k \nabla T) = \frac{1}{\omega^2}\frac{\partial}{\partial \omega}\left(\omega^2 k \frac{\partial T}{\partial \omega}\right) + \frac{1}{\omega^2 \sin \varphi}\frac{\partial}{\partial \varphi}\left(k \frac{\partial T}{\partial \varphi}\sin \varphi\right) + \frac{1}{\omega^2 \sin^2 \varphi}\frac{\partial}{\partial \theta}\left(k \frac{\partial T}{\partial \theta}\right)$$

$$\Phi = 2\mu\left\{\left[\left(\frac{\partial v_\omega}{\partial \omega}\right)^2 + \left(\frac{1}{\omega}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_\omega}{\omega}\right)^2 + \left(\frac{1}{\omega \sin \varphi}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\omega}{\omega} + \frac{v_\varphi \cot \varphi}{\omega}\right)^2\right]\right.$$
$$+ \frac{1}{2}\left[\omega \frac{\partial}{\partial \omega}\left(\frac{v_\varphi}{\omega}\right) + \frac{1}{\omega}\frac{\partial v_\omega}{\partial \varphi}\right]^2 + \frac{1}{2}\left[\frac{\sin \varphi}{\omega}\frac{\partial}{\partial \varphi}\left(\frac{v_\theta}{\sin \varphi}\right) + \frac{1}{\omega \sin \varphi}\frac{\partial v_\theta}{\partial \theta}\right]^2$$
$$\left. + \frac{1}{2}\left[\frac{1}{\omega \sin \varphi}\frac{\partial v_\omega}{\partial \theta} + \omega \frac{\partial}{\partial \omega}\left(\frac{v_\theta}{\omega}\right)\right]^2\right\} - \frac{2}{3}\mu(\nabla \cdot \mathbf{q})^2$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_\omega \frac{\partial}{\partial \omega} + \frac{v_\varphi}{\omega}\frac{\partial}{\partial \varphi} + \frac{v_\theta}{\omega \sin \varphi}\frac{\partial}{\partial \theta}$$

Equations (5-34) and (5-35) may be used to obtain the energy equation in spherical coordinates for a viscous incompressible fluid and for a nonviscous fluid, respectively, viz.,

(i) *Viscous Incompressible Fluid* (ρ, μ, and k are constant).

$$k\left[\frac{1}{\omega^2}\frac{\partial}{\partial \omega}\left(\omega^2 \frac{\partial T}{\partial \omega}\right) + \frac{1}{\omega^2 \sin \varphi}\frac{\partial}{\partial \varphi}\left(\sin \varphi \frac{\partial T}{\partial \varphi}\right) + \frac{1}{\omega^2 \sin^2 \varphi}\frac{\partial^2 T}{\partial \theta^2}\right] + \Phi_{(i)} = \rho C_\tau \frac{DT}{Dt} \tag{5-81}$$

where

$$\Phi_{(1)} = 2\mu\left\{\left(\frac{\partial v_\omega}{\partial \omega}\right)^2 + \left(\frac{1}{\omega}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_\omega}{\omega}\right)^2 + \left(\frac{1}{\omega \sin\varphi}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\omega}{\omega} + \frac{v_\varphi \cos\varphi}{\omega}\right)^2 + \frac{1}{2}\left[\omega\frac{\partial}{\partial \omega}\left(\frac{v_\varphi}{\omega}\right) + \frac{1}{\omega}\frac{\partial v_\omega}{\partial \varphi}\right]^2 + \frac{1}{2}\left[\frac{\sin\varphi}{\omega}\frac{\partial}{\partial \varphi}\left(\frac{v_\theta}{\sin\varphi}\right) + \frac{1}{\omega \sin\varphi}\frac{\partial v_\theta}{\partial \theta}\right]^2 + \frac{1}{2}\left[\frac{1}{\omega \sin\varphi}\frac{\partial v_\omega}{\partial \theta} + \omega\frac{\partial}{\partial \omega}\left(\frac{v_\theta}{\omega}\right)\right]^2\right\}$$

(ii) *Nonviscous Fluid.*

$$\frac{1}{\omega^2}\frac{\partial}{\partial \omega}\left(\omega^2 k\frac{\partial T}{\partial \omega}\right) + \frac{1}{\omega^2 \sin\varphi}\frac{\partial}{\partial \varphi}\left(k \sin\varphi\frac{\partial T}{\partial \varphi}\right) + \frac{1}{\omega^2 \sin^2\varphi}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right) = \rho\frac{Dh}{Dt} - \frac{Dp}{Dt} \tag{5-82}$$

For an adiabatic flow, i. e., $dQ = 0$, Eq. (5-81) reduces to

$$\rho\frac{Dh}{Dt} - \frac{Dp}{Dt} = 0 \tag{5-83}$$

Illustrative Example 5.6. Consider a two-dimensional incompressible steady flow field with velocity components in spherical coordinates (ω, φ, θ) given by

$$v_\omega = c_1\left(1 - \frac{3}{2}\frac{\omega_0}{\omega} + \frac{1}{2}\frac{\omega_0^3}{\omega^3}\right)\cos\varphi \qquad v_\theta = 0$$

$$v_\varphi = -c_1\left(1 - \frac{3}{4}\frac{\omega_0}{\omega} - \frac{\omega_0^3}{4\omega^3}\right)\sin\varphi \qquad \omega \geq \omega_0 > 0 \tag{a}$$

where c_1 and ω_0 are arbitrary constants. Is the equation of continuity satisfied?

Solution. According to Eq. (5-61) we have, for $v_\theta = 0$,

$$\frac{1}{\omega}\frac{\partial}{\partial \omega}(\omega^2 v_\omega) + \frac{1}{\sin\varphi}\frac{\partial}{\partial \varphi}(v_\varphi \sin\varphi) = 0 \tag{b}$$

Differentiating,

$$\frac{1}{\omega}\left(\omega^2\frac{\partial v_\omega}{\partial \omega} + 2\omega v_\omega\right) + \frac{1}{\sin\varphi}\left(\sin\varphi\frac{\partial v_\varphi}{\partial \varphi} + v_\varphi \cos\varphi\right) = 0$$

Simplifying,

$$\frac{\partial v_\omega}{\partial \omega} + 2\frac{v_\omega}{\omega} + \frac{1}{\omega}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_\varphi}{\omega}\frac{\cos\varphi}{\sin\varphi} = 0 \tag{c}$$

The derivatives of Eq. (a) with respect to ω and φ give, respectively

$$\frac{\partial v_\omega}{\partial \omega} = c_1\left(0 + \frac{3}{2}\frac{\omega_0}{\omega^2} - \frac{3\omega_0^3}{2\omega^4}\right)\cos\varphi$$

$$\frac{\partial v_\varphi}{\partial \varphi} = -c_1\left(1 - \frac{3\omega_0}{4\omega} - \frac{\omega_0^3}{4\omega^3}\right)\cos\varphi \tag{d}$$

Combining Eqs. (d) and (c), we obtain

$$\frac{3c_1\omega_0}{2\omega^2}\cos\varphi - \frac{3c_1\omega_0^3}{2\omega^4}\cos\varphi + \frac{2c_1}{\omega}\cos\varphi - \frac{3c_1\omega_0^2}{\omega^2}\cos\varphi$$
$$+ \frac{c_1\omega_0^3}{\omega^4}\cos\varphi - \frac{c_1}{\omega}\cos\varphi + \frac{3c_1\omega_0}{4\omega^2}\cos\varphi$$
$$+ \frac{c_1\omega_0^3}{4\omega^4}\cos\varphi - \frac{c_1}{\omega}\cos\varphi + \frac{3c_1\omega_0}{4\omega^2}\cos\varphi + \frac{c_1\omega_0^3}{4\omega^4}\cos\varphi = 0$$

Hence, Eq. (a) is a solution of the equation of continuity [Eq. (5–61)].

Illustrative Example 5.7. Consider an inviscid, incompressible, steady flow with negligible body force whose velocity components are

$$v_\omega = U_\infty\left(1 - \frac{R^3}{\omega^3}\right)\cos\varphi \tag{a}$$

$$v_\varphi = -U_\infty\left(1 + \frac{R^3}{2\omega^3}\right)\sin\varphi \tag{b}$$

$$v_\theta = 0 \tag{c}$$

in spherical coordinates where R is a constant. Is the equation of motion satisfied?

Solution. According to Eqs. (5–79), we have

$$\rho\left[\frac{\partial v_\omega}{\partial t} + v_\omega\frac{\partial v_\omega}{\partial\omega} + \frac{v_\varphi}{\omega}\frac{\partial v_\omega}{\partial\varphi} + \frac{v_\theta}{\omega\sin\varphi}\frac{\partial v_\omega}{\partial\theta} - \left(\frac{v_\varphi^2 + v_\theta^2}{\omega}\right)\right] = \rho X_\omega - \frac{\partial p}{\partial\omega} \tag{d}$$

$$\rho\left[\frac{\partial v_\varphi}{\partial t} + v_\omega\frac{\partial v_\varphi}{\partial\omega} + \frac{v_\varphi}{\omega}\frac{\partial v_\varphi}{\partial\varphi} + \frac{v_\varphi}{\omega\sin\varphi}\frac{\partial v_\omega}{\partial\theta} + \frac{v_\omega v_\varphi}{\omega} - \frac{v_\theta^2\cot\varphi}{\omega}\right]$$
$$= \rho X_\varphi - \frac{1}{\omega}\frac{\partial p}{\partial\varphi} \tag{e}$$

$$\rho\left[\frac{\partial v_\theta}{\partial t} + v_\omega\frac{\partial v_\theta}{\partial\omega} + \frac{v_\varphi}{\omega}\frac{\partial v_\theta}{\partial\varphi} + \frac{v_\theta}{\omega\sin\varphi}\frac{\partial v_\theta}{\partial\theta} + \frac{v_\theta v_\omega}{\omega} + \frac{v_\theta v_\varphi\cot\varphi}{\omega}\right]$$
$$= \rho X_\theta - \frac{1}{\omega\sin\varphi}\frac{\partial p}{\partial\theta} \tag{f}$$

With the aid of Eqs. (a), (b), and (c), Eqs. (d), (e), and (f) reduce to

$$\rho\left(v_\omega\frac{\partial v_\omega}{\partial\omega} + \frac{v_\varphi}{\omega}\frac{\partial v_\omega}{\partial\varphi} - \frac{v_\varphi^2}{\omega}\right) = -\frac{\partial p}{\partial\omega} \tag{g}$$

$$\rho\left(v_\omega\frac{\partial v_\varphi}{\partial\omega} + \frac{v_\varphi}{\omega}\frac{\partial v_\varphi}{\partial\varphi} + \frac{v_\omega v_\varphi}{\omega}\right) = -\frac{1}{\omega}\frac{\partial p}{\partial\varphi} \tag{h}$$

$$0 = -\frac{1}{\omega\sin\theta}\frac{\partial p}{\partial\theta} \tag{i}$$

Equation (i) implies that $p = p\ (\omega, \varphi)$ only. From Eq. (a) we obtain

$$\frac{\partial v_\omega}{\partial \omega} = U_\infty\left[0 - \frac{(-3)R^3}{\omega^4}\right]\cos\varphi = \left(\frac{3U_\infty R^3}{\omega^4}\right)\cos\varphi \tag{j}$$

and

$$\frac{\partial v_\varphi}{\partial \varphi} = -U_\infty\left(1 - \frac{R^3}{\omega^3}\right)\sin\varphi \tag{k}$$

Combining Eqs. (a), (b), (g), (j), and (k),

$$\rho\left\{U_\infty\left(1 - \frac{R^3}{\omega^3}\right)\cos\varphi \left[\frac{3U_\infty R^3}{\omega^4}\cos\varphi\right] - \frac{U_\infty}{\omega}\left(1 - \frac{R^3}{2\omega^3}\right)\sin\varphi\cdot\left[-U_\infty\left(1 - \frac{R^3}{\omega^3}\right)\sin\varphi\right] - \frac{1}{\omega}\left[U_\infty^2\left(1 + \frac{R^3}{2\omega^3}\right)^2\sin^2\varphi\right]\right\} = -\frac{\partial p}{\partial \omega}$$

Simplifying,

$$\frac{3U_\infty^2 R^3}{\omega^4}\left(1 - \frac{R^3}{\omega^3}\right)\cos^2\varphi + \frac{U_\infty^2}{\omega}\left(1 - \frac{R^3}{2\omega^3} - \frac{R^6}{2\omega^6}\right)\sin^2\varphi - \frac{U_\infty^2}{\omega}\left(1 + \frac{R^3}{\omega^3} + \frac{R^6}{4\omega^6}\right)\sin^2\varphi = -\frac{1}{\rho}\frac{\partial p}{\partial \omega}$$

or

$$\frac{3U_\infty^2 R^3}{\omega^4}\left(1 - \frac{R^3}{\omega^3}\right)\cos^2\varphi + \frac{U_\infty^2}{\omega}\left(1 - \frac{R^3}{2\omega^3} - \frac{R^6}{2\omega^6} - 1 - \frac{R^3}{\omega^3} - \frac{R^6}{4\omega^6}\right)\sin^2\varphi = -\frac{1}{\rho}\frac{\partial p}{\partial \omega}$$

or

$$\frac{3U_\infty^2 R^3}{\omega^4}\left(1 - \frac{R^3}{\omega^3}\right)\cos^2\varphi - \frac{3U_\infty^2 R^3}{2\omega^4}\left(1 + \frac{R^3}{2\omega^3}\right)\sin^2\varphi = -\frac{1}{\rho}\frac{\partial p}{\partial \omega} \tag{l}$$

From Eq. (b) we obtain

$$\frac{\partial v_\varphi}{\partial \omega} = -U_\infty\left(0 - \frac{3R^3}{2\omega^4}\right)\sin\varphi = \frac{3U_\infty R^3}{2\omega^4}\sin\varphi \tag{m}$$

and

$$\frac{\partial v_\varphi}{\partial \varphi} = -U_\infty\left(1 + \frac{R^3}{2\omega^3}\right)\cos\varphi \tag{n}$$

Combining Eqs. (a), (b), (h), (m), and (n),

$$U_\infty\left(1 - \frac{R^3}{\omega^3}\right)\cos\varphi\left(\frac{3U_\infty R^3}{2\omega^4}\right)\sin\varphi + \frac{1}{\omega}\left[-U_\infty\left(1 + \frac{R^3}{2\omega^3}\right)\sin\varphi\right]\left[-U_\infty\left(1 + \frac{R^3}{2\omega^3}\right)\cos\varphi\right] + \frac{1}{\omega}\left[U_\infty\left(1 - \frac{R^3}{\omega^3}\right)\cos\varphi\right]\left[-U_\infty\left(1 + \frac{R^3}{2\omega^3}\right)\sin\varphi\right] = -\frac{1}{\omega\rho}\frac{\partial p}{\partial \varphi} \tag{o}$$

Simplifying,

$$\frac{3U_\infty^2 R^3}{2\omega^4}\left(1-\frac{R^3}{\omega^3}\right)\sin\varphi\cos\varphi+\frac{U_\infty^2}{\omega}\left(1+\frac{R^3}{2\omega^3}\right)\sin\varphi\cos\varphi$$
$$-\frac{U_\infty^2}{\omega}\left(1-\frac{R^3}{2\omega^3}-\frac{R^6}{2\omega^6}\right)\sin\varphi\cos\varphi=-\frac{1}{\omega\rho}\frac{\partial p}{\partial\varphi}$$

or

$$\frac{3U_\infty^2 R^3}{2\omega^4}\left(1-\frac{R^3}{\omega^3}\right)\sin\varphi\cos\varphi$$
$$+\frac{U_\infty^2}{\omega}\left(1+\frac{R^3}{\omega^3}+\frac{R^6}{4\omega^6}-1+\frac{R^3}{2\omega^3}+\frac{R^6}{2\omega^6}\right)\sin\varphi\cos\varphi=-\frac{1}{\omega\rho}\frac{\partial p}{\partial\varphi}$$

or

$$\frac{3U_\infty^2 R^3}{2\omega^4}\left(1-\frac{R^3}{\omega^3}\right)\sin\varphi\cos\varphi$$
$$+\frac{3U_\infty^2 R^3}{2\omega^4}\left(1+\frac{R^3}{2\omega^3}\right)\sin\varphi\cos\varphi=-\frac{1}{\omega\rho}\frac{\partial p}{\partial\varphi}\qquad\text{(p)}$$

Differentiating Eq. (l) with respect to φ, we have

$$\frac{3U_\infty^2 R^3}{\omega^4}\left(1-\frac{R^3}{\omega^3}\right)(2)\cos\varphi(-\sin\varphi)$$
$$-\frac{3U_\infty^2 R^3}{2\omega^4}\left(1+\frac{R^3}{2\omega^3}\right)(2)\sin\varphi\cos\varphi=-\frac{1}{\rho}\frac{\partial^2 p}{\partial\omega\,\partial\varphi}$$

Simplifying,

$$-\frac{6U_\infty^2 R^3}{\omega^4}\left(1-\frac{R^3}{\omega^3}\right)\sin\varphi\cos\varphi$$
$$-\frac{3U_\infty^2 R^3}{\omega^4}\left(1+\frac{R^3}{2\omega^3}\right)\sin\varphi\cos\varphi=-\frac{1}{\rho}\frac{\partial^2 p}{\partial\omega\,\partial\varphi}$$

or

$$\left(-\frac{9U_\infty^2 R^3}{\omega^4}+\frac{9}{2}\frac{U_\infty^2 R^6}{\omega^7}\right)\sin\varphi\cos\varphi=-\frac{1}{\rho}\frac{\partial^2 p}{\partial\omega\,\partial\varphi}\qquad\text{(q)}$$

Differentiating Eq. (p) with respect to ω, we have

$$\frac{3}{2}U_\infty^2 R^3\left[\frac{-3}{\omega^4}-\frac{(-6)R^3}{\omega^7}\right]\sin\varphi\cos\varphi$$
$$+\frac{3}{2}U_\infty^2 R^3\left[\frac{-3}{\omega^4}+\frac{(-6)R^3}{2\omega^7}\right]\sin\varphi\cos\varphi=-\frac{1}{\rho}\frac{\partial^2 p}{\partial\omega\,\partial\varphi}$$

Simplifying,

$$-\frac{9}{2}\frac{U_\infty^2 R^3}{\omega^4}\left(1-\frac{2R^3}{\omega^3}\right)\sin\varphi\cos\varphi-\frac{9}{2}\frac{U_\infty^2 R^3}{\omega^4}\left(1+\frac{R^3}{\omega^3}\right)\sin\varphi\cos\varphi=$$

$$-\frac{1}{\rho}\frac{\partial^2 p}{\partial\omega\,\partial\varphi}$$

or

$$\left(-\frac{9U_\infty^2 R^3}{\omega^4}+\frac{9U_\infty^2 R^6}{2\omega^7}\right)\sin\varphi\cos\varphi=-\frac{1}{\rho}\frac{\partial^2 p}{\partial\omega\,\partial\varphi} \qquad \text{(r)}$$

We note that Eqs. (q) and (r) are identical. Hence, the equation of motion is satisfied.

NOMENCLATURE FOR CHAPTER FIVE

$\mathbf{A}$	vectoral area, Eq. (5-12b)
a_r, a_θ, a_z	acceleration components in cylindrical coordinates, Eqs. (5-42), (5-63), and Fig. 5.4
$a_\omega, a_\varphi, a_\theta$	acceleration components in spherical coordinates, Eq. (5-64) and Fig. 5.7
C_p	specific heat at constant pressure, Eq. (5-33)
C_T	specific heat at constant volume, Eq. (5-34)
E	internal energy per unit mass of gas, Eq. (5-22)
$\mathbf{f}$	elementary surface force, Eq. (5-12a) and Fig. 5.2
$\mathbf{F}$	resultant force, Eqs. (5-10), (5-11), (5-12), (5-43), (5-44), (5-47), (5-65), (5-69), and (5-72)
h	enthalpy per unit mass, Eq. (5-26)
k	thermal conductivity, Eq. (5-27)
p	pressure, Eq. (5-17)
$\mathbf{P}_x, \mathbf{P}_y, \mathbf{P}_z$	resultant of surface forces in Cartesian coordinates, Eq. (5-12b)
$\mathbf{P}_r, \mathbf{P}_\theta, \mathbf{P}_z$	resultant of surface forces in cylindrical coordinates, Eq. (5-46)
$\mathbf{P}_\omega, \mathbf{P}_\varphi, \mathbf{P}_\theta$	resultant of surface forces in spherical coordinates, Eq. (5-71)
$\mathbf{q}$	velocity vector, Eq. (5-4)
Q	heat added per unit mass of gas, Eq. (5-22)
r, θ, z	cylindrical coordinates, Fig. 5.3
R	gas constant, Eq. (5-37)
T	temperature, Eq. (5-27)
u, v, w	velocity components in Cartesian coordinates, Fig. 3.5
v_r, v_θ, v_z	velocity components in cylindrical coordinates, Fig. 5.4
$v_\omega, v_\varphi, v_\theta$	velocity components in spherical coordinates, Fig. 5.7a
x, y, z	Cartesian coordinates, Fig. 5.2
$\mathbf{X}$	body force per unit mass, Eqs. (5-11), (5-47), and (5-72)
$\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$	rate of shearing strain in Cartesian coordinates, Eq. (5-21)
$\gamma_{r\theta}, \gamma_{\theta z}, \gamma_{zr}$	rate of shearing strain in cylindrical coordinates, Eq. (5-51)
$\gamma_{\omega\varphi}, \gamma_{\varphi\theta}, \gamma_{\theta\omega}$	rate of shearing strain in spherical coordinates, Eq. (5-75)
$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$	rate of linear strain in Cartesian coordinates, Eq. (5-21)
$\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}$	rate of linear strain in cylindrical coordinates, Eq. (5-51)
$\varepsilon_{\omega\omega}, \varepsilon_{\varphi\varphi}, \varepsilon_{\theta\theta}$	rate of linear strain in spherical coordinates, Eq. (5-75)
μ	coefficient of viscosity, Eq. (5-17)

ν	kinematic viscosity, Eq. (5-19)
ρ	density, Eq. (5-1)
$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$	normal stresses in Cartesian coordinates, Eq. (5-15)
$\sigma_{xy}, \sigma_{yz}, \sigma_{zx}$	shearing stresses in Cartesian coordinates, Eq. (5-15)
$\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}$	normal stresses in cylindrical coordinates, Eq. (5-49) and Fig. 5.5
$\sigma_{r\theta}, \sigma_{\theta z}, \sigma_{zr}$	shearing stresses in cylindrical coordinates, Eq. (5-50) and Fig. 5.5
$\sigma_{\omega\omega}, \sigma_{\varphi\varphi}, \sigma_{\theta\theta}$	normal stresses in spherical coordinates, Eq. (5-74a) and Figs. 5.8 and 5.9
$\sigma_{\omega\varphi}, \sigma_{\varphi\theta}, \sigma_{\theta\omega}$	shearing stresses in spherical coordinates, Eq. (5-74b) and Figs. 5.8—5.11
τ	volume, Eq. (5-1)
Φ	dissipation function per unit volume, Eqs. (5-23), (5-56), and (5-80)
$\Phi_{(i)}$	dissipation function per unit volume for incompressible fluids, Eq. (5-34)
ω, φ, θ	spherical coordinates, Fig. 5.7a
∇^2	Laplace's operator, Eq. (5-53c) for cylindrical coordinates and Eq. (5-78c) for spherical coordinates
D/Dt	Euler's derivative, Eq. (5-56) for cylindrical coordinates and Eq. (5-80) for spherical coordinates

PROBLEMS

5.1. Does the one-dimensional incompressible flow given by

$$u(y) = Ay^2 + By + C$$

$$v = w = 0$$

where A, B, and C are arbitrary constants, satisfy the equation of continuity?

5.2. Does the two-dimensional incompressible flow given by

$$u(x, y) = c_1\left[\frac{x^2 + y^2 - c_2^2}{(x^2 + y^2 - c_2^2)^2 + 4c_2^2y^2}\right]$$

$$v(x, y) = 2c_1\left[\frac{xy}{(x^2 + y^2 - c_2^2) + 4c_2^2y^2}\right]$$

$$w = 0$$

where c_1 and c_2 are arbitrary constants, satisfy the continuity equation?

5.3. Does the three-dimensional incompressible flow given by

$$u(x, y, z) = \frac{kx}{(x^2 + y^2 + z^2)^{3/2}}$$

$$v(x, y, z) = \frac{ky}{(x^2 + y^2 + z^2)^{3/2}}$$

$$w(x, y, z) = \frac{kz}{(x^2 + y^2 + z^2)^{3/2}}$$

satisfy the equation of continuity? k is an arbitrary constant.

5.4. Show that the velocity field

$$u(y) = -\frac{h^2}{8\mu}\left(\frac{dp}{dx}\right)\left[1 - 4\left(\frac{y}{h}\right)^2\right] \qquad v = w = 0$$

satisfies the equation of motion for the two-dimensional steady flow of a viscous incompressible fluid with constant viscosity and constant pressure gradient.

5.5. Show that the velocity field

$$u(x, y) = B\frac{(x^2 - y^2)}{(x^2 + y^2)^2}, \qquad v(x, y) = \frac{2Bxy}{(x^2 + y^2)^2}, \qquad w = 0$$

satisfies the equation of motion for an inviscid incompressible flow. Determine the pressure associated with this velocity field. B is a constant.

5.6. Show that the velocity field

$$u(\eta) = U\left[1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta_1^2} d\eta_1\right]$$

satisfies the equation of motion

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}$$

where $\eta = y/2\sqrt{\nu t}$ and U and ν are constants.

5.7. Does the one-dimensional incompressible flow given by

$$v_r(r) = \frac{c_1}{r} \qquad v_\theta = v_z = 0$$

where $r > 0$ and c_1 is an arbitrary constant, satisfy the continuity equation?

5.8. Does the two-dimensional incompressible flow given by

$$v_r(r, \theta) = c_1\left(\frac{1}{r^2} - 1\right)\cos\theta$$

$$v_\theta(r, \theta) = c_1\left(\frac{1}{r^2} + 1\right)\sin\theta \qquad (r > 0)$$

where c_1 is an arbitrary nonzero constant, satisfy the continuity equation?

5.9. Does the three-dimensional incompressible flow given by

$$v_r(r, \theta, z) = \frac{1}{4\pi}\frac{(3r^2z - 2z^3 - r^3)}{(z^2 + r^2)^{7/2}}\cos 2\theta$$

$$v_\theta(r, \theta, z) = \frac{1}{2\pi}\frac{(z^2 - r^2)}{(z^2 + r^2)^{5/2}}\sin 2\theta$$

$$v_z(r, \theta, z) = \frac{5}{4\pi}\frac{zr^2}{(z^2 + r^2)^{5/2}}\cos 2\theta$$

satisfy the equation of continuity?

5.10. Show that the velocity field

$$v_r = 0, \qquad v_\theta = c_1 r + \frac{c_2}{r}, \qquad v_z = 0$$

satisfies the equation of motion

$$\frac{d^2 v_\theta}{dr^2} + \frac{d}{dr}\left(\frac{v_\theta}{r}\right) = 0$$

where c_1 and c_2 are constants.

5.11. Determine the pressure associated with the velocity field given in Prob. 5.10 by solving the equation

$$\rho \frac{v_\theta^2}{r} = \frac{dp}{dr}$$

5.12. The velocity components

$$v_r(r, \theta) = -U_\infty\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta(r, \theta) = U_\infty\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

satisfy the equation of motion for a two-dimensional inviscid incompressible flow. Find the pressure associated with this velocity field. U_∞ and a are constants.

5.13. Does the one-dimensional incompressible flow given by

$$v_\omega = \frac{c_1}{\omega^2}, \qquad v_\theta = v_\varphi = 0$$

where c_1 is an arbitrary constant and $\omega > 0$, satisfy the equation of continuity?

5.14. Does the two-dimensional incompressible flow given by

$$v_\omega = \frac{c_1}{\omega^2} + c_2 \cos\varphi, \qquad v_\varphi = -c_2 \sin\varphi, \qquad v_\theta = 0$$

where c_1 and c_2 are arbitrary constants and $\omega > 0$, satisfy the equation of continuity?

5.15. Does the three-dimensional incompressible flow given by

$$v_\omega = 100\left(1 - \frac{75}{\omega} + \frac{62{,}500}{\omega^3}\right)\cos\varphi$$

$$v_\varphi = -100\left(1 - \frac{37.5}{\omega} + \frac{31{,}250}{\omega^3}\right)\sin\varphi$$

$$v_\theta = 100(\varphi\omega)$$

satisfy the equation of continuity?

5.16. Given the velocity field of illustrative example 5.7, show that as $\omega \to \infty$, the resultant velocity is equal to U_∞.

5.17. Show that at any point on the sphere, $\omega = R$, the velocity is given by

$$v_\infty = 0, \qquad v_\varphi = -\frac{3}{2}U_\infty \sin\varphi$$

5.18. Show that the pressure distribution on the sphere is given by

$$P = P_\infty + \tfrac{1}{2}\rho U_\infty^2(1 - \tfrac{9}{4}\sin^2\varphi)$$

5.19. The velocity and temperature distributions for plane Poiseuille flow, satisfying the boundary conditions $T = T_0$ for $y = \pm h/2$, are given by

$$u = U_{\max}\left[1 - \left(\frac{2y}{h}\right)^2\right]$$

$$T(y) = T_0 + \mu\frac{U_{\max}^2}{3k}\left[1 - \left(\frac{2y}{h}\right)^4\right]$$

Show that the equation of energy [Eq. (5-34)] is satisfied.

5.20. Calculate the universal gas constant $\mathscr{R}$ or the gas constant for air in the following units:

(a) joules/°K moles (gm)
(b) calories/°K mole (gm)
(c) gm-m/°K moles (gm)
(d) gm-m/°K gm
(e) m^2/sec^2 °K
(f) ft^2/sec^2 °R

6

ONE-DIMENSIONAL, INVISCID INCOMPRESSIBLE FLOW

In the last chapter the fundamental differential equations of the viscous compressible fluid have been derived and discussed. Since these equations are nonlinear and there is no known general method for solving them at the present time, solutions of problems in fluid mechanics can only be found when the necessary mathematical simplifications can be made while still maintaining a physically realistic situation. Fortunately, however, a large amount of useful information can be obtained from the solutions of these equations which describe simple configurations of flow patterns.

In Chap. Three the motions in a fluid field have been discussed without regard to the forces producing them. In the present chapter the study of the dynamics of the frictionless motion of a one-dimensional incompressible fluid is considered. The two governing equations in this case, the continuity equation and the equation of motion, are obtained from the general fundamental equations given in Chap. Five by excluding those terms unrelated to the present problem. Although the material presented in this chapter deals with the most elementary phase of fluid dynamics, it provides results which are applicable to many important and practical problems. The study of the dynamics

of frictionless motion in a multidimensional incompressible fluid will be presented in the next chapter.

In § 6.1 the equation of continuity is derived for an incompressible fluid in one dimension by considering the conservation of mass of fluid flowing within a stream tube. The stream tube concept is introduced to clarify the derivation. It is also demonstrated that the same equation can be obtained by direct integration of the general continuity equation.

In § 6.2 the equation of motion is reduced from the general equation of motion and verified by a flow-element analysis. Conservation of momentum both along the flow path and normal to the flow path are considered.

In § 6.3 the Bernoulli equation is derived for a steady, inviscid, incompressible fluid.

In § 6.4 the application of the Bernoulli equation to practical flow problems is demonstrated. Among the examples discussed are the flow from a tank through a small orifice and the trajectory of a free jet.

In § 6.5 several methods of flow measurement are discussed. In addition, various instruments employed in flow measurement of an incompressible fluid are described.

In § 6.6 the momentum theorem is derived from the equations of continuity and motion. The usefulness of this theorem in the solution of certain flow problems is discussed.

In § 6.7 the momentum theorem is applied to several fluid flow problems. Included in these examples are discussions of the force exerted by a fluid jet on a plate and an explanation of the impulse turbine.

6.1. *The Equation of Continuity—Stream Tube Flow*

In § 3.3 a streamline was defined as the instantaneous locus of a line in the flow field to which the velocity vector is tangent. This method of description now becomes a convenient means of determining the one-dimensional statement of conservation of mass. In Fig. 6.1 a series of streamlines which form a closed conduit is shown. Since the streamlines cannot be crossed by the fluid, they act as a solid boundary with the fluid flowing within them. This is called a stream tube. The cross-sectional areas of this tube are small in comparison with its length so the velocity can be assumed constant across any sectional area, A_1. For an unsteady flow the stream tube changes shape with each succeeding instant of time. When the stream tube maintains the same shape for all time the flow is steady.

Consider a stream tube with two arbitrary cross sections, A_1 and A_2,

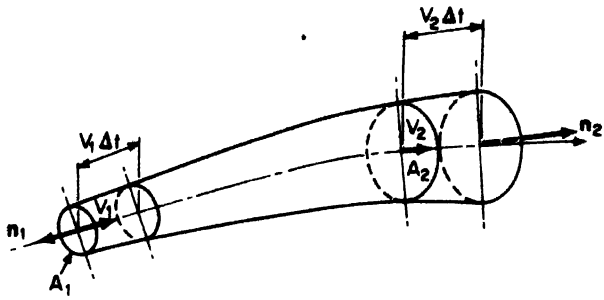


Figure 6.1

with the mass of fluid entering the tube through A_1 and leaving through A_2. In an increment of time Δt, the fluid particles which entered at A_1 have moved an infinitesimal distance $V_1\Delta t$, and those that were at A_2 have moved an amount $V_2\Delta t$. Since, by definition, no fluid can flow across the stream tube, the mass of fluid entering the stream tube at A_1 in time Δt must be exactly equal to the mass of fluid leaving at A_2 during the same interval of time. Mathematically, this statement can be expressed in the form

$$\rho V_1 A_1 \Delta t = \rho V_2 A_2 \Delta t$$

or

$$Q = V_1 A_1 = V_2 A_2 = VA = \text{Constant} \tag{6-1}$$

Equation (6-1) is called the equation of continuity. The constant Q represents the volume of fluid which passes through each cross section of the stream tube per unit time. It is called the volumetric flow rate and has dimensions of ft^3/sec.

Equation (6-1) can also be derived directly from the general continuity equation given in § 5.1. From Eq. (5-6) we can write the continuity equation as

$$\iiint \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q})\right] d\tau = 0 \tag{5-6}$$

However, in the present case the density is constant so that Eq. (5-6) reduces to

$$\rho \iiint (\nabla \cdot \mathbf{q})\, d\tau = 0 \tag{6-2a}$$

Applying Gauss's theorem to Eq. (6-2a) we have

$$\rho \iint \mathbf{q} \cdot d\mathbf{A} = 0 \tag{6-2b}$$

Equation (6–2b) represents the mass flowing per unit time through the surfaces which bound the volume τ. But for flow through a stream tube there can be no flow across any streamline. Therefore, the only areas involved in Eq. (6–2b) are the cross-sectional areas of the stream tube. Thus Eq. (6–2b) can be rewritten as

$$\rho \int\int \mathbf{q} \cdot d\mathbf{A} = \int\int \mathbf{q}_1 \cdot \mathbf{n}_1 \, dA_1 + \int\int \mathbf{q}_2 \cdot \mathbf{n}_2 \, dA_2 \qquad (6\text{–}2c)$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are the outwardly directed normal unit vectors to A_1 and A_2, respectively. From Fig. 6.1 it can be seen that the direction of $\mathbf{n}_2$ is positive and opposite to that of $\mathbf{n}_1$. Therefore, remembering that $\mathbf{q}$ is uniform over the cross section and perpendicular to it, we may integrate Eq. (6–2c) as follows:

$$\int\int \mathbf{q} \cdot d\mathbf{A} = \int\int_{A_2} q \, dA - \int\int_{A_1} q \, dA = V_2 A_2 - V_1 A_1 = 0 \qquad (6\text{–}3a)$$

or

$$Q = V_2 A_2 = V_1 A_1 = VA = \text{Constant} \qquad (6\text{–}3b)$$

Equation (6–3b) is identical to Eq. (6–1). For a stream tube with constant cross-sectional area, the continuity equation reduces to

$$V = \text{Constant} \qquad (6\text{–}3c)$$

It should be emphasized that Eqs. (6–3) are valid only for one-dimensional inviscid incompressible flow. They are particularly valuable, however, for determining the local velocities of stream tube type flows, such as nozzles and wind tunnels (low speed).

Illustrative Example 6.1. In Fig. 6.A is shown water flowing through a circular pipe (stream tube); it enters at A and leaves at C and D. If the velocity at B is 2 ft/sec, and the velocity at C is 9 ft/sec, calculate the velocities at A and D, and the volumetric flow rate. Friction will be ignored in this chapter.

Solution. According to the equation of continuity [Eq. (6–1)] we have

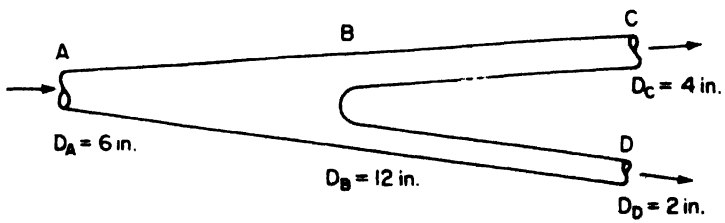


Figure 6.A

$$V_A \frac{\pi D_A^2}{4} = V_B \frac{\pi D_B^2}{4}$$

from which

$$V_A = 2\left(\frac{12}{6}\right)^2 = 8 \text{ ft/sec}$$

Similarly,

$$V_B \frac{\pi D_B^2}{4} = V_C \frac{\pi D_C^2}{4} + V_D \frac{\pi D_D^2}{4}$$

$$V_D = 2\left(\frac{12}{2}\right)^2 - 9\left(\frac{4}{2}\right)^2 = 72 - 36 = 36 \text{ ft/sec}$$

The volumetric flow rate is

$$\begin{aligned} Q = V_A \frac{\pi D_A^2}{4} &= V_C \frac{\pi D_C^2}{4} + V_D \frac{\pi D_D^2}{4} \\ &= 8\left(\frac{\pi \times 36}{4 \times 144}\right) = 9\left(\frac{\pi \times 16}{4 \times 144}\right) + 36\left(\frac{\pi \times 4}{4 \times 144}\right) \\ &= 1.571 \text{ ft}^3/\text{sec} \end{aligned}$$

6.2. *Equation of Motion—Euler's Equation*

The general vector equation defining the conservation of momentum for an inviscid fluid (Euler's equation) is given by Eq. (5–20) as

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla)\mathbf{q} = \mathbf{X} - \frac{\nabla p}{\rho} \tag{5–20}$$

For a one-dimensional flow along an arbitrary path, s, Eq. (5–20) reduces to

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s} = X_s - \frac{1}{\rho}\frac{\partial p}{\partial s} \tag{6–4}$$

Equation (6–4), the equation of motion of an inviscid fluid along a path s, is valid for both compressible and incompressible flows. It can also be derived by analysis of a stream tube element.

Let us consider an elementary section of a stream tube shown in Fig. 6.2 a. The mass of the stream tube element of length ds is $\rho\, dA\, ds$ and the acceleration of the fluid flowing along ds is DV/Dt. The forces acting on the element are of two types, the body force and the force exerted by the hydrostatic pressure on the ends of the tube. The total force along the path ds with tangential unit vector s is

$$\sum F = X_s \rho\, dA\, ds + p\, dA - \left(p + \frac{\partial p}{\partial s} ds\right) dA$$

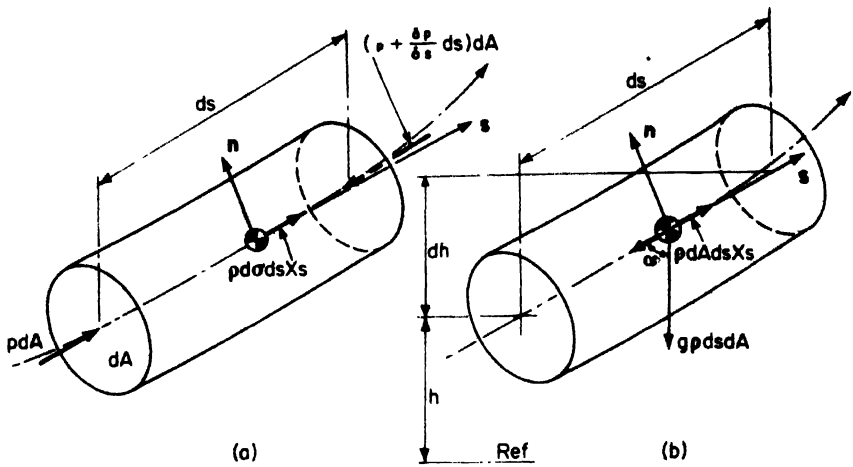


Figure 6.2

Thus, the equation of motion representing the conservation of momentum along the path, s, is

$$\frac{DV}{Dt}(\rho\, dA\, ds) = X_s \rho\, dA\, ds - \frac{\partial p}{\partial s} dA\, ds$$

or

$$\frac{DV}{Dt} = X_s - \frac{1}{\rho}\frac{\partial p}{\partial s} \tag{6-5a}$$

Now, by substitution from Eq. (3-17) we may write Eq. (6-5a) as

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s} = X_s - \frac{1}{\rho}\frac{\partial p}{\partial s} \tag{6-5b}$$

Equation (6-5b) is identical to Eq. (6-4). A useful form of Eq. (6-5b) is obtained by assuming that the body force per unit mass, X_s, is a gravitational force per unit mass. From Fig. 6.2b it is seen that

$$\rho\, dA\, ds\, X_s = -\rho\, dA\, ds\, g \cos\alpha$$

or

$$X_s = -g\cos\alpha$$

where

$$\cos\alpha = \frac{\partial h}{\partial s}$$

Substitution of the above expression for X_s into Eq. (6-5b) yields

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s} = -g\cos\alpha - \frac{1}{\rho}\frac{\partial p}{\partial s} \tag{6-5c}$$

For steady flow Eq. (6-5c) becomes

$$V\frac{\partial V}{\partial s} = -g\cos\alpha - \frac{1}{\rho}\frac{\partial p}{\partial s}$$

or

$$\frac{1}{2}\frac{\partial(V^2)}{\partial s} = -g\frac{\partial h}{\partial s} - \frac{1}{\rho}\frac{\partial p}{\partial s} \qquad (6\text{-}5d)$$

The equation of motion relative to the axis normal to the streamline can also be written from Eq. (5-20) by taking the component of the equation along the normal unit vector, **n** (Fig. 6.3a). The result is

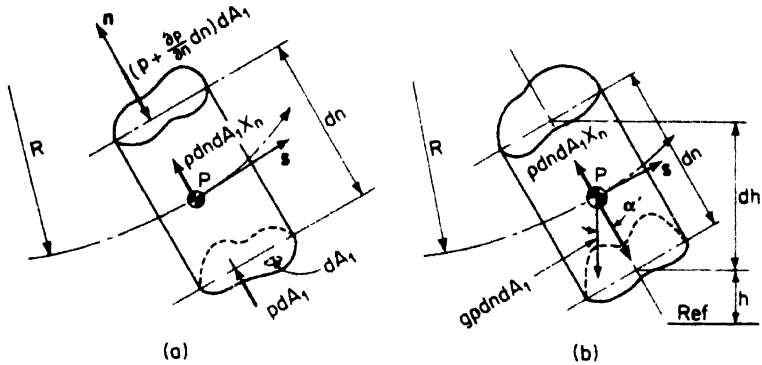


Figure 6.3

$$\frac{DV_n}{Dt} = X_n - \frac{1}{\rho}\frac{\partial p}{\partial n} \qquad (6\text{-}6)$$

In order to clarify the meaning of Eq. (6-6), let us consider the conservation of momentum of a fluid element along an axis normal to the direction of flow. In Fig. 6.3a the element of length is dn, and the element of the mass of the fluid, $\rho\, dn\, dA_1$, is acted upon by the body force per unit mass, X_n, and the pressure force exerted on the ends of the element. The centripetal acceleration of the fluid normal to the path of flow, s, which results from a path curvature, is V^2/R. Denoting the radius of curvature at P by R, the equation of motion along the normal **n** is

$$\rho\, dn\, dA_1 \frac{V^2}{R} = \rho\, dn\, dA_1 X_n + p\, dA_1 - \left(p + \frac{\partial p}{\partial n}dn\right) dA_1$$

or

$$\frac{V^2}{R} = X_n - \frac{1}{\rho}\frac{\partial p}{\partial n} \qquad (6\text{-}7a)$$

Now if the body force per unit mass X_n is a gravitational force (see Fig. 6.3b), Eq. (6–7a) becomes

$$\frac{V^2}{R} = -g\cos\alpha' - \frac{1}{\rho}\frac{\partial p}{\partial n} \tag{6–7b}$$

where

$$\cos\alpha' = \frac{\partial h}{\partial n}$$

It is interesting to note from Eq. (6–7b) that in the absence of body forces the pressure is constant across a straight streamline, $R = \infty$.

Equations (6–5) and (6–7) are Euler's equations of motion for one-dimensional flow which are used in the calculation of pressure-velocity relationships for an inviscid fluid flow.

6.3. *The Bernoulli Equation*

We shall now derive a special form of the equation of motion along a streamline, s. For steady flow of an incompressible fluid, Eq. (6–5d) becomes

$$\frac{\partial}{\partial s}\left(\frac{V^2}{2} + \frac{p}{\rho} + gh\right) = 0 \tag{6–8}$$

Integrating Eq. (6–8) along the streamline yields (when n is constant)

$$\frac{V^2}{2} + \frac{p}{\rho} + gh = \text{Constant} \tag{6–9a}$$

or

$$\frac{V^2}{2g} + \frac{p}{\gamma} + h = \text{Constant} \tag{6–9b}$$

where $\gamma = \rho g$. The constant of integration in Eqs. (6–9) take on a different value for each streamline in the flow. Equations (6–9) are two forms of the Bernoulli equation for a steady, inviscid, incompressible flow. Equation (6–9a) can be used to make several definitions of considerable importance. Assume that the gravitational body force is negligible. Then Eq. (6–9a) can be written

$$\tfrac{1}{2}\rho V^2 + p = \text{Constant} \tag{6–10a}$$

The term $\frac{1}{2}\rho V^2$ has units the same as those of pressure. However, this pressure can exist only if the fluid is in motion. Therefore, we define this term as

$$\tfrac{1}{2}\rho V^2 = p_d \tag{6–10b}$$

where p_d is called the *dynamic pressure* of the fluid.

We can draw yet another definition from Eq. (6–10a). In Fig. 6.4 is shown the inviscid streamline flow around a blunt body. Writing Eq.

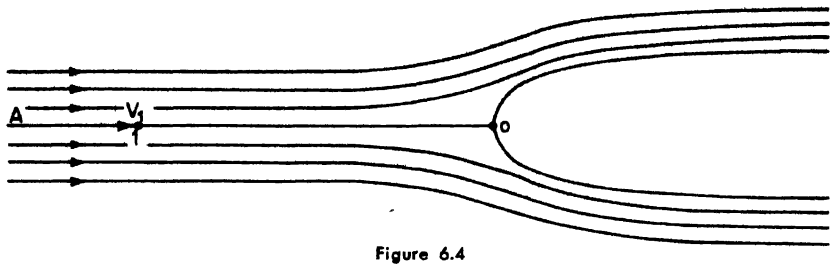


Figure 6.4

(6–10a) for the points 1 and 0 on streamline A, we have

$$\tfrac{1}{2}\rho V_1^2 + p_1 = \tfrac{1}{2}\rho V_0^2 + p_0 = \text{Constant} \qquad (6\text{–}10c)$$

However, on streamline A the flow is brought to rest at point 0; hence, $V_0 = 0$. This point of zero velocity is called the *stagnation point*. The pressure at point 0 is therefore called the *stagnation pressure* (or total pressure) and is given from Eq. (6–10c) by

$$p_0 = \tfrac{1}{2}\rho V_1^2 + p_1 \qquad (6\text{–}11)$$

Therefore, for an inviscid incompressible fluid with negligible body force the stagnation pressure is equivalent to the sum of the dynamic pressure and the static pressure, and is constant throughout the fluid.

Equation (6–9b) is often called the "head" equation because each term has the dimension of length (see § 2.2). Consider the flow of an inviscid fluid through the constriction in the pipe as shown in Fig. 6.5.

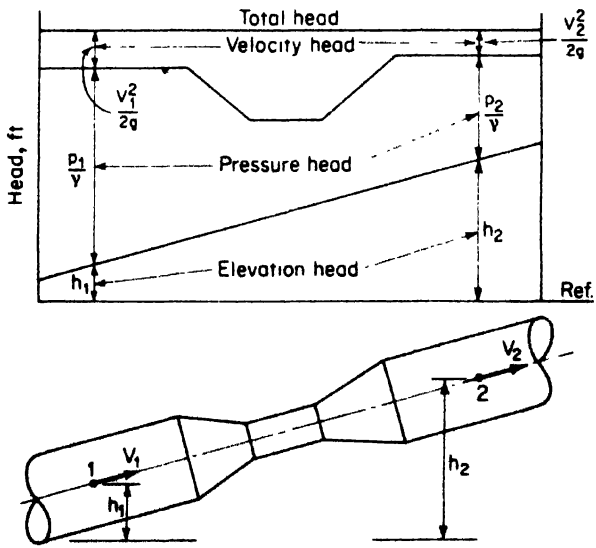


Figure 6.5

The graph indicates the variation of the elevation h, the static pressure head, p/γ, and the velocity head, $V^2/2g$, with position along the pipe. Notice that the total head stays constant in accordance with Eq. (6-9b), but the component heads vary as dictated by Eqs. (2-8) and (6-3b).

The preceding paragraphs contain a discussion of the Bernoulli equation for steady, incompressible flow. A discussion of the equation for unsteady, incompressible flow is contained in § 7.5.

6.4. *Applications of the Bernoulli Equation*

In the following examples the application of the Bernoulli equation to the solution of fluid flow problems will be illustrated. In each case the fluid will be assumed inviscid and incompressible.

a. Flow from a Tank through a Small Orifice

In Fig. 6.6 is shown a liquid-filled tank which is sealed except for a small orifice near its base. It is desired to determine the velocity of efflux of the liquid from the tank when the orifice is opened. Let the cross-sectional area of the tank be A_1 and that of the orifice be A_2.

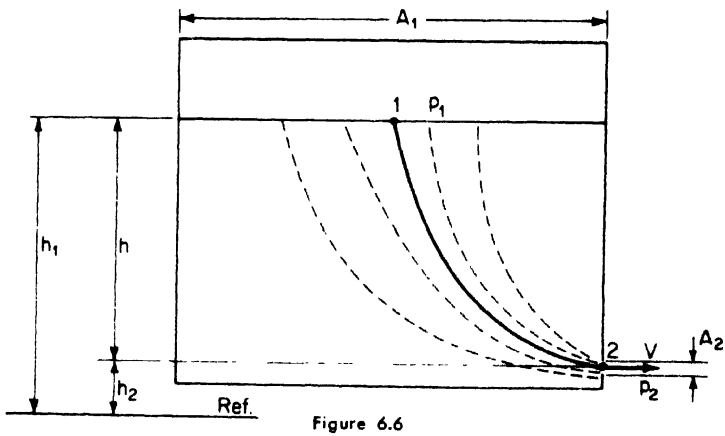


Figure 6.6

The line connecting point 1 on the liquid surface with point 2 in the jet represents a streamline of the flow. Writing Eq. (6-9a) between the points 1 and 2 we have

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = gh_2 + \frac{p_2}{\rho} + \frac{V_2^2}{2} \qquad \text{(a)}$$

However, from the figure we see that $h = h_1 - h_2$. In addition, from

the continuity equation [Eq. (6–3b)] the velocities can be related by

$$V_1 = \left(\frac{A_2}{A_1}\right) V_2 \tag{b}$$

Thus, substitution of Eq. (b) into Eq. (a) leads to

$$V_2 = \sqrt{\frac{2}{(1 - A_2^2/A_1^2)}\left(\frac{p_1 - p_2}{\rho} + gh\right)} \tag{c}$$

Equation (c) is the relation for the velocity of efflux from the tank through the orifice. Two special cases of this equation are of particular interest for the condition where $A_2 \ll A_1$. First, if the tank is vented to the atmosphere or has an open surface so that $p_1 = p_2$, the relation becomes

$$V_2 = \sqrt{2gh} \tag{d}$$

Thus *the velocity of efflux from the vented tank is equal to that of a rigid body falling freely from a height h.* This statement is known as *Torricelli's theorem*, after Evangelista Torricelli (1608–1647), the Italian mathematician.

In the second case, assume the tank with $A_2 \ll A_1$ to be pressurized to the extent that $(p_1 - p_2)/\rho \gg gh$. Then Eq. (c) becomes

$$V_2 = \sqrt{\frac{2(p_1 - p_2)}{\rho}} \tag{e}$$

Illustrative Example 6.2. Calculate the velocity of the water jet as shown in Fig. 6.6 for $p_2 = 14.7$ lb/in.^{2}, $p_1 = 30$ lb/in.^{2}, $A_2/A_1 = 0.01$, and $h = 10$ ft.

Solution. According to Eq. (c) in § 6.4a, we have

$$V_2 = \sqrt{\frac{2}{1 - (0.01)^2}\left[\frac{(30 - 14.7) \times 144}{1.94} + 32.2 \times 10\right]}$$

$$= \sqrt{2917.8} = 54.01 \text{ ft/sec}$$

If $p_1 = p_2$, the discharge velocity is

$$V_2 = \sqrt{2(32.2)(10)} = 25.38 \text{ ft/sec}$$

Experiments have shown that the actual jet velocity is somewhat smaller than the calculated V_2 above because friction is neglected. In order to take friction into account a correction factor may be introduced as

$$V_2 = C\sqrt{2gh}$$

where the so-called coefficient of discharge C can be determined experimentally. It varies from 0.95 to 1.0 for an opening of the type shown

in Fig. 6.6. For a sharp-edged opening, C is usually taken as low as 0.6.

b. Trajectory of a Free Jet

Consider the liquid jet of Fig. 6.7. If the atmospheric viscous effects are neglected, the jet can be considered as a streamline. Since the entire jet is in the atmosphere, $p_1 = p_2$ and the Bernoulli equation between the jet exit 1 and an arbitrary point 2 on the streamline is

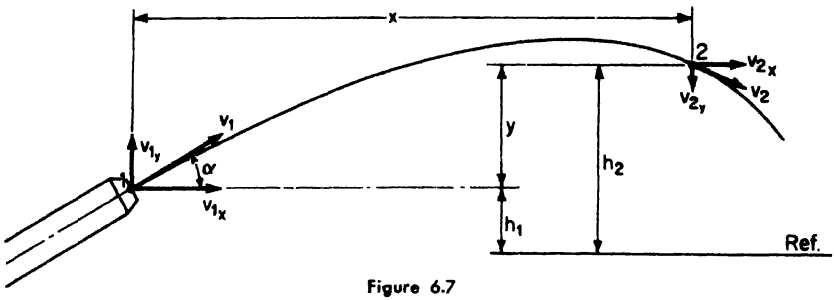


Figure 6.7

$$gh_1 + \frac{V_1^2}{2} = gh_2 + \frac{V_2^2}{2} \tag{a}$$

Now, suppose we want to determine the velocity of the jet at any point on its trajectory as a function of the jet flow rate, Q, the horizontal distance x to point 2, and the initial inclination angle, α. Assume the exit area of the nozzle to be A_1; then Eq. (a) becomes

$$V_2^2 = \left(\frac{Q}{A_1}\right)^2 - 2gy \tag{b}$$

where $Q = V_1 A_1$, the volume flow rate and $y = h_2 - h_1$. Now in order to evaluate y in terms of x and α, we must use the laws of kinematics. The relations between the velocity components at points 1 and 2 are given by

$$V_{2_x} = V_{1_x} \tag{c}$$

$$V_{2_y} = V_{1_y} - gt \tag{d}$$

The coordinates of the point 2 are found to be

$$x = V_{1_x} t \tag{e}$$

$$y = V_{1_y} t - \tfrac{1}{2} g t^2 \tag{f}$$

Eliminating t from Eqs. (e) and (f) we have for y the following expression:

$$y = \frac{V_{1_y}}{V_{1_x}} x - \frac{g}{2} \frac{x^2}{V_{1_x}^2} \tag{g}$$

But $V_{1_y}/V_{1_x} = \tan\alpha$, and $V_{1_x}^2 = V_1^2\cos^2\alpha$. Therefore, substitution of Eq. (g) into Eq. (b) yields

$$V_2^2 = \left(\frac{Q}{A_1}\right)^2 - 2g\left(x\tan\alpha - \frac{gx^2}{2V_1^2\cos^2\alpha}\right)$$

or, finally,

$$V_2 = \sqrt{\left(\frac{Q}{A_1}\right)^2 - 2g\left(x\tan\alpha - \frac{gx^2}{2V_1^2\cos^2\alpha}\right)} \qquad \text{(h)}$$

Illustrative Example 6.3. If the water jet in Fig. 6.7 is discharged from a nozzle (inclined at a 60° angle with the horizontal) at 20 ft/sec, calculate the horizontal distance required for the jet striking the ground which is 3 ft below the horizontal line of the nozzle. What is the velocity of the jet just before reaching the ground?

Solution. The horizontal and vertical components of the jet velocity at the nozzle exit are, respectively,

$$V_{1_x} = V_1\cos\alpha = 20(\tfrac{1}{2}) = 10 \text{ ft/sec}$$

$$V_{1_y} = V_1\sin\alpha = 20(0.866) = 17.32 \text{ ft/sec}$$

According to Eq. (g) we have

$$-3 = \frac{17.32}{10}x - \frac{g}{2}\frac{x^2}{(10)^2}$$

Simplifying,

$$x^2 - 10.74x - 18.63 = 0$$

The horizontal distance between the nozzle and jet striking the ground is

$$x = \frac{10.74}{2} + \frac{1}{2}\sqrt{(10.74)^2 + 4 \times 18.62} = 12.13 \text{ ft}$$

The jet velocity at $x = 12.27$, $y = -3$ can be obtained either from Eq. (b) or Eq. (h). Equation (b) gives

$$V_2 = \sqrt{(20)^2 - 2g(-3)} = \sqrt{593} = 24.24 \text{ ft/sec}$$

and Eq. (h) gives

$$V_2 = \sqrt{(20)^2 - 2g\left[(12.27)(\tan 60^\circ) - \frac{g(12.27)^2}{2(20)^2\cos^2 60^\circ}\right]}$$

$$= \sqrt{400 - 64.4(21.25 - 24.25)} = 24.24 \text{ ft/sec}$$

c. Pressure Distribution on a Protuberance in a Closed Channel

In this problem we want to find the pressure distribution on a semicylindrical protuberance in the closed channel shown in Fig. 6.8.

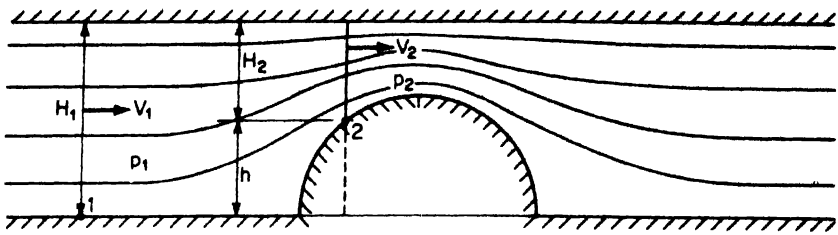


Figure 6.8

If it is assumed that the channel is completely filled with the moving fluid, Eq. (6–9a) can be employed to determine the pressure at any point on the semicylinder. Assume the channel to be of constant width so that the area at any section can be indicated by bH, where b represents the local channel width. In addition the elevation change will be assumed negligible. The Bernoulli equation between points 1 and 2 is

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2}$$

or

$$\frac{p_2 - p_1}{\rho} = \frac{V_1^2 - V_2^2}{2} \tag{a}$$

But from the continuity equation for constant channel width,

$$V_1 = \frac{bH_2}{bH_1} V_2 \tag{b}$$

However, from Fig. 6.8,

$$H_2 = H_1 - h$$

Therefore, substitution of this relation and Eq. (b) into Eq. (a) results in

$$\frac{p_2 - p_1}{\rho} = \frac{V_1^2}{2}\left[1 - \frac{H_1^2}{(H_1 - h)^2}\right]$$

or

$$\frac{p_2 - p_1}{\frac{1}{2}\rho V_1^2} = \frac{h}{H_1}\left[\frac{h/H_1 - 2}{(1 - h/H_1)^2}\right] \tag{c}$$

It will be noted that the left-hand side of Eq. (c) is a dimensionless ratio, the numerator being the difference in static pressure between the two stations and the denominator being the dynamic pressure of the undisturbed flow. This ratio is called the *pressure coefficient* (see Appendix C) and is defined by

$$C_p = \frac{p_2 - p_1}{\frac{1}{2}\rho V_1^2} \tag{d}$$

6.5. Flow Measurement

The principle of continuity and the Bernoulli equation are very useful in determining the average flow properties along a streamline flow. In fact they are widely applied in the design of various physical devices which provide an accurate means of measurement of the flow. Several of these devices will be discussed in detail in this section.

a. Pitot Tube

The stagnation pressure was defined in § 6.3 by

$$p_0 = p_1 + \tfrac{1}{2}\rho V_1^2 \tag{6-11}$$

A device for measuring the stagnation pressure is the Pitot tube. Physically, the Pitot tube is simply a piece of tubing usually bent at right angles with one longitudinal axis aligned with the flow and the other end connected to a manometer (see Fig. 6.9a). The stagnation

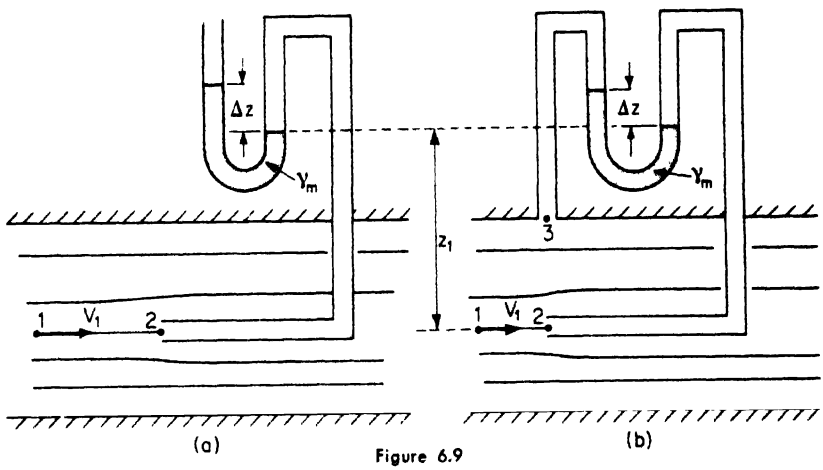


Figure 6.9

pressure can be calculated from the manometer deflection by application of the Bernoulli equation. For points 1 and 2 in Fig. 6.9a where $h_1 = h_2$, we have

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2}$$

But since the flow is brought to rest at 2 we have, after putting $p_2 = p_0$ and $V_2 = 0$,

$$p_0 = p_1 + \tfrac{1}{2}\rho V_1^2 \tag{6-11a}$$

In terms of the manometer deflection (Fig. 6.9a) and pressures, the static equilibrium relation gives

$$p_0 - \gamma z_1 = p_{atm} + \gamma_m \cdot \Delta z$$

or

$$p_0 = \gamma_m \cdot \Delta z + p_{atm} + \gamma z_1 \tag{6-12}$$

where $\gamma = \rho g$, p_{atm} = atmospheric pressure, γ_m is the specific weight of the manometer liquid.

Therefore, it is seen that knowledge of the stagnation pressure alone is not sufficient to determine the velocity of the flow. However, from Eq. (6-11) we see that the velocity can be determined by measuring the difference in the stagnation and static pressures. This can be achieved with the Pitot tube by applying local static pressure to the open side of the manometer (Fig. 6.9b). The static pressure of the flow is obtained from a wall connection such as that shown in Fig. 6.9b. The manometer deflection of such an arrangement is related to pressures in static equilibrium, i.e.,

$$p_0 - \gamma z_1 = p_1 - (z_1 + \Delta z)\gamma + \gamma_m \cdot \Delta z$$

where $p_2 = p_0$ and $p_1 = p_3$. Hence, the desired stream velocity can be obtained in terms of manometer deflection as follows:

$$V_1 = \sqrt{\frac{2(p_0 - p_1)}{\rho}} = \sqrt{2g \cdot \Delta z \left(\frac{\gamma_m}{\gamma} - 1\right)} \tag{6-13}$$

b. Pitot-Static Tube

It is often inconvenient (also inaccurate for measurement in nonuniform flows) to determine the velocity of a flow with a Pitot tube and wall connection. This difficulty in velocity measurement can be overcome by the use of a Pitot-static tube. The device, shown in Fig. 6.10, is merely a Pitot tube with a concentric outer chamber which is sealed from the effects of the direct flow but contains a ring of holes on its periphery for static pressure measurement. The stagnation pressure is measured at point 2 and the static pressure at point 3 as indicated in Fig. 6.10. The velocity in terms of the manometer deflection is given by Eq. (6-13). In order for the measured velocity to be that of the free stream, the Pitot-static tube must not be positioned near a boundary or protuberance in the flow which would alter the streamline pattern around the tube.

c. Venturi Meter

A Venturi meter is a device commonly used to measure the flow rate of a fluid in a conduit. It consists of a conduit of constant cross

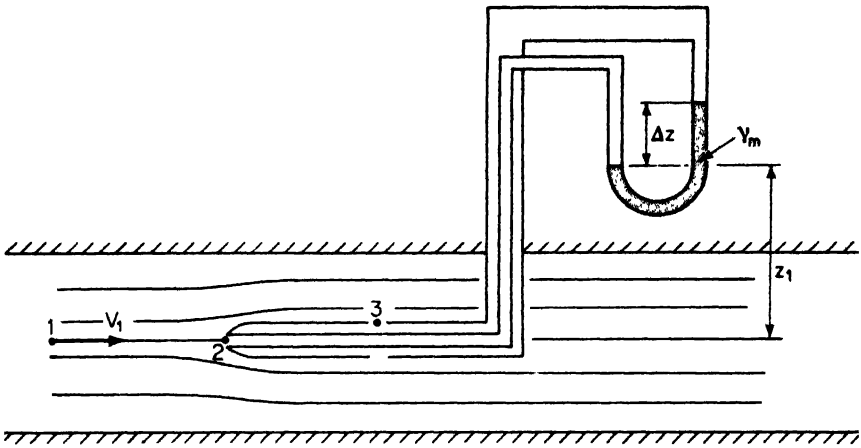


Figure 6.10

section tapering to a section of smaller cross section and then gradually expanding to the original cross section. A typical Venturi meter is shown schematically in Fig. 6.11. Rings of static pressure ports are placed at the entrance and exit of the converging section. As the flow proceeds from the larger to the smaller cross-sectional area, it is accelerated and a static pressure variation is produced on it. The measurement of this pressure change provides an accurate measure of the flow rate in the conduit. This can be shown theoretically by application of the continuity and Bernoulli equations. Let A_1 and A_2 be the cross-sectional areas of sections 1 and 2 shown in Fig. 6.11. Then,

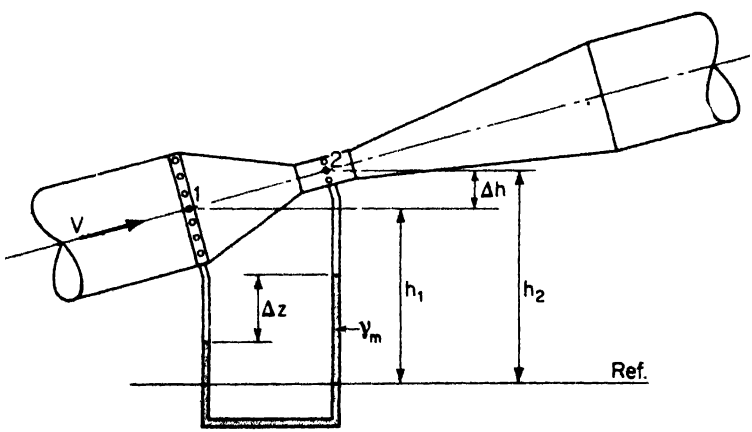


Figure 6.11

$$A_1 V_1 = A_2 V_2 \tag{6-1}$$

and

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gh_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gh_2 \tag{6-9a}$$

Solving Eq. (6–1) and (6–9a) for the velocity at 2, one finds

$$V_2 = \sqrt{\frac{2[(p_1 - p_2)/\rho - g(h_2 - h_1)]}{1 - (A_2/A_1)^2}} \tag{6-14}$$

Equation (6–14) is identical to Eq. (c) in § 6.4a. Hence, the flow rate at either section is given theoretically by

$$Q_{\text{theory}} = V_2 A_2 = A_2 \sqrt{\frac{2(\Delta p/\rho - g\Delta h)}{1 - (A_2/A_1)^2}} \tag{6-15a}$$

where

$$\Delta p = p_1 - p_2, \qquad \Delta h = h_2 - h_1, \qquad \gamma = \rho g$$

or in terms of manometer deflection (Fig. 6.11)

$$Q_{\text{theory}} = \frac{A_2}{\sqrt{1 - (A_2/A_1)^2}} \sqrt{2g(\Delta z)\left(\frac{\gamma_m}{\gamma} - 1\right)} \tag{6-15b}$$

where $\Delta z = (\Delta p/\gamma - \Delta h)[\gamma/(\gamma_m - \gamma)]$, the manometer deflection. Equations (6–15) represent the theoretical flow rate through the Venturi meter. However, in practice the flow rate is somewhat less than the theoretical value because of nonuniformity of the velocity distribution. Therefore, we may write

$$Q_{\text{theory}} = \frac{C_{d_v} \cdot A_2}{\sqrt{1 - (A_2/A_1)^2}} \sqrt{2g(\Delta z)\left(\frac{\gamma_m}{\gamma} - 1\right)} \tag{6-16}$$

where $C_{d_v} = Q_{\text{actual}}/Q_{\text{theory}}$ is the *discharge coefficient* which is determined experimentally for a given Venturi meter.

The Venturi meter is well adapted to pipe flow rate measurement because the gradual expanding section keeps boundary layer separation to a minimum, resulting in good pressure recovery across the meter. On the other hand, the length of the Venturi meter restricts its usefulness in a wide variety of applications.

Illustrative Example 6.4. The Venturi meter (Fig. 6.11) has an entrance of 6 in. diameter and a throat of 3 in. diameter whose center is 18 in. above the center of the entrance. Find the velocity of water at the throat when $p_1 - p_2 = 5$ lbf/in.^{2}. The coefficient of discharge C_{d_v} is 0.97. Find the value of the constant for the Venturi meter.

Solution. According to Eq. (6–16) we have

$$V_2 = \frac{C_{d_v}}{\sqrt{1 - (A_2/A_1)^2}} \sqrt{2g(\Delta z)\left(\frac{\gamma_{Hg}}{\gamma_w} - 1\right)}$$

where $C_d = 0.97$,

$$\frac{A_2}{A_1} = \left(\frac{3}{6}\right)^2 = \frac{1}{4}, \qquad \Delta z\left(\frac{\gamma_{Hg}}{\gamma_w} - 1\right) = \frac{5 \times 144}{62.4} - \frac{18}{12} = 10.04 \text{ ft}$$

Hence,

$$V_2 = \frac{0.97}{\sqrt{1 - (\frac{1}{4})^2}}\sqrt{2g(10.03)} = 25.4 \text{ ft/sec}$$

If Δz is given in inches, the constant for this Venturi tube is

$$V_2 = \frac{0.97}{\sqrt{1 - (\frac{1}{4})^2}}\sqrt{\frac{2g}{12}(13.6 - 1)} \cdot \sqrt{\Delta z} = 8.22\sqrt{\Delta z} \text{ ft/sec}$$

d. Orifice Plate

The disadvantage of the Venturi meter due to its excessive length can be overcome by the use of an orifice plate such as that shown in Fig. 6.12. The orifice plate consists of a plate with a sharp-edged hole

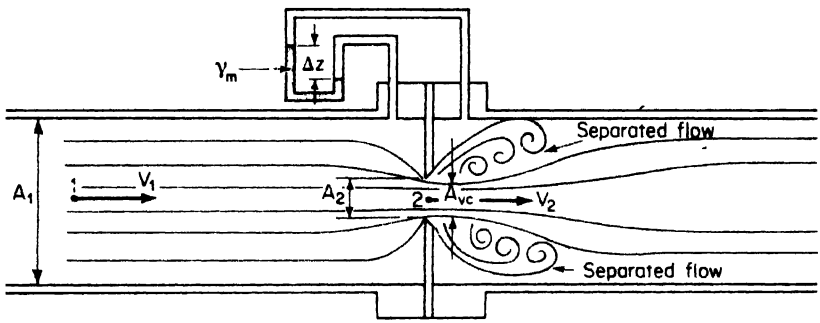


Figure 6.12

in its center. It is placed in a conduit so that the flow passes only through the hole at its center. Measurement of the pressure drop across the plate provides a means of determining the flow rate in the conduit. If we let A_1 be the conduit area and A_2 be the orifice hole area, Eqs. (6–1) and (6–9a) can be combined to give the following theoretical expression for the flow rate:

$$Q_{\text{theory}} = \frac{A_2}{\sqrt{1 - (A_2/A_1)^2}}\sqrt{\frac{2(p_1 - p_2)}{\rho}} \qquad (6\text{–}17a)$$

or in terms of manometer deflection,

$$Q_{\text{theory}} = \frac{A_2}{\sqrt{1 - (A_2/A_1)^2}}\sqrt{2g(\Delta z)\left(\frac{\gamma_m}{\gamma} - 1\right)} \qquad (6\text{–}17b)$$

In practice the flow rate is given by

$$V_{actual} = \frac{C_{d_0} A_2}{\sqrt{1 - (A_2/A_1)^2}} \sqrt{2g\Delta z \left(\frac{\gamma_m}{\gamma} - 1\right)} \tag{6-18}$$

where the discharge coefficient, C_{d_0}, here is a function of jet size as well as viscous effects. The jet-size effect is usually based on the area ratio A_{vc}/A_2, A_2 being the orifice hole area and A_{vc} being the area of the *vena contracta* (minimum jet area) just downstream of the orifice plate. The viscous effect is usually expressed in terms of a nondimensional parameter called the Reynolds number.

The major disadvantage in the application of orifice plates to flow measurement is that a sizeable pressure loss is incurred because of the flow separation downstream of the plate.

e. *Weir*

In many cases the flow rate of fluid with a free surface (as in an open channel or river) must be determined. In these situations the preceding devices are unsatisfactory for determining average flow rates because they are considerably smaller than the flow area involved and tend to yield local values of flow rate which may not be the average. This difficulty is circumvented by the use of weirs. There are two types of weirs, both of which depend on the same physical principles to provide flow measurement. A sharp-crested weir is a plate mounted normal to the flow direction and spanning the fluid stream (Fig. 6.13). The opening is usually a symmetrical geometric shape, such as a rectangle or a triangle.

The broad-crested weir (Fig. 6.14) of which a dam spillway is an example, may be beneath the water surface across the entire stream. In either case because of the flow pattern and viscous and turbulance effects, a mathematical analysis of weir flow is difficult to perform. However, some qûantitative results can be obtained as follows:

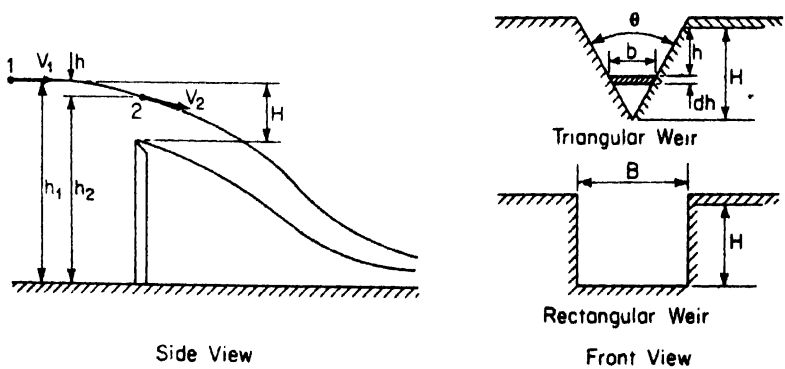


Figure 6.13

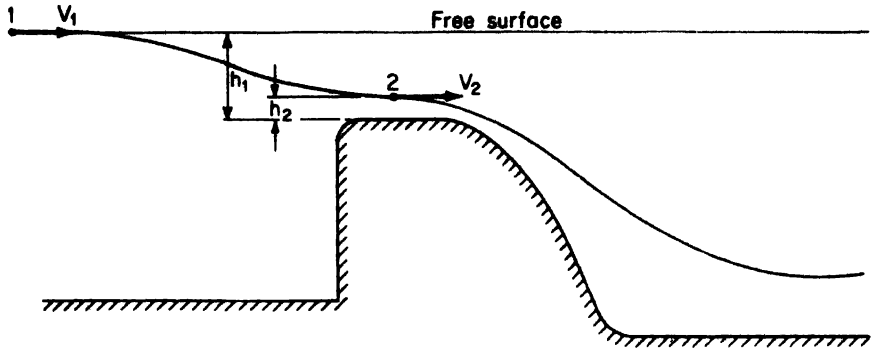


Figure 6.14

(i) *Sharp-crested Weir* (Fig. 6.13). The Bernoulli equation for any streamline in the flow between stations 1 and 2 is

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gh_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gh_2 \tag{6-9a}$$

But in this case, atmospheric pressure is applied to the entire free surface and $h_2 = h_1 - h$. Thus, the velocity at the weir plane is given by

$$V_2 = \sqrt{2gh + V_1^2}$$

The flow rate over the weir is then given by

$$dQ = V_2\, dA_2$$

Thus for a rectangular weir where $dA_2 = B\, dh$ we have

$$dQ_{\text{theory}} = B\sqrt{2gh + V_1^2}\, dh$$

$$Q_{\text{theory}} = B\int_0^H \sqrt{2gh + V_1^2}\, dh$$

or finally

$$Q_{\text{theory}} = \frac{2B}{3(2g)}[(2gH + V_1^2)^{3/2} - V_1^3] \tag{6-19}$$

In Eq. (6-19) both Q and V_1 are unknown, so a trial and error solution is necessary. If the body of water being measured is large in comparison with the weir opening, V_1 can be assumed negligible and Eq. (6-19) becomes

$$Q_{\text{theory}} = \frac{2B}{3}\sqrt{2g}\, H^{3/2}$$

or for the actual case

$$Q_{\text{actual}} = \frac{2BC_d}{3}\sqrt{2g}\, H^{3/2} \tag{6-20}$$

where C_{d_w} is a discharge coefficient. For the triangular weir or v-notch shown in Fig. 6.13 the flow rate is

$$Q_{\text{actual}} = \frac{8}{15} C_{d_w} \sqrt{2g} \tan \frac{\theta}{2} H^{5/2} \tag{6-21}$$

(ii) *Broad-crested Weir* (Fig. 6.14). The analysis of the broad-crested weir is parallel to that for the sharp-crested weir. Assuming uniform parallel flow at station 2 and that $V_1 \ll V_2$, we may write the Bernoulli equation between stations 1 and 2 as

$$gh_1 = \frac{V_2^2}{2} + gh_2$$

Therefore, the flow rate over a weir of width B is given by

$$Q_{\text{theory}} = Bh_2\sqrt{2g(h_1 - h_2)} \tag{6-22}$$

or

$$Q_{\text{actual}} = C_{d_w} Bh_2\sqrt{2g(h_1 - h_2)} \tag{6-23}$$

The maximum rate of discharge can be determined by letting $dQ/dh_2 = 0$. Performing this operation on Eq. (6-22) we have

$$\frac{dQ}{dh_2} = 0 = \sqrt{2g(h_1 - h_2)} - \frac{gh_2}{\sqrt{2g(h_1 - h_2)}}$$

or $h_2 = \frac{2}{3} h_1$ for maximum discharge rate. Substituting this expression into Eq. (6-22) results in

$$Q_{\max} = B\left(\frac{2h_1}{3}\right)^{3/2} \sqrt{g} \tag{6-24a}$$

The velocity corresponding to the maximum flow rate is

$$(V)_{Q_{\max}} = \sqrt{gh_2} \tag{6-24b}$$

This is known as the critical velocity, and it is equal to the velocity of propagation of a surface wave.

f. Elbow Meter

The simple device shown in Fig. 6.15 is called an elbow meter. It employs the principle of conservation of momentum normal to the flow direction to provide accurate flow rate information. Piezometer openings on the outside (point 1) and inside (point 2) of the elbow are connected to a differential manometer. A difference in pressure intensities at the two points is produced as the result of the centrifugal force at the bend. The flow rate can be determined analytically from this measured pressure difference by applying Eq. (6-7b) to the flow. The result is

$$\frac{V^2}{R} = -g \cos \alpha - \left(\frac{p_2 - p_1}{\rho D}\right) \tag{6-25}$$

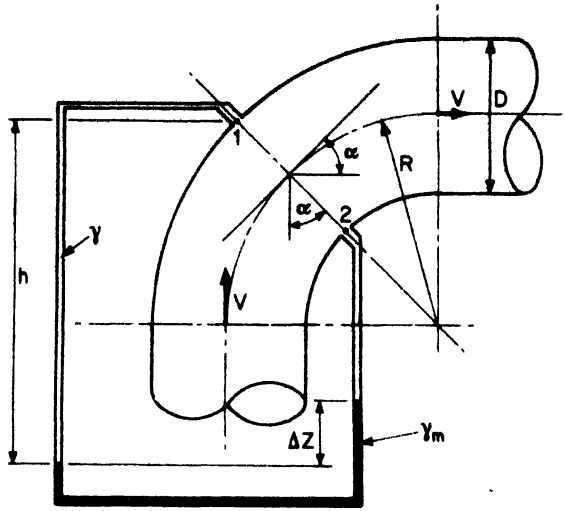


Figure 6.15

For an elbow in the horizontal plane Eq. (6–25) becomes

$$V^2 = \frac{R(p_1 - p_2)}{\rho D}$$

Hence the rate of flow is given by

$$Q = \frac{\pi D^2}{4}\sqrt{\frac{R(p_1 - p_2)}{\rho D}} \tag{6–26a}$$

Or, in terms of the manometer deflection, Δz, we have

$$Q = \frac{\pi D^2}{4}\sqrt{\frac{Rg(\Delta z)}{D}\left(\frac{\gamma_m}{\gamma} - 1\right)} \tag{6–26b}$$

g. Rotometer

A direct-reading meter for small diameter applications is the rotometer shown in Fig. 6.16. It consists of a float contained within a tube of gradually increasing diameter from inlet to outlet. Since the flow rate is

$$Q = AV$$

the flow through the rotometer is determined by the difference in cross-sectional areas of the float and the tube. However, the float is free to rise in the tube; thus, its vertical position is a measure of the flow rate through the tube. In practice a calibration scale is etched on the glass tube in units of flow rate. Also, to ensure the repeatability of readings,

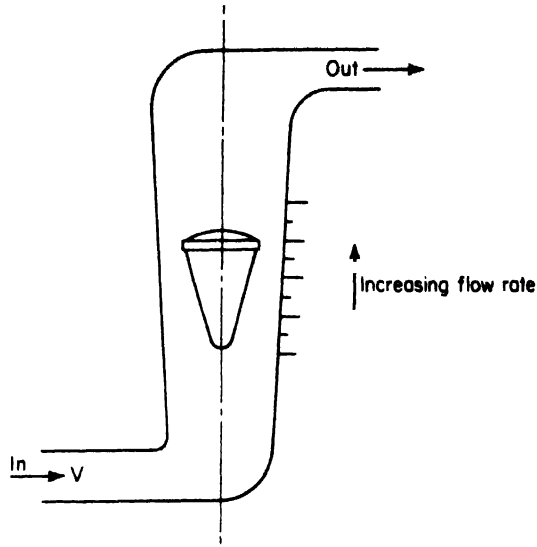


Figure 6.16

the float is designed to rotate in the presence of a flow so that its axis will remain positioned along the axis of the tube.

6.6. *The Momentum Theorem*

In § 6.2 we have discussed Euler's equation which is derived on the basis of the law of conservation of momentum with all of the forces acting on an infinitesimal fluid particle. The application of Euler's equation to a particular problem involves the solution of differential equations satisfying appropriate boundary conditions. In many technically important problems in fluid mechanics they can be solved when the behavior of fluid elements at the boundaries of a certain well-defined region in the flow fluid (control volume) is known, instead of the motion of individual particles as given by Euler's equation. This approach to problems is based on the momentum theorem which, in essence, is the same as the momentum principle in solid mechanics. The equation which describes the change of momentum of the fluid within the control volume (Fig. 6.17) may be obtained from Euler's equation in the following manner: Multiplication of Eq. (6–5c) by ρA results in

$$\rho A \frac{\partial V}{\partial t} + \rho A V \frac{\partial V}{\partial s} = -A \frac{\partial p}{\partial s} - \rho g A \frac{\partial h}{\partial s} \tag{6-27}$$

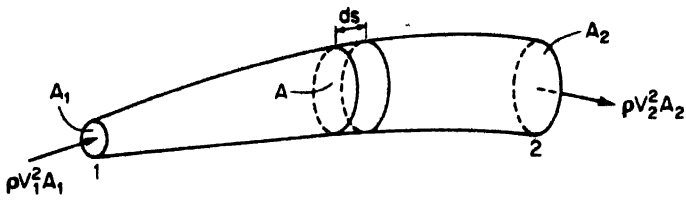


Figure 6.17

But from Eq. (5-1) for a stream tube, the continuity equation becomes

$$\frac{D}{Dt}(\rho A \Delta s) = \frac{\partial}{\partial t}(\rho A \Delta s) + V\frac{\partial}{\partial s}(\rho A \Delta s) = 0$$

Now, making the substitution $\Delta s = V\Delta t$ into the preceding equation we have

$$\frac{\partial}{\partial t}(\rho A)\Delta s + V\frac{\partial}{\partial s}(\rho A V)\Delta t = 0 \tag{6-28a}$$

or

$$V\frac{\partial}{\partial t}(\rho A) + V\frac{\partial}{\partial s}(\rho A V) = 0 \tag{6-28b}$$

Addition of Eq. (6-28b) to Eq. (6-27) results in

$$\frac{\partial}{\partial t}(\rho A V) + \frac{\partial}{\partial s}(\rho A V^2) = -\frac{\partial}{\partial s}(pA) + p\frac{\partial A}{\partial s} - \rho g A\frac{\partial h}{\partial s} \tag{6-29}$$

Integrating the equation above with respect to s between any two stations for incompressible flow gives

$$\rho\int_1^2 \frac{\partial(AV)}{\partial t}\,ds + \rho(V_2^2 A_2 - V_1^2 A_1) = (p_1 A_1 - p_2 A_2) + \int_1^2 p\,dA - \rho g\int_1^2 A\,dh \tag{6-30}$$

The left-hand side of Eq. (6-30) is the rate of change of momentum in space between sections 1 and 2 which consists of the contribution from nonstationary changes within the space and the contribution from the transport of momentum into the space through the end sections, respectively. The right-hand side is the total force in the s-direction, due to the pressure on the ends and the walls and to gravity, respectively.

For steady flow with gravity neglected, Eq. (6-30) becomes

$$\rho Q(V_2 - V_1) = (p_1 A_1 - p_2 A_2) + \int_1^2 p\,dA \tag{6-31}$$

where

$$Q = V_1 A_1 = V_2 A_2$$

6.7. Applications of the Momentum Theorem

The usefulness of the momentum theorem is not readily apparent from its mathematical formulation. The application of this theorem in the following examples will serve to illustrate its advantages in the solution of practical problems.

a. Pressure Exerted on a Plate by a Free Jet

Consider a jet of fluid directed at the inclined plate shown in Fig. 6.18. Suppose we are interested in determining the force necessary to hold the plate in equilibrium against the pressure of the jet. For a

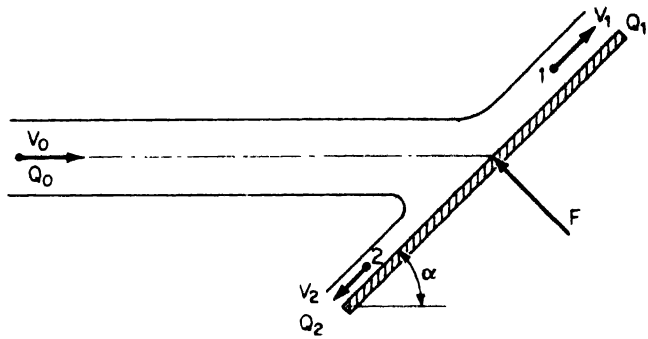


Figure 6.18

free jet the static pressure is the same for all points in the jet. Thus the velocities can be related by

$$V_0 = V_1 = V_2 \tag{a}$$

In addition, for an inviscid fluid there can be no shearing force parallel to the plate surface; thus, the reaction force F is normal to the plate surface. From the momentum theorem this force must be equal to the rate of change of momentum normal to the plate surface. For this case Eq. (6-31) becomes

$$F = \rho V_0 Q_0 \sin \alpha \tag{b}$$

where $Q_0 = V_0 A$ and A is the cross-sectional area of the jet. It is interesting to note that the division of flow along the plate is uneven. The magnitudes of the flow rates along the plate can be determined by a consideration of the momentum theorem parallel to the plate. In this case Eq. (6-31) can be written

$$(\rho Q_1 V_1 - \rho Q_2 V_2) - \rho V_0 Q_0 \cos \alpha = 0 \tag{c}$$

Also, the continuity equation stipulates that

$$Q_0 = Q_1 + Q_2 \tag{d}$$

Combination of Eqs. (a), (c), and (d) leads to the following results for the flow rates:

$$Q_1 = \frac{Q_0}{2}(1 + \cos\alpha) \tag{e}$$

$$Q_2 = \frac{Q_0}{2}(1 - \cos\alpha) \tag{f}$$

Illustrative Example 6.5. Calculate the force exerted by a jet of water $\frac{3}{4}$ in. in diameter which strikes a flat plate at an angle of 30° to the normal of the plate with a velocity of 30 ft/sec if (a) the plate is stationary, (b) the plate is moving in the direction of the jet with a velocity of 10 ft/sec.

Solution. (a) The angle α shown in Fig. 6.18 is $90 - 30 = 60°$; hence the normal force [Eq. (b)] is

$$F = \frac{62.4}{g}\left[(30)\frac{\pi}{4}\left(\frac{3}{4 \times 12}\right)^2\right](30)\sin 60° = 4.63 \text{ lbf}$$

(b) The change of velocity on impact in this case is

$$V_0 - V_{\text{plate}} = 20 \text{ ft/sec}$$

The normal force is

$$F = \frac{62.4}{g}\left[30 \times \frac{\pi}{4}\left(\frac{3}{4 \times 12}\right)^2\right](20)\sin 60° = 3.09 \text{ lbf}$$

b. Jet Discharge Propulsion

In § 6.4 Torricelli's theorem for the efflux velocity from an orifice in a large tank was derived. The momentum theorem can now be applied to this example to determine the propulsive force created by the orifice flow. In Fig. 6.19 is shown a large pressurized tank with an orifice of area A_2. We assume that $A_2 \ll A_1$ and that $gh \ll (p_1 - p_2)/\rho$. Then the velocity of the jet is given by

$$V_2 = \sqrt{\frac{2(p_1 - p_2)}{\rho}} \tag{a}$$

and

$$Q = A_2 V_2 \tag{b}$$

Let F represent the force necessary to hold the tank in equilibrium. Then Eq. (6–31) becomes

$$\rho Q(V_2 - 0) = F \tag{c}$$

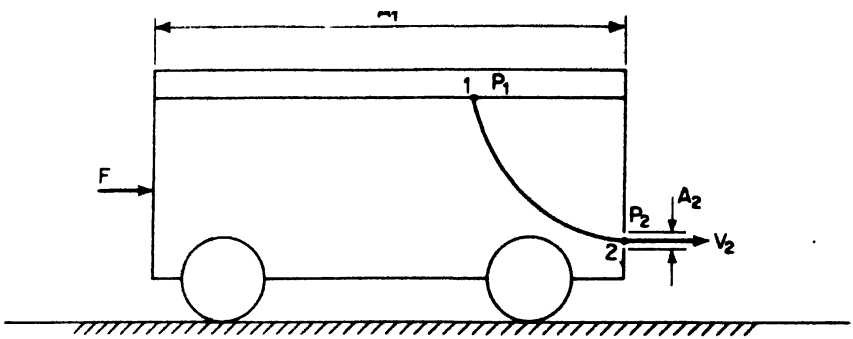


Figure 6.19

Substitution of Eqs. (a) and (b) into Eq. (c) leads to

$$F = 2A_2(p_1 - p_2) \tag{d}$$

Therefore, the propulsive force or thrust induced by the jet is given by

$$T = -F = 2A_2(p_2 - p_1) \tag{e}$$

c. Force Exerted by a Flowing Fluid on a Contracting Pipe Bend

The change of momentum of a fluid flowing through a pipe bend induces a force on the pipe. Consider the pipe bend shown in Fig. 6.20. The fluid enters the bend with velocity V_1 through area A_1 and leaves with a velocity V_2 through area A_2 after having been turned through an angle α. Let F be the force required to hold the pipe in equilibrium against the pressure of the fluid. F_x and F_y are the component forces in the negative x- and y-directions, respectively. Equation (6–31) can now be employed to determine the magnitude of the force F. Along the x-direction we have

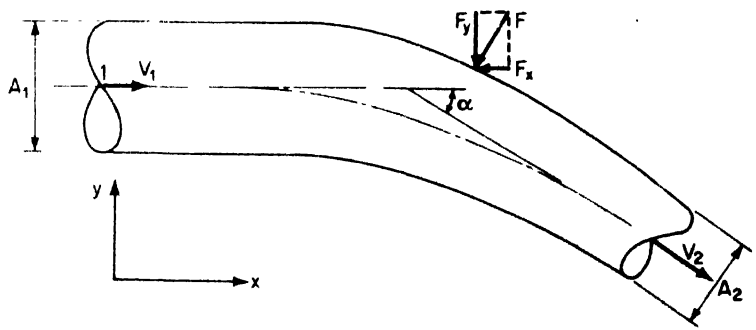


Figure 6.20

$$\rho Q(V_2 \cdot \cos\alpha - V_1) = -F_x + p_1 A_1 - p_2 A_2 \cdot \cos\alpha \qquad \text{(a)}$$

and in the y-direction we have

$$\rho Q(-V_2 \cdot \sin\alpha + 0) = -F_y + p_2 A_2 \cdot \sin\alpha \qquad \text{(b)}$$

Then the total restraining force is given by

$$\begin{aligned} F = \sqrt{F_x^2 + F_y^2} = \{&\rho^2 Q^2 (V_1^2 + V_2^2 - 2V_1 V_2 \cdot \cos\alpha) \\ &+ p_1^2 A_1^2 + p_2^2 A_2^2 - 2p_1 p_2 A_1 A_2 \cos\alpha + 2\rho Q\,[p_1 V_1 A_1 \\ &+ p_2 V_2 A_2 - (p_2 V_1 A_2 + p_1 V_2 A_1)\cos\alpha]\}^{1/2} \end{aligned} \qquad \text{(c)}$$

But, from the equation of continuity the velocities are related by

$$V_2 = \frac{A_1}{A_2} V_1$$

Thus, Eq. (c) becomes

$$\begin{aligned} F = \Big\{&\rho^2 Q^2 V_1^2 \Big(1 - 2\frac{A_1}{A_2}\cos\alpha + \frac{A_1^2}{A_2^2}\Big) + 2\rho Q^2 \Big[p_1 + p_2 \\ &- \Big(p_1 \frac{A_1}{A_2} + p_2 \frac{A_2}{A_1}\Big)\cos\alpha\Big] + p_1^2 A_1^2 \Big[1 + \Big(\frac{p_2}{p_1}\Big)^2 \Big(\frac{A_2}{A_1}\Big)^2 \\ &- 2\Big(\frac{p_2}{p_1}\Big)\Big(\frac{A_2}{A_1}\Big)\cos\alpha\Big]\Big\}^{1/2} \end{aligned} \qquad \text{(d)}$$

For equal areas ($A_1 = A_2$) Eq. (d) reduces to

$$\begin{aligned} F = \Big\{&2\rho Q^2 (1 - \cos\alpha)(\rho V_1^2 + p_1 + p_2) \\ &+ p_1^2 A_1^2 \Big[1 + \Big(\frac{p_2}{p_1}\Big)^2 - 2\Big(\frac{p_2}{p_1}\Big)\cos\alpha\Big]\Big\}^{1/2} \end{aligned} \qquad \text{(e)}$$

and, if the bend is 90°, the force for the constant area bend becomes

$$F = \Big\{2\rho Q^2 (\rho V_1^2 + p_1 + p_2) + p_1^2 A_1^2 \Big[1 - \Big(\frac{p_1}{p_2}\Big)^2\Big]\Big\}^{1/2} \qquad \text{(f)}$$

d. The Impulse Turbine

An application of the momentum theorem of great interest is that of the impulse turbine. In this case the momentum change resulting from the turning of a fluid jet by a vane is employed to turn a turbine wheel and generate power. We can analyze this problem by application of Eq. (6–31) to the blade shown in Fig. 6.21a. Let F be the force exerted on the blade by the fluid and v be the velocity of the blade. The relative velocity of the jet entering the vane is given by

$$V_e = V_1 - v \qquad \text{(a)}$$

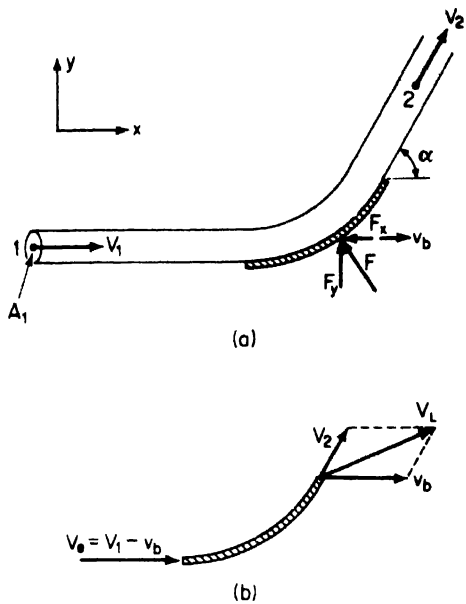


Figure 6.21

Since the same pressure acts on the entire flow, the relative velocity leaving the blade is also given by Eq. (a), or

$$V_2 = V_1 - v_b \tag{b}$$

Now, in order to determine the actual velocity of the jet leaving the blade it is necessary to take the vector sum of V_2 and v_b (Fig. 6.21b). Let the velocity of the fluid leaving the blade be given by V_L. Then

$$V_L^2 = V_{L_x}^2 + V_{L_y}^2$$

and

$$V_{L_x} = V_2 \cdot \cos \alpha + v_b \tag{c}$$

$$V_{L_y} = V_2 \cdot \sin \alpha \tag{d}$$

Now the force exerted on the vane by the fluid can be easily determined by application of Eq. (6-31). In the x-direction we have

$$-F_x = \rho Q(V_2 \cdot \cos \alpha + v_b - V_1)$$

or

$$F_x = \rho A_1(V_1 - v_b)^2(1 - \cos \alpha) \tag{e}$$

where $Q = A_1(V_1 - v_b)$ and A_1 is the cross-sectional area of the jet. In the y-direction the force component is given by

$$F_y = \rho A_1(V_1 - v_b)^2 \cdot \sin \alpha \tag{f}$$

The only force which can generate power for a vane fixed to a turbine wheel is F_x. Therefore, for a frictionless turbine we can write the power generated (or work done on moving vanes) as

$$P_w = F_x v_b = \rho A_1 v_b (V_1 - v_b)^2 (1 - \cos \alpha) \tag{g}$$

If a series of vanes are so arranged on the periphery of a wheel so that the entire jet, directed tangentially to the circumference, is striking either one vane or another successively, the work done per second is

$$P_w = \rho A_1 V_1 (V_1 - v_b)(1 - \cos \alpha) v_b \tag{h}$$

Illustrative Example 6.6. Determine the force components due to the water jet for a moving vane of Fig. 6.B.

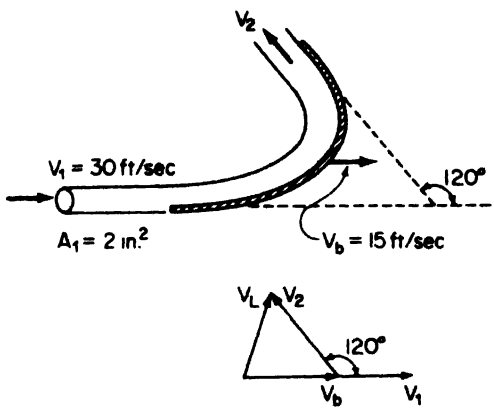


Figure 6.8

Solution. The actual velocity of the jet leaving the vane (see Fig. 6.B) is

$$V_{L_x} = V_2 \cos 120^\circ + v_b = -(V_1 - v_b) \cos 60^\circ + v_b$$

$$= 15 - \frac{15}{2} = 7.5 \text{ ft/sec}$$

$$V_{L_y} = V_2 \sin 120^\circ = (V_1 - v_b) \sin 60^\circ = 13 \text{ ft/sec}$$

The force components are:

$$F_x = \rho A_1 (V_1 - v_b)^2 (1 - \cos 120^\circ) = \frac{62.4}{72g} (15)^2 (1.5) = 9.08 \text{ lbf}$$

$$F_y = \left(\frac{62.4}{g}\right)\left(\frac{2}{144}\right)(15)^2 (0.866) = 5.24 \text{ lbf}$$

The force components on the vane are 9.08 lbf in the positive x-direc-

tion and 5.24 lbf in the negative y-direction. The power generated by the vane is

$$p = F_x \cdot v_b = 9.08 \times 15 = 136.5 \text{ ft-lbf/sec}$$

NOMENCLATURE FOR CHAPTER SIX

A	area
A_{vc}	area of the *vena contracta*, § 6.5d and Fig. 6.12
b	channel width, § 6.4b
C_{d_o}	discharge coefficient of orifice plate, § 6.5 [Eq. (6-18)] and Fig. 6.12
C_{d_v}	discharge coefficient of Venturi meter, § 6.5c
C_{d_w}	discharge coefficient of the weir, § 6.5e
C_p	pressure coefficient, § 6.4c [Eq. (d)]
F	reaction force, § 6.7a
g	gravitational acceleration, § 6.2
h	distance measured from reference line, Fig. 6.2b
H	height of the channel, Fig. 6.8
$\mathbf{n}$	unit vector normal to an arbitrary path s, Fig. 6.2
p	pressure, Eq. (6-5a)
p_d	dynamic pressure, Eq. (6-10b)
p_0	stagnation pressure (or total pressure), Eq (6-11)
P_w	power generated by the vane, § 6.7d
$\mathbf{q}$	velocity vector, Eq. (6-2a)
Q	flow rate (volumetric flow), Eq. (6-1)
s	arbitrary path, Fig. 6.2
T	thrust, § 6.7b
V	velocity component along the path s
X	body force per unit mass, Eq. (6-6)
X_s	gravitational force per unit mass, Eq. (6-5b)
x, y, z	Cartesian coordinates
γ	specific weight, Eq. (6-9b)
γ_m	specific weight of manometer liquid, Eq. (6-12) and Fig. 6.9
ρ	density, Eq. (6-1)
τ	volume, Eq. (6-2a)

PROBLEMS

6.1. Water flows through a pipe of length l which tapers from the entrance radius r_1 to the exit radius r_2. If the entrance velocity is V_1, and the relation between r_2 and r_1 is given by $r_2 = r_1 \pm ml$ where m is the slope, verify that the exit velocity V_2 is

$$V_2 = V_1\left[1 \pm \frac{2m(l/r_1) + m^2(l/r_1)^2}{1 + 2m(l/r_1) + m^2(l/r_1)^2}\right],$$

6.2. In Prob. 6.1 if $V_1 = 10$ ft/sec, $V_2 = 6$ ft/sec, $r_1 = 6$ in., and $l = 6$ ft, find the radius r_2 at the exit section.

6.3. A 6 in. diameter pipe forks (see Fig. 6.A), one branch having a 3 in. diameter and the other a 4 in. diameter. If the velocities of water in the 3 in. and 4 in. pipes are 4 ft/sec and 2 ft/sec, respectively, calculate the velocity in feet per second and volumetric flow rate in gallons per second in the 6 in. diameter pipe.

6.4. Determine the discharge in Fig. 6.6 for

(a) $p_1 = 10$ psig and $h = 15$ ft, $\gamma = 64$ lbf/ft^3, and $A_2 = 4$ in.^{2}

(b) $p_1 = 10$ psia, $p_2 = 29.5$ in. mercury, $\gamma = 51.2$ lbf/ft^3, and $A_2 = 4$ in.^{2}

6.5. Calculate the height h shown in Fig. P.6.5 when the discharge is 1.0 ft^3/sec.

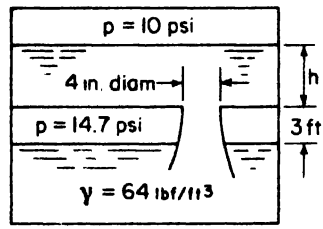


Figure P.6.5

6.6. Determine the discharge from a 1 in. diameter nozzle in Fig. 6.7 if $x = 20$ ft, $y = -5$ ft, and $\alpha = 45°$ above the horizontal.

6.7. Determine the location of the maximum height of the trajectory of the jet in Prob. 6.6. Calculate the velocity of the jet when it reaches the maximum height.

6.8. Calculate the horizontal distance required for the jet striking the ground which is 2 ft. above the horizontal line of the nozzle. The jet is inclined at a 30° angle with the horizontal at a velocity of 25 ft/sec.

6.9. Plot the pressure coefficient C_p versus the angular position of the semicylindrical protuberance in a closed channel shown in Fig. 6.8. The channel is completely filled with water. The height H_1 is 2 ft, and the radius of the semicylinder is 1 ft.

6.10. Using the Bernoulli equation and the condition of the static equilibrium of the manometer, show that Eq. (6-13) can be used in determining the velocity by a Pitot-static tube shown in Fig. 6.10. If water is used as the manometer fluid, determine the manometer deflection when the velocity of the air stream is 100 ft/sec.

6.11. A water Venturi meter has a throat diameter of 4 in. and a pipe diameter of 7 in. Calculate the velocity at the throat if the deflection of the mercury manometer which connects the pipe and the throat is 6 in. The coefficient of discharge is 0.96.

6.12. A Venturi meter has its axis vertical, the inlet and throat diameter ratio being 2.5. The throat is 12 in. above the inlet and the coefficient of discharge is 0.97. Determine the pressure difference between the inlet and throat when the velocity of water at the inlet is 6 ft/sec.

6.13. A Venturi meter with an inlet diameter of 10 in. is connected to an alcohol manometer which measures the pressure difference between inlet and throat. If the rate of flow passing through the meter is 100 ft^3/sec of gas ($\gamma = 0.062$ lbf/ft^3), calculate the diameter of the throat when the manometer deflection is 3 in. Alcohol has a specific gravity of 0.8.

6.14. Using the continuity equation, the Bernoulli equation, and the condition of the static equilibrium of the manometer, derive the flow rate across the orifice plate as given in Eq. (6-17b).

6.15. An orifice plate consists of a circular orifice (Fig. 6.12) in a 9 in. diameter pipe and has a coefficient of discharge of 0.65. If the mercury manometer which connects the two sides of the orifice plate has a deflection of 20 in., calculate the diameter of the orifice hole when the rate of flow of water in the pipeline is 2.25 ft^3/sec.

6.16. In Fig. 6.13, if $H = 0.833$ ft, $B = 9$ ft, and the coefficient of discharge $C_{d_w} = 0.6$, what is the rate of flow of water? If we want to keep 3 per cent accuracy in the calculation, what is the maximum V_1 which can be assumed negligible in Eq. (6-19)?

6.17. Derive the expression for the theoretical rate of flow [$C_{d_w} = 1.0$ in Eq. (6-21)] for a triangular weir (Fig. 6.13). If $\theta = 90°$, $C_{d_w} = 0.625$, and $Q = 1500$ gal/min of water, calculate H for the triangular weir.

6.18. In Fig. 6.14, if $B = 20$ ft and $h_1 = 2$ ft, plot the theoretical rate of discharge as a function of h_2/h_1. Determine graphically the value of Q_{max} and check this result with Eq. (6-24a).

6.19. In Fig. 6.15, if $R = 18$ in., $\alpha = 45°$, and $D = 8$ in., determine the rate of flow of water across the elbow when the mercury manometer has a deflection of 3 in.

6.20. A jet of water 2 in. in diameter discharges 1.0 ft^3/sec. Calculate the force required to move a flat plate toward the jet with a velocity of 25 ft/sec. The jet is perpendicular to the plate.

6.21. A jet of water 1 in. in diameter strikes a flat plate at an angle 30° to the normal of the plate with a velocity of 30 ft/sec. Determine the velocity of the plate, moving parallel to itself, if the normal force exerted by the jet is 2.5 lb.

6.22. A rectangular plate 4 in. wide and 10 in. long hangs vertically from hinges at its top edge. A jet of water 1 in. in diameter with a velocity of 30 ft/sec strikes the plate at its center. Determine the weight of the plate if the plate stays in equilibrium after deflecting 20° from its original position (Fig. P. 6.22).

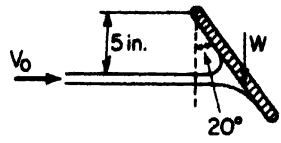


Figure P.6.22

6.23. In Fig. 6.19 the pressurized tank contains 500 gal of water and has an empty weight of 200 lbf. If $A_2 \doteq 10$ in.^{2}, $p_2 = 14.7$ lbf/in.^{2}, determine the minimum p_1 necessary to set the tank in motion. The friction in the wheels of the tank is 3 per cent of the total weight of the tank.

6.24. In Fig. 6.20 the pipeline gradually reduces from 18 in. to 10 in. in diameter and $\alpha = 30°$. Determine the magnitude and direction of the force exerted on the bend when the velocity V_1 is 10 ft/sec.

6.25. A horizontal straight pipe gradually reduces in diameter from 24 in. to 12 in. Determine the total longitudinal thrust exerted on the pipe if the pressure at the larger end is 50 lbf/in.^{2} and the velocity of the water is 8 ft/sec.

6.26. Water is discharged through two identical nozzles at opposite ends of an arm which is pivoted at its center. If the velocity of discharge relative to the nozzle is 25 ft/sec and the diameter is 1 in., calculate the torque which must be applied to hold the arm stationary (Fig. P.6.26).

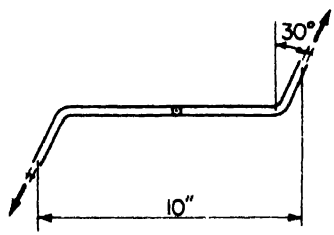


Figure P.6.26

6.27. A jet of water having a diameter of 1 in. and a velocity of 20 ft/sec is deflected through an angle of 60° by a fixed, curved vane. Calculate the force components exerted.

6.28. A nozzle discharges 1 ft^3/sec of water with a velocity of 30 ft/sec. The jet strikes a curved vane moving with a velocity of 10 ft/sec in the same direction as the jet. If the jet is deflected 45° as it leaves the vane, calculate the components of force exerted.

6.29. Assuming the velocity v_b of the vane as the only variable in Prob. 6.28, determine the maximum work done per second on the moving vane.

6.30. A jet of water having a diameter of 1 in. and a velocity of 40 ft/sec strikes a series of vanes so arranged on the periphery of a wheel that the entire jet is deflected through an angle of 170°. Calculate the maximum amount of work per unit time that can be done.

7

TWO- AND THREE-DIMENSIONAL INVISCID INCOMPRESSIBLE FLOW

The mechanics of an incompressible and inviscid fluid in two dimensions and three dimensions (axial symmetry) which will be discussed in the present chapter have both academic value and application to a variety of problems in the engineering field. It is believed that the concepts of two- and three-dimensional flows are in many ways similar so that it is advantageous to discuss them concurrently. The first part of this chapter (§7.1–§7.7) deals with the governing equations and the concept of basic theorems as follows:

In § 7.1 the equation of continuity for a three-dimensional, incompressible fluid is reduced from the general form in Chap. Five as well as derived in Cartesian coordinates.

In § 7.2 the Euler equation is found from the simplification of the Navier-Stokes equation in Chap. Five. It is also derived in the conventional manner.

In § 7.3 the circulation theorems of Stokes and Kelvin are derived and discussed. Several examples of flow involving circulation are presented.

In § 7.4 the concept of irrotational flow is discussed. Its consequence in the existence of the velocity potential function is explained.

In § 7.5 the Bernoulli equation is derived by integration of Euler's equation.

In § 7.6 the momentum theorem is derived by vector methods in the form of general coordinates.

In § 7.7 the moment of momentum theorem is also derived in the form of general coordinates.

In the second part (§ 7.8–§ 7.13), the Laplace equation is derived from the kinematics of irrotational flow of an ideal fluid. The stream function is defined as a consequence of the continuity equation and some simple flow examples are also discussed.

In § 7.8 the Laplace equation and harmonic functions are discussed. In particular the significance of the Laplace equation in fluid mechanics is pointed out. The general equation is also presented in cylindrical and spherical coordinates.

In § 7.9 the two-dimensional stream function is derived and discussed. The relationship of the stream function to the velocity components in both Cartesian and polar coordinates is presented. In addition the implications of a harmonic stream function are discussed.

In § 7.10 a convenient graphical method of analysis known as the flow net is discussed. In this section the roles of the velocity potential and the stream function in flow description are illustrated.

In § 7.11 the three-dimensional stream function for axially symmetric flows is derived. The velocity components for cylindrical and spherical coordinates are presented, and the relationship between the stream function and the velocity potential for axially symmetric flow is shown.

In § 7.12 certain simple two-dimensional flow examples are discussed. In particular ϕ, ψ, and the velocity components are derived for the uniform rectilinear flow, source-sink flow, vortex, doublet, and dipole.

In § 7.13 the three-dimensional forms of the uniform rectilinear flow, source-sink, and doublet are analyzed in detail.

The third part (§ 7.14–§ 7.19) deals with the motion of solid bodies in an ideal fluid. The pressure distribution on the bodies is found from the Bernoulli equation after the velocity field has been determined from the stream function.

In § 7.14 Rankine's method of constructing streamlines and equipotential lines is illustrated.

In § 7.15 the superposition of point source flow and rectilinear flow in both the two- and three-dimensional cases is discussed.

In § 7.16 the superposition of a point source and a point sink with a rectilinear flow in three dimensions is discussed. This is shown to result in a closed surface which is called the Rankine body.

In § 7.17 the superposition of line sources with a rectilinear flow in three dimensions is discussed. This flow pattern is shown to be useful in simulating streamline bodies such as airship hulls.

In § 7.18 the superposition of a doublet and a rectilinear flow in both the two- and three-dimensional cases is discussed. In the three-dimensional case this combination represents a uniform flow past a sphere. In the two-dimensional case the flow around a circular cylinder is simulated by the combination.

In § 7.19 the superposition of a vortex and a doublet with a rectilinear flow for the two-dimensional case is discussed. This is demonstrated to be the flow about a circular cylinder with circulation. The implications of the results of this analysis on the formulation of the airfoil theory are discussed.

A. BASIC EQUATIONS AND CONCEPTS OF FLOW

7.1. Equation of Continuity

For an incompressible fluid the equation of continuity is given in Eq. (5–9b) as follows:

$$\nabla \cdot \mathbf{q} = 0 \tag{5–9b}$$

With reference to Fig. 7.1, the equation above can be easily derived in a Cartesian coordinate system. The net amount of fluid flowing through the surface $\Delta y\, \Delta z$ in a unit time is

$$\frac{\partial u}{\partial x} \Delta x\, \Delta y\, \Delta z$$

The net outflows in the y and z-directions are similar. Therefore,

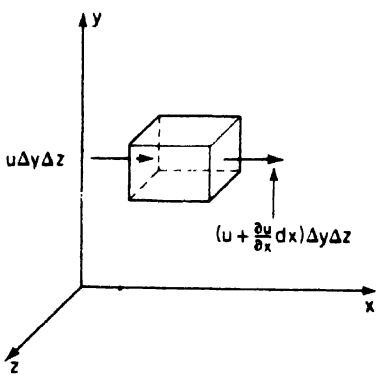


Figure 7.1

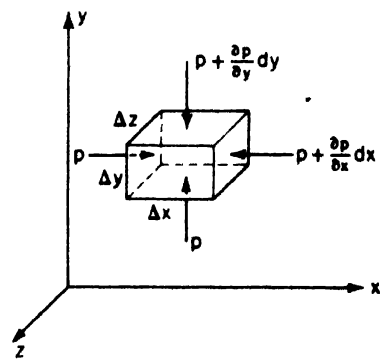


Figure 7.2

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\nabla\cdot\mathbf{q}=0 \tag{7-1}$$

7.2. Eulerian Equations of Motion

Euler's equation of motion given in Eq. (5-20) is valid for both compressible and incompressible flows

$$\frac{\partial \mathbf{q}}{\partial t}+\mathbf{q}\cdot\nabla\mathbf{q}=\mathbf{X}-\frac{\nabla p}{\rho} \tag{5-20}$$

In Cartesian coordinates Eq. (5-20) may also be derived as follows: Consider a fluid element of dimensions $\Delta x\,\Delta y\,\Delta z$ (Fig. 7.2). The forces acting on this fluid element consist of the body force and the normal pressure force exerted by the fluid around it. Let X_x be the body force per unit mass in the x-direction and $\Delta M_x=\rho\,\Delta x\,\Delta y\,\Delta z\,u$ be the momentum along the x-direction. According to Newton's second law we have

$$\frac{D}{Dt}(\Delta M_x)=-\frac{\partial p}{\partial x}\Delta x\,\Delta y\,\Delta z+X_x\rho\,\Delta x\,\Delta y\,\Delta z$$

or

$$\frac{Du}{Dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+X_x \tag{7-2a}$$

Similarly,

$$\frac{Dv}{Dt}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+X_y \tag{7-2b}$$

$$\frac{Dw}{Dt}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+X_z \tag{7-2c}$$

Eqs. (7-2a), (7-2b), and (7-2c) can be condensed into one vector equation which is identical to Eq. (5-20).

For a conservative system (see Ref. 32) the body force can be derived from a potential ϕ_f in such a way that

$$X_x=-\frac{\partial\phi_f}{\partial x},\qquad X_y=-\frac{\partial\phi_f}{\partial y},\qquad X_z=-\frac{\partial\phi_f}{\partial z}$$

Then

$$\mathbf{X}=\mathbf{i}X_x+\mathbf{j}X_y+\mathbf{k}X_z=-\left(\mathbf{i}\frac{\partial\phi_f}{\partial x}+\mathbf{j}\frac{\partial\phi_f}{\partial y}+\mathbf{k}\frac{\partial\phi_f}{\partial z}\right)=-\nabla\phi_f \tag{7-3}$$

so that Eq. (5-20) becomes

$$\frac{\partial \mathbf{q}}{\partial t}+(\mathbf{q}\cdot\nabla)\mathbf{q}=-\nabla\phi_f-\frac{\nabla p}{\rho} \tag{7-4}$$

7.3. Circulation Theorem

a. Circulation Concept

Consider a closed curve C placed in a moving fluid. The circulation around the contour C is defined as the summation of the velocity components tangential to C times the elemental length of the contour dr at every point on the contour. Let the velocity field be $\mathbf{q}$ at any arbitrary point and $d\mathbf{r}$ be the directed element of arc at the same point; then the circulation Γ is given by

$$\Gamma = \oint_C \mathbf{q} \cdot d\mathbf{r} = \oint_C q \cos\theta \, dr = \oint_C (u\,dx + v\,dy + w\,dz) \qquad (7\text{-}5)$$

where the symbol $\oint$ denotes a line integral taken around a closed curve in a counterclockwise direction (Fig. 7.3). The concept of circulation presented here is similar to the work done in a vector force field discussed in solid mechanics. In a vector force field, the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \qquad (7\text{-}6)$$

represents the work done in moving around the closed circuit C.

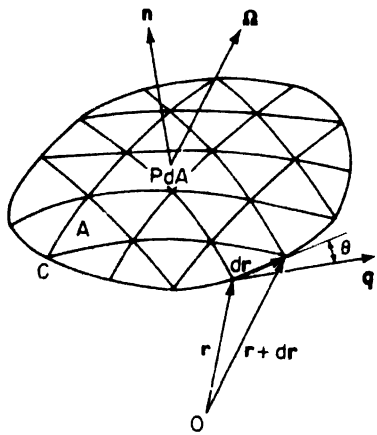


Figure 7.3

b. Stokes' Theorem

Let A be a surface bounded by the contour C and $\mathbf{n}$ be the unit vector normal to this surface at point P (Fig. 7.3). Stokes' theorem may then be expressed by

$$\Gamma = \oint_C \mathbf{q} \cdot d\mathbf{r} = \iint_A \boldsymbol{\Omega} \cdot \mathbf{n} \, dA \qquad (7\text{-}7)$$

The importance of the equation above is that it relates the circulation along a closed circuit C to the vorticity within the surface bounded by that line.

In order to prove the equation above, let us divide the space A (Fig. 7.3) into a very large number of infinitesimally small triangles ΔA. If lines are drawn from the vertices of these triangles parallel to the x, y, and z-axes, a series of infinitesimally small tetrahedrons is formed. Figure 7.4 shows one of these elementary tetrahedrons, $PABC$,

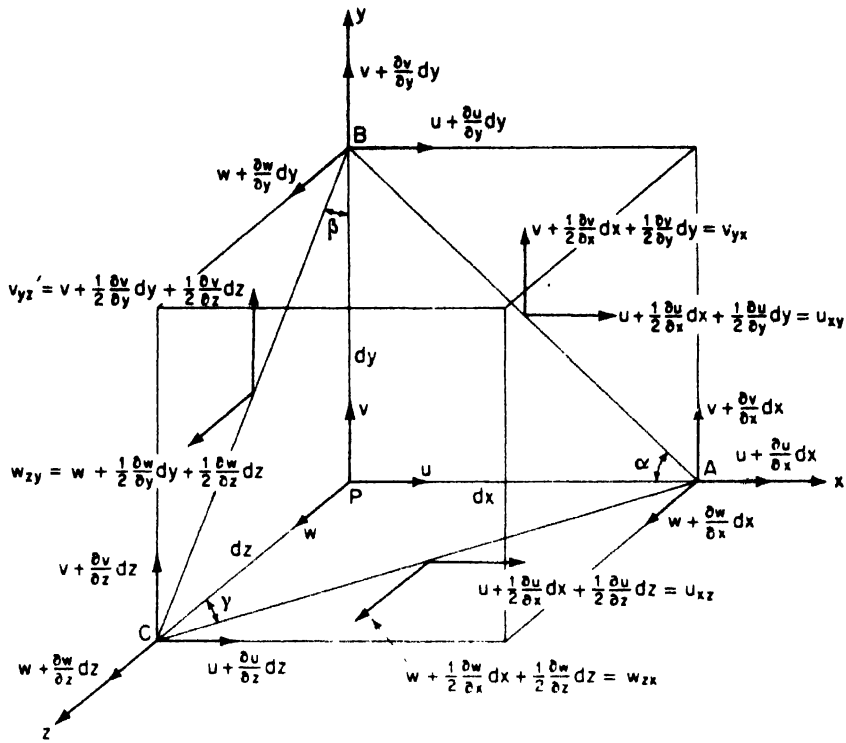


Figure 7.4

with edges PA, PB, and PC equal to dx, dy, and dz, respectively, and AB, BC, and CA equal to $dx/\cos\alpha = dy/\sin\alpha$, $dy/\cos\beta = dz/\sin\beta$, $dz/\cos\gamma = dx/\sin\gamma$, respectively. The velocity components at the midpoints of AB, BC, and CA which contribute to the circulation have the following values (they are calculated from the velocity components at the vertices, P, A, B, and C):

$$
\begin{aligned}
u_{xy} &= u + \frac{1}{2}\frac{\partial u}{\partial x} dx + \frac{1}{2}\frac{\partial u}{\partial y} dy \\
u_{xz} &= u + \frac{1}{2}\frac{\partial u}{\partial x} dx + \frac{1}{2}\frac{\partial u}{\partial z} dz \\
v_{yx} &= v + \frac{1}{2}\frac{\partial v}{\partial y} dy + \frac{1}{2}\frac{\partial v}{\partial x} dx \\
v_{yz} &= v + \frac{1}{2}\frac{\partial v}{\partial y} dy + \frac{1}{2}\frac{\partial v}{\partial z} dz \\
w_{zx} &= w + \frac{1}{2}\frac{\partial w}{\partial z} dz + \frac{1}{2}\frac{\partial w}{\partial x} dx
\end{aligned}
\tag{7-8}
$$

$$w_{xy} = w + \frac{1}{2}\frac{\partial w}{\partial z}dz + \frac{1}{2}\frac{\partial w}{\partial y}dy$$

Accordingly, the circulation along the sides of the triangle *ABCA* is given by (circulation is considered positive if it rotates according to the right-handed screw rule with normal outward)

$$\begin{aligned} d\Gamma &= -u_{xy}\cos\alpha(AB) + v_{yx}\sin\alpha(AB) - v_{yz}\cos\beta(BC) \\ &\quad + w_{zy}\sin\beta(BC) - w_{zx}\cos\gamma(CA) + u_{zx}\sin\gamma(CA) \\ &= -\frac{1}{2}\frac{\partial u}{\partial y}dx\,dy + \frac{1}{2}\frac{\partial u}{\partial z}dz\,dx + \frac{1}{2}\frac{\partial v}{\partial x}dx\,dy - \frac{1}{2}\frac{\partial v}{\partial z}dz\,dy \\ &\quad - \frac{1}{2}\frac{\partial w}{\partial x}dx\,dz + \frac{1}{2}\frac{\partial w}{\partial y}dy\,dz \\ &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)dy\,dz + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dz\,dx \\ &\quad + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy \end{aligned} \tag{7-9}$$

But

$$\begin{aligned} 2\boldsymbol{\Omega}\cdot\mathbf{n}\,dA &= 2(\mathbf{i}\Omega_x + \mathbf{j}\Omega_y + \mathbf{k}\Omega_z)\cdot\left(\mathbf{i}\frac{dy\,dz}{2} + \mathbf{j}\frac{dz\,dx}{2} + \mathbf{k}\frac{dx\,dy}{2}\right) \\ &= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial x}\right)dy\,dz + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dz\,dx \\ &\quad + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy \end{aligned} \tag{7-10}$$

Hence,

$$d\Gamma = \boldsymbol{\Omega}\cdot\mathbf{n}\,dA \tag{7-11}$$

In the same manner the circulation around all other elementary triangles, dA, can be formed for the entire surface A, and since the circulation along the elementary sides common to two triangles cancels, the remaining circulation will be that around the closed contour C. Hence the Stokes' theorem expressed in Eq. (7-7) is verified.

c. Kelvin's Theorem, Constancy of Circulation

In an ideal homogeneous fluid the circulation is independent of time along a closed fluid line when the external forces are conservative and are derived from a single-valued potential. This fundamental theorem is due to Lord Kelvin.* The proof of Kelvin's theorem can be made by finding the time derivative of the circulation [Eq. (7-5)]

*This theorem is sometimes called Thomson's theorem since Lord Kelvin's given name was William Thomson.

as follows:

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint_c \mathbf{q} \cdot d\mathbf{r} \tag{7-12}$$

Since the integration above is accomplished at constant time, it is proper to reverse the order of integration and differentiation. Hence, Eq. (7-12) can be written

$$\frac{D\Gamma}{Dt} = \oint_c \frac{D}{Dt}(\mathbf{q} \cdot d\mathbf{r}) \tag{7-12a}$$

but

$$\frac{D}{Dt}(\mathbf{q} \cdot d\mathbf{r}) = d\mathbf{r} \cdot \frac{D\mathbf{q}}{Dt} + \mathbf{q} \cdot d\mathbf{q} \tag{7-13}$$

By scalar multiplication of each term in the equation of motion [Eq. (7-4)] by $d\mathbf{r}$,

$$d\mathbf{r} \cdot \frac{D\mathbf{q}}{Dt} = -d\mathbf{r} \cdot \nabla\phi_f - \frac{1}{\rho} d\mathbf{r} \cdot \nabla p = -d\phi_f - \frac{dp}{\rho} \tag{7-14}$$

where the body force $\mathbf{X}$ is derived from a potential ϕ_f. Substituting Eq. (7-14) into Eq. (7-13), we obtain

$$\frac{D}{Dt}(\mathbf{q} \cdot d\mathbf{r}) = -d\phi_f - \frac{dp}{\rho} + \mathbf{q} \cdot d\mathbf{q} \tag{7-15}$$

Integrating the expression above along the closed contour C results in

$$\frac{D\Gamma}{Dt} = -\left[\phi_f + \frac{p}{\rho} - \frac{q^2}{2}\right]_c \tag{7-16}$$

If, as assumed, ϕ_f, p, and $\mathbf{q}$ are single-valued functions* of $\mathbf{r}$, then the right-hand side of Eq. (7-16) is equal to zero. Equation (7-16) is the rate of change of circulation along a closed curve and Kelvin's theorem has been proved.

This theorem is valid for any kind of flow of an inviscid fluid as long as forces are conservative. The proof of Kelvin's theorem as applied to the compressible fluid will be discussed in § 11.6.

Illustrative Example 7.1. Calculate the circulation Γ of a circulatory flow ($v_r = v_z = 0$) with constant vorticity ($\Omega_z = \text{const.}$) from both the circulation concept and Stokes' theorem.

Solution. The circulatory flow with constant vorticity is shown in Fig. 7.A. From the vorticity equation, Eq. (3-29), we have

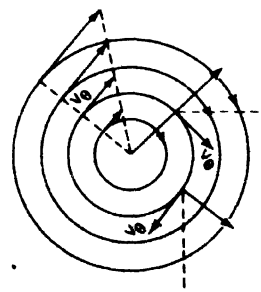


Figure 7.A

*If for every value of x in the interval, $y = f(x)$ has only one value, y is a single-valued function of x in the interval; if y has more than one value for each value of x, it is a multiple-valued function of x.

$$\Omega_z = \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} = \text{Constant} \tag{a}$$

and from Eq. (5–41), the continuity equation reduces to

$$\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0 \tag{b}$$

Hence,

$$v_\theta = f(r) \tag{c}$$

According to Eq. (7–5) the circulation Γ is given by

$$\Gamma = \oint_C \mathbf{q} \cdot d\mathbf{r} \tag{d}$$

Substituting Eq. (c) into Eq. (d) we obtain

$$\Gamma = \int_0^{2\pi} v_\theta r \, d\theta = \int_0^{2\pi} f(r)\, r \, d\theta = 2\pi r f(r) \tag{e}$$

On the other hand Stokes' theorem is given by [Eq. (7–7)]

$$\Gamma = \iint_A \boldsymbol{\Omega} \cdot \mathbf{n} \, dA \tag{f}$$

With the aid of Eq. (a), Eq. (f) becomes

$$\Gamma = \int_0^{2\pi}\int_0^r \Omega_z r \, dr \, d\theta = \int_0^{2\pi}\int_0^r \left(\frac{dv_\theta}{dr} + \frac{v_\theta}{r}\right) r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^r \frac{d}{dr}(v_\theta r)\, dr \, d\theta = \int_0^{2\pi} v_\theta r \, d\theta = 2\pi r v_\theta \tag{g}$$

Since $v_\theta = f(r)$, Eqs. (e) and (g) are identical and Stokes' theorem is satisfied. The circulatory velocity is [from Eqs. (g) and (a)]

$$v_\theta = \frac{\Gamma}{2\pi r} \tag{h}$$

Illustrative Example 7.2. In Fig. 7.B is shown the motion in concentric circles. In this case we consider a flow for which an integral portion of fluid bounded by a circle (the core) is undergoing rigid-body motion with constant angular velocity Ω_z while a concentric outer portion of the fluid represents a vortex-free circulatory flow. Calculate the tangential velocity in both regions.

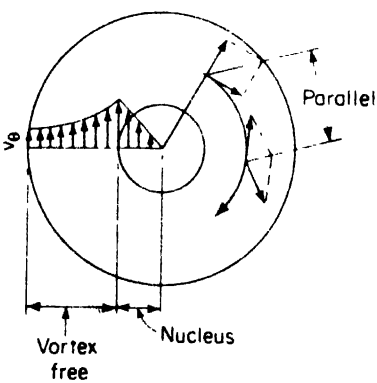


Figure 7.B

Solution. From illustrative example 7.1 we have

$$\Omega_z = \frac{dv_\theta}{dr} + \frac{v_\theta}{r} \tag{a}$$

$$v_\theta = f(r) \tag{b}$$

In the outer region of the flow, we have

$$\Omega_z = 0 \tag{c}$$

Hence, Eq. (a) becomes

$$\frac{dv_\theta}{dr} + \frac{v_\theta}{r} = 0 \tag{d}$$

Integrating,

$$\ln v_\theta + \ln r = \ln C$$

or

$$v_\theta = \frac{C}{r} \tag{e}$$

This represents a vortex-free circulatory flow. In the inner region, Eq. (a) becomes

$$\frac{dv_\theta}{dr} + \frac{v_\theta}{r} = A_1 \tag{f}$$

where A_1 is a constant. Equation (f) can be written as

$$\frac{1}{r}\frac{d}{dr}(v_\theta r) = A_1 \tag{g}$$

Integrating,

$$v_\theta r = A_1 \frac{r^2}{2}$$

or

$$v_\theta = Ar \tag{h}$$

The constant of integration is made zero from the condition $v_\theta = 0$ at $r = 0$. Equation (h) represents either a rotating solid body or a fluid rotating like a solid body in the nucleus. This concept is employed in the two-dimensional airfoil theory wherein vortex-free flow exists outside the boundary of a solid body such as an airfoil.

7.4. Velocity Potential—Irrotational Flow

The movement of ideal fluids is, in general, of two types, each differing from the other both physically and mathematically. They are called rotational and irrotational flows. The nature of rotational flow has been discussed in § 3.5 and § 7.3. The importance of irrotational flow can be seen as follows: From the definition of circulation we have

$$\Gamma = \oint_C \mathbf{q} \cdot d\mathbf{r} = \iint_A \mathbf{n} \cdot \boldsymbol{\Omega}\, dA \tag{7-7}$$

if

$$\boldsymbol{\Omega} = \nabla \times \mathbf{q} = 0, \qquad \Gamma = 0$$

Let us consider a closed contour C with circuit *ABDEA* in the fluid (Fig. 7.5). The circuit *ABDEA* in this case must lie in a simply connected region. This means that the closed contour must include only fluid and can be contracted to a point of the region without ever passing out of the region. Examples of simply connected regions are the region interior or exterior to a sphere and the region between two concentric spheres. Now Eq. (7–7) can be written in the following form:

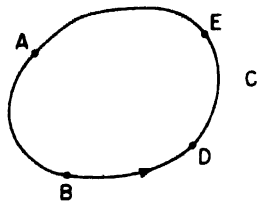


Figure 7.5

$$\int_{ABDEA} \mathbf{q} \cdot d\mathbf{r} = 0 = \int_{ABD} \mathbf{q} \cdot d\mathbf{r} + \int_{DEA} \mathbf{q} \cdot d\mathbf{r}$$

Hence,

$$\int_{ABD} \mathbf{q} \cdot d\mathbf{r} = \int_{AED} \mathbf{q} \cdot d\mathbf{r} = \int_{A}^{D(x,y,z)} \mathbf{q} \cdot d\mathbf{r} = \int_{A}^{D(x,y,z)} d\phi = \phi_D - \phi_A \tag{7–17}$$

where ϕ_D is a scalar quantity of space whose value depends only on the position D (and on the fixed point A) and not on the path from A to D. From Eq. (7–17) it follows that

$$\mathbf{q} \cdot d\mathbf{r} = d\phi = d\mathbf{r} \cdot \nabla\phi \tag{7–18a}$$

or

$$u\,dx + v\,dy + w\,dz = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \tag{7–18b}$$

Hence,

$$\mathbf{q} = \nabla\phi \tag{7–19a}$$

or

$$u = \frac{\partial\phi}{\partial x}, \qquad v = \frac{\partial\phi}{\partial y}, \qquad w = \frac{\partial\phi}{\partial z} \tag{7–19b}$$

Thus when the motion is irrotational the velocity vector is the gradient of a scalar function of position, ϕ. The scalar function ϕ is called the velocity potential. We can then see that the necessary condition of the existence of a velocity potential in a fluid is irrotational motion. Equation (7–19a) can be further verified by

$$\nabla \times \mathbf{q} = \nabla \times (\nabla\phi) = 0 \tag{7–20}$$

The consideration above is at a certain instant independent of time; however, the existence of the velocity potential is not limited to steady flow.

Illustrative Example 7.3. Give examples of irrotational and rotational flows.

Solution. (a) In Fig. 7.C is shown parallel flow with uniform velocity. In this case we have the following velocity components:

$$u = Ax, \qquad v = 0, \qquad w = 0 \tag{a}$$

The vorticity about the z-axis is

$$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{b}$$

Hence, the flow is irrotational. Furthermore, the fluid elements undergo no distortion, but translation only.

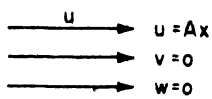


Figure 7.C

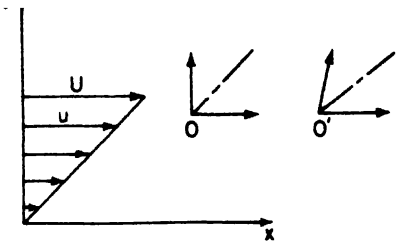


Figure 7.D

(b) A two-dimensional shear flow is shown in Fig. 7.D (see Chap. Eight). In this case the velocity components are

$$u = Ay, \qquad v = 0, \qquad w = 0 \tag{c}$$

The vorticity and shearing stress are, respectively,

$$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -A \tag{d}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = A \tag{e}$$

It can be seen that in a two-dimensional shear flow, every fluid element experiences both rotation and distortion. The rotation of the fluid elements characterizes a vortex flow but does not necessarily imply the rotation of the whole field.

7.5. Integration of the Equations of Motion—Bernoulli's Equation

In order to introduce the velocity potential ϕ in Euler's equation of motion when the flow is irrotational, the convective terms in Eq. (5–20) are expanded as follows:

$$\nabla(\mathbf{q}_1 \cdot \mathbf{q}_2) = \mathbf{q}_1 \cdot \nabla\mathbf{q}_2 + \mathbf{q}_2 \cdot \nabla\mathbf{q}_1 + \mathbf{q}_1 \times (\nabla \times \mathbf{q}_2) + \mathbf{q}_2 \times (\nabla \times \mathbf{q}_1)$$

For $\mathbf{q}_1 = \mathbf{q}_2 = \mathbf{q}$, we have

$$\nabla(\mathbf{q} \cdot \mathbf{q}) = 2\mathbf{q} \cdot \nabla\mathbf{q} + 2\mathbf{q} \times (\nabla \times \mathbf{q})$$

or

$$(\mathbf{q} \cdot \nabla)\mathbf{q} = \tfrac{1}{2}\nabla q^2 - \mathbf{q} \times \boldsymbol{\Omega} \tag{7-21}$$

Substituting Eq. (7-21) into Eq. (5-20), we obtain

$$\frac{\partial \mathbf{q}}{\partial t} - \mathbf{q} \times \boldsymbol{\Omega} = \mathbf{X} - \frac{\nabla p}{\rho} - \nabla\left(\frac{q^2}{2}\right) = -\nabla\left(\phi_f + \frac{p}{\rho} + \frac{q^2}{2}\right) \tag{7-22}$$

a. Steady Motion

For a steady motion Eq. (7-22) becomes

$$\mathbf{q} \times \boldsymbol{\Omega} = \nabla\left(\phi_f + \frac{p}{\rho} + \frac{q^2}{2}\right)$$

or

$$(\mathbf{q} \times \boldsymbol{\Omega}) \cdot d\mathbf{r} = d\left(\phi_f + \frac{p}{\rho} + \frac{q^2}{2}\right) \tag{7-23}$$

If we integrate the equation above along a streamline, the term $(\mathbf{q} \times \boldsymbol{\Omega}) \cdot d\mathbf{r}$ vanishes and Eq. (7-23) becomes

$$\phi_f + \frac{p}{\rho} + \frac{q^2}{2} = C, \tag{7-24}$$

where C, is a constant along a streamline but varies from streamline to streamline.

b. Irrotational Flow

For an irrotational flow, $\boldsymbol{\Omega} = 0$, we can rewrite the first term on the left-hand side of Eq. (7-22) in terms of the velocity potential.

$$\frac{\partial \mathbf{q}}{\partial t} = \frac{\partial(\nabla\phi)}{\partial t} = \nabla\frac{\partial \phi}{\partial t}$$

Hence Eq. (7-22) becomes

$$\nabla\left(\frac{\partial \phi}{\partial t} + \phi_f + \frac{p}{\rho} + \frac{q^2}{2}\right) = 0 \tag{7-25a}$$

By integration of the equation above, after taking the dot product of $d\mathbf{r}$, we have

$$\frac{\partial \phi}{\partial t} + \frac{q^2}{2} + \frac{p}{\rho} + \phi_f = F(t) \tag{7-25b}$$

where $F(t)$ is an arbitrary function of time.

In steady, irrotational flow the equation above reduces to Bernoulli's equation for incompressible fluids,

$$\frac{q^2}{2} + \frac{p}{\rho} + \phi_f = \mathrm{C} \tag{7-26}$$

where the constant C has the same value for the entire fluid regime. Equation (7–26) reduces to Eq. (6–9a) if we put $q = V$ and $\phi_f = gh$.

7.6. *The Momentum Theorem*

In § 6.6 the momentum equation for a one-dimensional flow has been derived, and its advantage in the application to physical problems has been demonstrated. In a similar manner the momentum equation in a three-dimensional flow can be derived for a control volume through which the fluid flows.

From Eq. (5–13) the law of conservation of momentum may be written as

$$\iiint \rho \frac{D\mathbf{q}}{Dt}\, d\tau = \mathbf{F}_S + \mathbf{F}_B \tag{7-27}$$

where $\mathbf{F}_S$ and $\mathbf{F}_B$ are the resultants of the surface and body forces, respectively. The term on the left-hand side of Eq. (7–27) represents the rate change of momentum, and its integrand can be expanded as follows:

$$\begin{aligned}\rho \frac{D\mathbf{q}}{Dt} &= \rho\left[\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial(u\mathbf{q})}{\partial x} + \frac{\partial(v\mathbf{q})}{\partial y} + \frac{\partial(w\mathbf{q})}{\partial z} - \mathbf{q}(\nabla \cdot \mathbf{q})\right] \\ &= \rho\left\{\frac{\partial \mathbf{q}}{\partial t} + \mathbf{i}\left[\frac{\partial(u^2)}{\partial x} + \frac{\partial(uv)}{\partial y} + \frac{\partial(uw)}{\partial z}\right] + \mathbf{j}\left[\frac{\partial(uv)}{\partial x} + \frac{\partial(v^2)}{\partial y}\right.\right. \\ &\quad \left.\left. + \frac{\partial(vw)}{\partial z}\right] + \mathbf{k}\left[\frac{\partial(uw)}{\partial x} + \frac{\partial(vw)}{\partial y} + \frac{\partial(w^2)}{\partial z}\right] - \mathbf{q}(\nabla \cdot \mathbf{q})\right\}\end{aligned} \tag{7-28a}$$

Since

$$\begin{aligned}\mathbf{i}\nabla \cdot (u\mathbf{q}) &= \mathbf{i}\left[\frac{\partial(u^2)}{\partial x} + \frac{\partial(uv)}{\partial y} + \frac{\partial(uw)}{\partial z}\right] \\ \mathbf{j}\nabla \cdot (v\mathbf{q}) &= \mathbf{j}\left[\frac{\partial(uv)}{\partial x} + \frac{\partial(v^2)}{\partial y} + \frac{\partial(vw)}{\partial z}\right] \\ \mathbf{k}\nabla \cdot (w\mathbf{q}) &= \mathbf{k}\left[\frac{\partial(uw)}{\partial x} + \frac{\partial(vw)}{\partial y} + \frac{\partial(w^2)}{\partial z}\right]\end{aligned} \tag{7-28b}$$

and

$$\nabla \cdot \mathbf{q} = 0$$

Eq. (7–28) becomes

$$\rho \frac{D\mathbf{q}}{Dt} = \rho \frac{\partial \mathbf{q}}{\partial t} + \rho[\mathbf{i}\nabla \cdot (u\mathbf{q}) + \mathbf{j}\nabla \cdot (v\mathbf{q}) + \mathbf{k}\nabla \cdot (w\mathbf{q})] \tag{7-29}$$

or

$$\iiint \rho \frac{D\mathbf{q}}{Dt} d\tau = \iiint \rho \frac{\partial \mathbf{q}}{\partial t} d\tau + \iiint \rho[\mathbf{i}\nabla \cdot (u\mathbf{q}) + \mathbf{j}\nabla \cdot (v\mathbf{q}) + \mathbf{k}\nabla \cdot (w\mathbf{q})] d\tau \tag{7-30}$$

According to Gauss's theorem (Appendix A), Eq. (7-30) may be written as

$$\iiint \rho \frac{D\mathbf{q}}{Dt} d\tau = \iiint \rho \frac{\partial \mathbf{q}}{\partial t} d\tau + \iint \rho[(\mathbf{i}u) + (\mathbf{j}v) + (\mathbf{k}w)](\mathbf{q} \cdot \mathbf{n}) dA$$

or

$$\iiint \rho \frac{D\mathbf{q}}{Dt} d\tau = \iiint \rho \frac{\partial \mathbf{q}}{\partial t} d\tau + \iint \rho \mathbf{q}(\mathbf{q} \cdot \mathbf{n}) dA \tag{7-31}$$

where $\mathbf{n}$ is the normal directed outward. Denoting the direction cosines of the normal to dA by l, m, and n, and remembering that

$$(\mathbf{q} \cdot \mathbf{n}) dA = (lu + mv + nw) dA = dQ \tag{7-32}$$

represents the volumetric flow dQ passing through the element of surface dA, then the momentum equation becomes

$$\iiint \rho \frac{\partial \mathbf{q}}{\partial t} d\tau + \iint \rho \mathbf{q}\, dQ = \mathbf{F}_S + \mathbf{F}_B \tag{7-33}$$

or, in components,

$$\iiint \rho \frac{\partial u}{\partial t} d\tau + \iint \rho u\, dQ = (F_S)_x + (F_B)_x \tag{7-34a}$$

$$\iiint \rho \frac{\partial v}{\partial t} d\tau + \iint \rho v\, dQ = (F_S)_y + (F_B)_y \tag{7-34b}$$

$$\iiint \rho \frac{\partial w}{\partial t} d\tau + \iint \rho w\, dQ = (F_S)_z + (F_B)_z \tag{7-34c}$$

Illustrative Example 7.4 Consider the vertical flight of a rocket in a frictionless medium. If the rocket motor is assumed to operate under steady conditions so that the chamber pressure and the effective exhaust velocity do not change with time, calculate the velocity of the rocket at any instant t with reference to a fixed reference.

Solution. Let m_0 be the total mass of the rocket system at the beginning of launching. The mass of the rocket system at any time t after firing is given by

$$m = m_0 - \alpha t \tag{a}$$

where α is the mass of fuel consumed per unit time. If A_e, ρ_e, p_e, and

v_e are chosen to denote the cross-sectional area, the mass density of the discharged gas, the discharge pressure, and the velocity at the exit, then

$$\alpha = \rho_e A_e v_e \tag{b}$$

Since the rocket operates under steady conditions, the momentum theorem [Eq. (7–34)] gives (control volume shown in Fig. 7.E is assumed attached to the rocket)

$$m\frac{dv}{dt} = \rho_e A_e v_e^2 + p_e A_e - mg \tag{c}$$

The first two terms on the right-hand side of Eq. (c) are usually referred to as the thrust and the last term is the gravitational force. Substituting Eqs. (a) and (b) into Eq. (c) we find

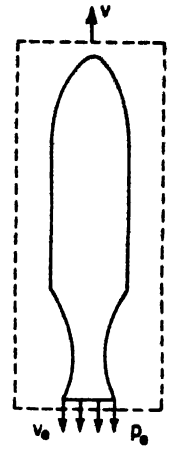


Figure 7.E

$$(m_0 - \alpha t)\frac{dv}{dt} = \alpha v_e + p_e A_e^* - (m_0 - \alpha t)g \tag{d}$$

or

$$dv = \left[\frac{\alpha v_e + p_e A_e}{m_0 - \alpha t} - g\right] dt \tag{e}$$

Integrating,

$$\begin{aligned} v &= -\left(\frac{\alpha v_e + p_e A_e}{\alpha}\right)[\ln (m_0 - \alpha t) - \ln m_0] - gt \\ &= \left(v_e + \frac{p_e A_e}{\alpha}\right) \ln \left(\frac{m_0}{m_0 - \alpha_0 t}\right) - gt \end{aligned} \tag{f}$$

Equation (f) represents the velocity of the rocket at any instant t after launching.

7.7. The Moment of Momentum Theorem

In the fundamental theorem of mechanics of discrete particles, Newton's law for moment of momentum states that the rate of change of the moment of momentum about a point or axis is equal to the moment of the external forces. Based on this law we can derive the equation of the moment of momentum from the Euler equation.

Let O be the origin of any reference frame, $\mathbf{r}$ the position vector of a point P, and $\rho\, d\tau$ the elementary mass at P (Fig. 7.6). The moment of the element of inertial force about the origin is

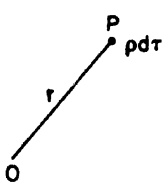


Figure 7.6

$$\mathbf{r} \times \frac{D\mathbf{q}}{Dt}\rho\, d\tau = \left(\mathbf{r} \times \frac{D\mathbf{q}}{Dt} + \mathbf{q} \times \mathbf{q}\right)\rho d\tau = \left(\mathbf{r} \times \frac{D\mathbf{q}}{Dt} + \frac{D\mathbf{r}}{Dt} \times \mathbf{q}\right)\rho\, d\tau$$
$$= \frac{D}{Dt}(\mathbf{r} \times \mathbf{q})\rho\, d\tau \tag{7-35}$$

Hence the rate change of the moment of momentum is

$$\iiint \mathbf{r} \times \frac{D\mathbf{q}}{Dt}\rho\, d\tau = \iiint \frac{D}{Dt}(\mathbf{r} \times \mathbf{q})\rho\, d\tau \tag{7-36}$$

The moment of the resultant of surface and body forces at the origin O are $\mathbf{M}_S$ and $\mathbf{M}_B$, respectively. According to Newton's law for moment of momentum, the equation of the moment of momentum is

$$\iiint \frac{D}{Dt}(\mathbf{r} \times \mathbf{q})\rho\, d\tau = \mathbf{M}_S + \mathbf{M}_B \tag{7-37}$$

but

$$\frac{D}{Dt}(\mathbf{r} \times \mathbf{q}) = \frac{\partial}{\partial t}(\mathbf{r} \times \mathbf{q}) + \frac{\partial(u\mathbf{r} \times \mathbf{q})}{\partial x} + \frac{\partial(v\mathbf{r} \times \mathbf{q})}{\partial y}$$
$$+ \frac{\partial(w\mathbf{r} \times \mathbf{q})}{\partial z} - (\mathbf{r} \times \mathbf{q})[\nabla \cdot \mathbf{q}]$$
$$= \frac{\partial \mathbf{R}}{\partial t} + \frac{\partial(u\mathbf{R})}{\partial x} + \frac{\partial(v\mathbf{R})}{\partial y} + \frac{\partial(w\mathbf{R})}{\partial z} \tag{7-38}$$

where $\mathbf{R} = \mathbf{r} \times \mathbf{q}$ is used for simplification. Following the expansion given in Eq. (7-28b) and expressing $\mathbf{R} = \mathbf{i}R_x + \mathbf{j}R_y + \mathbf{k}R_z$, we obtain

$$\mathbf{i}\nabla \cdot (R_x\mathbf{q}) = \mathbf{i}\left[\frac{\partial(R_x u)}{\partial x} + \frac{\partial(R_x v)}{\partial y} + \frac{\partial(R_x w)}{\partial z}\right]$$
$$\mathbf{j}\nabla \cdot (R_y\mathbf{q}) = \mathbf{j}\left[\frac{\partial(R_y u)}{\partial x} + \frac{\partial(R_y v)}{\partial y} + \frac{\partial(R_y w)}{\partial z}\right] \tag{7-39}$$
$$\mathbf{k}\nabla \cdot (R_z\mathbf{q}) = \mathbf{k}\left[\frac{\partial(R_z u)}{\partial x} + \frac{\partial(R_z v)}{\partial y} + \frac{\partial(R_z w)}{\partial z}\right]$$

Hence Eq. (7-38) becomes

$$\frac{D}{Dt}(\mathbf{r} \times \mathbf{q}) = \frac{\partial}{\partial t}(\mathbf{r} \times \mathbf{q}) + [\mathbf{i}\nabla \cdot (R_x\mathbf{q}) + \mathbf{j}\nabla \cdot (R_y\mathbf{q}) + \mathbf{k}\nabla \cdot (R_z\mathbf{q})] \tag{7-40}$$

or

$$\iiint \rho \frac{D}{Dt}(\mathbf{r} \times \mathbf{q})d\tau = \iiint \rho \frac{\partial}{\partial t}(\mathbf{r} \times \mathbf{q})d\tau$$
$$+ \iiint \rho[\mathbf{i}\nabla \cdot (R_x\mathbf{q}) + \mathbf{j}\nabla \cdot (R_y\mathbf{q}) + \mathbf{k}\nabla \cdot (R_z\mathbf{q})]\, d\tau \tag{7-41}$$

Introducing Gauss's theorem, Eq. (7–41) may be written as

$$\iiint \rho \frac{D}{Dt}(\mathbf{r} \times \mathbf{q})\, d\tau = \iiint \rho \frac{\partial}{\partial t}(\mathbf{r} \times \mathbf{q})\, d\tau$$

$$+ \iint \rho[\mathbf{i}R_x + \mathbf{j}R_y + \mathbf{k}R_z](\mathbf{q} \cdot \mathbf{n})\, dA$$

$$= \iiint \rho \frac{\partial}{\partial t}(\mathbf{r} \times \mathbf{q})\, d\tau + \iint \rho(\mathbf{r} \times \mathbf{q})\, dQ \qquad (7\text{–}42)$$

Finally the equation for the moment of momentum becomes

$$\iiint \rho \frac{\partial(\mathbf{r} \times \mathbf{q})}{\partial t} d\tau + \iint \rho(\mathbf{r} \times \mathbf{q})\, dQ = \mathbf{M}_S + \mathbf{M}_B \qquad (7\text{–}43)$$

or, in components with $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$,

$$\iiint \rho \frac{\partial}{\partial t}(yw - zv)\, d\tau + \iint \rho(yw - zv)\, dQ = (M_S)_x + (M_B)_x$$

$$\iiint \rho \frac{\partial}{\partial t}(zu - xw)\, d\tau + \iint \rho(zu - xw)\, dQ = (M_S)_y + (M_B)_y \qquad (7\text{–}44)$$

$$\iiint \rho \frac{\partial}{\partial t}(xv - yu)\, d\tau + \iint \rho(xv - yu)\, dQ = (M_S)_z + (M_B)_z$$

Illustrative Example 7.5. In Fig. 7.F is shown a lawn sprinkler which is held in a stationary position while water is discharging at the exit pipe area A. Assuming the discharge velocity is constant and R is much larger than the pipe diameter, calculate the torque in the rotor.

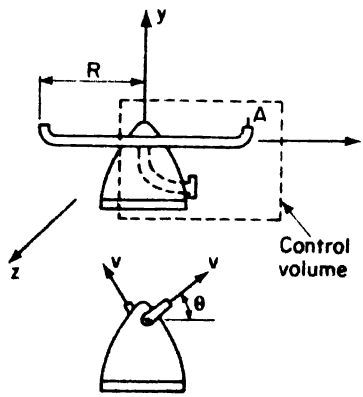


Figure 7.F

Solution. The velocity vector relative to the nonrotating frame (see control volume) is

$$\mathbf{q} = \mathbf{i}_z v \cos\theta \qquad (a)$$

and the position vector $\mathbf{r} = \mathbf{i}_z R$. Hence,

$$\mathbf{r} \times \mathbf{q} = (\mathbf{i}_z R) \times (\mathbf{i}_v v \cos\theta) = -\mathbf{i}_y R v \cos\theta \tag{b}$$

Since $\mathbf{r} \times \mathbf{q}$ is a constant and $\mathbf{M}_B = 0$, the equation for the moment of momentum [Eq. (7–43)] becomes

$$T = \rho(\mathbf{r} \times \mathbf{q})Q \tag{c}$$

Substituting Eq. (b) and $Q = 2vA$ into Eq. (c), we obtain the torque on the rotor,

$$T = -2\rho R v^2 A \cos\theta \tag{d}$$

Illustrative Example 7.6. In order to apply the equations of moment of momentum [Eq. (7–43)] or angular momentum to flow through turbines or compressors, cylindrical coordinates may be adopted. In the case of a turbine, the blades derive a torque from the flow and the compressor, the torque is imposed on the blades, and thus the angular momentum of the flow is increased. If the axis of rotation of the impeller of a compressor (or the runner of a turbine) is taken to be the z-axis, the velocity vector in the $r - \theta$ plane is

$$\mathbf{q} = \mathbf{i}_r v_r + \mathbf{i}_\theta v_\theta \tag{a}$$

and

$$\mathbf{r} \times \mathbf{q} = \mathbf{i}_r r \times (\mathbf{i}_r v_r + \mathbf{i}_\theta v_\theta) = \mathbf{i}_z r v_\theta \tag{b}$$

Hence Eq. (7–43) can be written for steady flow,

$$T_z = (M_z)_z = \int\int \rho r v_\theta \, dQ \tag{c}$$

but

$$dQ = \mathbf{n} \cdot \mathbf{q} \, dA = q \cos\theta \, dA \tag{d}$$

where θ is the angle between $\mathbf{n}$ and $\mathbf{q}$. Finally the torque becomes

$$T_z = \int\int v_\theta r \rho q \cos\theta \, dA \tag{e}$$

In Fig. 7.G(a) is shown the top view of a turbine which is also the control volume. The fluid flowing over the blade exerts a torque on the shaft which is rotating at 400 rpm. If water enters the blade radially with a velocity of 100 ft/sec, calculate the horsepower developed by the turbine. The passage between blades is $\frac{1}{2}$ ft.

Solution. The control volume is taken between the entrance and the exit. Since the velocities are assumed to be uniform over the entrance and exit areas, $\mathbf{n} \perp \mathbf{q}$ over other areas of the control volume. The angular momentum equation [Eq. (e)] can be written:

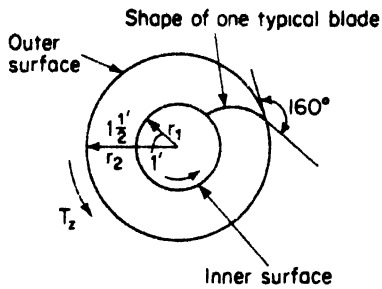


Figure 7.G(a)

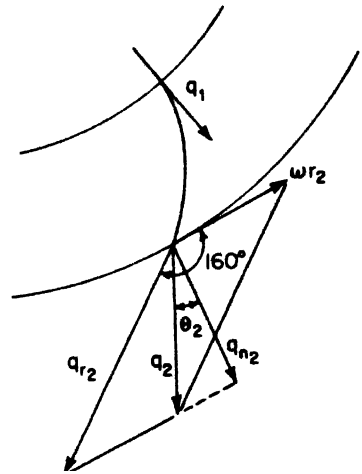


Figure 7.G(b)

$$T_z = v_{\theta_2} r_2 A_2 \rho q_2 \cos\theta_2 - v_{\theta_1} r_1 A_1 \rho q_1 \cos\theta_1 = v_{\theta_2} r_2 A_2 \rho q_2 \cos\theta_2 \qquad \text{(f)}$$

where

$$\theta_1 = \frac{\pi}{2}$$

With reference to Fig. 7.G(b) and the continuity equation, we have

$$A_1 q_1 \cos\theta_1 = A_2 q_2 \cos\theta_2$$

or

$$q_2 \cos\theta_2 = 100(1)\frac{A_1}{A_2} = 100\frac{r_1}{r_2} = 66.67$$

but

$$q_{n_2} = q_2 \cos\theta_2 = q_{r_2} \cos(160^\circ - 90^\circ)$$

Hence,

$$q_{r_2} = \frac{66.67}{\cos 70^\circ} = \frac{66.67}{0.342} = 195 \text{ ft/sec}$$

The tangential component of the absolute velocity q_2 is then

$$v_{\theta_2} = \omega r_2 - q_{r_2} \sin(160^\circ - 90^\circ)$$

$$= \frac{2\pi(400)(\frac{3}{2})}{60} - 195 \sin 70^\circ$$

$$= 20\pi - 183 = -120.2 \text{ ft/sec}$$

With the values above for v_{θ_2} and $q_2 \cos\theta_2$ Eq. (f) becomes

$$T_z = (-120.2)(\tfrac{3}{2})[(2\pi)(\tfrac{3}{2})(\tfrac{1}{2})](1.94)(66.67) = -109{,}500 \text{ ft-lbf}$$

which represents the torque opposing the motion. The power developed by the turbine is

$$\frac{T_z \omega}{550} = \frac{190{,}500}{550}\left[2\pi \frac{(400)}{60}\right] = 8310 \text{ hp}$$

B. SIMPLE FLOWS

7.8. Laplace's Equation

a. Laplace's Equation in Cartesian Coordinates

If we substitute the gradient of the velocity potential for the velocity vector into the equation of continuity, the following second order partial differential equation is obtained:

$$\nabla^2 \phi = 0 \tag{7-45a}$$

or in Cartesian coordinates,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{7-45b}$$

This equation, though often called the velocity potential equation in this application, is generally known as Laplace's equation, and its solution ϕ is called a harmonic function. When the equation is written in spherical coordinates, the solution comes out in terms of functions known as solid spherical harmonics, of which zonal harmonics is an example. These types of solutions will be illustrated later.

The solution of Laplace's equation has been attempted by many important mathematicians during the last century, and yet there is no general analytic method for its solution. The solution of the equation is important because it can be applied not only in fluid dynamics but also in other fields such as electromagnetics and heat conduction. Since Eq. (7-45a) has the property that it is linear, the superposition of different solutions of velocity potential is valid here.

Equations (7-45a) and (7-26) give the solutions for ϕ and p which are the two variables in an irrotational and incompressible flow.

b. Boundary Conditions

When fluid is in contact with a solid surface at rest (or with another unmixed fluid), the boundary conditions (see Fig. 7.7) which the solutions of Laplace's equation must satisfy are

(i) $$\mathbf{n} \cdot \mathbf{q} = \frac{\partial \phi}{\partial n} = 0 \tag{7-46}$$

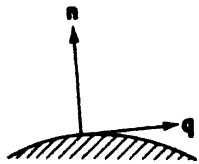


Figure 7.7

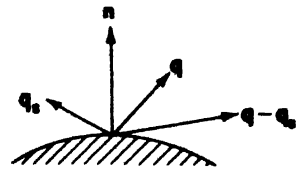


Figure 7.8

(ii) The pressure shall be continuous at a common boundary of fluids.

Condition (i) is a kinematical relation (Fig. 7.8) which means that the fluid and the surface with which contact is taking place must have the same normal velocity. Thus if $\mathbf{q}$ and $\mathbf{q}_s$ are the velocities of the fluid and the moving solid,

$$\mathbf{n} \cdot \mathbf{q} = \mathbf{n} \cdot \mathbf{q}_s$$

or

$$\mathbf{n} \cdot (\mathbf{q} - \mathbf{q}_s) = 0 \tag{7-47}$$

When $\mathbf{q}_s = 0$, i.e., the solid surface is at rest, Eq. (7–47) reduces to Eq. (7–46). Hence, we may conclude from Eq. (7–46) that the velocity of the fluid is tangential to the surface for an inviscid fluid.

Condition (ii) is a physical relation which can be interpreted by considering an infinitesimal surface dA common to the two fluids. If p_1 and p_2 are the pressures in the two fluids, we have

$$\mathbf{n}p_1 \cdot d\mathbf{A} = \mathbf{n}p_2 \cdot d\mathbf{A}$$

or

$$p_1 = p_2 \tag{7-48}$$

The relation above must be modified when surface tension is present.

Illustrative Example 7.7. Show that the velocity potential

$$\phi = \frac{a}{2}(x^2 + y^2 - 2z^2) \tag{a}$$

satisfies the Laplace equation and represents the flow against a fixed plane wall.

Solution. From Eq. (a) the velocity components and their respective derivatives are found as follows:

$$u = \frac{\partial \phi}{\partial x} = ax, \qquad v = \frac{\partial \phi}{\partial y} = ay, \qquad w = \frac{\partial \phi}{\partial z} = -2az \tag{b}$$

and

$$\frac{\partial u}{\partial x} = \frac{\partial^2 \phi}{\partial x^2} = a, \qquad \frac{\partial v}{\partial y} = \frac{\partial^2 \phi}{\partial y^2} = a, \qquad \frac{\partial w}{\partial z} = \frac{\partial^2 \phi}{\partial z^2} = -2a \tag{c}$$

Substituting Eq. (c) into Eq. (7–45b), we obtain

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = a + a - 2a = 0 \tag{d}$$

Hence the Laplace equation is satisfied. The streamlines in the y-z plane are given by the differential equation [Eq. (3–10)]

$$\frac{dz}{w} = \frac{dy}{v} \tag{e}$$

Combining Eqs. (e) and (b), we have

$$-\frac{dz}{2z} = \frac{dy}{y} \tag{f}$$

which has a solution in the form

$$y^2 z = \text{Constant} \tag{g}$$

Equation (g) is often called a cubic hyperbola. The result of this solution is the flow against a fixed plane wall which is shown in Fig. 7.H.

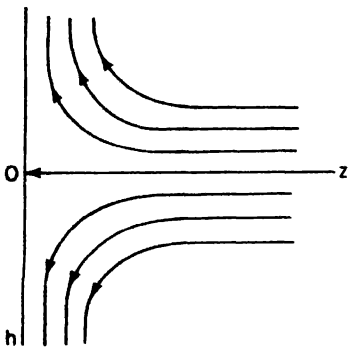


Figure 7.H

c. Laplace's Equation in Cylindrical Coordinates

The Laplace equation in terms of cylindrical coordinates (see Appendix A) can be derived as follows:

$$\begin{aligned} d\mathbf{s} \cdot \mathbf{q} = d\mathbf{s} \cdot \nabla\phi = d\phi &= \frac{\partial\phi}{\partial r}\,dr + \frac{\partial\phi}{\partial\theta}\,d\theta + \frac{\partial\phi}{\partial z}\,dz \\ &= (\mathbf{i}_r\,dr + \mathbf{i}_\theta r\,d\theta + \mathbf{i}_z\,dz) \cdot (\mathbf{i}_r v_r + \mathbf{i}_\theta v_\theta + \mathbf{i}_z v_z) \\ &= v_r\,dr + v_\theta r\,d\theta + v_z\,dz \end{aligned} \tag{7–49}$$

Hence,

$$v_r = \frac{\partial\phi}{\partial r}, \qquad v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \qquad v_z = \frac{\partial\phi}{\partial z} \tag{7–50}$$

From Eq. (5–41), replacing the velocity components by Eq. (7–50), we have

$$\nabla^2\phi = \nabla \cdot \mathbf{q} = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = 0 \tag{7–51}$$

d. Laplace's Equation in Spherical Coordinates

In a similar manner to that employed in § 7.8c, the Laplace equation in terms of spherical coordinates (see Appendix A) is obtained:

$$d\phi = \frac{\partial \phi}{\partial \omega} d\omega + \frac{\partial \phi}{\partial \varphi} d\varphi + \frac{\partial \phi}{\partial \theta} d\theta$$

$$= (\mathbf{i}_\omega\, d\omega + \mathbf{i}_\varphi \omega\, d\varphi + \mathbf{i}_\theta \omega \sin\varphi\, d\theta) \cdot (\mathbf{i}_\omega v_\omega + \mathbf{i}_\varphi v_\varphi + \mathbf{i}_\theta v_\theta)$$

$$= v_\omega\, d\omega + v_\varphi \omega\, d\varphi + v_\theta \omega \sin\varphi\, d\theta \tag{7-52}$$

Hence,

$$v_\omega = \frac{\partial \phi}{\partial \omega}, \qquad v_\varphi = \frac{1}{\omega} \frac{\partial \phi}{\partial \varphi}, \qquad v_\theta = \frac{1}{\omega \sin\varphi} \frac{\partial \phi}{\partial \theta} \tag{7-53}$$

From Eq. (5–61), substituting Eq. (7–53) for the velocity components we have

$$\nabla^2 \phi = \nabla \cdot \mathbf{q}$$

$$= \frac{1}{\omega^2} \frac{\partial}{\partial \omega}\left(\omega^2 \frac{\partial \phi}{\partial \omega}\right) + \frac{1}{\omega^2 \sin\varphi} \frac{\partial}{\partial \varphi}\left(\sin\varphi \frac{\partial \phi}{\partial \varphi}\right) + \frac{1}{\omega^2 \sin^2\varphi} \frac{\partial^2 \phi}{\partial \theta^2}$$

$$= 0 \tag{7-54}$$

The general solution of $\nabla^2 \phi = 0$ with zonal symmetry ($\partial/\partial\theta = 0$) is

$$\phi_n = A_n \frac{P_n(\cos\varphi)}{\omega^{n+1}} \tag{7-55}$$

where P_n are Legendre's polynomials or zonal harmonics (Fig. 7.9). For $n = 0$ and $n = 1$ we have, respectively,

$$P_0(\cos\varphi) = 1 \qquad \text{and} \qquad P_1(\cos\varphi) = \cos\varphi \tag{7-56}$$

Hence,

$$\phi_0 = \frac{A_0}{\omega} \tag{7-57}$$

and

$$\phi_1 = \frac{A_1 \cos\varphi}{\omega^2} \tag{7-58}$$

where A_0 and A_1 are arbitrary constants. The specific values for these two constants will be discussed later.

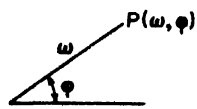


Figure 7.9

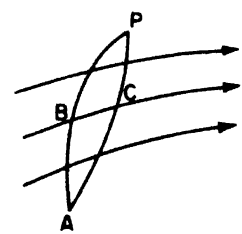


Figure 7.10

7.9. Stream Function in Two-Dimensional Motion

Let $A(x_0, y_0)$ be a fixed point in the x-y plane which has unit thickness in the z-direction and $P(x, y)$ an arbitrary point in the same plane (Fig. 7.10). Now, two arbitrary paths ABP and ACP are drawn between the points P and A. As a consequence of the condition of continuity, there is no fluid created or destroyed within the region bounded by these curves. Hence, the rate of fluid flow entering the region across the curve ABP is equal to the flow rate across the curve ACP. The term *flux* for the rate of flow will be used hereafter. The flux across the curve ABP is equal to the flux across any curve joining A to P. Since the point A is fixed, the flux is a function of the position of P and the time t. If the flux is denoted by ψ, ψ is a function of the position of P and the time, i.e.,

$$\psi = \psi(x, y, t) \tag{7-59}$$

The function ψ is called the stream function and has the dimensions of volume per unit length per unit time. It must be borne in mind that the existence of this function is a consequence only of the conditions for the continuity and incompressibility* of the fluid, so a stream function is valid for a viscous fluid.

Now consider two points P_1 and P_2 and two curves drawn from them to the fixed point A. Let ψ_1, ψ_2 represent the values of the stream function at points P_1, P_2, respectively (Fig. 7.11). Then, the flux across the curve AP_2 is equal to the flux across the curve AP_1 plus that across the curve P_1P_2. Hence, the flux across the curve P_1P_2 is $\psi_2 - \psi_1$. It can be easily seen that if the reference point A is replaced by another point A_1, the value of the stream function $\psi_2 - \psi_1$ changes by a constant, namely the flux across A_1A. When the points P_1 and P_2 are points of the same streamline (not necessarily coincident), the flux across P_1P_2 is $\Delta\psi = \psi_2 - \psi_1 = 0$. Since by definition there is no flow across a streamline, the stream function is constant along a streamline. The

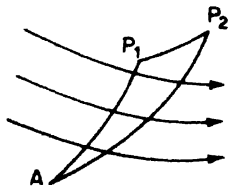


Figure 7.11

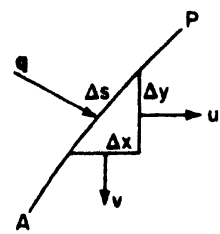


Figure 7.12

*ψ for compressible flow is given in § 11.2a.

velocity component in any direction can be derived from the stream function. In Cartesian coordinates the velocity components in the x and y-directions are found from the following consideration. Let Δs be an infinitesimal length of the curve AP whose components in the x and y-directions are Δx and Δy, respectively (Fig. 7.12). The flux across the curve AP is

$$\psi = \int_{AP} \mathbf{q} \cdot \mathbf{n} ds = \int_{AP} (u\, dy - v\, dx) \tag{7-60}$$

or the flux across the arc Δs is

$$d\psi = u\, dy - v\, dx \tag{7-61}$$

Hence, the velocity components u, v parallel to the axes are given by

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \tag{7-62}$$

Upon the substitution of the expressions for the velocity components ($w = 0$) into the continuity equation [Eq. (7-1)] we have

$$\frac{\partial^2 \psi}{\partial x\, \partial y} - \frac{\partial^2 \psi}{\partial x\, \partial y} = 0 \tag{7-63}$$

In other words the continuity equation is automatically satisfied. This is analogous to the case wherein the existence of the velocity potential ϕ automatically satisfies the condition of irrotationality.

Equation (7-62) is valid for both rotational and irrotational flow. For an irrotational flow ($\Omega_z = 0$) Laplace's equation for ψ is obtained:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{7-64}$$

Hence, the solution ψ is a harmonic function. Relations between the stream function and the velocity potential are found by equating the expressions for the velocity components:

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{7-65}$$

in Cartesian coordinates and

$$\frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} \tag{7-66}$$

in polar coordinates. Hence,

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad v_\theta = -\frac{\partial \psi}{\partial r} \tag{7-67}$$

The Laplace equation for ψ in polar coordinates can be obtained from the condition of irrotational flow as follows:

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0 \tag{7-68}$$

Illustrative Example 7.8. Show that the velocity vector $\mathbf{q}$ is everywhere tangent to lines in the x-y plane along which $\psi(x, y) = \text{Constant}$.

Solution. The stream function $\psi = \psi(x, y)$ in general can be expressed by

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \tag{a}$$

Along lines $\psi(x, y) = \text{Constant}$, $d\psi = 0$ and Eq. (1) becomes

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0 \tag{b}$$

Recalling the definition of stream function,

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \tag{c}$$

Combining Eqs. (b) and (c), we have

$$v\,dx + u\,dy = 0$$

or

$$\frac{dy}{dx} = \frac{v}{u} \tag{d}$$

Equation (d) implies that $\mathbf{q}$ is tangent to lines $\psi = \text{Constant}$ (Fig. 7.I).

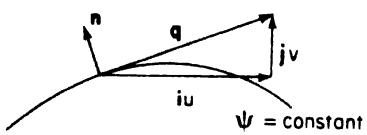


Figure 7.I

Furthermore,

$$\mathbf{n} \cdot \mathbf{q} = 0$$

because $\mathbf{n}$ is normal to the direction of the streamlines. Hence, the boundary condition of Eq. (7–46) is satisfied.

7.10. The Flow Net

In § 7.9 streamlines heve been defined as lines along which ψ is constant. Similarly, lines along which ϕ is constant are called potential lines. It will now be shown that the potential lines and the streamlines are orthogonal. From Eqs. (7–18) and (7–61) we have, respectively, for potential lines and streamlines

$$d\phi = 0 = u\,dx + v\,dy \tag{7–18}$$

$$d\psi = 0 = u\,dy - v\,dx \tag{7–61}$$

Therefore, the slopes of a potential line and a streamline are, respectively,

$$\left(\frac{dy}{dx}\right)_{\phi=\text{Constant}} = -\frac{u}{v}$$

$$\left(\frac{dy}{dx}\right)_{\psi=\text{Constant}} = \frac{v}{u}$$

Since these slopes are negative reciprocals, the lines are perpendicular to one another. Thus, the potential lines and streamlines form an orthogonal set of lines which completely describes the flow in a two-dimensional incompressible, irrotational flow field. Such a net is called a flow net (Fig, 7.13).

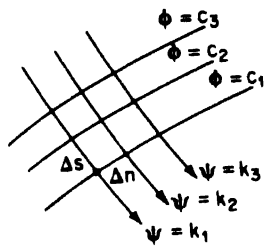


Figure 7.13

The premise in this analysis is that as the spacing between potential lines and streamlines approaches zero in the limit, the lines forming the flow net become equally spaced (i.e., a square net is formed). For small finite line spacing the net approximates square spacing. Thus, if the potential line spacing is denoted by Δs and the streamline spacing is denoted by Δn for a flow with velocity q, we have:

$$q \doteq \frac{\Delta\phi}{\Delta s} \doteq \frac{\Delta\psi}{\Delta n}$$

It can be seen, then, that the spacing in the flow net becomes a direct indication of the magnitude of the flow velocity. As the spacing becomes larger, the velocity decreases (and from the Bernoulli equation the pressure increases), whereas decreasing net spacing indicates increasing velocity and decreasing pressure.

Illustrative Example 7.9. As an example of the flow net, consider the uniform rectilinear flow of Fig. 7.J. In this case the flow net is made up of perfect squares which maintain the same size throughout. Therefore, the pressure and velocity are constant. Figure 7.K shows the uniform flow around a circular cylinder. At large distances from the cylinder, the net is almost square, and there is only a slight change in the net size. As the flow approaches the cylinder, however,

Illustrative Example 7.8. Show that the velocity vector **q** is everywhere tangent to lines in the x-y plane along which $\psi(x, y) = \text{Constant}$.

Solution. The stream function $\psi = \psi(x, y)$ in general can be expressed by

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \tag{a}$$

Along lines $\psi(x, y) = \text{Constant}$, $d\psi = 0$ and Eq. (1) becomes

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0 \tag{b}$$

Recalling the definition of stream function,

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \tag{c}$$

Combining Eqs. (b) and (c), we have

$$v\,dx + u\,dy = 0$$

or

$$\frac{dy}{dx} = \frac{v}{u} \tag{d}$$

Equation (d) implies that **q** is tangent to lines $\psi = \text{Constant}$ (Fig. 7.I). Furthermore,

$$\mathbf{n} \cdot \mathbf{q} = 0$$

because **n** is normal to the direction of the streamlines. Hence, the boundary condition of Eq. (7-46) is satisfied.

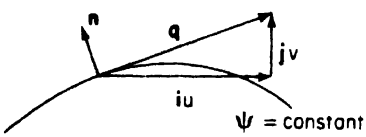


Figure 7.I

7.10. The Flow Net

In § 7.9 streamlines heve been defined as lines along which ψ is constant. Similarly, lines along which ϕ is constant are called potential lines. It will now be shown that the potential lines and the streamlines are orthogonal. From Eqs. (7-18) and (7-61) we have, respectively, for potential lines and streamlines

$$d\phi = 0 = u\,dx + v\,dy \tag{7-18}$$

$$d\psi = 0 = u\,dy - v\,dx \tag{7-61}$$

Therefore, the slopes of a potential line and a streamline are, respectively,

$$\left(\frac{dy}{dx}\right)_{\phi=\text{Constant}} = -\frac{u}{v}$$

$$\left(\frac{dy}{dx}\right)_{\psi=\text{Constant}} = \frac{v}{u}$$

Since these slopes are negative reciprocals, the lines are perpendicular to one another. Thus, the potential lines and streamlines form an orthogonal set of lines which completely describes the flow in a two-dimensional incompressible, irrotational flow field. Such a net is called a flow net (Fig, 7.13).

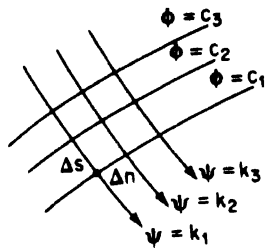


Figure 7.13

The premise in this analysis is that as the spacing between potential lines and streamlines approaches zero in the limit, the lines forming the flow net become equally spaced (i.e., a square net is formed). For small finite line spacing the net approximates square spacing. Thus, if the potential line spacing is denoted by Δs and the streamline spacing is denoted by Δn for a flow with velocity q, we have:

$$q \doteq \frac{\Delta\phi}{\Delta s} \doteq \frac{\Delta\psi}{\Delta n}$$

It can be seen, then, that the spacing in the flow net becomes a direct indication of the magnitude of the flow velocity. As the spacing becomes larger, the velocity decreases (and from the Bernoulli equation the pressure increases), whereas decreasing net spacing indicates increasing velocity and decreasing pressure.

Illustrative Example 7.9. As an example of the flow net, consider the uniform rectilinear flow of Fig. 7.J. In this case the flow net is made up of perfect squares which maintain the same size throughout. Therefore, the pressure and velocity are constant. Figure 7.K shows the uniform flow around a circular cylinder. At large distances from the cylinder, the net is almost square, and there is only a slight change in the net size. As the flow approaches the cylinder, however,

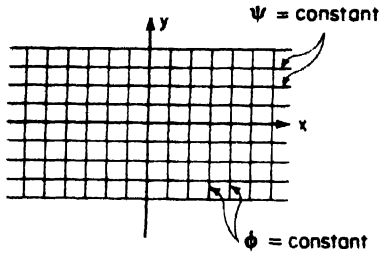


Figure 7.J

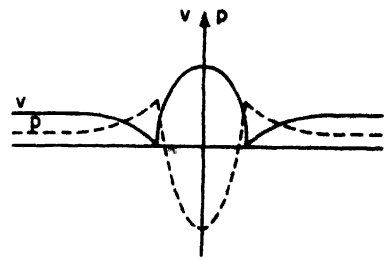


Figure 7.L

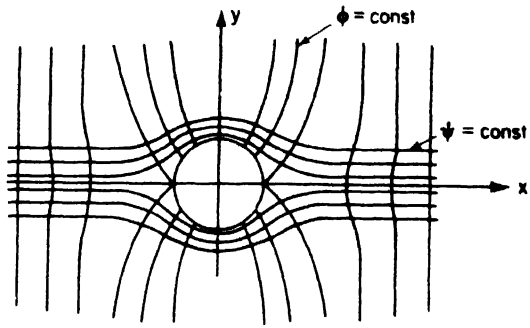


Figure 7.K

the streamlines turn away from the x-axis, creating large net spacing in the vicinity of the stagnation point. Then as the flow bends back toward the x-axis, the net size decreases. From this pattern alone we can deduce the form of the velocity and pressure distributions for the flow shown in Fig. 7.L.

The actual construction of a flow net is a trial and error procedure. The first step is to sketch in streamlines for the flow which are spaced closely enough together to yield an adequate picture of the flow. Next, the potential lines are sketched in with the same spacing, being orthogonal everywhere to the streamlines. For every flow which can be described by ψ and ϕ there is a unique solution. Therefore, several readjustments of the flow pattern are usually necessary before an accurate flow net is achieved.

7.11. *Stream Function in Three-Dimensional Motion*

The simplest case of motion in three dimensions is that in which the motion is the same in every plane containing a certain axis. This type of flow is called axially symmetric flow and is similar to the two-dimensional case in that a stream function can be defined. The

stream function defined only for three-dimensional flow with axial symmetry is called Stokes' stream function.

Let z be the axis of symmetry and 0_1 be a fixed point on the axis of symmetry (Fig. 7.14a). Consider an arbitrary point P lying in the $z - r$ plane (Fig. 7.14b) and connected to 0_1 by an arbitrary curve. If we rotate the curve $0_1 P$ about the axis of symmetry, a surface of revolution is generated. The flow through the surface of revolution is then a function of the position of P. If the flux across the surface of revolution is denoted by $2\pi\psi$, the function ψ at a given time depends on the position of P, i.e., $\psi = \psi(z, r)$. When the path from 0_1 to P is changed, the value of ψ is unaltered. It can be seen (Fig. 7.14b) that the flow across 0_1P is the same as that across 0_2P because there is no flow across the axis of symmetry, $0_1 0_2$. Hence, when the point P lies on the axis, the stream function is zero.

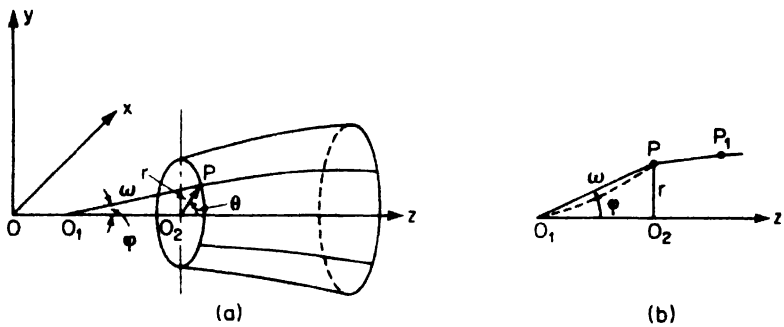


Figure 7.14

If there is no flux across the curve joining P and P_1, then the value of the stream function at point P is equal to that at the point P_1 (Fig. 7.14b). In other words the curve joining the points P and P_1 is a streamline. The surface generated by rotating the streamline PP_1 about the axis of symmetry is called a stream surface.

To find the relation between ψ and the velocity components, let us consider the flow across an infinitesimal surface of revolution in cylindrical coordinates (Fig. 7.15a). If ψ_P and ψ_{P_1} denote the values of the stream function at P and P_1, the flux across the surface generated by the revolution about the z-axis of the line PP_1 is

$$2\pi r v_n \Delta S = 2\pi \Delta\psi = 2\pi(\psi_{P_1} - \psi_P) = 2\pi(v_z \Delta r - v_r \Delta z)r \qquad (7\text{–}69)$$

Proceeding to the limit and solving for velocity components, we find

$$v_n = \frac{1}{r}\frac{\partial\psi}{\partial s}, \qquad v_z = \frac{1}{r}\frac{\partial\psi}{\partial r}, \qquad v_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \qquad v_\theta = 0 \qquad (7\text{–}70)$$

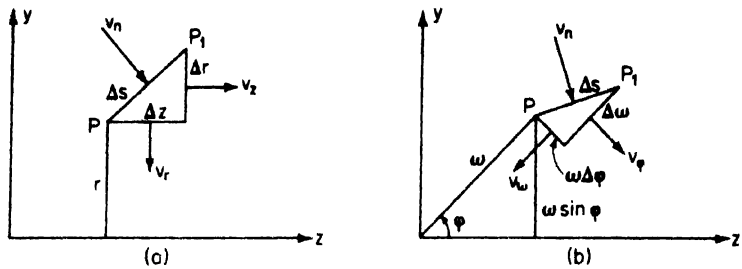


Figure 7.15

The relations between the stream function [Eq. (7-70)] and the velocity potential in cylindrical coordinates [Eq. (7-50)] are

$$\frac{\partial\phi}{\partial z} = \frac{1}{r}\frac{\partial\psi}{\partial r}, \qquad \frac{\partial\phi}{\partial r} = -\frac{1}{r}\frac{\partial\psi}{\partial z} \tag{7-71}$$

In spherical coordinates (Fig. 7.15b) the flux across the surface of revolution of the line PP_1 is

$$2\pi\Delta\psi = 2\pi\omega\sin\varphi\ [-v_\varphi\Delta\omega - v_\omega(-\omega\Delta\varphi)]$$

The velocity components are

$$v_\omega = \frac{1}{\omega^2\sin\varphi}\frac{\partial\psi}{\partial\varphi}, \qquad v_\varphi = -\frac{1}{\omega\sin\varphi}\frac{\partial\psi}{\partial\omega}, \qquad v_\theta = 0 \tag{7-72}$$

The relations between the stream function [Eq. (7-72)] and the velocity potential in spherical coordinates [Eq. (7-53)] are

$$\frac{\partial\phi}{\partial\omega} = \frac{1}{\omega^2\sin\varphi}\frac{\partial\psi}{\partial\varphi}, \qquad \frac{\partial\phi}{\partial\varphi} = -\frac{1}{\sin\varphi}\frac{\partial\psi}{\partial\omega} \tag{7-73}$$

7.12. Two-Dimensional Flow Examples

In order to describe flow patterns of physical significance it will be desirable to examine the characteristics of a number of important simple flows of irrotational motion. The velocity potential and stream function of these flows will be derived, and they may be readily verified to be harmonic functions by substituting them into Laplace's equation.

a. Rectilinear Flow

If a uniform flow is given parallel to the positive direction of the x-axis, the expressions for velocity components, velocity potential, and stream function are

$$u = U_\infty = \text{Constant}, \qquad v = 0 \tag{7-74}$$

then

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad 0 = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{7-75}$$

and

$$\phi = U_\infty x = U_\infty r \cos \theta \tag{7-76}$$

$$\psi = U_\infty y = U_\infty r \sin \theta \tag{7-77}$$

The streamlines are all straight lines parallel to the x-axis, and the potential lines are straight lines parallel to the y-axis.

Similarly, the expressions for velocity components, velocity potential, and stream function can be obtained for a uniform flow parallel to the positive direction of the y-axis as follows;

$$u = 0, \qquad v = V_\infty = \text{Constant} \tag{7-78}$$

Then

$$0 = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad V_\infty = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{7-79}$$

and

$$\phi = V_\infty y \tag{7-80}$$

$$\psi = -V_\infty x \tag{7-81}$$

The streamlines in this case are straight lines parallel to the y-axis, and the potential lines are straight lines parallel to the x-axis. It can easily be shown that both the stream function and the velocity potential in the cases above are harmonic functions which satisfy the Laplace equation.

b. Source and Sink— Flow

In a two-dimensional flow, a source at point O (Fig. 7.16) is conceived of as a point from which fluid emanates in equal amounts along radial paths. Hence, the lines of flow in this case will be straight radial lines, i.e.,

$$v_\theta = 0, \qquad v_r = v_r(r) \tag{7-82}$$

From considerations of conservation of mass in a steady incompressible flow [see Eq. (5-41)], the following equation must be satisfied:

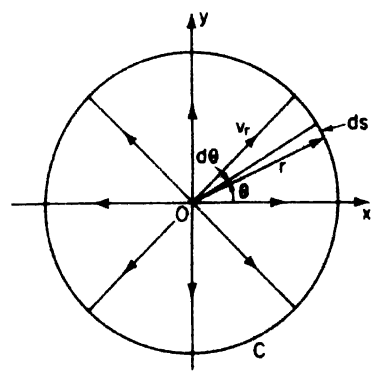


Figure 7.16

$$\nabla \cdot \mathbf{q} = \frac{dv_r}{dr} + \frac{v_r}{r} = 0 \tag{7-83}$$

The solution of Eq. (7-83) yields

$$\ln (v_r r) = \text{Constant}$$

or

$$v_r = \frac{B}{r} \tag{7-84}$$

where B is a constant to be determined.

Now let Q be the quantity of fluid per second flowing out through curve C surrounding the source. Then Q must be equal to the total flow across any circle of radius r with its center at the source (Fig. 7.16), i.e.,

$$Q = \int_C \mathbf{n} \cdot \mathbf{q}\, ds = \int_0^{2\pi} v_r r\, d\theta \tag{7-85}$$

Substituting v_r from Eq. (7-84) into the expression above we have

$$B = \frac{Q}{2\pi} \tag{7-86}$$

where Q is defined as the source strength. Thus

$$v_r = \frac{Q}{2\pi r}, \qquad v_\theta = 0 \tag{7-87}$$

A sink is simply the reverse of a source. It is a point into which fluid flows radially in equal amounts from all directions in the field. Algebraically, a sink is represented as the negative of a source.

The stream function is found by the direct integration of the velocity components v_r and v_θ of Eqs. (7-67) and (7-87):

$$\psi = \frac{Q}{2\pi}\theta + f(r) \tag{7-88}$$

and

$$\psi = f(\theta) \tag{7-89}$$

Comparing the two expressions above for ψ, one finds $f(r)$ equal to a constant which can be made to vanish if the line from which θ is measured is chosen as the zero streamline. Hence,

$$\psi = \frac{Q}{2\pi}\theta \tag{7-90}$$

From the vorticity equation [Eq. (3-29)] we have

$$\Omega_z = \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\theta}{r} \tag{7-91}$$

Substitution of the velocity components from Eqs. (7-87) into the equation above results in:

$$\Omega_z = 0 \tag{7-92}$$

Therefore, the velocity potential exists in the case of source and sink flows. From Eq. (7-50)

$$v_r = \frac{\partial \phi}{\partial r}, \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

Hence,

$$\phi = \int \frac{Q}{2\pi r}\, dr + f_1(\theta) = \frac{Q}{2\pi} \ln r + f_1(\theta) \tag{7-93}$$

and

$$\phi = g(r) \tag{7-94}$$

Comparing the two expressions above for ϕ, we have

$$\phi = \frac{Q}{2\pi} \ln r + \text{Constant} \tag{7-95}$$

where $Q > 0$ for a source and $Q < 0$ for a sink.

Again it can by shown that both the velocity potential and stream function are harmonic functions by substituting Eqs. (7-90) and (7-95) into the Laplace equation [Eqs. (7-51) and (7-68)].

c. Vortex Flow

In § 7.4 motion in concentric circles was given as an example for irrotational flow. In the example it was illustrated that a circulatory flow free from rotation surrounds a core which acts like a rotating solid body. Since the flow in this case (Fig. 7.17) is irrotational flow with circular streamlines [$v_r = 0$, $v_\theta = v_\theta(r)$] it must satisfy the vorticity equation [Eq. (3-29)]

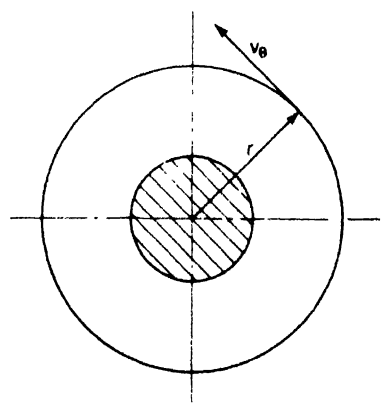


Figure 7.17

$$\frac{v_\theta}{r} + \frac{dv_\theta}{dr} = 0$$

The solution of the equation above gives

$$\ln (v_\theta r) = \text{Constant}$$

or

$$v_\theta = \frac{A}{r} \tag{7-96}$$

where the constant A is to be determined.

From the definition of circulation we have

$$\Gamma = \oint_C \mathbf{q} \cdot d\mathbf{s} = \int_0^{2\pi} v_\theta r \, d\theta = 2\pi A \tag{7-97}$$

Equation (7-97) results in

$$A = \frac{\Gamma}{2\pi} \tag{7-98}$$

Hence, the velocity components in the vortex flow are

$$v_\theta = \frac{\Gamma}{2\pi r}, \qquad v_r = 0 \tag{7-99}$$

The velocity potential is

$$\phi = \int v_\theta r \, d\theta + \text{Constant} = \frac{\Gamma}{2\pi}\theta + \text{Constant} \tag{7-100}$$

The continuity equation in polar coordinates,

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0 \tag{5-41}$$

is satisfied by the velocity components in this case. Hence, the stream function can be obtained from

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad \text{and} \qquad v_\theta = -\frac{\partial \psi}{\partial r} \tag{7-67}$$

and is given by

$$\psi = -\int v_\theta \, dr + \text{Constant} = -\frac{\Gamma}{2\pi} \ln r + \text{Constant} \tag{7-101}$$

The stream function and velocity potential given in Eqs. (7-100) and (7-101), respectively, are harmonic functions.

d. Doublet

Let a source and sink of equal strength Q be placed at a distance $2a$ apart along the x-axis (Fig. 7.18). From any point $P(r, \theta)$ lines r and r' are drawn to the source and sink, respectively. The angles θ and θ' are measured from the positive x-axis to lines r and r', respectively. Since the velocity potential for source or sink flow [Eq. (7-95)] satisfies the Laplace equation [Eq. (7-75)], we can obtain the velocity potential of the combined flow by superposition of the source and sink flows (see Illustrative Example 7.12),

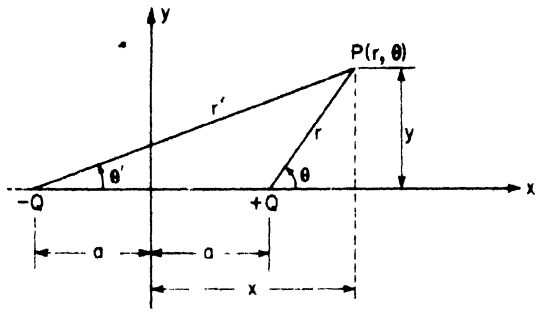


Figure 7.18

$$\phi = \frac{Q}{2\pi}(\ln r - \ln r') + \text{Constant} = -\frac{Q}{2\pi}\ln\frac{r'}{r} + \text{Constant} \quad (7\text{-}102)$$

Similarly, the stream function for the combined flow is

$$\psi = \frac{Q}{2\pi}(\theta - \theta') \quad (7\text{-}103)$$

The family of streamlines for ψ = Constants is shown in Fig. 7.19. The streamline contour equation can be easily derived with the aid of the following trigonometric relations

$$\tan(\theta - \theta') = \frac{\tan\theta - \tan\theta'}{1 + \tan\theta\tan\theta'}$$

where $\tan\theta = y/(x - a)$ and $\tan\theta' = y/(x + a)$ can be seen from Fig. 7.18. In terms of Cartesian coordinates, we have

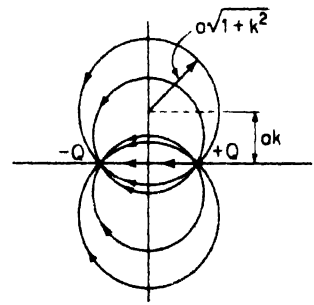


Figure 7.19

$$\tan(\theta - \theta') = \frac{2ay}{x^2 + y^2 - a^2}$$

Upon substitution of the expression above into Eq. (7-103), one obtains

$$\tan\frac{2\pi\psi}{Q} = \frac{2ay}{x^2 + y^2 - a^2} \quad (7\text{-}104a)$$

or

$$x^2 + (y - ka)^2 = (1 + k^2)a^2 \quad (7\text{-}104b)$$

where $k = \cot(2\pi\psi/Q)$ is an arbitrary constant. Equation (7-104b) is the equation of a circle with center on the y-y axis a distance ak from the origin and with radius equal to $a\sqrt{1 + k^2}$. Hence, the streamlines are coaxial circles passing through Q and $-Q$. The flow is directed from source to sink. In a similar manner the potential lines are given by ϕ = Constant as follows:

$$y^2 + \left(x + \frac{C+1}{C-1}a\right)^2 = \left(a\frac{2\sqrt{C}}{C-1}\right)^2 \tag{7-105}$$

where $C = e^{4\pi\phi/Q}$ is an arbitrary constant.

Hence, the potential lines are also circles with center along the x-x axis a distance $a(C+1)/(C-1)$ from the origin and with radius $2a\sqrt{C}/(C-1)$

Now consider a very important flow pattern that results when a source and a sink of equal strength are allowed to approach each other while the product of their strength and distance apart is held constant. This combination of a source and sink of infinite strength at an infinitesimal distance apart is called a doublet of strength μ. In order to develop the equations describing the doublet, let

$$r' = r + \Delta r, \qquad \theta' = \theta - \Delta\theta, \qquad \text{and} \qquad 2a = \Delta s$$

Substituting the expressions above into Eq. (7-102) we have

$$\phi = -\frac{Q}{2\pi}\ln\left(1 + \frac{\Delta r}{r}\right) + \text{Constant}$$

$$= -\frac{Q}{2\pi}\left[\frac{\Delta r}{r} - \frac{1}{2}\left(\frac{\Delta r}{r}\right)^2 + \cdots\right] + \text{Constant} \tag{7-106}$$

Since Δs, Δr, and $\Delta\theta$ are infinitesimally small quantities, we neglect all of the second order terms. Substituting

$$r^2 = (\Delta s)^2 + (r + \Delta r)^2 - 2\Delta s\ (r + \Delta r)\cos(\theta - \Delta\theta)$$

$$\frac{\Delta r}{r} = \frac{\Delta s}{r}\cos\theta$$

into Eq. (7-106), we obtain the velocity potential for the doublet

$$\phi = \lim_{\substack{\Delta s \to 0 \\ Q \to \infty}}\left[-\frac{Q\Delta s}{2\pi r}\cos\theta\right] + \text{Constant}$$

$$= -\frac{\mu}{2\pi r}\cos\theta + \text{Constant} \tag{7-107}$$

where $\mu = Q\Delta s$ is the strength of the doublet.

The velocity components in this case can be obtained from Eq. (7-107) as follows:

$$v_r = \frac{\partial\phi}{\partial r} = \frac{\mu\cos\theta}{2\pi r^2}$$

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\mu\sin\theta}{2\pi r^2} \tag{7-108}$$

The stream function of a doublet can be obtained from the velocity components above,

$$\psi = -\int v_\theta\, dr + \text{Constant} = \frac{\mu\sin\theta}{2\pi r} + \text{Constant}$$

or

$$\psi = \int v_r r \, d\theta + \text{Constant} = \frac{\mu \sin \theta}{2\pi r} + \text{Constant} \tag{7-109}$$

where the constant can be made to vanish if the zero streamline is chosen where $\theta = 0$.

For any arbitrary constant value of ψ, Eq. (7-109) can be rewritten as

$$r = r_0 \sin \theta \tag{7-110}$$

where $r_0 = \mu/2\pi\psi_c$ and ψ_c is an arbitrary constant for ψ. The equation above represents a circle which passes through the origin with center on the y-axis and a diameter of r_0. Similarly, for any arbitrary constant value of ϕ, Eq. (7-107) can be rewritten as

$$r = R_0 \cos \theta \tag{7-111}$$

where $R_0 = -\mu/2\pi\phi_c$ and ϕ_c is an arbitrary constant for ϕ. Equation (7-111) represents a circle which passes through the origin with center on the x-axis and a diameter of R_0. The families of streamlines and potential lines for doublet flow are shown in Figs. 7.20a and 7.20b, respectively.

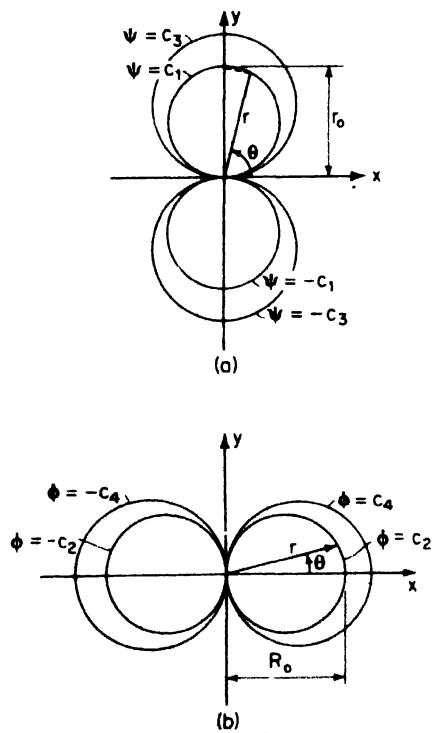


Figure 7.20

e. Vortex Pair—Dipole

In analogy to the doublet the two-dimensional vortex pair is defined by a pair of vortices of equal strength Γ but of opposite rotation separated by a distance Δs (Fig. 7.21). If we let $\Delta s \to 0$ and $\Gamma \to \infty$ in such a manner that $\nu = \Gamma \Delta s$, a dipole* of constant strength ν is obtained.

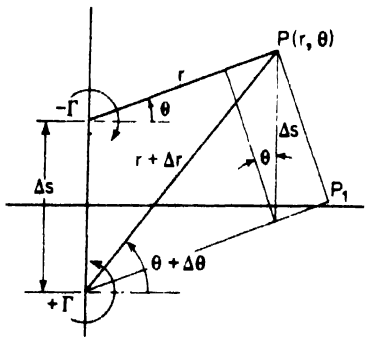


Figure 7.21

The velocity potential of the combined flow in the present case can be obtained by superposition of the vortex flows [Eq. (7-100)]:

$$\phi = -\frac{\Gamma}{2\pi}\theta + \frac{\Gamma}{2\pi}(\theta + \Delta\theta) + \text{Constant}$$

$$= \frac{\Gamma}{2\pi}\Delta\theta + \text{Constant} \qquad (7\text{-}112)$$

Referring to Fig. 7.21 the following trigonometric relation is noticed:

$$PP_1 = \Delta s \cos\theta = (r + \Delta r)\sin\Delta\theta \doteq r\Delta\theta$$

or

$$\Delta\theta = \frac{\Delta s \cos\theta}{r} \qquad (7\text{-}113)$$

With the relation above, Eq. (7-112) can be expressed in terms of Δs as follows:

$$\phi = \lim_{\substack{\Delta s \to 0 \\ \Gamma \to \infty}} \left(\frac{\Gamma \Delta s}{2\pi r}\cos\theta\right) + \text{Constant}$$

$$= \frac{\nu \cos\theta}{2\pi r} + \text{Constant} \qquad (7\text{-}114)$$

It is interesting to note that the velocity potential of the dipole is identical to that of the doublet if the dipole strength ν is replaced by the doublet strength $-\mu$. Consequently, the velocity components and stream function of a dipole are identical to Eqs. (7-108) and (7-109), respectively, if μ is replaced by $-\nu$:

$$v_r = -\frac{\nu \cos\theta}{2\pi r^2}, \qquad v_\theta = -\frac{\nu \sin\theta}{2\pi r^2} \qquad (7\text{-}115)$$

and

$$\psi = -\frac{\nu \sin\theta}{2\pi r} + \text{Constant} \qquad (7\text{-}116)$$

*The electric dipole is defined by two equal and opposite charges separated by an infinitesimal distance.

Both the streamlines and the potential lines are identical to those of the doublet given in Figs. 7.20a and 7.20b, respectively.

Illustrative Example 7.10. Verify that the stream function ψ and velocity potential ϕ of a two-dimensional vortex flow satisfies the Laplace equation.

Solution. For an irrotational flow the Laplace equation in ψ can be obtained from Eqs. (3–29) and (7–67) as follows:

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0 \tag{a}$$

From Eq. (7–101) the stream function of a vortex flow is given by

$$\psi = -\frac{\Gamma}{2\pi}\ln r + c \tag{b}$$

Substituting Eq. (b) into Eq. (a) we find

$$\frac{\Gamma}{2\pi}\frac{1}{r^2} - \frac{\Gamma}{2\pi}\frac{1}{r^2} + 0 = 0 \tag{c}$$

According to Eq. (7–51) we have

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \tag{d}$$

The velocity potential of the vortex flow is

$$\phi = \frac{\Gamma}{2\pi}\theta \tag{e}$$

Substituting Eq. (e) into Eq. (d) we find

$$\nabla^2\phi = 0 \tag{f}$$

7.13. Three-Dimensional Axially Symmetric Flow Examples

a. Uniform Flow

If a uniform flow U_∞ is given parallel to the positive direction of the z-axis (Fig. 7.14b), the expressions for velocity components are

$$v_z = U_\infty = \text{Constant}, \qquad v_r = 0, \qquad v_\theta = 0$$

According to Eqs. (7–50) and (7–70), one finds

$$U_\infty = \frac{\partial\phi}{\partial z} = \frac{1}{r}\frac{\partial\psi}{\partial r}, \qquad 0 = -\frac{1}{r}\frac{\partial\psi}{\partial z} = \frac{\partial\phi}{\partial r}$$

Therefore, the expressions for velocity potential and stream function are obtained.

$$\phi = \int U_\infty \, dz + \text{Constant} = U_\infty z + \text{Constant} = U_\infty \omega \cos\varphi + \text{Constant} \tag{7-117}$$

$$\psi = \int U_\infty r \, dr + \text{Constant} = \frac{U_\infty r^2}{2} + \text{Constant} = \frac{U_\infty}{2}\omega^2 \sin^2\varphi + \text{Constant} \tag{7-118}$$

The velocity components in spherical coordinates are [Eq. (7-53)]

$$v_\omega = \frac{\partial \phi}{\partial \omega} = U_\infty \cos\varphi$$

$$v_\varphi = \frac{1}{\omega}\frac{\partial \phi}{\partial \varphi} = -U_\infty \sin\varphi \tag{7-119}$$

$$v_\theta = 0$$

Equation (7-118) can also be obtained from the definition of the three-dimensional stream function, i.e.,

$$2\pi\psi = \pi r^2 U_\infty$$

or

$$\psi = \frac{r^2}{2}u = \frac{U_\infty}{2}\omega^2 \sin^2\varphi$$

where $\varphi = 0$ corresponds to the zero streamline.

The velocity components in the present case satisfy both the continuity equation and the zero vorticity equation; hence, the velocity potential and the stream function are harmonic functions.

b. Radial Flow—Source or Sink

A simple source in three-dimensional flow is a point from which fluid emanates in equal amounts in all directions (Fig. 7.22). Hence, the lines of flow in this case will be straight radial lines, i.e.,

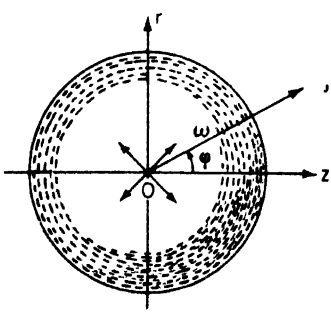


Figure 7.22

$$v_\omega = v_\omega(\omega), \qquad v_\varphi = 0,$$

$$v_\theta = 0, \qquad \frac{\partial}{\partial \varphi} = 0$$

From the continuity equation [Eq. (5-61)]

$$\nabla \cdot \mathbf{q} = \frac{1}{\omega^2}\frac{\partial}{\partial \omega}(\omega^2 v_\omega) = 0$$

Therefore

$$\omega^2 v_\omega = K \quad \text{or} \quad v_\omega = \frac{K}{\omega^2} \tag{7-120}$$

where K is a constant. Now let Q be the quantity of fluid flowing out of the source which, by the condition of continuity, would be equal to the quantity of fluid flowing through a sphere of radius R. Thus

$$Q = \iint_A \mathbf{n} \cdot \mathbf{q}\, dA = \iint_{sphere} v_\omega\, dA = \frac{K}{R^2}\left[2\int_0^{2\pi}\int_0^{\pi/2} (R\, d\varphi) R \sin\varphi\, d\theta\right] = 4\pi K \tag{7-121a}$$

The constant K is given by

$$K = \frac{Q}{4\pi} \tag{7-121b}$$

The quantity Q is called the strength of the source and $-Q$ is the strength of the sink.

The stream function is found by direct integration of the velocity components [Eq. (7-72)],

$$\psi = \int v_\omega \omega^2 \sin\varphi\, d\varphi + \text{Constant} \tag{7-122a}$$

Upon substitution of Eqs. (7-120) and (7-121b) for v_ω, we have

$$\psi = -\frac{Q}{4\pi}\cos\varphi + \text{Constant} \tag{7-122b}$$

From the vorticity equation [Eq. (3-32c)] we have found

$$\Omega_\theta = \frac{\partial v_\varphi}{\partial \omega} - \frac{1}{\omega}\frac{\partial v_\omega}{\partial \varphi} + \frac{v_\varphi}{\omega} = 0$$

Therefore, a velocity potential exists in the case of sources and sinks in three-dimensional flows. The velocity potential may be obtained from Eq. (7-53) and is given by

$$\phi = \int v_\omega\, d\omega + \text{Constant}$$

or

$$\phi = \int \frac{Q}{4\pi\omega^2}\, d\omega + \text{Constant} = -\frac{Q}{4\pi\omega} + \text{Constant} \tag{7-123}$$

where $Q > 0$ for a source and $Q < 0$ for a sink. It can easily be verified that both ϕ and ψ in three-dimensional flow are harmonic functions.

c. Doublet

As in the two-dimensional case, a doublet in three-dimensional flow is a combination of source and sink of infinite strength at an infinitesimal distance apart, with the product of their strength and distance apart remaining finite and constant. The

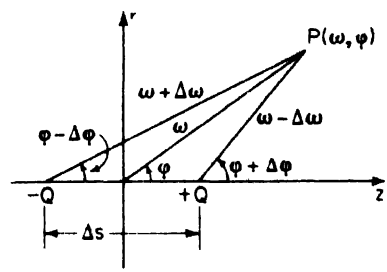


Figure 7.23

velocity potential of a three-dimensional doublet flow can be obtained by the superposition of the source and sink flows (Fig. 7.23):

$$\phi = -\frac{Q}{4\pi(\omega - \Delta\omega)} - \frac{-Q}{4\pi(\omega + \Delta\omega)} + \text{Constant}$$

$$= -\frac{Q}{4\pi\omega}\left[\frac{1}{1 - \Delta\omega/\omega} - \frac{1}{1 + \Delta\omega/\omega}\right] + \text{Constant}$$

$$= -\frac{Q(2\Delta\omega)}{4\pi\omega^2} + \text{Constant} \tag{7-124}$$

But the trigonometric relation in Fig. 7.23 gives

$$\Delta\omega = \frac{\Delta s}{2}\cos(\varphi - \Delta\varphi) = \frac{\Delta s}{2}\cos\varphi + \cdots$$

Hence,

$$\phi = -\lim_{\substack{\Delta s \to 0 \\ Q \to \infty}}\left(\frac{Q\Delta s}{4\pi\omega^2}\cos\varphi\right) + \text{Constant}$$

$$= -\frac{\mu}{4\pi\omega^2}\cos\varphi + \text{Constant} \tag{7-125}$$

where $\lim_{\substack{\Delta s \to 0 \\ Q \to \infty}} Q\Delta s = \mu$ is the doublet strength.

The velocity components of the doublet, obtained by direct integration of Eq. (7-53), are

$$v_\omega = \frac{\partial\phi}{\partial\omega} = \frac{\mu}{2\pi\omega^3}\cos\varphi \tag{7-126a}$$

$$v_\varphi = \frac{1}{\omega}\frac{\partial\phi}{\partial\varphi} = \frac{\mu}{4\pi\omega^3}\sin\varphi \tag{7-126b}$$

The stream function in this case is, from Eq. (7-72),

$$\psi = \int v_\omega\omega^2\sin\varphi\, d\varphi + \text{Constant}$$

$$= \frac{\mu}{4\pi\omega}\sin^2\varphi + \text{Constant} \tag{7-127}$$

Since the velocity potential and the stream function are obtained by the superposition of source and sink flows, they are harmonic functions. In Fig. 7.24 are shown streamlines and potential lines for a three-dimensional doublet.

Illustrative Example 7.11. Verify that the stream function ψ and velocity potential ϕ of a three-dimensional (axially symmetric) doublet flow satisfy the Laplace equation.

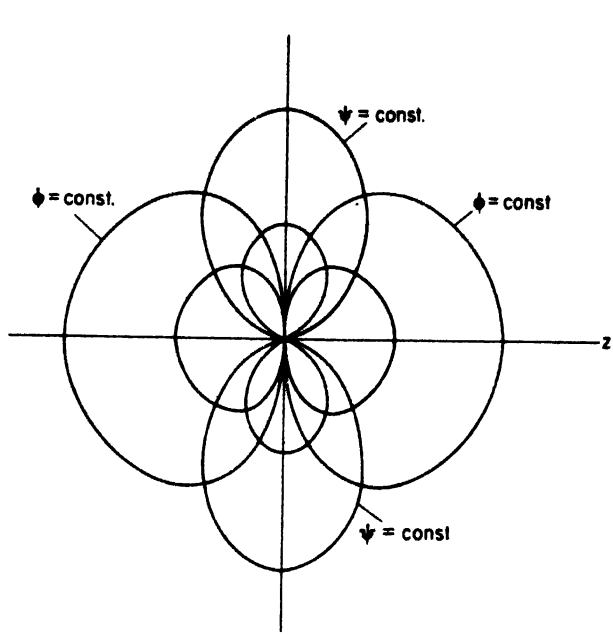


Figure 7.24

Solution. From Eqs. (3-32c) and (7-72) we have

$$\nabla^2\psi = \frac{1}{\sin\varphi}\frac{\partial^2\psi}{\partial\omega^2} + \frac{1}{\omega^2}\frac{\partial}{\partial\varphi}\left(\frac{1}{\sin\varphi}\frac{\partial\psi}{\partial\varphi}\right) = 0 \tag{a}$$

The stream function of doublet flow [Eq. (7-127)] is

$$\psi = \frac{\mu}{4\pi\omega}\sin^2\varphi + C \tag{b}$$

Substituting Eq. (b) into Eq. (a), one finds

$$\nabla^2\psi = -\frac{\mu}{2\pi}\frac{1}{\omega^3}\sin\varphi + \frac{\mu}{4\pi\omega^3}\frac{\partial}{\partial\varphi}\left(-2\cos\varphi\right) = 0 \tag{c}$$

From Eq. (7-53) we have

$$\nabla^2\phi = \frac{\partial}{\partial\omega}\left(\omega^2\frac{\partial\phi}{\partial\omega}\right) + \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial\phi}{\partial\varphi}\right) = 0 \tag{d}$$

The velocity potential in this case is given by Eq. (7-125) as

$$\phi = -\frac{\mu}{4\pi\omega^2}\cos\varphi + C \tag{e}$$

Combining Eqs. (d) and (e), we find

$$\nabla^2\phi = \frac{\partial}{\partial\omega}\left(\frac{\mu}{2\pi\omega}\right)\cos\varphi + \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\frac{\mu}{4\pi\omega^2}\sin^2\varphi\right)$$

$$= -\frac{\mu}{2\pi}\frac{\cos\varphi}{\omega^2} + \frac{\mu}{2\pi}\frac{\cos\varphi}{\omega^2} = 0 \qquad \text{(f)}$$

C. MOTION OF SOLID BODIES IN A FLUID

The formal solution of the type of flow problems under discussion consists of a solution of the Laplace equation that satisfies the prescribed boundary conditions. In general this direct analytical method is very difficult if not impossible. For a two-dimensional, irrotational, incompressible flow the theory of complex variables is employed in finding flow patterns that satisfy given boundary conditions. For example, the flow about a given airfoil may be determined by conformal mapping of the flow about a circle. The application of the method of conformal mapping to plane-flow problems is beyond the scope of this text.

The simple-flow examples given in § 7.12 and § 7.13 are solutions of Laplace's equation. These solutions are obtained by indirect methods, i.e., by investigation of various functions of flow which satisfy the Laplace equation and prescribed boundary conditions. Since Laplace's equation is linear and the dependent variables ϕ and ψ occur only linearly, the method of superposition can be employed in this case. This method of building up solutions by superposition is the very basis of the general theory of linear differential equations.* For instance, if we denote the velocity potentials for uniform flow and source flow by ϕ_1 and ϕ_2 which are solutions of the Laplace equation, a third velocity potential can be constructed by simple superposition, $\phi_3 = \phi_1 + \phi_2$, which also satisfies Laplace's equation. Based on the principle of superposition, some complicated flow patterns of physical significance will be obtained by combination of the elementary flow patterns which were discussed in § 7.12 and § 7.13.

Illustrative Example 7.12. If ψ_1, ψ_2 are solutions of the Laplace equation, show that $\psi_3 = \psi_1 + \psi_2$ is a solution of the Laplace equation.

Solution. Consider $\psi_1 = Ax$ where A is a constant. It can be easily verified that

$$\nabla^2\psi_1 = \frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial y^2} = \frac{\partial}{\partial x}(A) + 0 = 0$$

Similarly, we can show that $\psi_2 = x^2 - y^2$ satisfies

*For example, the methods of Fourier series, Laplace transforms, etc.

$$\nabla^2\psi_2 = \frac{\partial^2\psi_2}{\partial x^2} + \frac{\partial^2\psi_2}{\partial y^2} = 2 - 2 = 0$$

Now we will prove that $\psi_3 = \psi_1 + \psi_2 = Ax + x^2 - y^2$ is also a solution of the Laplace equation. Substituting ψ_3 into the Laplace equation we find

$$\nabla^2\psi_3 = \frac{\partial^2\psi_3}{\partial x^2} + \frac{\partial^2\psi_3}{\partial y^2} = \frac{\partial}{\partial x}(A + 2x) + \frac{\partial}{\partial y}(-2y) = 2 - 2 = 0$$

7.14. Rankine's Method of Constructing Streamlines

If a stream function ψ can be expressed as the sum of two harmonic functions ψ_a and ψ_b, we can construct the streamlines of ψ from those of ψ_a and ψ_b by a graphical method. This method of determining ψ is attributed to Rankine.

Let us choose a small constant, h. The lines $\psi_a = -2h, -h, 0, h, 2h, \ldots$ and $\psi_b = -2h, -h, 0, h, 2h, \ldots$ are drawn and form a network resembling parallelograms. In any particular cell, say cell A (Fig. 7.25),

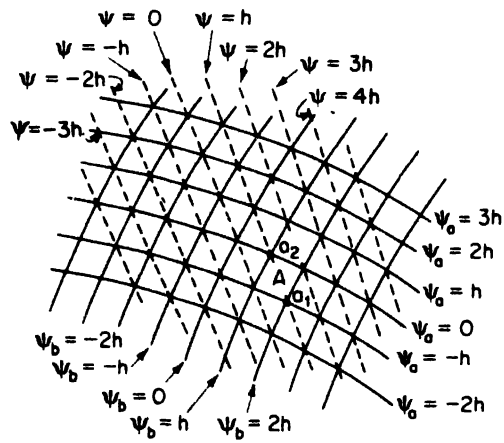


Figure 7.25

the boundaries are formed by the streamlines $\psi_a = 0$, $\psi_b = h$, $\psi_a = -h$, and $\psi_b = 2h$, and ψ has the value $\psi_a + \psi_b = h$ on diagonally opposite vertices, a_1 and a_2. In general the streamlines ψ can be constructed by joining successive opposite vertices as shown by the dotted lines in Fig. 7.25. In a similar manner the construction of potential lines can be made when the potential function ϕ can be expressed as the sum of two harmonic functions, ϕ_a and ϕ_b.

7.15. Superposition of Source and Rectilinear Flow

a. Two-Dimensional Case

The stream function for the uniform flow of a fluid with velocity U_∞ in the direction of the positive x-axis is given by the expression [Eq. (7–77)]

$$\psi = U_\infty y + \text{Constant}$$

If the uniform flow is superimposed on a source of strength Q at the origin [Fig. 7.26a and Eq. (7–88)], the stream function of the combined flow is

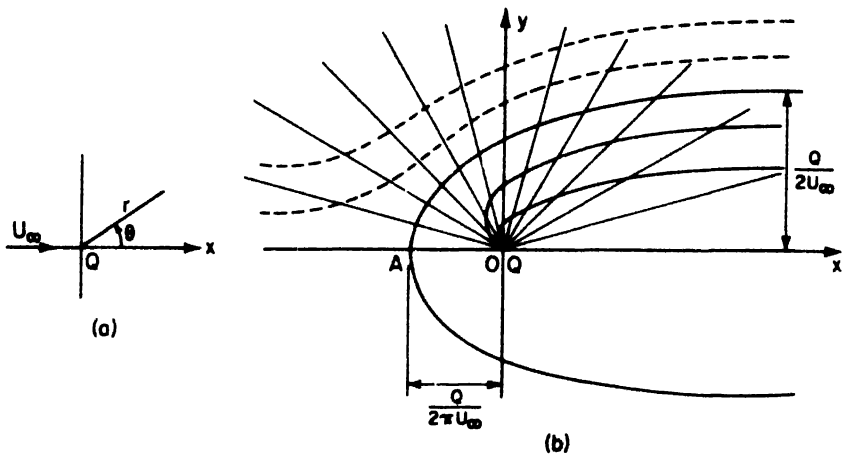


Figure 7.26

$$\psi = \frac{Q}{2\pi}\theta + U_\infty y + \text{Constant} \tag{7–128}$$

and the streamlines shown in Fig. 7.26b can be easily drawn by Rankine's method.

The velocity potential [Eqs. (7–76) and (7–95)] and the velocity components are

$$\phi = \frac{Q}{2\pi}\ln r + U_\infty x + \text{Constant} \tag{7–129}$$

$$u = \frac{\partial\phi}{\partial x} = \frac{Q}{2\pi}\frac{x}{r^2} + U_\infty = \frac{Q}{2\pi r}\cos\theta + U_\infty \tag{7–130}$$

$$v = \frac{\partial\phi}{\partial y} = \frac{Q}{2\pi}\frac{y}{r^2} = \frac{Q}{2\pi r}\sin\theta \tag{7–131}$$

In Fig. 7.26b we see that the streamlines are symmetrical about the x-axis. The stagnation point can be determined from Eq. (7–130) along the negative x-axis where $u = 0$, $\theta = \pi$, and $r = -x$; hence Eq. (7–130) gives

$$x = -\frac{Q}{2\pi U_\infty} \tag{7–132}$$

The location of the stagnation point is denoted by A in Fig. 7.26b. Now let us consider the streamline $\psi = \psi_0$ along the negative x-axis. From Eq. (7–128), we have

$$\psi_0 = \left[\frac{Q}{2\pi}\theta + U_\infty r \sin\theta\right]_{\theta=\pi} + \text{Constant} = \frac{Q}{2} + \text{Constant} \tag{7–133}$$

Substituting the value of ψ_0 above into Eq. (7–128) the equation for the streamline, $\psi = \psi_0$, is obtained:

$$\frac{Q}{2\pi}\theta + U_\infty r \sin\theta = \frac{Q}{2}$$

or

$$y = r\sin\theta = \frac{Q}{2U_\infty}\left(1 - \frac{\theta}{\pi}\right) \tag{7–134}$$

Equation (7–134) represents the streamline passing through the stagnation point A (for convenience, it is called the zero streamline) and divides the flow into two regions. Hence, this dividing streamline can be replaced by a solid wall (because there is no flow across this zero streamline). The source is enclosed in the solid wall and is of no interest to us. The external uniform stream in the presence of this wall is given by Eqs. (7–128) to (7–131).

The maximum value of the height of the wall occurs at $\theta = 0$ and $x \to \infty$ which gives [Eq. (7–134)]

$$(y)_{x\to\infty} = \frac{Q}{2U_\infty} \tag{7–135}$$

Therefore, the height of the wall approaches $Q/2U_\infty$ asymptotically. Physically, a source in a uniform stream can be interpreted as the flow in the neighborhood of one end of a long bluff body in a stream. In this connection it is interesting to note that if a long bluff body is taken as a cliff, the stagnation point A would be the most sheltered place when a strong wind blows toward a cliff.

b. Three-Dimensional Case

If we superimpose a source of strength Q at the origin and a uniform flow of a fluid with velocity U_∞, then the combined stream

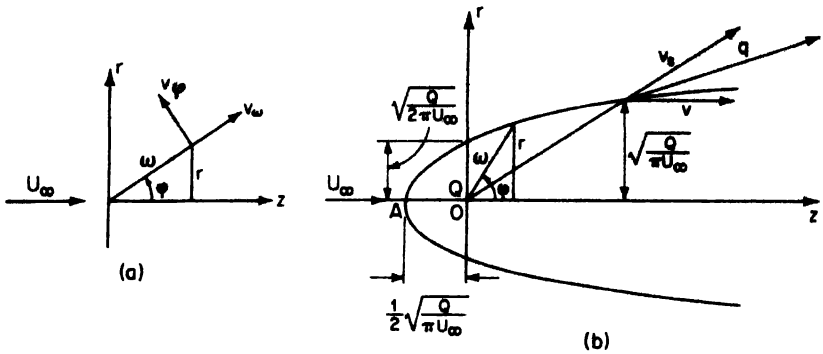


Figure 7.27

function is [Fig. 7.27a and Eqs. (7–118) and (7–122b)]

$$\psi = -\frac{Q}{4\pi}\cos\varphi + \frac{1}{2}U_\infty\omega^2\sin^2\varphi + \text{Constant} \tag{7–136}$$

The velocity components of the combined flow are [Eqs. (7–72) and Fig. 7.27a]

$$v_\omega = \frac{1}{\omega^2\sin\varphi}\frac{\partial\psi}{\partial\varphi} = \frac{Q}{4\pi\omega^2} + U_\infty\cos\varphi \tag{7–137}$$

$$v_\varphi = -\frac{1}{\omega\sin\varphi}\frac{\partial\psi}{\partial\omega} = -U_\infty\sin\varphi \tag{7–138}$$

The stagnation point along the negative z-axis where $\varphi = \pi$ is such that $v_\omega = 0$, $v_\varphi = 0$, or

$$(\omega)_{\varphi=\pi} = \sqrt{\frac{Q}{4\pi U_\infty}} \tag{7–139}$$

Along the negative direction of the z-axis ($\varphi = \pi$) we have

$$\psi_0 = \frac{Q}{4\pi} + \text{Constant} \tag{7–140}$$

The stream surface which passes through the stagnation point A is therefore, [Eqs. (7–136) and (7–140)]

$$-\frac{Q}{4\pi}\cos\varphi + \frac{1}{2}U_\infty\omega^2\sin^2\varphi = \frac{Q}{4\pi} \tag{7–141}$$

Equation (7–141) is the dividing stream surface which separates the flow into two regions, and it may be expressed in the form

$$r^2 = \frac{Q}{2\pi U_\infty}(1 + \cos\varphi) \tag{7–142}$$

with maximum value ($\varphi = 0$ and $z \to \infty$)

$$(r)_{z\to\infty} = \sqrt{\frac{Q}{\pi U_\infty}} = r_0 \tag{7–143}$$

Figure 7. 27b, which is plotted from Eq. (7–142), is called a half streamline-body (blunt-nosed cylindrical body) whose maximum diameter at $z \to \infty$ is $2\sqrt{Q/\pi U_\infty} = 2r_0$.

The pressure equation can be obtained from the Bernoulli equation

$$p = p_\infty + \frac{\rho U_\infty^2}{2} - \frac{\rho q^2}{2} \tag{7–144}$$

where

$$q^2 = v_s^2 + U_\infty^2 + 2U_\infty v_s \cos\varphi$$

The velocity v_s is the radial velocity due to the source alone and takes the form [Eq. (7–120)]

$$v_s = \frac{Q}{4\pi\omega^2} = \frac{\pi U_\infty r_0^2}{4\pi\omega^2} = U_\infty \left(\frac{r_0}{2\omega}\right)^2 \tag{7–145}$$

Hence,

$$q^2 = U_\infty^2 \left[\left(\frac{r_0}{2\omega}\right)^4 + 1 + 2\left(\frac{r_0}{2\omega}\right)^2 \cos\varphi\right] \tag{7–146}$$

and the pressure at any point is given by

$$p = p_\infty - \frac{\rho U_\infty^2}{2}\left[\left(\frac{r_0}{2\omega}\right)^4 + 2\left(\frac{r_0}{2\omega}\right)^2 \cos\varphi\right] \tag{7–147a}$$

On the surface of the half streamline-body we have, from Eq. (7–142),

$$r^2 = \frac{r_0^2}{2}(1 + \cos\varphi) = r_0^2 \cos^2\left(\frac{\varphi}{2}\right) \tag{7–148a}$$

and

$$\frac{r_0}{\omega} = \frac{r/\cos(\varphi/2)}{r/\sin\varphi} = 2\sin\left(\frac{\varphi}{2}\right) \tag{7–148b}$$

$$\left(\frac{r_0}{\omega}\right)^2 = 4\sin^2\left(\frac{\varphi}{2}\right) = 4\left[1 - \left(\frac{r}{r_0}\right)^2\right] \tag{7–148c}$$

Substituting Eqs. (7–148a) and (7–148c) into Eq. (7–147a), the pressure at any point on the surface of the half streamline-body is obtained:

$$p = p_\infty + \frac{\rho U_\infty^2}{2}\left[3\left(\frac{r}{r_0}\right)^4 - 4\left(\frac{r}{r_0}\right)^2 + 1\right] \tag{7–147b}$$

The pressure coefficient is

$$C_p = \frac{p - p_\infty}{(1/2)\rho U_\infty^2} = \left[3\left(\frac{r}{r_0}\right)^2 - 1\right]\left[\left(\frac{r}{r_0}\right)^2 - 1\right] \tag{7–147c}$$

and the relation between r/r_0 and z/r_0 is given by

$$\left(\frac{z}{r_0}\right)^2 = \frac{[2(r/r_0)^2 - 1]^2}{4[1 - (r/r_0)^2]} \tag{7–148d}$$

The pressure distribution on the surface of a half streamline-body is shown in Fig. 7.28.

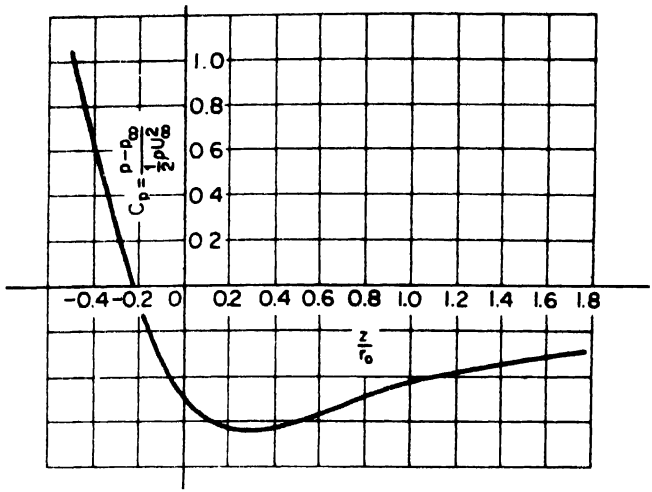


Figure 7.28

Illustrative Example 7.13. Find the total force on a half streamline-body (as shown in Fig. 7.M).

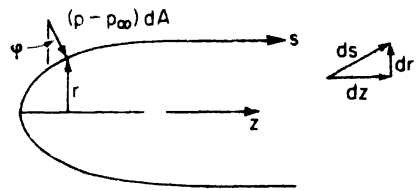


Figure 7.M

Solution. The horizontal component force acting on the element of the body is

$$dF_x = [(p - p_\infty)\, dA] \sin \varphi = (p - p_\infty)(2\pi r\, ds) \sin \varphi = 2\pi(p - p_\infty) r\, dr \tag{a}$$

The total horizontal force acting on the body is

$$F_x = \int 2\pi(p - p_\infty) r\, dr = \pi \int_0^1 (p - p_\infty) r_0^2\, d\left(\frac{r}{r_0}\right)^2 \tag{b}$$

According to Eq. (7-147b) we have

$$(p - p_\infty) = \frac{\rho U_\infty^2}{2}\left[3\left(\frac{r}{r_0}\right)^4 - 4\left(\frac{r}{r_0}\right)^2 + 1\right] \tag{c}$$

where

$$r_0 = \sqrt{\frac{Q}{\pi U_\infty}} = (r)_{x\to\infty}$$

Substituting Eq. (c) into Eq. (b) we find

$$F_x = \frac{\pi r_0^2 \rho U_\infty^2}{2} \int_0^1 \left[3\left(\frac{r}{r_0}\right)^4 - 4\left(\frac{r}{r_0}\right)^2 + 1\right] d\left(\frac{r}{r_0}\right)^2$$

Integrating,

$$F_x = \frac{\pi r_0^2 \rho U_\infty^2}{2} \left[\left(\frac{r}{r_0}\right)^6 - 2\left(\frac{r}{r_0}\right)^4 + \left(\frac{r}{r_0}\right)\right]_0^1 = 0$$

Since the body is symmetrical about the z-axis, the vertical components of force are canceled out.

7.16. Superposition of Source and Sink with Rectilinear Flow—The Rankine Body

Let a source and sink of equal strength in spherical coordinates be located at equal distances from the origin along the negative and the positive directions of the z-axis, respectively (Fig. 7.29). If a uniform rectilinear flow in the positive direction of the z-axis is superimposed on

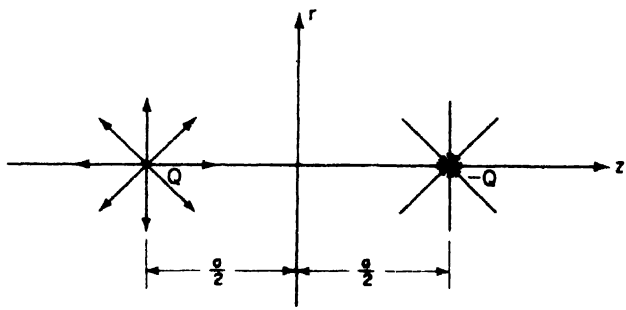


Figure 7.29

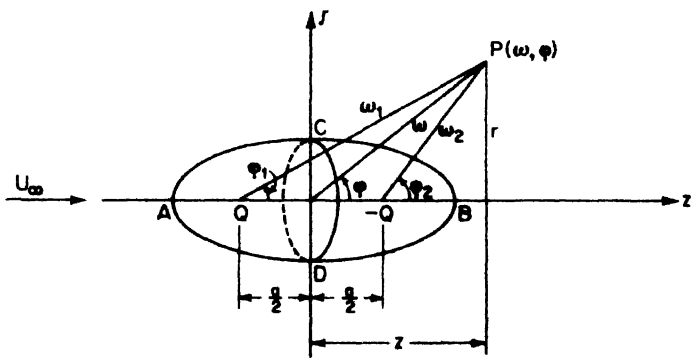


Figure 7.30

the source-sink combination, a closed contour in a uniform flow is obtained (Fig. 7.30). The stream function for the body, a Rankine oval, is [Eqs. (7–122b) and (7–118)]

$$\psi = \frac{Q}{4\pi}(\cos\varphi_2 - \cos\varphi_1) + \frac{U_\infty}{2}\omega^2 \sin^2\varphi \tag{7–149}$$

The velocity potential for the combined flow is [Eqs. (7–117) and (7–123)]

$$\phi = -\frac{Q}{4\pi}\left(\frac{1}{\omega_1} - \frac{1}{\omega_2}\right) + U_\infty \omega \cos\varphi \tag{7–150}$$

Referring to Fig. 7. 30 we see that $\varphi_1 = \varphi_2 = 0$ or $\varphi_1 = \varphi_2 = \pi$ when P is on the z-axis excluding points between source and sink. According to Eq. (7–149), $\psi = 0$ contains the entire z-axis excluding the part between source and sink, and the equation of this dividing streamline then becomes

$$r^2 + \frac{Q}{2\pi U_\infty}(\cos\varphi_2 - \cos\varphi_1) = 0 \tag{7–151}$$

A dividing stream surface is generated from the dividing streamline by rotation about the z-axis. This stream surface which is symmetrical about the plane $z = 0$ is called the Rankine oval body. The velocity components for the combined flow are

$$v_\omega = \frac{\partial\phi}{\partial\omega} = -\frac{Q}{4\pi}\left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right) + U_\infty \cos\varphi \tag{7–152}$$

$$v_\varphi = \frac{1}{\omega}\frac{\partial\phi}{\partial\varphi} = -U_\infty \sin\varphi \tag{7–153}$$

The stagnation points A and B (Fig. 7.30) where $v_\omega = v_\phi = 0$ are given as follows:

$$U_\infty \sin\varphi = U_\infty \frac{r}{\omega} = 0$$

or

$$r = 0$$

and

$$\frac{\omega_1^2 - \omega_2^2}{\omega_1^2\omega_2^2} = \frac{4\pi U_\infty}{Q}\cos\varphi$$

$$\frac{z_0}{(z_0^2 - a^2/4)^2} = \frac{2\pi U_\infty}{aQ} \tag{7–154}$$

Hence, $\pm z_0$ determines the length of the Rankine oval.

To find the maximum ordinate r_0 which is located in the stream

surface along the y-axis, let $\varphi_2 = 180^\circ - \varphi_1$ and thus Eq. (1–151) gives

$$r_0^2 = \frac{Q}{\pi U_\infty} \cos \varphi_1 = \frac{(a/2)(Q/\pi U_\infty)}{\sqrt{r_0^2 + (a/2)^2}} \tag{7–155}$$

7.17. Superposition of Rectilinear Flow and Line Sources

In § 7.13c we have shown that a source and a sink in a three-dimensional flow situated along the z-axis can be combined to form a new type of flow. If we superimpose a series of sources placed along the z-axis, the stream function of the combined sources is

$$\psi = -\frac{Q_0}{4\pi} \cos \varphi_0 - \frac{Q_1}{4\pi} \cos \varphi_1 - \frac{Q_2}{4\pi} \cos \varphi_2 - \cdots \tag{7–156}$$

where $Q_0, Q_1, Q_2, \cdots$ are the individual strengths of the sources which are situated at angles $\varphi_0, \varphi_1, \varphi_2, \cdots$ from a field point $P(\omega, \varphi)$.

If we replace the above series of sources of finite strength by a continuous distribution of sources along the z-axis, a line source is obtained. Let $q(\xi)$ be the source strength per unit length, then $q(\xi)d\xi$ represents infinitesimal source strength at $z = \xi$. The variable ξ is introduced to represent the source distribution beginning at $z = 0$ and extending to $z = l$ (Fig. 7.31). The stream function at any point

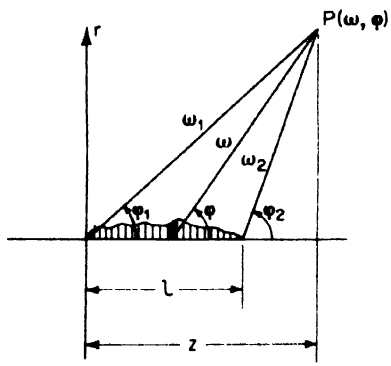


Figure 7.31

$P(\omega, \varphi)$ for the elemental source is

$$d\psi = -\frac{q(\xi)\, d\xi}{4\pi} \cos \varphi \tag{7–157}$$

and for the distributed sources from $\xi = 0$ to $\xi = l$ is

$$\psi = -\frac{1}{4\pi}\int_0^l q(\xi)\cos\varphi\, d\xi + \text{Constant} \tag{7-158}$$

where $q(\xi)$ and $\cos\varphi$ are functions of ξ.

If the strength of source $q(\xi) = q$ is a constant, the integrand in Eq. (7-158) can be transformed as follows:

$$-\cos\varphi\, d\xi = -\frac{(z-\xi)}{\omega}d\xi \tag{7-159}$$

Since z and r are fixed in the integral of Eq. (7-158), we have

$$\omega^2 = (z-\xi)^2 + r^2 \tag{7-160a}$$

and

$$\omega\, d\omega = -(z-\xi)\, d\xi$$

Hence,

$$-\cos\varphi\, d\xi = d\omega \tag{7-160b}$$

In terms of ω, the stream function given in Eq. (7-158) becomes

$$\psi = \frac{q}{4\pi}\int_{\omega_1}^{\omega_2} d\omega + \text{Constant} = \frac{q}{4\pi}(\omega_2 - \omega_1) + \text{Constant}$$

$$= \frac{Q}{4\pi l}(\omega_2 - \omega_1) + \text{Constant} \tag{7-161}$$

where $Q = ql$.

Equation (7-161) may be used to calculate the flow around a body shaped like an airship. In this case, a combined flow consisting of a uniform stream in the positive direction of the z-axis with a point source Q at the origin and a uniform line sink (a negative line source) of total strength $-Q$ extending from $z = 0$ to $z = l$ yields a stream function

$$\psi = \frac{1}{2}U_\infty \omega^2 \sin^2\varphi - \frac{Q}{4\pi l}(\omega_2 - \omega_1) - \frac{Q}{4\pi}\cos\varphi_1 \tag{7-162}$$

The streamline for which $\psi = 0$ coincides with the axis of symmetry, i.e., the z-axis, and with a surface of revolution which completely surrounds the source-sink combination, and there is no net flow across it. The equation of this surface is

$$\frac{U_\infty \omega^2}{2}\sin^2\varphi = \frac{Q}{4\pi}\left(\frac{\omega_2 - \omega_1}{l} + \cos\varphi_1\right) \tag{7-163a}$$

or

$$\frac{U_\infty r^2}{2} = \frac{Q}{4\pi}\left(\frac{\omega_2 - \omega_1}{l} + \frac{z}{\omega_1}\right) \tag{7-163b}$$

In Fig. 7.32 is shown a cross section of the surface of revolution resulting from the source-sink and uniform flow combination mentioned above. The streamlines of the source- and line-sink combination are shown

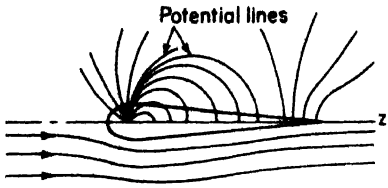


Figure 7.32

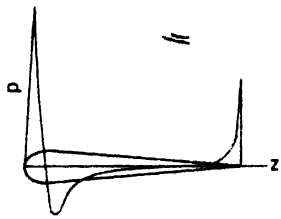


Figure 7.33

on the upper half of the body while the lower half depicts the streamlines of the resultant flow. The pressure at any point on the surface can be calculated by the use of Bernoulli's equation [Eq. (7–144)] and is shown in Fig. 7.33.

The discussion above is an inverse problem. In general, $q(\xi)$ in Eq. (7–158) must be determined for a given problem by satisfying the boundary conditions. The solution is usually obtained numerically, and in most cases the integral in Eq. (7–158) is approximated by a sum like Eq. (7–156) with a finite number of sources and sinks. Application of the boundary conditions, at the same number of boundary points, leads to a set of simultaneous equations for the source and sink strengths.* The direct problem in this case is to find analytical expressions for a given airship shape. This involves the solution of an integral equation, i.e., Eq. (7–158), which is enormously more difficult.

7.18. Superposition of Rectilinear Flow and Doublet

a. Three-Dimensional Motion-Sphere in a Uniform Stream

In Fig. 7.34a is shown the development of a uniform flow and a doublet combination. The doublet is so oriented that the source is placed upstream.

The stream function for a doublet in a uniform flow of magnitude U_∞ in the positive z-direction is

$$\psi = -\frac{\mu}{4\pi\omega}\sin^2\varphi + \frac{1}{2}U_\infty\omega^2\sin^2\varphi + C \tag{7–164}$$

where $-\mu$ is the doublet strength with source placed upstream. Let us consider the streamline $\psi = \psi_0 = C$, i.e.,

$$\frac{1}{2}U_\infty\omega^2 - \frac{\mu}{4\pi\omega}\sin^2\varphi = 0 \tag{7–165a}$$

or

*NACA TM 574.

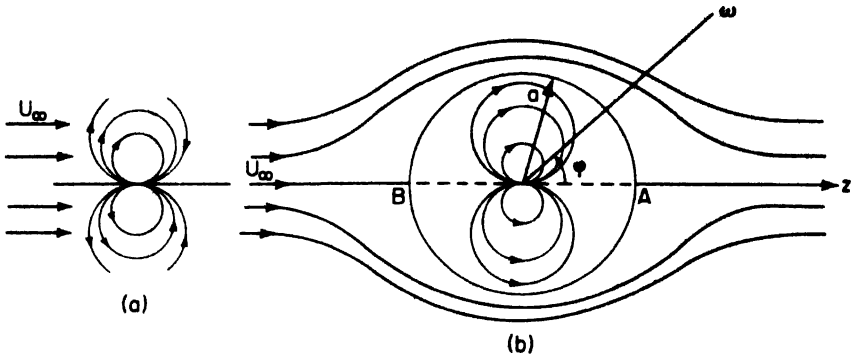


Figure 7.34

$$\left(\omega^3 - \frac{\mu}{2\pi U_\infty}\right) \sin^2 \varphi = 0 \tag{7-165b}$$

The equation above is satisfied if either $\sin \varphi = 0$ or the quantity in the parenthesis is zero, that is, when

$$\varphi = 0 \quad \text{or} \quad \pi$$

and

$$\omega = \left(\frac{\mu}{2\pi U_\infty}\right)^{1/3} \tag{7-166a}$$

But the value of ω at points A and B on the z-axis (Fig. 7.34b) is the radius of the sphere a; it follows that

$$(\omega)_{\varphi=0} = a = \left(\frac{\mu}{2\pi U_\infty}\right)^{1/3} \tag{7-166b}$$

Hence, ψ_0 represents the streamline for a sphere of radius a and the z-axis. The sphere of radius a may be taken as a solid boundary as there is no flow across a streamline. For $\omega > a$, the flow is outside the sphere. With the new relation of μ and a, Eq. (7-164) becomes

$$\psi = \frac{U_\infty}{2}\left(\omega^2 - \frac{a^3}{\omega}\right) \sin^2 \varphi \tag{7-167}$$

which is the stream function for a sphere of radius a in uniform flow of magnitude U_∞ in the direction of $\varphi = 0$ (positive z-axis).

Introducing the value above for the strength of a doublet μ, the velocity potential and the velocity components in the radial and tangential directions for the sphere in a rectilinear flow are obtained as follows:

$$\phi = U_\infty \omega \left(\frac{a^3}{2\omega^3} + 1\right) \cos \varphi \tag{7-168}$$

$$v_\omega = \frac{\partial \phi}{\partial \omega} = U_\infty \left(1 - \frac{a^3}{\omega^3}\right) \cos \varphi \tag{7-169}$$

$$v_\varphi = \frac{1}{\omega} \frac{\partial \phi}{\partial \varphi} = -U_\infty \left(\frac{a^3}{2\omega^3} + 1\right) \sin \varphi \tag{7-170}$$

The velocity components at the surface of a sphere, $\omega = a$, are

$$v_\omega = \left(\frac{\partial \phi}{\partial \omega}\right)_{\omega = a} = 0 \tag{7-171a}$$

$$v_\varphi = \left(\frac{1}{\omega} \frac{\partial \phi}{\partial \varphi}\right)_{\omega = a} = -\tfrac{3}{2} U_\infty \sin \varphi \tag{7-171b}$$

Hence the stagnation points occur at $\varphi = 0$ and $\varphi = \pi$ and the maximum velocities occur at $\varphi = \pi/2$ and $\varphi = 3\pi/2$ with a value

$$|v_\varphi|_{max} = \tfrac{3}{2} U_\infty \tag{7-172}$$

The pressure at any point on the sphere can be calculated by the use of Bernoulli's equation and Eq. (7-171b):

$$p = p_\infty + \frac{\rho}{2} U_\infty^2 - \frac{\rho}{2} v_\varphi^2$$

$$= p_\infty + \frac{\rho}{2} U_\infty^2 \left(1 - \frac{9}{4} \sin^2 \varphi\right) \tag{7-173}$$

The pressure distribution as calculated from Eq. (7-173) is shown in Fig. 7.35.

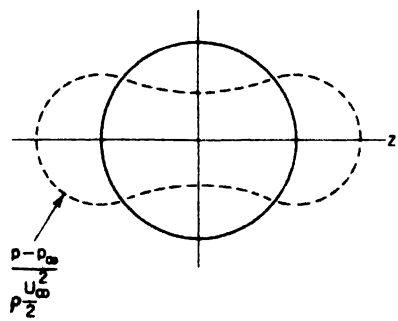


Figure 7.35

From the symmetry of the pressure distribution it is evident that there is no force or moment acting on the sphere in a rectilinear flow. This is a serious discrepancy with actual results for viscous fluids (see § 8.7).

Illustrative Example 7.14. A hemisphere lying on a flat plate (not fastened) is shown in Fig. 7.N. What density of the material of the hemisphere is required in order to keep it staying on the plate ?

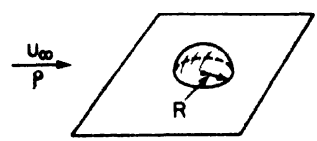


Figure 7.N

Solution. Let ρ_h be the density of the hemisphere and ρ the density of the air,

The weight of the hemisphere is

$$W_h = \tfrac{2}{3}\rho_h g\pi R^3$$

The buoyancy force is

$$B = \tfrac{2}{3}\rho g\pi R^3$$

The lift force acting on the hemisphere is

$$\int\int (p - p_\infty) \sin\varphi \cos\theta \, dA \tag{a}$$

According to Eq. (7–173) we have

$$p - p_\infty = \frac{\rho}{2}U_\infty^2\left(1 - \frac{9}{4}\sin^2\varphi\right) \tag{b}$$

and

$$dA = (R\sin\varphi \, d\varphi)(R\, d\theta) \tag{c}$$

Substituting Eqs. (b) and (c) into Eq. (a), we find

$$\begin{aligned} L &= -\int_0^{\pi}\int_{-\pi/2}^{\pi/2} \frac{\rho U_\infty^2}{2}\left(1 - \frac{9}{4}\sin^2\varphi\right)R^2\sin^2\varphi\cos\theta \, d\varphi \, d\theta \\ &= -\frac{\rho U_\infty^2}{2}\int_0^{\pi}\left(1 - \frac{9}{4}\sin^2\varphi\right)\sin^2\varphi\,(2R^2)\, d\varphi \\ &= -\rho R^2 U_\infty^2\left(\frac{\pi}{2} - \frac{27}{32}\pi\right) = \frac{11}{32}\pi\rho R^2 U_\infty^2 \end{aligned} \tag{d}$$

Equating Eq. (d) with W_h and B, we obtain

$$\frac{2}{3}\rho_h g\pi R^3 = \frac{2}{3}\rho g\pi R^3 + \frac{11}{32}\pi\rho R^2 U_\infty^2$$

or

$$\frac{\rho_h}{\rho} = \left(1 + \frac{33}{64}\frac{U_\infty^2}{Rg}\right) \tag{e}$$

Hence ρ_h must be equal to or greater than

$$\rho\left(1 + \frac{33}{64}\frac{U_\infty^2}{Rg}\right)$$

b. Two-Dimensional Case—A Cylinder in a Uniform Stream

In § 7.12a and § 7.12d the nature of uniform flow and flow due to a doublet, respectively, have been discussed. In analogy to the three-dimensional motion if a doublet is placed in a uniform flow of magnitude U_∞ in the positive direction of the x-axis (Fig. 7.36), the combined stream function becomes

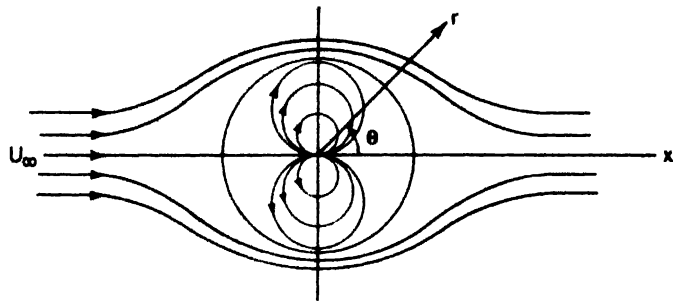


Figure 7.36

$$\psi = U_\infty r \sin\theta - \frac{\mu \sin\theta}{2\pi r} + C \tag{7-174}$$

where $-\mu$ is the doublet strength with source placed upstream. For the streamline $\psi = \psi_0 = C$ the equation above reduces to

$$\left(U_\infty - \frac{\mu}{2\pi r^2}\right) \sin\theta = 0 \tag{7-175}$$

Equation (7–175) is satisfied if $\sin\theta = 0$ or the quantity in the parenthesis is zero. Thus we have $\theta = 0$ or π and

$$r = \left(\frac{\mu}{2\pi U_\infty}\right)^{1/2} = R \tag{7-176}$$

Hence ψ_0 represents the streamline for a circle of radius R and the x-axis. The cylinder of radius R may be taken as a solid boundary. The stream function and the velocity potential for uniform flow around a circular cylinder of radius R, by substitution of the value of μ, are

$$\psi = U_\infty \left(r - \frac{R^2}{r}\right) \sin\theta \tag{7-177}$$

$$\phi = U_\infty \left(r + \frac{R^2}{r}\right) \cos\theta \tag{7-178}$$

The tangential and radial components of velocity are

$$v_\theta = -\frac{\partial \psi}{\partial r} = -U_\infty \left(1 + \frac{R^2}{r^2}\right) \sin\theta \tag{7-179}$$

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U_\infty \left(1 - \frac{R^2}{r^2}\right) \cos\theta \tag{7-180}$$

It can easily be shown that both ψ and ϕ in Eqs. (7–177) and (7–178) are solutions of Laplace's equation. The boundary conditions which must be satisfied are

(i) $$v_r = \frac{\partial \phi}{\partial r} = 0 \quad \text{at} \quad r = R \tag{7-181a}$$

(ii) $$\left.\begin{aligned} u &= U_\infty \\ v &= 0 \end{aligned}\right\} \quad \text{at} \quad r \to \infty \tag{7-181b}$$

Condition (i) is evident by substituting $r = R$ in Eq. (7-180) and condition (ii) can be verified for $r \to \infty$ as follows (Fig. 7.37):

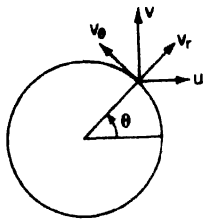


Figure 7.37

$$\begin{aligned} u &= v_r \cos\theta - v_\theta \sin\theta \\ &= U_\infty \cos^2\theta + U_\infty \sin^2\theta \\ &= U_\infty \\ v &= v_r \sin\theta + v_\theta \cos\theta \\ &= U_\infty(\cos\theta \sin\theta - \cos\theta \sin\theta) \\ &= 0 \end{aligned} \tag{7-182}$$

From Eq. (7-182) the tangential velocity component on the surface of the cylinder ($r = R$) is obtained:

$$v_\theta = -2U_\infty \sin\theta \tag{7-183}$$

The negative sign indicates v_θ is in the clockwise direction. Equation (7-183) reveals that stagnation points in this case are located at $\theta = 0$ and π, and at $\theta = \pm\pi/2$ the value of the maximum velocity is

$$|v_\theta|_{max} = 2U_\infty \tag{7-184}$$

The pressure at any point on the surface of the cylinder can be calculated by employing Bernoulli's equation between infinity and the local points; thus

$$p = p_\infty + \frac{\rho}{2}U_\infty^2 - \frac{\rho}{2}v_\theta^2 \tag{7-185}$$

For points on the cylinder, by substituting the value of v_θ from Eq. (7-183),

$$p - p_\infty = \frac{\rho U_\infty^2}{2}(1 - 4\sin^2\theta) \tag{7-186}$$

The equation above gives the values and locations of the maximum, minimum, and zero pressures, respectively, as follows:

$$\frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2} = 1 \qquad \text{at} \qquad \theta = 0 \text{ and } \pi \tag{7-187a}$$

$$\frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2} = -3 \qquad \text{at} \qquad \theta = \frac{\pi}{2} \text{ and } \frac{3\pi}{2} \tag{7-187b}$$

$$\frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2} = 0 \qquad \text{at} \qquad \theta = \pm\frac{\pi}{6} \text{ and } \pm\frac{5\pi}{6} \tag{7-187c}$$

From the symmetry of the pressure distribution it is easy to show that there are no lift and drag forces acting on the cylinder in a recti-

linear flow. By integration of the y and x-components of the pressure force per unit length over the cylinder, we have

$$L = \int_0^{2\pi} -(p - p_\infty)R\, d\theta \sin\theta \tag{7-188a}$$

or, using the relation given in Eq. (7-186),

$$L = -\frac{\rho U_\infty^2}{2}\int_0^{2\pi}(1 - 4\sin^2\theta)\sin\theta\, d\theta = 0 \tag{7-188b}$$

and

$$D = \int_0^{2\pi}(p - p_\infty)R\, d\theta \cos\theta$$

$$= \frac{\rho U_\infty^2}{2}\int_0^{2\pi}(1 - 4\sin^2\theta)\cos\theta\, d\theta = 0 \tag{7-189}$$

The integrand in Eq. (7-188a) is given a negative sign because the normal direction of the pressure force is inward and it contributes a negative lift. The result for drag on the cylinder provides us with an interesting historical sidelight. The great French mathematician, D' Alembert, was the first to develop a rational theory of calculating the resistance acting on a body immersed in a uniform stream of nonviscous fluids. His mathematical proof that drag was zero was contradicted by physical evidence. As a result, the nonexistence of drag on a body in a uniform stream has since been called D' Alembert's paradox.

Modification of this paradox (for a long time regarded as an unrealistic science) led to the rational theory of lift which was developed later by Joukowski and Lanchester. In the next section it will be demonstrated that the flow of a uniform stream about a cylinder must be accompanied by a circulation to produce a lift force.

7.19. Superposition of Vortex, Rectilinear Flow and Doublet in a Two-Dimensional Case

Let us now examine the superposition of a vortex to the doublet in a uniform flow. The combination results in flow around a circular cylinder with circulation (clockwise direction for Γ is assumed). In § 7.12c the nature of vortex flow was discussed. Combining a vortex flow [§ 7.12c] and the flow past a cylinder given in § 7. 18b, we have

$$\phi = U_\infty\left(r + \frac{R^2}{r}\right)\cos\theta - \frac{\Gamma}{2\pi}\theta + \text{Constant} \tag{7-190}$$

$$\psi = U_\infty\left(r - \frac{R^2}{r}\right)\sin\theta + \frac{\Gamma}{2\pi}\ln r + \text{Constant} \tag{7-191}$$

The velocity components in the tangential and radial directions are

$$v_\theta = -\frac{\partial \psi}{\partial r} = -U_\infty\left(1 + \frac{R^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r} \tag{7-192}$$

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U_\infty\left(1 - \frac{R^2}{r^2}\right)\cos\theta \tag{7-193}$$

Since the radial velocity, v_r, remains unchanged from the case without circulation and the tangential velocity, v_θ, does not alter the condition at infinity, the boundary conditions described in the preceding section are satisfied in this case. It can be seen that Laplace's equation is satisfied with both ψ and ϕ here.

The tangential velocity components on the surface of the cylinder $r = R$) can be obtained from Eq. (7-192),

$$v_\theta = -2U_\infty \sin\theta - \frac{\Gamma}{2\pi R} \tag{7-194}$$

For the stagnation values of θ, v_θ must vanish:

$$\sin(-\theta_s) = \frac{\Gamma}{4\pi R U_\infty} = \sin(\pi + \theta_s) \tag{7-195}$$

where $\Gamma/4\pi U_\infty R \leq 1$. When the circulation becomes $4\pi R U_\infty$, the two stagnation points coincide at $r = R$ and $\theta = -\pi/2$. For $\Gamma > 4\pi R U_\infty$ the stagnation point moves away from the cylinder in the direction of the negative y-axis. This can be shown by the following analysis.

The condition for a stagnation point away from the surface of the cylinder ($r > R$) is that both the velocity components must vanish, thus from Eqs. (7-192) and (7-193),

$$0 = -U_\infty\left(1 + \frac{R^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r} \tag{7-196a}$$

$$0 = U_\infty\left(1 - \frac{R^2}{r^2}\right)\cos\theta \tag{7-196b}$$

Equation (7-196b) requires $\theta = \pi/2$ or $3\pi/2$. But Eq. (7-196a) can only be satisfied with $\theta = 3\pi/2$ because Γ cannot be negative. Hence Eq. (7-196a) gives

$$U_\infty\left(1 + \frac{R^2}{r^2}\right) = \frac{\Gamma}{2\pi r}$$

or

$$r = \frac{\Gamma}{4\pi U_\infty} + \frac{1}{2}\sqrt{\left(\frac{\Gamma}{2\pi U_\infty}\right)^2 - 4R^2} \qquad \text{at} \qquad \theta = \frac{3\pi}{2} \tag{7-197}$$

Stagnation points for several values of Γ are shown in Fig. 7.38.

The pressure at any point in the field of flow may again be found from Bernoulli's equation:

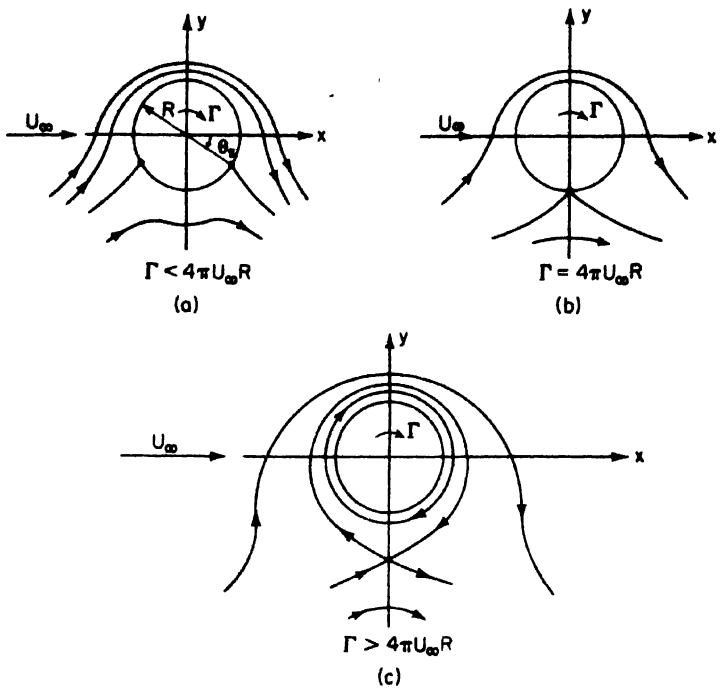


Figure 7.38

$$p - p_\infty = \frac{\rho U_\infty^2}{2} - \frac{\rho}{2}\left(2U_\infty \sin\theta + \frac{\Gamma}{2\pi r}\right)^2 \tag{7-198}$$

The drag on the cylinder of radius R again is zero because of the symmetry of the flow pattern about the y-axis [$\sin\theta = \sin(\pi - \theta)$] The lift per unit length on the cylinder, however, becomes

$$L = -\int_0^{2\pi}\left[\frac{\rho U_\infty^2}{2} - \frac{\rho}{2}\left(2U_\infty \sin\theta + \frac{\Gamma}{2\pi R}\right)^2\right]\sin\theta\, R\, d\theta \tag{7-199a}$$

Since

$$\int_0^{2\pi} \sin\theta\, d\theta = \int_0^{2\pi} \sin^3\theta\, d\theta = 0,$$

Eq. (7-199a) becomes

$$L = \frac{\rho U_\infty \Gamma}{\pi}\int_0^{2\pi} \sin^2\theta\, d\theta = \frac{\rho U_\infty \Gamma}{\pi}\int_0^{2\pi} \frac{1}{2}(1 - \cos 2\theta)\, d\theta = \rho U_\infty \Gamma \tag{7-199b}$$

Hence, the lift is directly proportional to the density of the fluid, the approach velocity, and the circulation. Equation (7-199b) is generally known as the Kutta-Joukowski theorem and may be stated: A right cylinder of any cross-sectional shape at rest in a uniform flow of

velocity U_∞, with circulation Γ around the cylinder, undergoes a force per unit length $\rho\Gamma U_\infty$ perpendicular to the direction of the flow. The formal proof of the Kutta-Joukowski theorem for an arbitrary airfoil may be found in texts of aerodynamic theory (see Ref. 10). The section lift coefficient is given by

$$C_l = \frac{L}{\rho(U_\infty^2/2)(2R)} = 4\pi \sin\theta_s \tag{7-200}$$

where θ_s is the angular position of the stagnation point.

The lift or thrust per unit length given in Eq. (7-199b) is referred to as the *Magnus effect*. The Flettner rotor ship (Fig. 7.39) was designed to utilize this principle by employing a rotating cylinder which was mounted to produce a propulsive force.

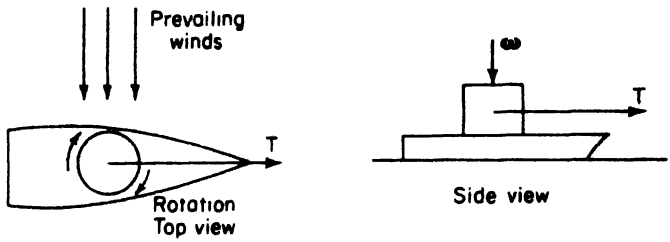


Figure 7.39

The Magnus effect is also used to good advantage by baseball pitchers. Figure 7.40 indicates the types of spin put on the ball to achieve a rise, drop, and curve, respectively.

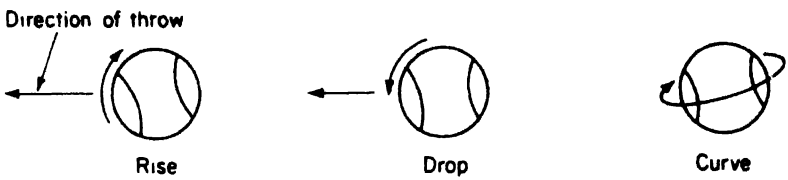


Figure 7.40

Illustrative Example 7.15. In Fig. 7.O is shown the flow around a circular cylinder (consisting of two halves) with circulation. Determine the force required to hold the two halves together. The weight of the cylinder is neglected.

Solution. The drag acting on the semicircular cylinder per unit length is

$$D_h = \int_A (p - p_\infty)\, dA = \int (p - p_\infty)(1)\, dy$$

$$= \int (p - p_\infty) R \cos\theta\, d\theta \tag{a}$$

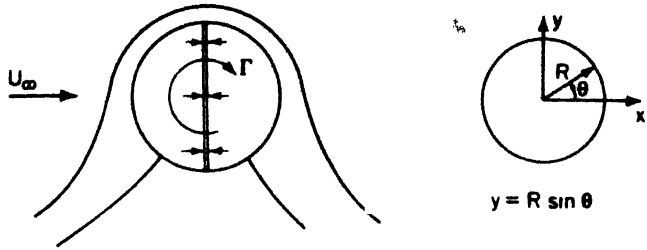


Fig. 7.O

Substituting Eq. (7-198) for $(p - p_\infty)$ into Eq. (a), we find:

$$D_h = \int_{-\pi/2}^{\pi/2} \left[\frac{\rho U_\infty^2}{2} - \frac{\rho}{2}\left(2U_\infty \sin\theta + \frac{\Gamma}{2\pi r}\right)^2 \right] R\cos\theta \, d\theta$$

$$= R\int_{-\pi/2}^{\pi/2} \left[\frac{\rho U_\infty^2}{2}\cos\theta - \frac{\rho}{2}\left(4U_\infty^2 \sin^2\theta + \frac{2U_\infty \sin\theta}{\pi R}\Gamma + \frac{\Gamma^2}{4\pi^2 R^2}\right)\cos\theta \right] d\theta$$

$$= 2R\left(\frac{\rho U_\infty^2}{2} - \frac{\rho\Gamma^2}{8\pi^2 R^2}\right) - \frac{4R\rho U_\infty^2}{3} - 0$$

$$= -\left(\frac{R\rho U_\infty^2}{3} + \frac{\rho\Gamma^2}{4\pi^2 R}\right) \qquad \text{(b)}$$

The drag given in Eq. (b) tends to pull the two halves apart. Hence an equal and opposite force is required to hold the two halves together.

NOMENCLATURE FOR CHAPTER SEVEN

A	area, Eq. (7-7)
C	contour, Eq. (7-5)
C_l	section lift coefficient, Eq. (7-200)
D	drag, Eq. (7-189)
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors of Cartesian coordinates, Eq. (7-3)
L	lift, Eq. (7-188a)
l, m, n	direction cosines, Eq. (7-32)
$\mathbf{n}$	unit vector normal to surface A
p	pressure, Eq. (7-2a)
p_∞	free-stream pressure, Eq. (7-144)
Q	(1) flow rate, Eq. (7-32); (2) source strength, Eqs. (7-87) and (7-122b)
$\mathbf{q}$	velocity vector, Eq. (7-1)
$q(\xi)$	source strength per unit length, Eq. (7-157)
r	length of the contour C, Eq. (7-5)
$\mathbf{r}$	position vector, Eq. (7-5)
r, θ, z	cylindrical coordinates

U_∞	velocity in uniform flow, Eq. (7-74)
u, v, w	velocity components in Cartesian coordinates, Eq. (7-1)
v_r, v_θ, v_z	velocity components in cylindrical coordinates, Eqs. (7-67) and (7-70)
$v_\omega, v_\varphi, v_\theta$	velocity components in spherical coordinates, Eqs. (7-53) and (7-72)
x, y, z	Cartesian coordinates
$\mathbf{X}$	body force per unit mass, Eq. (7-3)
Γ	circulation, Eq. (7-5)
θ_s	angular position of the stagnation point, Eq. (7-200)
μ	strength of the doublet, Eqs. (7-107) and (7-127)
ν	strength of the dipole, Eq. (7-115)
ρ	density, Eq. (7-2a)
ϕ	velocity potential, Eq. (7-19a)
ϕ_f	force potential, Eq. (7-3)
ψ	stream function, Eq. (7-59)
ω, φ, θ	spherical coordinates
$\boldsymbol{\Omega}$	vorticity vector, Eq. (7-7)

PROBLEMS

7.1. Evaluate the line integral of the velocity $u = \sqrt{y}$, $v = x - y$ over the following paths

(a) $x = t$, $y = t$ (c) $x = t$, $y = t^2$

(b) $x = t^2$, $y = t$ (d) $x = t$, $y = t^3$

where $0 \leq t \leq 1$.

7.2. Evaluate the line integral of the velocity $u = x^2y$, $v = x^2 - y^2$ over the paths

(a) $y = 3x^2$ (b) $y = 3x$

where $0 \leq x \leq 1$, $0 \leq y \leq 3$.

7.3. Find the circulation about the square enclosed by the lines $x = \pm 2$, $y = \pm 2$ for the flow $u = x + y$, $v = x^2 - y$.

7.4. Verify the result of Prob. 7.3 by using Stokes' theorem.

7.5. Evaluate the circulation along the closed path $x^2 + y^2 = 1$ when $\mathbf{q} = -4y\mathbf{i} + 4v\mathbf{j}$.

7.6. Investigate the possibility of irrotational flows for the following cases:

(a) $v = w = 0$, $u = f_1(y)$

(b) $v = w = 0$, $u = f_2(x, y)$

7.7. Determine the relation between v_r, v_θ, and ϕ in polar coordinates.

7.8. Show that the following velocity field satisfies the condition for irrotationality.

$$u = \frac{(y^2 - x^2)}{(x^2 + y^2)^2}, \qquad v = \frac{-2xy}{(x^2 + y^2)^2}, \qquad w = 0$$

7.9. For an irrotational flow determine the following velocity components in polar coordinates:

(a) v_r when $v_\theta = 0$

(b) v_θ when $v_r = 0$

7.10. An irrotational velocity field is given as $\mathbf{q} = \mathbf{i}y + \mathbf{j}(x + 2y)$. Determine the required function of the velocity potential ϕ.

7.11. Find the velocity potential $\phi(r, \theta)$ or $\phi(x, y)$ for

(a) flow in Prob. 7.9

(b) flow with $u = \text{Constant}$, $v = 0$

(c) flow with $v = \text{Constant}$, $u = 0$

7.12. Find the velocity vector $\mathbf{q}$ for the following velocity potential ϕ:

(a) $\phi = A(x^2 - y^2)$ (c) $\phi = (x^2 + y^2)/(x - y)$

(b) $\phi = Axy$ (d) $\phi = A \cdot \arcsin(y/x)$

7.13. Show that the equations of motion

$$\frac{D\mathbf{q}}{Dt} = \mathbf{X} - \frac{1}{\rho}\nabla p$$

can be transformed into the forms

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}q^2\right) + w\Omega_y - v\Omega_z = X_x - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{dt} + \frac{\partial}{\partial y}\left(\frac{1}{2}q^2\right) + u\Omega_z - w\Omega_x = X_y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial z}\left(\frac{1}{2}q^2\right) + v\Omega_x - u\Omega_y = X_z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

where $\mathbf{\Omega} = \mathbf{i}\Omega_x + \mathbf{j}\Omega_y + \mathbf{k}\Omega_z$, $\mathbf{q} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$, and $\mathbf{X} = \mathbf{i}X_x + \mathbf{j}X_y + \mathbf{k}X_z$. Reduce the three equations above to a single vector equation,

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) + \mathbf{\Omega} \times \mathbf{q} = \mathbf{X} - \frac{1}{\rho}\nabla p$$

7.14. Show that for an irrotational flow the equation of motion,

$$\nabla\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}q^2 + \frac{p}{\rho}\right) = 0$$

is identical to the following equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

7.15. Find the pressure $p(r, \theta)$ or $p(x, y)$ for each of the flows in Prob. 7.11. Denote p at infinity in each case by p_∞.

7.16. A stationary rocket is discharging gases at a temperature of 5,000°F at a velocity of 6000 ft/sec. The mass flow rate of the gases is 55 slug/sec and

the diameter of the exit cross section is 2 ft. If the nozzle exit pressure p_e is 18 lbf/in.^{2} and the atmospheric pressure p_a is 14.7 lbf/in.^{2}, determine the thrust of the rocket.

7.17. Consider a jet plane of mass m flying at a speed v. Let v_e be the discharge jet speed relative to the engine, p_e the discharge pressure, and A_e the discharge area of the jet nozzle. If the difference of air entering the jet engine and gas leaving the engine is neglected, use the momentum theorem to obtain an equation in the direction of the motion.

7.18. Consider a rocket in vertical ascent with initial weight of 600,000 lbf of which 70% is fuel. Assume that the fuel consumption is at a constant rate, g is constant at 32.2 ft/sec^2, and the air resistance is negligible. If the gas discharge velocity v_e = 10,000 ft/sec, find:

(a) the burning time for the rocket

(b) the speed of the rocket when the fuel is completely consumed, and its height above sea level.

The discharge pressure is assumed equal to the atmospheric pressure.

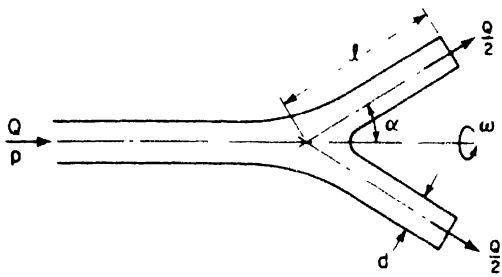


Figure P.7.19

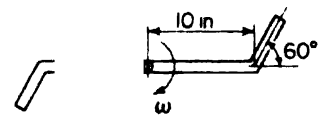


Figure P.7.20

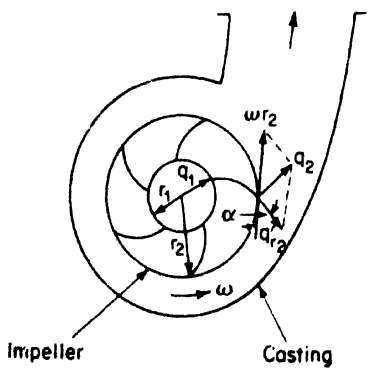


Figure P.7.21

7.19. In Fig. P. 7-19 a pipe is shown branching into legs which are symmetrical about the axis of rotation. The whole system is rotated at a constant angular velocity Ω. Assuming there is no incoming angular momentum of the mass flow, calculate the torque about the axis of rotation as shown.

7.20. A symmetrical sprinkler has a total volumetric flow of 25 gal/min of water and nozzle tips of $\frac{1}{4}$ in. in diameter. If friction is neglected, find the speed of rotation of the sprinkler shown in Fig. P. 7.20.

7.21. The centrifugal pump shown in Fig. P. 7.21 is rotating at 75 rpm. If water enters the impeller radially with a velocity of 10 ft/sec, calculate the work done on the fluid by the impeller ($r_2 = 2$ ft, thickness of impeller $= \frac{3}{8}$ in., and $\alpha = 30°$).

7.22. In Fig. P. 7.22 is shown a simple impulse turbine. Using the expression in Eq. (c) in illustrative example 7. 6, show that the torque T_z exerted on the wheel is

$$T_z = Q\rho r(q_1 - \omega r)(1 + \cos\theta)$$

where Q is the discharge of the fluid jet, and ω is the angular speed.

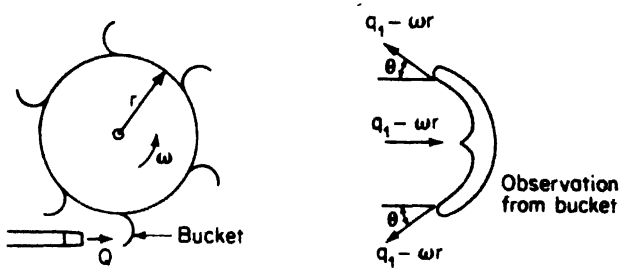


Figure P.7.22

7.23. Verify that $\nabla^2\phi = 0$ for the flows of Prob. 7.11.

7.24. Determine which of the following functions is a possible velocity potential for inviscid incompressible flow.

(a) $\sin(\pi x/l) + \sin(\pi y/l)$
(b) $\sin(\pi xy/l^2)$
(c) arc tan $[2yc/(x^2 + y^2 - c^2)]$
(d) $A(r + a^2/r)\cos\theta - (B/2\pi)\theta$
(e) $A\omega(a^3/2\omega^3 + 1)\cos\varphi$

7.25. Find the relation between v_r, v_θ, and $\psi(r, \theta)$ from the following relation: $d\psi = u\,dy - v\,dx$

7.26. Find the stream function $\psi(x, y)$ or $\psi(r, \theta)$ for each of the flows:

(a) flows in Prob. 7.9
(b) flow with $u =$ Constant, $v = 0$
(c) flow with $v =$ Constant, $u = 0$

7.27. Verify $\nabla^2\psi = 0$ for each of the stream functions in Prob. 7.26.

7.28. Which of the following stream functions ψ are possible irrotational flow fields?

(a) $\psi = 2Ax^2y^2$
(b) $\psi = Ax + By^2$
(c) $\psi = -Ay/(x^2 + y^2)$
(d) $\psi = A\ln(xy^2)$
(e) $\psi = A\ln(x/y)$

7.29. Verify that the following stream functions ψ are possible irrotational flow fields:

(a) $\psi = -(c/r)\sin\theta$
(b) $\psi = c \ln r$
(c) $\psi = c(r - A/r)\sin\theta$
(d) $\psi = (c/w)\sin^2\varphi$
(e) $\psi = c(\omega^2 - A/\omega)\sin^2\varphi$

7.30. Sketch lines of ψ = constant and ϕ = Constant for flows

(a) u = Constant, $v = 0$
(b) v = Constant, $u = 0$
(c) $v_r = c/r$, $v_\theta = 0$
(d) $v_\theta = c/r$, $v_r = 0$

7.31. Verify that the stream function ψ of two-dimensional rectilinear flow, source (or sink) flow, doublet flow, and dipole flow satisfy $\nabla^2\psi = 0$.

7.32. Verify that the velocity potential ϕ of two-dimensional rectilinear flow, source flow, doublet flow, and dipole flow satisfy $\nabla^2\phi = 0$.

7.33. The velocity potential ϕ of the stagnation flow is given by $\phi = x^2 - y^2$. Find the expression for stream function and plot the lines of constant ϕ and ψ for the flow.

7.34. Verify that the stream function ψ of the three-dimensional (axially symmetric) rectilinear flow and source (or sink) flow satisfy $\nabla^2\psi = 0$.

7.35. Verify that the velocity potential ϕ of the three-dimensional (axially symmetric) rectilinear flow and source (or sink) satisfy $\nabla^2\phi = 0$.

7.36. Verify that $\psi = [(A/\omega^2)\cos\varphi + B\omega^2]\sin^2\varphi$ satisfies $\nabla^2\psi = 0$, and find the corresponding velocity potential.

7.37. The velocity potentials $\phi_1 = x^2 - y^2$ and $\phi_2 = \sqrt{r}\cos(\theta/2)$ are solutions of the Laplace equation. Verify that $\phi_3 = (x^2 - y^2) + \sqrt{r}\cos(\theta/2)$ satisfies $\nabla^2\phi_3 = 0$.

7.38. Verify that the velocity potential and the stream function of a two-dimensional source in uniform flow satisfy the respective Laplace equations.

7.39. In Fig. P. 7.39 is shown a streamlined nose. Find an approximate equation of the streamline passing through the stagnation point A for $(h/d) > 1$.

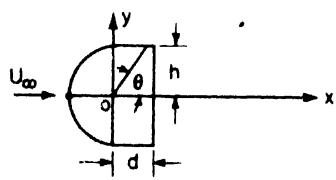


Figure P.7.39

7.40. Find the velocity potential of a half streamline-body and verify that it satisfies the Laplace equation.

7.41. Verify that the stream function of a half streamline-body satisfies the Laplace equation.

7.42. Plot the pressure distribution on the surface of a half streamline-body. Use C_p vs. z/r_0.

7.43. Verify that the velocity potential and stream function of a Rankine oval body satisfy the respective Laplace equations.

7.44. Find the source strength Q and the distance between the source and equal sink for flow of a uniform stream of 30 ft/sec around a Rankine oval body 4 ft long and 2 ft thick.

7.45. A source strength 25 ft^3/sec and a sink of equal strength are located on the z-axis, 2 ft apart. If this combination is superimposed with a rectilinear flow of 50 ft/sec in the negative z-direction, determine the size of the Rankine oval body formed by this flow.

7.46. Find the stream function ψ and velocity potential ϕ for the flow around a closed contour formed by the superposition of a two-dimensional source and sink with rectilinear flow. The source and equal sink are located at equal distances from the origin along the negative and positive x-axis, respectively.

7.47. Verify that the stream function ψ and the velocity potential ϕ of Prob. 7.46 satisfy the respective Laplace equations.

7.48. Verify that there is no drag force acting on the sphere in a rectilinear flow.

7.49. Verify that the stream function ψ and the velocity potential ϕ for a three-dimensional doublet in a uniform flow satisfy the respective Laplace equations.

7.50. A steel hemisphere is laid on its bottom in a water tank. What minimum velocity is required to separate the hemisphere from the bottom of the tank? The radius of the hemisphere is 1 ft and the height of the water is 2 ft.

7.51. Show that the pressure distribution in the z-direction can be expressed by

$$p_z = \rho \frac{U_\infty^2}{2}\left(2\frac{a^3}{z^3} - \frac{a^6}{z^6}\right)$$

for a sphere in a rectilinear flow.

7.52. Verify that the stream function ψ and the velocity potential ϕ for uniform flow around a circular cylinder of radius R satisfy their respective Laplace equations.

7.53. Verify that there are no lift and drag forces acting on a cylinder in a rectilinear flow.

7.54. Plot the pressure coefficient on the surface of a cylinder between the stagnation points. What pressures do you measure when $C_p = 1$ and $C_p = 0$?

7.55. Verify that the stream function and the velocity potential for flow around a circular cylinder with circulation satisfy their respective Laplace equations.

7.56. Plot the pressure coefficient C_p versus θ about a cylinder of radius 1.0 ft which is moving in the negative direction with a velocity 120 ft/sec. The circulation around the cylinder is 600 ft²/sec.

7.57. If $U_\infty = 120$ ft/sec, $R = 5$ ft, $\Gamma = 600$ ft²/sec, and the constant $= -\Gamma/2\pi \ln R$ are given, sketch the streamlines for flow around a circular cylinder with circulation.

8

LAMINAR FLOW OF VISCOUS INCOMPRESSIBLE FLUIDS

In the preceding two chapters we have seen that by neglecting the viscous effect in the fluid a very extensive and elegant mathematical theory for the flow of inviscid incompressible fluids has been developed. However, this theory fails completely to account for the drag of a body moving in a fluid (§ 7.18b, D'Alembert's paradox). The fundamental assumptions which we have made in the theory of an ideal fluid (§ 1.4a) are: (1) Only normal pressure exists over the surface of a solid body immersed in the moving fluid. (2) In addition to zero normal velocity at the contact surface of a solid and the moving fluid, there is also a relative tangential velocity, or a velocity of slip.

Actually, all real fluids are in general viscous, having characteristics which cause them to show a certain resistance to alterations of form. In § 1.3d the concepts of viscosity and shearing resistance of fluids have been briefly discussed. It was stated (§ 4.3) that in a fluid there is no resistance to deformation under applied pressure, but only to the time rate of deformation per unit length (velocity gradient). Hence, the rate of change of strain of a fluid is proportional to the shearing stress applied. The proportionality constant in the shearing

motion of a fluid is then the viscosity of the fluid. If the shearing stress is linearly proportional to the rate of strain, the fluid is called a Newtonian fluid. A non-Newtonian fluid (§ 1.4g) is one for which the dependence of shearing stress on rate of strain is nonlinear; however, when the rate of strain is zero, the shearing stress is also zero.

In solid mechanics we consider a material to be perfectly elastic if it undergoes a finite deformation when a load in tension, compression, or shear is applied. The deformation vanishes as the load is removed. Since the deformation in this case is finite and constant for a given load, it cannot have any rate of deformation. As long as the applied stress does not exceed the yield stress of the material, the proportionality of the stress and the strain is constant, or we may say that the material is within its elastic limit. When the applied stress exceeds the yield stress of the material, the proportionality of the stress and the strain is no longer constant, and a continuous deformation associated with a certain rate appears. If the ratio of the stress and rate of strain is linear, the material is ideally plastic. The distinguishing differences in shearing stress and rate of strain relationships for solids and fluids are shown in Fig. 8.1.

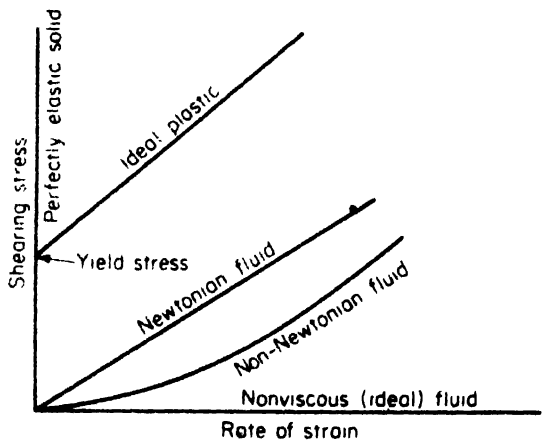


Figure 8.1

It is interesting to note from the figure that the behavior of a nonviscous fluid for which the shearing stress is zero will be represented by the abscissa. The ordinate represents the perfectly elastic material because its deformation is finite and constant. Both curves for the Newtonian fluid and the ideal plastic solid have constant slope;

however, the latter behaves like a perfectly elastic material within the yield stress. The curve which represents the non-Newtonian fluid displays the variable coefficient of viscosity, μ (§ 1.4g).

There are two distinct types of viscous fluid, the molecular and the molar. The first type is most clearly distinguishable when the motion is steady and the fluid moves in layers parallel to a wall. This is called streamline or laminar flow. The second type predominates when the motion is so irregular that it may be described as turbulent or tumultuous (see Chap Ten). In this chapter and in Chap Nine we are concerned only with the laminar flow.

In § 8.1 the significance of the Reynolds number, the viscous parameter, is discussed based on the similarity of flows.

In § 8.2 the coefficient of viscosity is discussed from the point of view of the kinetic theory. It brings out the physical character of the Reynolds number which is the ratio of the measured velocity to the characteristic velocity of the molecules times the ratio of the corresponding lengths.

In § 8.3 the flow between two parallel flat plates is discussed. This includes both the Couette and plane Poiseuille flows.

In § 8.4 the flows through a straight pipe and between two coaxial pipes are discussed.

In § 8.5 the flow between two concentric rotating cylinders is presented. The method of calculating wall shearing stress is given.

In § 8.6 applications of the parallel flow theory to the measurement of viscosity and the hydrodynamics of bearing lubrication are presented.

In § 8.7 the steady flow around a sphere for very small Reynolds numbers is discussed. Stokes' drag coefficient is obtained and the stream function for the flow is calculated.

In § 8.8 the unsteady motion of a flat plate which is started impulsively from rest is presented. The arbitrary thickness of the boundary layer is defined, and its relation with the kinematic viscosity is given.

8.1. *Similarity of Flows—the Reynolds Number*

The concepts and general relationships of the stresses and rates of strain have been discussed in detail in Chap Four. For a viscous incompressible flow the equations of motion (Navier-Stokes equations) are given in the vector form as follows:

$$\rho \frac{\partial \mathbf{q}}{\partial t} + \rho(\mathbf{q} \cdot \nabla)\mathbf{q} = \rho \mathbf{X} - \nabla p + \mu \nabla^2 \mathbf{q} \tag{5-19}$$

Let q_∞, p_∞, l, and $t_0 = l/q_\infty$ denote the characteristic reference

magnitudes of velocity and pressure at infinity, a typical linear dimension of the body and the time scale equal to l/q_∞, respectively. We can now introduce the following nondimensional quantities:

$$t^* = \frac{t}{t_0}, \qquad \Delta^* = l\nabla, \qquad \mathbf{q}^* = \frac{\mathbf{q}}{q_\infty}, \qquad p^* = \frac{p}{p_\infty}, \qquad \mathbf{X}^* = \frac{\mathbf{X}}{X_0} \tag{8-1}$$

The Navier-Stokes equation, Eq. (5-19), in the nondimensional variables becomes

$$\frac{\partial \mathbf{q}^*}{\partial t^*} + (\mathbf{q}^* \cdot \nabla^*)\mathbf{q}^* = \left(\frac{X_0}{q_\infty^2/l}\right)\mathbf{X}^* - \left(\frac{p_\infty}{\rho q_\infty^2}\right)\nabla^* p^* + \left(\frac{\mu q_\infty/l}{\rho q_\infty^2}\right)\nabla^{*2}\mathbf{q}^* \tag{8-2}$$

The coefficient of the left-hand side of Eq. (8-2) is of the order unity. If X_0 represents the gravitational force per unit mass, the coefficient of the first term on the right-hand side of Eq. (8-2) may be interpreted as the ratio of the gravitational force to the dynamic force which is related to the inverse of the Froude number, F, i.e.,

$$\mathrm{F} = \frac{q_\infty^2}{gl} = \frac{(mq_\infty^2)/l}{mg} = \frac{mq_\infty^2/l}{X_0} \tag{8-3}$$

The Froude number is a very useful parameter in open-channel flow when the flow is mainly due to gravity. In general, the Froude number is very large, since the gravitational force is negligibly small in comparison to the dynamic and pressure forces acting on the fluid particles. Hence, we may neglect entirely the body force in Eq. (8-2). The quantity $p_\infty/\rho q_\infty^2$ in Eq. (8-2) represents the ratio of pressure to twice the dynamic pressure (see § 6.3) which is unimportant for the similarity of flows because of the constant density in the incompressible flow. The coefficient of the last term on the right-hand side of Eq. (8-2) is, however, very important, and its inverse value is known as the Reynolds number, Re, i.e.,

$$\mathrm{Re} = \frac{\rho q_\infty^2}{\mu q_\infty/l} = \frac{q_\infty l}{\nu} \tag{8-4}$$

Hence, the Reynolds number may be interpreted as the ratio of the dynamic pressure to the shearing stress (or the inertia force to the friction force) and is a parameter for viscosity.

The significance of the Reynolds number can be seen in the requirement of similarity of flows (or dynamical similarity) of an incompressible viscous fluid. In order for flows around two bodies to be completely similar, not only must the bodies be geometrically similar, but also the Reynolds numbers in the two flow patterns must be equal in magnitude. In other words, the geometrical similarity requires that all dimensions must be scaled alike by a scale factor $\alpha = l_1/l_2$, where l_1 and l_2 are two corresponding lengths in systems of flow 1 and 2. The dynamical similarity of the two flow patterns re-

quires the establishment of the time scale $\beta = t_1/t_2$ and force scale $\gamma = X_1/X_2$ in the two systems.

Let us consider two geometrically similar systems subject only to friction and inertia forces. In order that dynamical similarity can be assured, the force scale in the two systems must be established. From the inertia forces, we have

$$\gamma = \frac{\rho_1 l_1^3 (l_1/t_1^2)}{\rho_2 l_2^3 (l_2/t_2^2)} = \frac{\rho_1 \alpha^4}{\rho_2 \beta^2} \tag{8-5a}$$

and from the frictional forces,

$$\gamma = \frac{\mu_1 \left(\frac{l_1/t_1}{l_1}\right) l_1^2}{\mu_2 \left(\frac{l_2/t_2}{l_2}\right) l_2^2} = \frac{\mu_1 \alpha^2}{\mu_2 \beta} \tag{8-5b}$$

However, for dynamical similarity both values of γ must be equal, so that in Eqs. (8-5a) and (8-5b)

$$\frac{\rho_1 \alpha^4}{\rho_2 \beta^2} = \frac{\mu_1 \alpha^2}{\mu_2 \beta}$$

or

$$\frac{q_1 l_1}{\nu_1} = \frac{q_2 l_2}{\nu_2} \tag{8-6}$$

where q_1 and q_2 are the corresponding velocities in the two systems and $q_1/q_2 = \alpha/\beta$. The relation given in Eq. (8-6) is known as Reynolds' law of dynamical similarity. It is apparent that the significance of the Reynolds number, the viscous parameter, is a means of reducing the multitude of flow and fluid scales into a single parameter. This is a great saving in effort.

The particularly important application of dynamical similarity is model testing. For two geometrically similar bodies the flow around the test body and the flow around the prototype can be considered the same or similar if the Reynolds numbers in the two cases are equal. If the same fluid is used on model and prototype, the products $q_1 l_1$ and $q_2 l_2$ in Eq. (8-6) must be the same in order to maintain dynamical similarity.

It can be seen from Eq. (8-6) that the Reynolds number is inversely proportional to the kinematic viscosity ν, everything else being constant. Thus, large viscosities correspond to low Reynolds numbers and vice versa.

In the case of flows over two geometrically similar bodies in viscous compressible fluids (§ 1.4c) we must have three additional similarity parameters, the Mach number, the ratio of specific heats, and the Prandtl number.

8.2. *Viscosity from the Point of View of the Kinetic Theory*

Before discussing the exchange of momentum in gases it might be helpful to illustrate an example of exchange of momentum in mechanics. Consider two open freight cars loaded with coal moving in the same direction on parallel tracks at slightly different speeds. Coal is exchanged between the two cars simultaneously. The slowly moving car is continually being struck by pieces of coal traveling slightly faster than the car; hence, there results an additional forward force on that car. Simultaneously, for an opposite effect there is an additional backward force on the faster car. The two opposite forces produced on the respective cars are equivalent to the frictional forces exerted by the cars on each other if the sides of the two cars were rubbing together. Thus, we see that the effect of the rate of momentum transport is equivalent to a frictional force.

In a similar manner we can explain that in gases the shearing stress is the result of the rate of momentum transport of the molecules due to their random motion. According to the Newtonian law of friction (§ 1.3d) a typical velocity distribution for flows between two parallel plates is shown in Fig. 8.2. The dotted line A-A in the figure represents an imaginary surface within the gas at an arbitrary height above the lower plate. Let u represent the velocity of the gas toward the right, along the layer A-A, and du/dy the velocity gradient at this layer. Since molecules are in random motion with an average velocity c all the time, they are continuously crossing the dotted surface A-A both from above and below. We shall assume that at its last collision before crossing A-A, each molecule acquires a flow

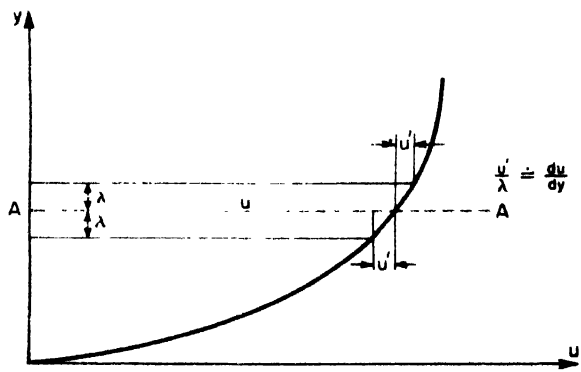


Figure 8.2

c

velocity toward the right, at layers a distance λ from the layer *A-A* (Fig. 8.2). The length λ is called the mean free path which is the mean value of the distance that molecules can travel between collisions.

Now we are concerned with only those molecules that after their last collisions are concentrated at layers above and below layer *A-A*, a distance 2λ apart. Let the flow velocity at the layer *A-A* be u; then the flow velocities at layers above and below *A-A* are $u + u'$ and $u - u'$, respectively. The molecules now at the layer above strike at the layer *A-A* with a velocity c and a net rate of transport of momentum results. According to Newton's second law we can equate the net rate of transport of momentum, per unit area, to the shearing force per unit area, i.e.,

$$\tau_{yx} = \sum \frac{\text{mass}}{\text{unit volume}} u'c \tag{8-7}$$

Let us assume that one-third of the molecules at this layer move in the x-direction and one-half of each of these thirds acquires the flow velocity $u + u'$. In terms of this assumption Eq. (8-7) becomes

$$\tau_{yx} = \frac{1}{6} Nm\lambda \frac{du}{dy} c \tag{8-8}$$

where N is the number of molecules per unit volume, m the mass of a single molecule, and $u' = \lambda (du/dy)$ (Fig. 8.2). A similar amount of momentum is transferred from the layer below layer *A-A*, and thus the total shearing stress is the sum of the net rate of transport of momentum from layers above and below layer *A-A*:

$$\tau_{yx} = \frac{1}{3} Nmc\lambda \frac{du}{dy} = \frac{1}{3} \rho c\lambda \frac{du}{dy} \tag{8-9}$$

where $\rho = Nm$.

Comparing Eq. (8-9) with Newton's law of friction [Eq. (1-7)] we see that

$$\mu = \frac{1}{3} \rho c\lambda \tag{8-10}$$

or

$$\nu = \frac{1}{3} c\lambda$$

Equation (8-10) brings out the physical character of the Reynolds number, namely that it represents the ratio of the measured velocity to the characteristic velocity of the molecules times the ratio of the corresponding lengths. Thus

$$\text{Re} = \frac{q_\infty l}{\nu} = \frac{q_\infty}{\frac{1}{3}c} \left(\frac{l}{\lambda}\right) \tag{8-11}$$

The treatment of viscosity above from the point of view of the kinetic theory of gases is rather simple and elementary; however, the expression is correct except for the numerical constant. The correct treatment of viscosity can be made by formulating the basic relation between the viscosity and the molecular quantities in which the Maxwellian distribution function is introduced. The more exact expression for μ is

$$\mu = 0.499\rho c\lambda \tag{8-12}$$

For air at 25°C and standard atmospheric pressure, the value of μ can be calculated from Eq. (8–12) as follows:

$$\mu = 0.499(0.002378)(1680)(2.2 \times 10^{-7})$$
$$= 4.38 \times 10^{-7} \frac{\text{lbf-sec}}{\text{ft}^2}$$

while Eq. (8–10) yields

$$\mu = 2.93 \times 10^{-7} \frac{\text{lbf-sec}}{\text{ft}^2}$$

The corresponding measured value of μ is 3.86×10^{-7} lbf-sec/ft^2.

8.3. Flow Between Parallel Flat Plates

For a viscous incompressible fluid in steady flow the Navier-Stokes equations with negligible body forces in Cartesian coordinates are (§ 5.2)

$$\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\
u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \\
u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)
\end{aligned} \tag{5-19}$$

The continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{5-9b}$$

Equations (5–19) are nonlinear second-order differential equations, and there is no known general method for solving them. However, there are several simple physical problems which enable one to reduce Eqs. (5–19) into linear differential equations. Hence some exact solutions of the Navier-Stokes equations in special cases are obtained. The present problem is one of these special cases.

Consider the two-dimensional steady laminar flow of a viscous incompressible fluid between two parallel straight plates (Fig. 8.3). Let x be the direction of flow, y the direction normal to the flow, and the width of the plates parallel to the z-direction be large com-

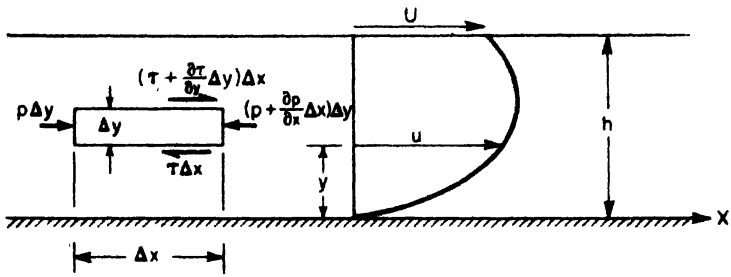


Figure 8.3

pared with the distance h between the plates. For this simple configuration, we have

$$u = u(y), \qquad v = 0, \qquad w = 0, \qquad \frac{\partial}{\partial z} = 0 \tag{8-13}$$

where the velocity component u is independent of x since the continuity equation [Eq. (5–9b)] becomes $\partial u/\partial x = 0$. The second equation of Eqs. (5–19) reduces to

$$\frac{\partial p}{\partial y} = 0 \tag{8-14}$$

so that the pressure depends on x only. Thus the Navier-Stokes equations [Eqs. (5–19)] become

$$\frac{dp}{dx} = \mu \frac{d^2u}{dy^2} \tag{8-15}$$

Since u is a function of y alone, the left-hand side of Eq. (8-15) can be either a function of y or a constant. But Eq. (8–14) shows that the pressure p cannot be a function of y; it follows that the term dp/dx must be a constant.

Equation (8–15) can also be derived from the equilibrium condition of the flow such as shown in Fig. 8.3. The only forces acting on a thin lamina $\Delta x \Delta y$ of unit width are the pressure force and the frictional force. The equation of motion for the lamina in the x-direction is

$$p\Delta y - \left(p + \frac{\partial p}{\partial x}\Delta x\right)\Delta y - \tau_{yz}\Delta x + \left(\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y}\Delta y\right)\Delta x = 0$$

After simplifying,

$$\frac{\partial \tau_{yx}}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}$$

or

$$\mu \frac{d^2 u}{dy^2} = \frac{dp}{dx} \tag{8-15}$$

where $\tau_{yx} = \mu(\partial u/\partial y)$. Integrating Eq. (8-15) with respect to y, the general solution of u is obtained:

$$u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + Ay + B \tag{8-16}$$

where A and B are arbitrary constants to be determined by the boundary conditions of each specific type of flow.

a. Couette Flow

In Fig. 8.3 the case of steady flow in a channel with two parallel flat plates is shown. The so-called Couette flow is the flow between two parallel flat plates, one of which is at rest and the other moving with a velocity U parallel with the fixed plate. The two arbitrary constants in Eq. (8-16) can now be determined from the boundary conditions

$$y = 0: \ u = 0; \qquad y = h: \ u = U \tag{8-17}$$

which leads to the following results,

$$A = \frac{U}{h} - \frac{1}{2\mu}\frac{dp}{dx}h; \qquad B = 0 \tag{8-18}$$

Replacing the constants A and B in Eq. (8-16) by the results in Eq. (8-18) gives

$$u = U\frac{y}{h} - \frac{h^2}{2\mu}\frac{dp}{dx}\frac{y}{h}\left(1 - \frac{y}{h}\right) \tag{8-19}$$

Equation (8-19) is the equation for generalized Couette flow. In nondimensional form the velocity distribution in Eq. (8-19) becomes

$$\frac{u}{U} = \frac{y}{h} + \alpha\frac{y}{h}\left(1 - \frac{y}{h}\right) \tag{8-20}$$

where

$$\alpha = \frac{h^2}{2\mu U}\left(-\frac{dp}{dx}\right) \tag{8-21}$$

is the nondimensional pressure gradient. For $\alpha > 0$, the pressure is decreasing in the direction of flow; hence, the velocity is positive

over the whole width between the plates. The pressure is increasing in the direction of flow when $\alpha < 0$, and the reverse flow begins to occur near the stationary wall as the value of α becomes less than -1. The reverse flow near the stationary wall is due to the influence of the adverse pressure gradient surpassing the action of the viscous force in that region. One must understand that the fluid motion in the positive x-direction (Fig. 8.3) is entirely due to action of the viscous force when $\alpha = 0$. This particular case is known as simple Couette flow and the velocity distribution as reduced from Eq. (8-20) is

$$= \frac{y}{h} \tag{8-22}$$

The velocity distribution of the Couette flow is plotted in Fig. 8.4 as a function of the distance from the stationary wall for various non-dimensional pressure gradients α.

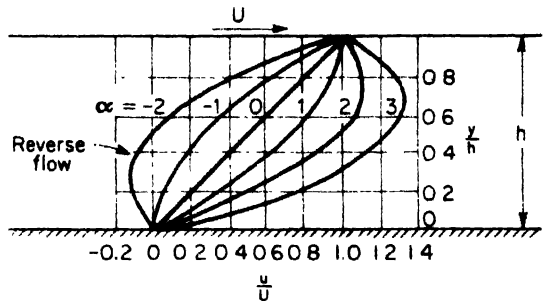


Figure 8.4

(i) *The Average and Maximum Velocities.* The average velocity distribution of the Couette flow between two parallel straight walls can be calculated by the expression

$$u_{av} = \frac{1}{h}\int_0^h u\,dy \tag{8-23}$$

Substituting Eq. (8-20) for u, the equation above becomes

$$u_{av} = \int_0^1 \left[U\frac{y}{h} + \alpha U\left(\frac{y}{h} - \frac{y^2}{h^2}\right)\right] d\left(\frac{y}{h}\right)$$

$$= \left(\frac{1}{2} + \frac{\alpha}{6}\right)U \tag{8-24}$$

In the case of simple Couette flow the velocity distribution increases linearly from zero at the stationary wall to the value of U at the moving plate and the average velocity is given by Eq. (8-24) to be

$U/2$. When the nondimensional pressure gradient is equal to -3, the average velocity becomes zero. This means there is actually no flow across the passage between plates because the influence of the adverse pressure gradient is in balance with the dragging action of the viscous force.

The volumetric flow Q per unit width is

$$Q = hu_{av} = \left(\frac{1}{2} + \frac{\alpha}{6}\right)Uh \tag{8-25}$$

The maximum or minimum velocity occurs at the position of y where the slope of the velocity is equal to zero. From Eq. (8-20) the derivative of the velocity is

$$\frac{du}{dy} = \frac{U}{h} + \frac{\alpha U}{h}\left(1 - 2\frac{y}{h}\right) \tag{8-26}$$

The position of the maximum or minimum velocity is

$$\frac{y}{h} = \frac{1}{2} + \frac{1}{2\alpha} \tag{8-27}$$

It is interesting to note that the maximum velocity for $\alpha = 1$ occurs at $y/h = 1$ and the minimum velocity for $\alpha = -1$ at $y/h = 0$. This implies that for $\alpha = -1$ the velocity gradient at the stationary wall is zero and it becomes negative (reverse flow) for any value of α less than -1. Equation (8-27) breaks down when $-1 < \alpha < 1$ because the maximum and minimum values for y/h have already been reached at $\alpha = 1$ and $\alpha = -1$, respectively.

The values of maximum and minimum velocities can be obtained by substituting the value of y from Eq. (8-27) into Eq. (8-20):

$$u_{max} = \frac{U}{4}\frac{(1+\alpha)^2}{\alpha} \qquad \text{for} \qquad \alpha \geq 1$$

and (8-28)

$$u_{min} = \frac{U}{4}\frac{(1+\alpha)^2}{\alpha} \qquad \text{for} \qquad \alpha \leq -1$$

(ii) *The Shearing Stress.* From Eq. (8-26) the shearing stress distribution in a Couette flow can be obtained, and it is

$$\tau_{yx} = \mu\frac{du}{dy} = \mu\frac{U}{h} + \frac{\mu U\alpha}{h}\left(1 - \frac{2y}{h}\right) \tag{8-29}$$

For a simple Couette flow when $\alpha = 0$ the shearing stress remains constant across the passage between plates. At the position $y/h = \frac{1}{2}$, the second term on the right-hand side of Eq. (8-29) becomes zero; hence, the shearing stress is independent of α at the center of the passage. The shearing stress at the stationary wall is positive for

$\alpha > -1$ and negative for $\alpha < -1$. Since the velocity gradient at the stationary wall is zero for $\alpha = -1$, the value of the shearing stress is zero for this condition.

b. Plane Poiseuille Flow

In the previous case we have discussed the flow between two parallel flat plates, one of which is stationary and the other moving with a uniform velocity. If the two parallel plates are both stationary, the fully developed flow (Fig. 8.5) between the plates requires the boundary conditions

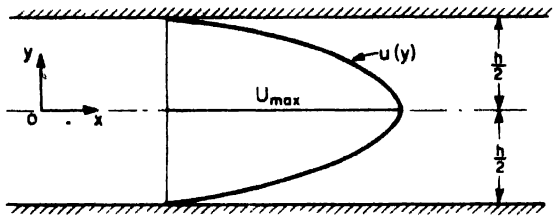


Figure 8.5

$$y = \pm \frac{h}{2} \qquad u = 0 \tag{8-30a}$$

where the x-axis is along the center between two plates. Hence the constants in Eq. (8–16) are

$$B = -\frac{h^2}{8\mu}\frac{dp}{dx}; \qquad A = 0 \tag{8-30b}$$

The velocity distribution across the channel between the two plates is obtained from Eq. (8–16) with constants determined in Eq. (8–30b) as follows:

$$u = -\frac{h^2}{8\mu}\frac{dp}{dx}\left[1 - 4\left(\frac{y}{h}\right)^2\right] \tag{8-31}$$

Equation (8–31) indicates that the velocity profile of the fully developed laminar flow between two parallel plates is parabolic. This type of flow with parabolic velocity distribution (Fig. 8.5) is generally referred to as plane Poiseuille flow.

(i) *The Maximum and Average Velocities.* The maximum velocity of the plane Poiseuille flow can be readily calculated from Eq. (8–31) by letting $y = 0$:

$$u_{max} = -\frac{h^2}{8\mu}\frac{dp}{dx} \tag{8-32}$$

The average velocity as defined in Eq. (8-23) can be used to calculate the average velocity in the plane Poiseuille flow as follows:

$$\begin{aligned} u_{av} &= \frac{1}{h}\int_{-h/2}^{h/2} u\,dy \\ &= \frac{u_{max}}{h}\int_{-h/2}^{h/2}\left(1 - 4\frac{y^2}{h^2}\right)dy \\ &= \frac{2}{3}u_{max} \end{aligned} \tag{8-33}$$

Combining Eqs. (8-32) and (8-33) the pressure drop becomes

$$\frac{dp}{dx} = -12\mu\frac{u_{av}}{h^2} \tag{8-34}$$

(ii) *The Shearing Stress.* The shearing stress at the wall for the plane Poiseuille flow can be determined from the velocity gradient as follows:

$$\left(\tau_{yx}\right)_{h/2} = -\mu\left(\frac{du}{dy}\right)_{h/2} = -\frac{h}{2}\frac{dp}{dx} = 4\mu\frac{u_{max}}{h} \tag{8-35}$$

The local frictional coefficient C_f is defined by

$$C_f = \frac{(\tau_{yx})_{h/2}}{\rho(u_{av}^2/2)} \tag{8-36}$$

Combining Eqs. (8-34), (8-35), and (8-36) the frictional coefficient for laminar flow between two stationary plates is

$$\begin{aligned} C_f &= \frac{-(h/2)}{\rho(u_{av}^2/2)}\left(-12\mu\frac{u_{av}}{h^2}\right) = \frac{12\nu}{u_{av}h} \\ &= \frac{12}{\mathrm{Re}} \end{aligned} \tag{8-37}$$

where $\mathrm{Re} = u_{av}h/\nu$ is the Reynolds number of the flow based on the average velocity and the channel height. Experiments show that Eq. (8-37) is valid for laminar flows up to Re ~ 2,300. Beyond this critical Reynolds number, transition to turbulent flows occurs. This phenomenon will be described in Chap. Ten.

Illustrative Example 8.1. Water at 70°F flows between two large parallel plates at a distance of $\frac{1}{16}$ in. apart. If the average velocity is 0.5 ft/sec, find (a) the maximum velocity, (b) the pressure drop, (c) the wall shearing stress, and (d) the frictional coefficient.

Solution. $\mu = 2.05 \times 10^{-5}$ lbf-sec/ft^2

(a) The maximum velocity is given by Eq. (8-33) as

$$u_{max} = \frac{3}{2}u_{av} = \frac{3}{2}(0.5) = 0.75 \text{ ft/sec}$$

(b) According to Eq. (8–34) the pressure drop is

$$\frac{dp}{dx} = -12\mu\frac{u_{av}}{h^2} = -12(2.05 \times 10^{-5})\frac{0.5}{(1/16)^2(1/144)}$$
$$= -4.54 \text{ lbf/ft}^3$$

(c) The wall shearing stress is [Eq. (8–35)]

$$(\tau_{yz})_{h/2} = -\frac{h}{2}\frac{dp}{dx} = -\frac{1}{32}\left(\frac{1}{12}\right)(-4.54) = 0.01745 \text{ lbf/ft}^2$$

(d) The frictional coefficient is

$$C_f = \frac{12}{\text{Re}} = \frac{12}{(u_{av}h/\nu)} = \frac{12}{0.5 \times [1/(16)\,(12)]/(1.059 \times 10^{-5})}$$
$$= \frac{12}{245} = 0.0489$$

where the Reynolds number Re = 245.

8.4. Steady Flow in Pipes

a. Flow Through a Pipe—The Hagen-Poiseuille Flow

The laminar flow without body forces through a long straight pipe of circular cross section with axial symmetry (Fig. 8.6) is similar to the preceding case of plane Poiseuille flow. Let z be the direction of flow along the axis of the pipe and r denote the radial direction measured outwards from the z-axis. The velocity components (v_r and v_θ) in radial and tangential directions, respectively, are zero. Under these flow conditions the equation of continuity [Eq. (5–41)] reduces to

$$\frac{\partial v_z}{\partial z} = 0 \tag{5–41a}$$

or

$$v_z = v_z(r) \tag{8–38}$$

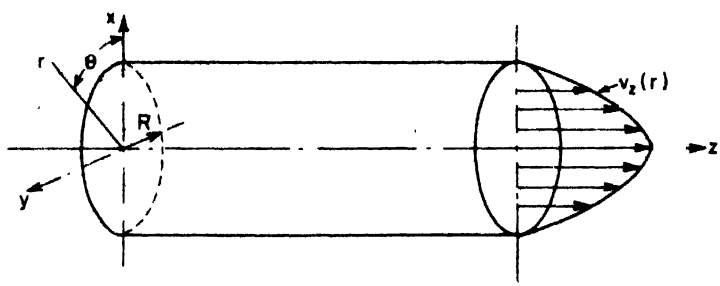


Figure 8.6[3]

From the Navier-Stokes equations in cylindrical coordinates, Eqs. (5-54a) and (5-54b) reduce to

$$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial \theta} = 0 \tag{8-39}$$

so that the pressure depends on z only. The Navier-Stokes equation for the axial direction [Eq. (5-54c)] becomes

$$\mu\left(\frac{d^2 v_z}{dr^2} + \frac{1}{r}\frac{dv_z}{dr}\right) = \frac{dp}{dz} \tag{8-40}$$

The term $\rho(Dv_z/Dt)$ in Eq. (5-54c) vanishes because $v_z = v_z(r)$ and $v_\theta = v_r = 0$. Equation (8-40) is the balance of the shearing force and pressure force so that no acceleration or deceleration takes place. Since in the present case we have shown that $v_z = v_z(r)$ and $p = p(z)$, Eq. (8-40) cannot be valid unless the pressure is a linear function of z. Under this condition the solution of Eq. (8-40) yields the velocity gradient and the velocity, respectively,

$$r\frac{dv_z}{dr} = \frac{1}{2\mu}\frac{dp}{dz}r^2 + A \tag{8-41a}$$

$$v_z = \frac{1}{4\mu}\frac{dp}{dz}r^2 + A \ln r + B \tag{8-41b}$$

where the constants A and B are to be determined from the boundary conditions. The first boundary condition is found from the symmetry of the flow which requires

$$r = 0 \qquad \frac{dv_z}{dr} = 0 \tag{8-42}$$

This condition gives $A = 0$ from Eq. (8-41a). The second boundary condition is the no-slip condition at the wall, i.e.,

$$r = R \qquad v_z = 0 \tag{8-43}$$

With this condition the constant B is obtained from Eq. (8-41b):

$$B = -\frac{1}{4\mu}\frac{dp}{dz}R^2 \tag{8-44}$$

Substitution of the values of the constants A and B into Eq. (8-41b) gives the axial velocity distribution of the Hagen-Poiseuille flow through a pipe,

$$v_z = -\frac{R^2}{4\mu}\frac{dp}{dz}\left[1 - \left(\frac{r}{R}\right)^2\right] \tag{8-45}$$

which has the form of a paraboloid of revolution.

Equation (8-45) can also be obtained from the condition of equilibrium in the z-direction which requires that the shearing force acting on the circumferential area of the cylinder be equal to the pressure

force acting on the cross section of the cylinder. At any radial position, the equilibrium condition can be expressed by

$$\tau_{rz}(2\pi r)l = (p_1 - p_2)\pi r^2$$

or

$$\tau_{rz} = \left(\frac{p_1 - p_2}{l}\right)\frac{r}{2} \tag{8-46}$$

Substitution of the value of $\tau_{rz} = -\mu(dv_z/dr)$ into Eq. (8-46) leads to

$$\frac{dv_z}{dr} = -\left(\frac{p_1 - p_2}{\mu l}\right)\frac{r}{2} \tag{8-47}$$

The solution of Eq. (8-47) with the aid of the no-slip condition at the wall [Eq. (8-43)] becomes

$$v_z = \frac{R^2}{4\mu}\left(\frac{p_1 - p_2}{l}\right)\left[1 - \left(\frac{r}{R}\right)^2\right] \tag{8-48}$$

Equations (8-44) and (8-48) are identical since

$$\frac{p_1 - p_2}{l} = -\frac{dp}{dz}$$

The velocity distribution given in Eq. (8-45) can only occur when the flow reaches a fully developed state and remains laminar. In other words the flow is initially, directly after a pipe entrance, almost uniform across the diameter of the pipe and the velocity distribution is not fully transformed from an almost uniform profile to a parabolic profile until the flow has traveled about 100 pipe-diameters length from the entrance. Once the velocity distribution becomes parabolic it remains unchanged from there on in the flow direction; hence, it is called fully developed flow. As long as the Reynolds number $\mathrm{Re} = 2v_z R/\nu$ has a value less than the critical Reynolds number, the laminar flow can be maintained. According to experiments the critical Reynolds number is approximately equal to 2300 as in the case of plane Poiseuille flow.

(i) *The Maximum and Average Velocities.* The maximum velocity in this case occurs at the center of the pipe where $r = 0$. This can be easily obtained from Eq. (8-45),

$$(v_z)_{\max} = -\frac{R^2}{4\mu}\frac{dp}{dz} \tag{8-49}$$

where $dp/dz < 0$. The average velocity in the Hagen-Poiseuille flow, $v_{z_{av}}$, can be obtained from

$$\begin{aligned} v_{z_{av}} &= \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R v_z\, r\, dr\, d\theta \\ &= \frac{(v_z)_{\max}}{2} = -\frac{R^2}{8\mu}\frac{dp}{dz} \end{aligned} \tag{8-50}$$

In the plane Poiseuille case the ratio u_{av}/u_{max} is $\frac{2}{3}$. The volumetric flow is

$$Q = \pi R^2 v_{z_{av}} = \frac{\pi}{8\mu} R^4 \left(-\frac{dp}{dz}\right) \qquad (8\text{–}51)$$

(ii) *The Shearing Stress.* The shearing stress in the Hagen-Poiseuille flow can be calculated from Eqs. (8–50) and (8–51), i.e.,

$$\tau_{rz} = -\mu \frac{dv_z}{dr} \qquad (8\text{–}52)$$

From the solution of the velocity distribution in Eq. (8–45), the shearing stress at the wall becomes, after differentiation,

$$(\tau_{rz})_R = -\frac{R}{2}\frac{dp}{dz} = 2\mu \frac{(v_z)_{max}}{R} \qquad (8\text{–}53)$$

where $(\tau_{rz})_R$ is the shearing stress at the wall. The local frictional coefficient C_f for laminar flow through a circular pipe can be expressed as follows:

$$(\tau_{rz})_R = C_f \rho \frac{v_{z_{av}}^2}{2} = 2\mu \frac{(v_z)_{max}}{R}$$

or

$$C_f = \frac{16}{\mathrm{Re}} \qquad (8\text{–}54)$$

where $\mathrm{Re} = v_{z_{av}}(2R)/\nu$. In engineering applications (hydraulics) the Darcy (or Chezy) resistance coefficient of pipe flow is often used. This coefficient is defined by the nondimensional pressure gradient

$$f = -\frac{d\bar{p}}{d\bar{z}} \qquad (8\text{–}55)$$

where $\bar{p} = p/(\rho v_{z_{av}}^2/2)$ and $\bar{z} = z/2R$. Equation (8–55) can be simplified as follows:

$$f = \frac{2R}{\rho(v_{z_{av}}^2/2)}\left(\frac{8\mu v_{z_{av}}}{R^2}\right) = \frac{64}{\mathrm{Re}} \qquad (8\text{–}56)$$

Comparing Eqs. (8–54) and (8–56) we obtain the following relation between the conventional frictional coefficient and that of Darcy,

$$f = 4C_f \qquad (8\text{–}57)$$

The frictional coefficient C_f given in Eq. (8–54) has been verified experimentally by G. Hagen. The excellent agreement between the theoretical results and experimental data on the resistance coefficient for laminar flow through pipes is shown in Fig. 8.7. Since the Hagen-Poiseuille parabolic velocity distribution is an exact solution of the Navier-Stokes equations, it can be said that the Navier-Stokes equa-

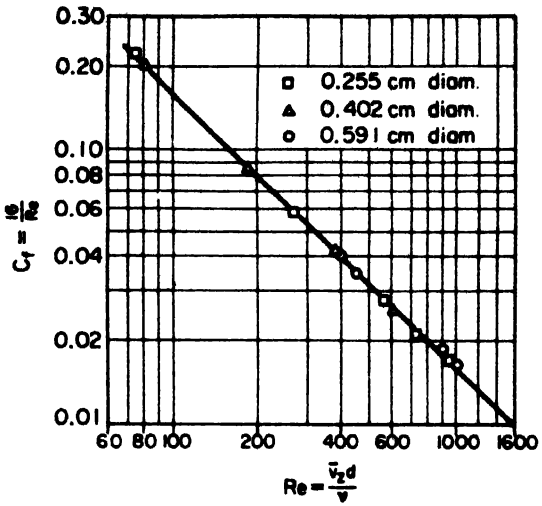


Figure 8.7

tions have been verified to hold at least for the case of parallel flow.

b. Flow Between Two Coaxial Cylinders

The velocity distribution of a viscous fluid which flows steadily parallel to the axis in the annular space between two coaxial cylinders of radii r_1 and r_2 (Fig. 8.8) is given in Eq. (8–41b):

$$v_z = \frac{1}{4\mu}\frac{dp}{dz}(r^2 + A_1 \ln r + B_1) \tag{8–41b}$$

The boundary conditions (see Fig. 8.8) are

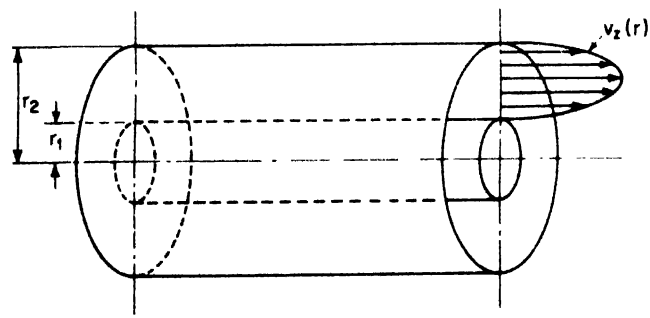


Figure 8.8

$$v_z = 0 \qquad \text{at } r = r_1 \quad \text{and} \quad r = r_2 \tag{8-58}$$

The two constants of integration in Eq. (8-41b) are determined from the boundary conditions,

$$A_1 = -\frac{(n^2 - 1)\, r_1^2}{\ln n}$$

$$B_1 = \frac{(n^2 - 1)\, r_1^2}{\ln n} \ln r_1 - r_1^2$$

where $n = r_2/r_1$. The velocity distribution in the annular space between two coaxial cylinders becomes

$$v_z = -\frac{1}{4\mu}\frac{dp}{dz}\left[(r_1^2 - r^2) + \frac{(n^2 - 1)\, r_1^2}{\ln n} \ln \frac{r}{r_1}\right] \tag{8-59}$$

The rate of volumetric flow in this case is

$$\begin{aligned} Q &= \int_0^{2\pi} \int_{r_1}^{nr_1} v_z r \, dr \, d\theta \\ &= \frac{\pi}{4\mu}\frac{dp}{dz}\left[\frac{r^4}{2} - r_1^2 r^2 - \frac{(n^2 - 1)\, r_1^2}{\ln n}\left(\ln \frac{r}{r_1} - \frac{1}{2}\right) r^2\right]_{r_1}^{nr_1} \\ &= -\frac{\pi r_1^4}{8\mu}\frac{dp}{dz}\left[(n^4 - 1) - \frac{(n^2 - 1)^2}{\ln n}\right] \end{aligned} \tag{8-60}$$

The average velocity in the annulus can be determined from Eq. (8-60):

$$v_{z_{av}} = \frac{Q}{\pi(n^2 - 1) r_1^2} = -\frac{r_1^2}{8\mu}\frac{dp}{dz}\left[(n^2 + 1) - \frac{n^2 - 1}{\ln n}\right] \tag{8-61}$$

The shearing stress at the wall of the inner cylinder and outer cylinder are, respectively,

$$(\tau_{rz})_{r_1} = \left(\mu \frac{dv_z}{dr}\right)_{r_1} = -\frac{r_1}{4}\frac{dp}{dz}\left(\frac{n^2 - 1}{\ln n} - 2\right) \tag{8-62a}$$

$$(\tau_{rz})_{r_2} = -\left(\mu \frac{dv_z}{dr}\right)_{r_2} = -\frac{r_1}{4}\frac{dp}{dz}\left(2n - \frac{n^2 - 1}{n}\frac{1}{\ln n}\right) \tag{8-62b}$$

It can be seen from the two equations above that the shearing stresses at both walls are positive; however, the velocity gradient at the wall of the outer cylinder is negative (Fig. 8.8).

Illustrative Example 8.2. (a) Determine the maximum value of the velocity profile in the annular space between two coaxial cylinders. (b) If $r_1 = 2$ in., $r_2 = 3$ in., and the volumetric flow of water, $Q = 0.2$ ft^3/sec, calculate (i) the pressure drop, (ii) the maximum value of v_{z_0} and (iii) the shearing stress at the wall of both cylinders.

Solution. (a) The velocity distribution in the annular space between two coaxial cylinders is [Eq. (8-59)]

$$v_z = -\frac{1}{4\mu}\left(\frac{dp}{dz}\right)\left[(r_1^2 - r^2) + \frac{(n^2-1)\,r_1^2}{\ln(n)}\ln\left(\frac{r}{r_1}\right)\right] \tag{a}$$

Differentiating Eq. (a) with respect to r, we obtain

$$\frac{dv_z}{dr} = -\frac{1}{4\mu}\left(\frac{dp}{dz}\right)\left[-2r + \frac{(n^2-1)\,r_1^2}{\ln(n)}\left(\frac{1}{r}\right)\right] \tag{b}$$

Setting Eq. (b) equal to zero and solving for r one obtains

$$-2r + \frac{(n^2-1)\,r_1^2}{\ln(n)}\left(\frac{1}{r}\right) = 0$$

or

$$r = \pm r_1\left[\frac{(n^2-1)}{2\ln(n)}\right]^{1/2} \tag{c}$$

Eq. (c) represents the radial location of the maximum value of v_z. Combining Eqs. (a) and (b) the maximum velocity is obtained as

$$v_{z_{max}} = -\frac{1}{4\mu}\left(\frac{dp}{dz}\right)^2\left[r_1^2 - \frac{(n^2-1)\,r_1^2}{2\ln(n)} + \frac{(n^2-1)\,r_1^2}{\ln(n)}\ln\left(\frac{n^2-1}{2\ln(n)}\right)^{1/2}\right]$$

$$= \frac{r_1^2}{4\mu}\left(\frac{dp}{dz}\right)\left\{1 - \frac{(n^2-1)}{2\ln(n)}\left[1 - \ln\left(\frac{n^2-1}{2\ln(n)}\right)\right]\right\} \tag{d}$$

(b) (i) From Eq. (8-60) we find (assuming $\mu = 2.05 \times 10^{-5}$ lbf-sec/ft^2)

$$0.2 = -\frac{\pi(\frac{1}{6})^4}{8(2.05 \times 10^{-5})}\left(\frac{dp}{dz}\right)\left[(1.5)^4 - 1 - \frac{\{(1.5)^2 - 1\}^2}{\ln(1.5)}\right]$$

$$= 2.955\left(\frac{dp}{dz}\right)$$

Hence,

$$\frac{dp}{dz} = -0.0677 \text{ lbf/ft}^3$$

(ii) The maximum velocity is obtained from Eq. (d) as follows:

$$v_{z_{max}} = -\left(\frac{1}{6}\right)^2 \frac{1}{4(2.05 \times 10^{-5})}(-0.0677)$$

$$\times \left\{1 - \frac{1.25}{2\ln(1.5)}\left[1 - \ln\left(\frac{1.25}{2\ln(1.5)}\right)\right]\right\} = 2.786 \text{ ft/sec}$$

(iii) The wall shearing stresses are [Eq. (8-62)]

$$(\tau_{rz})_{r_1} = -\frac{1}{4}\left(\frac{1}{6}\right)(-0.0677)\left(\frac{1.25}{\ln(1.5)} - 2\right)$$

$$= 0.00586 \text{ lbf/ft}^2$$

$$(\tau_{rz})_{r_2} = -\frac{1}{4}\left(\frac{1}{6}\right)(-0.0677)\left[3.0 - \left(\frac{1.25}{1.5}\right)\frac{1}{\ln(1.5)}\right]$$

$$= 0.00266 \text{ lbf/ft}^2$$

8.5. Flow Between Two Concentric Rotating Cylinders

Consider the flow between two concentric rotating cylinders with radii r_1 and r_2, respectively (Fig. 8.9). Assume the flow to be peripheral so that we have only the tangential component of velocity v_θ. Let ω_1 and ω_2 be the steady angular velocities of the inner and outer cylinders, respectively. According to Eq. (5–41) the continuity equation becomes

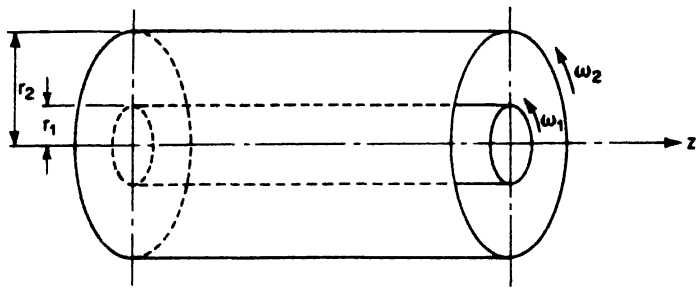


Figure 8.9

$$\frac{\partial v_\theta}{\partial \theta} = 0 \qquad \text{or} \qquad v_\theta = v_\theta(r) \tag{8–63}$$

The Navier-Stokes equations [Eqs. (5–54)] for the present case reduces to

$$\rho \frac{v_\theta^2}{r} = \frac{dp}{dr} \tag{8–64}$$

and

$$\frac{d^2 v_\theta}{dr^2} + \frac{d}{dr}\left(\frac{v_\theta}{r}\right) = 0 \tag{8–65}$$

Since v_θ is a function of r only, it follows from axial symmetry that the pressure p in Eq. (8–64) must be either a function of r or a constant. Hence, the term $\partial p/\partial \theta$ does not appear in Eq. (8–65). The first integration of Eq. (8–65) yields

$$\frac{dv_\theta}{dr} + \frac{v_\theta}{r} = 2C_1$$

or

$$\frac{1}{r}\frac{d}{dr}(rv_\theta) = 2C_1 \tag{8-66b}$$

The integration of Eq. (8-66) results in

$$v_\theta = C_1 r + \frac{C_2}{r} \tag{8-67}$$

where C_1 and C_2 are constants of integration to be determined. The boundary conditions of the flow are

$$\begin{aligned} v_\theta &= r_1\omega_1 \qquad \text{at} \qquad r = r_1 \\ v_\theta &= r_2\omega_2 \qquad \text{at} \qquad r = r_2 \end{aligned} \tag{8-68}$$

The solution given in Eq. (8-67) which satisfies the boundary conditions in (8-68) is

$$v_\theta = \frac{1}{r_2^2 - r_1^2}\left[(\omega_2 r_2^2 - \omega_1 r_1^2)r - \frac{r_1^2 r_2^2}{r}(\omega_2 - \omega_1)\right] \tag{8-69}$$

where

$$C_1 = \omega_1 + \frac{r_2^2}{r_2^2 - r_1^2}(\omega_2 - \omega_1)$$

$$C_2 = -\frac{r_1^2 r_2^2}{r_2^2 - r_1^2}(\omega_2 - \omega_1)$$

The radial pressure resulting from the peripheral motion can be determined from Eqs. (8-64) and (8-69):

$$\begin{aligned} p = p_1 + \frac{\rho}{(r_2^2 - r_1^2)^2}\Bigg[&(\omega_2 r_2^2 - \omega_1 r_1^2)^2\left(\frac{r^2 - r_1^2}{2}\right) \\ &- 2r_1^2 r_2^2(\omega_2 - \omega_1)(\omega_2 r_2^2 - \omega_1 r_1^2)\ln\frac{r}{r_1} \\ &- \frac{r_1^4 r_2^4}{2}(\omega_2 - \omega_1)^2\left(\frac{1}{r^2} - \frac{1}{r_1^2}\right)\Bigg] \end{aligned} \tag{8-70}$$

where

$$p = p_1 \qquad \text{at} \qquad r = r_1$$

The shearing stress in this case can be determined from Eqs. (5-50) and (5-51):

$$\tau_{r\theta} = \mu\left[r\frac{d}{dr}\left(\frac{v_\theta}{r}\right)\right] \tag{8-71}$$

Substituting Eq. (8-69) into Eq. (8-71) yields

$$\tau_{r\theta} = \frac{2\mu}{r_2^2 - r_1^2}\frac{r_1^2 r_2^2}{r^2}(\omega_2 - \omega_1) \tag{8-72}$$

At the walls of the outer cylinder the shearing stress is

$$(\tau_{r\theta})_{r=r_1} = 2\mu(\omega_2 - \omega_1)\frac{1}{(r_2/r_1)^2 - 1} \tag{8-73}$$

Similarly, the shearing stress at the walls of the inner cylinder can be obtained from Eq. (8–72) by letting $r = r_1$.

8.6. Applications of the Parallel Flow Theory

Some of the results of § 8.4 and § 8.5 may be applied toward the solution of many interesting and important practical problems. Two basic problems of importance which will be discussed in this section are the measurement of viscosity and the hydrodynamics of lubrication.

a. The Measurement of Viscosity

(i) Measuring the flow through a capillary tube was the first method used for determining viscosities of fluids. Equation (8–51) which gives the relationship between the rate of volumetric flow and the pressure gradient, the fourth power of the radius of the tube, and the coefficient of viscosity μ, was verified experimentally by Hagen and later independently by Poiseuille. Since some distance is required for the fluid to develop its parabolic velocity distribution after it enters the capillary tube, the pressure measured must be where the flow has developed. This is shown in Fig. 8.10. If the radius R and

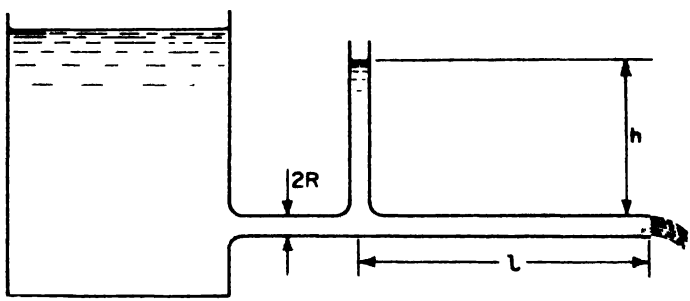


Figure 8.10

the length l of the tube are given, the difference of pressure between two points at an axial distance l apart is determined, and the volume of flow per unit time is measured, the coefficient of viscosity μ can be calculated from Eq. (8–51) as follows:

$$\mu = \frac{\pi R^4}{8Q}\frac{p_1 - p_2}{l} = \frac{\pi R^4}{8Q} g\rho \left(\frac{h}{l}\right) \tag{8-74}$$

where g is the acceleration due to gravity and ρ the density of the fluid. An adaptation of the capillary tube for industrial purposes is called the Saybolt viscometer.

(ii) Another type of viscometer is a mechanical device of rotating concentric cylinders (Fig. 8.11). A narrow space between the two cylinders is filled with the fluid. The outer cylinder of radius r_2 is arranged to rotate at a constant velocity ω_2 with respect to the stationary inner cylinder which is suspended by a torsional spring. The torque T transmitted by the fluid to the inner cylinder causes a torsional strain of the restraining spring. With reference to Fig. 8.11 it is clear that the measured torque results from the shearing forces of the fluid which resist the motion of the outer cylinder. However, the shearing forces exist between both the circumferential surfaces and the bottom surfaces of the two concentric cylinders.

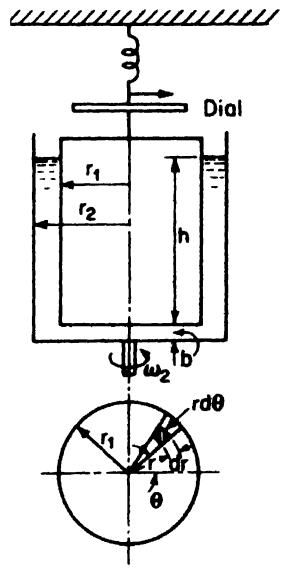


Figure 8.11

According to Eq. (8–72) the shearing stress at the circumferential surface of the stationary inner cylinder ($\omega_1 = 0$) is

$$(\tau_{r\theta})_{r_1} = \frac{2\mu}{r_2^2 - r_1^2} r_2^2 \omega_2 \tag{8-75}$$

Since the gap $r_2 - r_1$ is much smaller than r_2, Eq. (8–75) can be approximately written

$$(\tau_{r\theta})_{r_1} = \frac{2\mu r_2^2 \omega_2}{2r_2(r_2 - r_1) - (r_2 - r_1)^2} \simeq \frac{\mu r_2 \omega_2}{r_2 - r_1} \tag{8-76}$$

The torque on the inner cylinder due to $(\tau_{r\theta})_{r_1}$ is

$$T_{r_1} = (2\pi r_1 h)(r_1)(\tau_{r\theta})_{r_1} = 2\pi\mu h\left(\frac{r_1^2 r_2}{r_2 - r_1}\right)\omega_2 \tag{8-77}$$

where h is the height of the fluid level measured from the bottom of the inner cylinder.

Now, let the small gap between the bottoms of the two cylinders be b. The shearing stress developed in the fluid within the gap b follows the law of Couette flow [Eq. (8–29)], i.e.,

$$(\tau_{z\theta})_b = \mu \frac{r\omega_2}{b} \tag{8-78}$$

Hence the elemental torque on an infinitesimal surface of the bottom of the inner cylinder is

$$dT_b = (\tau_{z\hat{\theta}})_b (r\, d\theta\, dr) r$$

$$= \frac{\mu\omega_2}{b} r^3\, dr\, d\theta$$

The torque T_b on the bottom of the inner cylinder becomes

$$T_b = \frac{\mu\omega_2}{b} \int_0^{2\pi} \int_0^{r_1} r^3\, dr\, d\theta = \frac{\pi}{2b} \mu\omega_2 r_1^4 \tag{8-79}$$

The total torque T in the torsional spring is the sum of T_{r_1} and T_b:

$$T = T_{r_1} + T_b = 2\pi\mu h \left(\frac{r_1^2 r_2}{r_2 - r_1}\right)\omega_2 + \frac{\pi}{2b}\mu\omega_2 r_1^4$$

$$= 2\pi\mu\omega_2 r_1^2 \left(\frac{r_2 h}{r_2 - r_1} + \frac{r_1^2}{4b}\right) \tag{8-80}$$

Hence the coefficient of viscosity can be determined in terms of the total torque, the angular velocity, and the geometry of the viscometer.

$$\mu = \frac{2b(r_2 - r_1)\, T}{\pi r_1^2 \omega_2 [4hbr_2 + r_1^2(r_2 - r_1)]} \tag{8-81}$$

Illustrative Example 8.3. Determine the viscosity of oil for the following two cases: (a) A horizontal pipe having a diameter of 2 in. and a length of 20 ft. Oil is discharged (sp gr 0.93) through the pipe at a rate of 400 lbf/min under a pressure difference of 2.5 psi. (b) The concentric-cylinder viscometer (Fig. 8.11) has the dimensions: $r_1 = 3$ in., $r_2 = 3.02$ in., $b = 0.015$ in., and $h = 8$ in. The total torque is 36 in.-lbf at a rotational speed of 30 rpm.

Solution. (a) The volumetric rate of oil flow in the pipe is

$$Q = \frac{400}{(60)(0.93)(62.4)} = 0.115 \text{ ft}^3/\text{sec}$$

According to Eq. (8-74) we have

$$\mu = \frac{\pi R^4}{8Q}\left(\frac{p_1 - p_2}{l}\right)$$

$$= \frac{\pi(\frac{1}{12})^4}{8(0.115)}\left(\frac{2.5 \times 144}{20}\right) = 0.00297 \text{ lbf-sec/ft}^2$$

(b) The coefficient of viscosity in this case can be determined from Eq. (8-81) as follows:

$$\mu = \frac{2(0.015/12)(0.02/12)(36/12)}{\pi(3/12)^2(30 \times 2\pi/60)\,[4(8/12)(0.015/12)(3.02/12) - (3/12)^2(0.02/12)]}$$

$$= 0.00294 \text{ lbf-sec/ft}^2$$

b. Hydrodynamics of Bearing Lubrication

The application of the hydrodynamic theory of Couette flow (§ 8.3a) in the study of lubrication can be illustrated best by an example of the slipper bearing (Fig. 8.12a). The slipper bearing consists of a short sliding block moving over a stationary guide and inclined at a small angle with respect to the stationary pad. The gap between the sliding block and the pad is always much smaller than the length of the block and is filled with a lubricant, usually oil. In Fig. 8.12a an inclined block of infinite length is shown (in the transverse direction with respect to the motion) moving with a velocity U parallel to an infinite stationary pad and carrying a load P. The problem now is one in two dimensions which was first developed by Reynolds[1] and published in 1886.

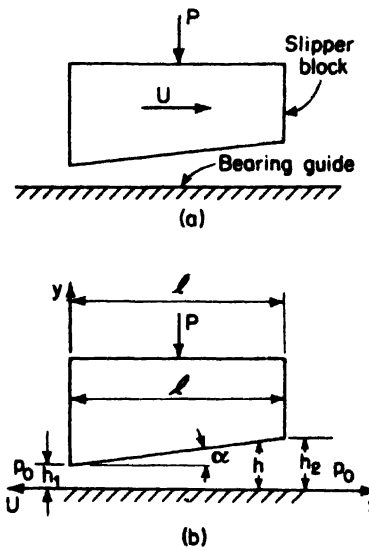


Figure 8.12

In order to make the motion steady a system of coordinates is chosen in which the slipper block is stationary and the pad moves with a velocity $-U$ (Fig. 8.12b). Since the slipper block is inclined relative to the guide, a pressure difference is set up in the gap between the slipper and the guide. At high velocities, extreme pressure difference can be created to support heavy loads in the direction normal to the guide.

For a two-dimensional flow and a small inclination angle α, we have

$$v \ll u \tag{8-82a}$$

$$w = 0, \qquad \frac{\partial}{\partial z} = 0 \tag{8-82b}$$

The expression in (8–82a) is deduced from the equation of continuity [Eq. (5–9b)] as $h(x) \ll l$. According to the conditions above Eqs. (5–19) reduce to

$$\rho u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{\partial p}{\partial y} = 0 \tag{8-83}$$

Equation (8–83) can be further simplified by the following comparison of the order of magnitude of the terms:

$$\frac{\partial^2 u/\partial x^2}{\partial^2 u/\partial y^2} = \frac{(U/l^2)\dfrac{\partial^2(u/U)}{\partial(x/l)^2}}{(U/h^2)\dfrac{\partial^2(u/U)}{\partial(y/h)^2}} \sim \left(\frac{h}{l}\right)^2$$

and

$$\frac{\rho u(\partial u/\partial x)}{\mu(\partial^2 u/\partial y^2)} = \frac{U^2(\rho/l)(u/U)\dfrac{\partial(u/U)}{\partial(x/l)}}{\mu(U/h^2)\dfrac{\partial^2(u/U)}{\partial(y/h)^2}} \sim \frac{Ul}{\nu}\left(\frac{h}{l}\right)^2$$

These results lead to the conclusion that the normal viscous-stress gradient is negligibly small in comparison with the shearing-stress gradient. Since the ratio $h/l \sim 10^{-3}$ in most lubrication problems and the value of Reynolds number referring to the length of the block seldom exceeds 2.5×10^4 (laminar flow), the inertia-force term in Eq. (8–83) can be safely neglected. Finally the equation governing the motion of the lubricant between the slipper and the guide is

$$\frac{dp}{dx} = \mu \frac{\partial^2 u}{\partial y^2} \tag{8–15}$$

which is identical to the case of flow between parallel flat plates (§ 8.3) except the pressure gradient is a function of x here instead of being a constant. Using the notation of Fig. 8.12b, the boundary conditions are

$$\begin{aligned} &p = p_0: && x = 0 \ (h = h_1) \quad \text{and} \quad x = l \ (h = h_2) \\ &u = -U: && y = 0 \\ &u = 0: && y = h \end{aligned} \tag{8–84}$$

The solution of Eq. (8–15) is

$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + f_1(x)y + f_2(x) \tag{8–85}$$

where

$$f_1(x) = \frac{U}{h} - \frac{h}{2\mu}\frac{dp}{dx}$$

$$f_2(x) = -U$$

are determined from the boundary conditions. With the determined values for $f_1(x)$ and $f_2(x)$, the velocity distribution of Eq. (8–85) becomes

$$u = -U\left(1 - \frac{y}{h}\right) - \frac{h^2}{2\mu}\frac{dp}{dx}\frac{y}{h}\left(1 - \frac{y}{h}\right) \tag{8–86}$$

The rate of volumetric flow in every section, Q, is obtained from the equation of continuity as follows:

$$Q = \int_0^h u\,dy = \text{Constant}$$

$$= -\frac{Uh}{2} - \frac{h^3}{12\mu}\frac{dp}{dx} \tag{8-87}$$

Or, solving for dp/dx,

$$\frac{dp}{dx} = -\frac{12\mu}{h^3}\left(\frac{Uh}{2} + Q\right) \tag{8-88}$$

Before we can calculate the value of the pressure in the gap it is necessary to relate the parameters U and Q to h and x. According to the boundary conditions (8-84), the integration of Eq. (8-88) between $x = 0$ and $x = l$ yields

$$\int_0^l \frac{dp}{dx}dx = 0 = -12\mu\left(\frac{U}{2}\int_0^l \frac{dx}{h^2} + Q\int_0^l \frac{dx}{h^3}\right)$$

or

$$Q = -\frac{U}{2}\frac{\int_0^l \frac{dx}{h^2}}{\int_0^l \frac{dx}{h^3}} \tag{8-89}$$

But from the geometry shown in Fig. 8.12b, we obtain

$$h = h_1 + (h_2 - h_1)\frac{x}{l} \qquad \text{and} \qquad \tan\alpha = \frac{h_2 - h_1}{l} \tag{8-90}$$

or

$$x = (h - h_1)\cot\alpha$$

Substituting the linear relation of x and h in Eq. (8-90) into Eq. (8-89), it follows that

$$Q = -\frac{h_1 h_2}{h_1 + h_2}U = -\frac{Uh_0}{2} \tag{8-91}$$

Combining Eqs. (8-91) and (8-88), the pressure gradient becomes

$$\frac{dp}{dx} = -\frac{6\mu U}{h^3}(h - h_0) \tag{8-92}$$

where $h_0 = 2h_1h_2/(h_1 + h_2)$. Equation (8-92) indicates that

$$\begin{aligned} \frac{dp}{dx} &> 0 \quad \text{at} \quad h = h_1 \\ \frac{dp}{dx} &< 0 \quad \text{at} \quad h = h_2 \\ \frac{dp}{dx} &= 0 \quad \text{at} \quad h = h_0 \end{aligned} \tag{8-93}$$

The pressure distribution in the gap (Fig. 8.13) can now be calculated from Eq. (8–92) with boundary conditions (8–84) satisfied:

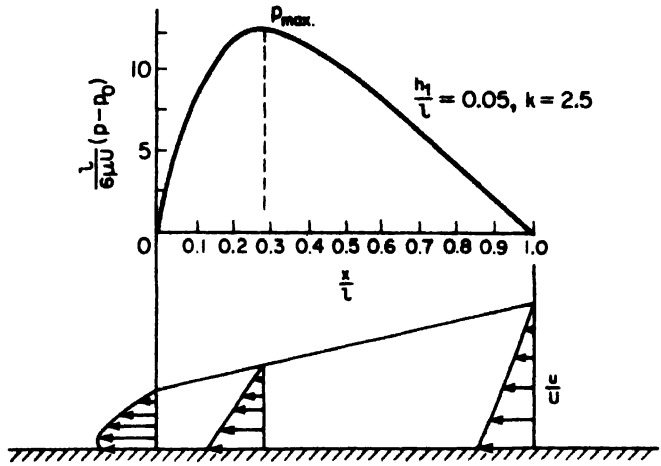


Figure 8.13

$$p = p_0 - \frac{6\mu Ul}{h^2(h_2^2 - h_1^2)}(h - h_1)(h - h_2) \tag{8–94}$$

Inserting the value of $h = h_0$ into Eq. (8–94), the value of maximum pressure is obtained:

$$p_{max} = p_0 + \frac{3\mu Ul(h_2 - h_1)}{2h_1h_2(h_2 + h_1)} \tag{8–95}$$

The location of the maximum pressure is obtained from Eq. (8–90) by putting $h = h_0$:

$$x_{p_{max}} = (h_0 - h_1)\frac{l}{h_2 - h_1}$$

or

$$\left(\frac{x}{l}\right)_{p_{max}} = \frac{h_1}{h_2 + h_1} \tag{8–96}$$

The total pressure force (bearing load) which the fluid exerts on the bearing guide can be calculated by integrating Eq. (8–94) over the entire surface of the slipper block. For a unit depth of the slipper the total load is

$$P = \int_0^l (p - p_0)\,dx = -\frac{6\mu Ul}{h_2^2 - h_1^2}\int_0^l \frac{(h - h_1)(h - h_2)}{h^2}\,dx$$

$$= -\frac{6\mu Ul^2}{(h_2 + h_1)(h_2 - h_1)^2}\int_{h_1}^{h_2}\left(1 - \frac{h_1 + h_2}{h} + \frac{h_1 h_2}{h^2}\right)dh$$

$$= \frac{6\mu Ul^2}{h_1^2(k-1)^2}\left[\ln k - \frac{2(k-1)}{k+1}\right] \tag{8-97}$$

where $k = h_2/h_1$. It is interesting to note that $P = 0$ when $k = 1$ and P becomes maximum when $k = 2.2$. The value of the maximum load is

$$P_{\max} \doteq 0.16\,\mu U\left(\frac{l}{h_1}\right)^2 \tag{8-98}$$

The resistance force on the bearing guide can be calculated from the wall shearing stress, i.e.,

$$F = \int_0^l \tau_0\,dx \tag{8-99}$$

The shearing stress at the bearing guide can be evaluated from Eqs. (8-86), (8-88), and (8-91) as follows:

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_0 = \mu\left(\frac{4U}{h} + \frac{6Q}{h^2}\right)$$

$$= \mu U\left(\frac{4}{h} - \frac{6h_1h_2}{h_1 + h_2}\frac{1}{h^2}\right) \tag{8-100}$$

Substituting τ_0 from Eq. (8-100) into Eq. (8-99) the resultant of the shearing stresses becomes

$$F = \frac{\mu Ul}{h_2 - h_1}\int_{h_1}^{h_2}\left(\frac{4}{h} - \frac{6h_1h_2}{h_1 + h_2}\frac{1}{h^2}\right)dh$$

$$= \frac{\mu Ul}{(k-1)h_1}\left[4\ln k - \frac{6(k-1)}{k+1}\right] \tag{8-101}$$

For $k = 2.2$, the value of F is

$$F_{P_{\max}} = 0.75\,\mu U\frac{l}{h_1} \tag{8-102}$$

The coefficient of friction is

$$\left(\frac{F}{P}\right)_{\max} = 4.7\frac{h_1}{l} \tag{8-103}$$

and it is independent of viscosity. For a normal bearing the value of the ratio h/l is of the order of 10^{-3}; hence, the coefficient of friction is in the neighborhood of 5×10^{-3}. This value is about $\frac{1}{100}$th to $\frac{1}{20}$th of the value of the coefficient of friction of a solid.

The results of a slipper bearing can be applied to the journal type

bearing (Fig. 8.14) with slight modifications. In this case the clearance between journal and bearing must be much smaller than the radius of the journal, and the eccentricity of the journal must have a component perpendicular to the direction of the load P. Then $p_1 > p_2$ and the equilibrium exists. The total load per unit length of the journal, in analogy to Eq. (8–97), is

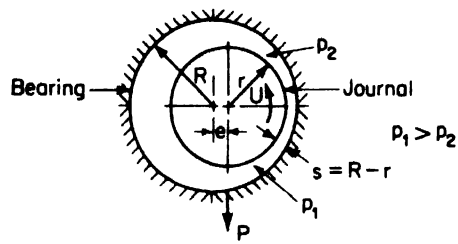


Figure 8.14

$$\frac{P}{2\mu\pi U}\left(\frac{s}{r}\right)^2 = \frac{6\varepsilon}{(2+\varepsilon^2)\sqrt{1-\varepsilon^2}} \tag{8–104}$$

where s is the clearance between the journal and bearing, r the radius of the journal, and ε the ratio of the eccentricity e and the clearance s. The nondimensional quantity on the right-hand side of Eq. (8–104) is known as the bearing constant (also called the Sommerfield number).

Illustrative Example 8.4. A slipper bearing consists of a sliding block moving over a stationary guide and inclined at a small angle with respect to the stationary pad (Fig. 8.12). Let $\mu = 0.002$ lbf-sec/ft^2, $U = 3$ ft/sec, $l/h_1 = 500$, and $k = 2.0$. If the thickness $h_0 = 5 \times 10^{-4}$ in. is given, calculate the maximum pressure.

Solution. According to Eq. (8–91), we have

$$h_0 = \frac{2h_1h_2}{h_1+h_2} = \frac{2h_2}{1+(h_2/h_1)} = \frac{2}{3}h_2$$

$$h_2 = \frac{3}{2}h_0 = 7.5 \times 10^{-4} \text{ in.}$$

From Eq. (8–95) we have

$$p_{max} - p_0 = \frac{3}{2}\mu\frac{U}{h_2}\left(\frac{l}{h_1}\right)\frac{(h_2/h_1)-1}{(h_2/h_1)+1} = \frac{(1.5)(0.002)(3)(500)(1/3)}{(7.5/12)\times 10^{-4}}$$

$$= 24{,}000 \text{ lbf/ft}^2$$

8.7. Steady Flow Around a Sphere—Theory of Very Slow Motion

In § 7.18a we have studied the motion of a sphere in an inviscid fluid and found that there is no force or moment acting on the sphere in a rectilinear flow. For fluids of high viscosity or flowing at very slow speeds ($\mathrm{Re} < 1$) the inertia-force terms on the left-hand side of Eq. (5-19) can be neglected in comparison with the friction-force terms. If the body force is also neglected and steady flow is assumed, the Navier-Stokes equations, Eq. (5-19), and the continuity equation, Eq. (5-9b), reduce, respectively, to

$$\nabla p = \mu \nabla^2 \mathbf{q} \tag{8-105}$$

$$\nabla \cdot \mathbf{q} = 0 \tag{8-106}$$

If the divergence of Eq. (8-105) is taken, it becomes

$$\nabla^2 p = \mu \nabla \cdot (\nabla^2 \mathbf{q}) = \mu \nabla^2 (\nabla \cdot \mathbf{q}) = 0 \tag{8-107}$$

Equation (8-107) indicates that the pressure p satisfies the Laplace equation; hence, for a very slow motion the pressure is a harmonic function.

Equations (8-105)–(8-107) may be applied to the problem of a steady uniform flow around a sphere at rest (Fig. 8.15a). This problem was first solved by Stokes[2] and is often referred to as Stokes' law. Let the origin of the coordinate system be chosen at the center of the sphere, the z-axis be the direction of uniform flow far away from the sphere, and $\omega = (x^2 + y^2 + z^2)^{1/2}$ be the distance of an arbitrary point from the origin. The boundary conditions which must be satisfied by the flow are:

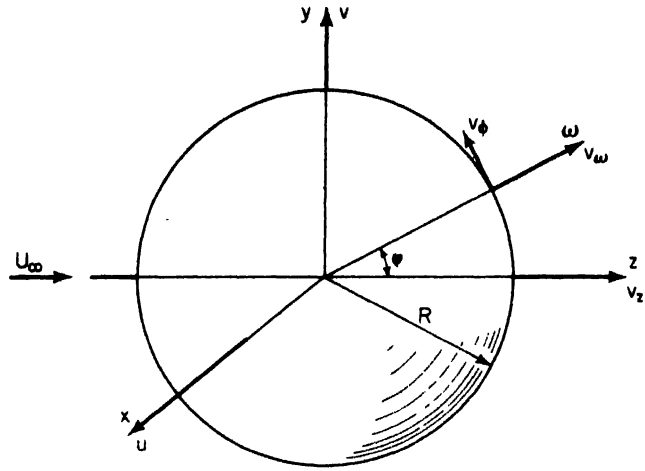
$$\begin{aligned} w = u = v = 0 \qquad & \text{at} \quad \omega = R \\ w = U_\infty, \quad u = v = 0, \quad p = p_\infty \quad & \text{at} \quad \omega = \infty \end{aligned} \tag{8-108}$$

where R is the radius of the sphere and U_∞ the uniform velocity far away from the sphere.

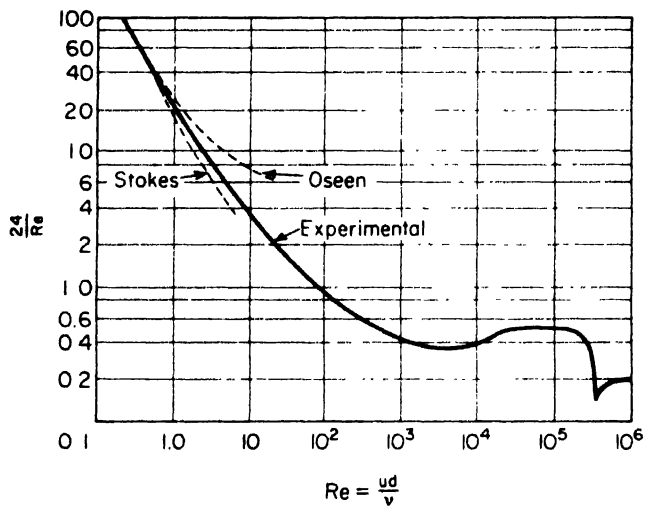
According to Eq. (7-58) the solution of Eq. (8-107) satisfying the required properties is

$$p - p_\infty = -\frac{A \cos \varphi}{\omega^2} = -\frac{Az}{\omega^3} \tag{8-109}$$

where A is a constant to be determined from the boundary conditions [Eq. (8-108)]. Substituting Eq. (8-109) into Eq. (8-105), one obtains



(a)



(b)

Figure 8.15

$$\nabla^2 w = \frac{A}{\mu}\left(\frac{3z^2}{\omega^5} - \frac{1}{\omega^3}\right)$$

$$\nabla^2 v = \frac{A}{\mu}\frac{3zy}{\omega^5} \qquad (8\text{-}110)$$

$$\nabla^2 u = \frac{A}{\mu}\frac{3zx}{\omega^5}$$

Although the details of the solutions of Eqs. (8–110) are beyond the scope of the text, the results of the solutions are

$$w = U_\infty\left[\frac{3}{4}\frac{Rz^2}{\omega^3}\left(\frac{R^2}{\omega^2}-1\right)+1-\frac{3}{4}\frac{R}{\omega}-\frac{1}{4}\frac{R^3}{\omega^3}\right]$$

$$v = U_\infty\left[\frac{3}{4}\frac{Rzy}{\omega^3}\left(\frac{R^2}{\omega^2}-1\right)\right] \tag{8–111}$$

$$u = U_\infty\left[\frac{3}{4}\frac{Rzx}{\omega^3}\left(\frac{R^2}{\omega^2}-1\right)\right]$$

and the pressure is

$$p - p_\infty = -\frac{3}{2}\mu U_\infty R\left(\frac{z}{\omega^3}\right) \tag{8–112}$$

It can be readily verified that the velocity components in Eqs. (8–111) satisfy Eqs. (8–110) and the boundary conditions (8–108). The pressures at the leading and trailing stagnation points of the sphere ($z = \mp R$) are, respectively,

$$p_{\mp R} - p_\infty = \pm\frac{3}{2}\frac{\mu U_\infty}{R}$$

This indicates that the fluid exerts a force in the z-direction. The total force (drag) acting on the sphere moving relative to an undisturbed stream along the z-axis is equal to the surface integral of the normal pressure and the tangential shearing stresses acting on it. The normal pressure acting on the surface of the sphere is [Eqs. (5–74a) and (5–75)]

$$(\sigma_{\omega\omega})_{\omega=R} = \left[-(p-p_\infty)+2\mu\frac{\partial v_\omega}{\partial\omega}\right] \tag{8–113}$$

and the corresponding shearing stress [Eqs. (5–74b) and (5–75)] is

$$(\sigma_{\omega\varphi})_{\omega=R} = \mu\left[\omega\frac{\partial}{\partial\omega}\left(\frac{v_\varphi}{\omega}\right)+\frac{1}{\omega}\frac{\partial v_\omega}{\partial\varphi}\right]_{\omega=R} \tag{8–114}$$

The radial and tangential components of velocity v_ω and v_φ, respectively, can be calculated in terms of u, v, w as follows [Eq. (5–64) and Fig. 5.7a]:

$$v_\omega = \left(u\frac{x}{r}+v\frac{y}{r}\right)\sin\varphi + w\cos\varphi$$

$$= u\frac{x}{\omega}+v\frac{y}{\omega}+w\frac{z}{\omega} \tag{8–115a}$$

$$v_\varphi = -w\sin\varphi+\left(u\frac{x}{r}+v\frac{y}{r}\right)\cos\varphi$$

$$= -w\frac{r}{\omega}+u\frac{xz}{\omega r}+v\frac{yz}{\omega r} \tag{8–115b}$$

where $r^2 = x^2 + y^2$. Substituting Eqs. (8–111) into Eqs. (8–115a) and (8–115b), we have

$$v_\omega = U_\infty \cos\varphi\left(1 - \frac{3}{2}\frac{R}{\omega} + \frac{1}{2}\frac{R^3}{\omega^3}\right) \tag{8–116a}$$

$$v_\varphi = -U_\infty \sin\varphi\left(1 - \frac{3R}{4\omega} - \frac{R^3}{4\omega^3}\right) \tag{8–116b}$$

With the values of radial and tangential components of velocity in Eqs. (8–116), the normal and shearing stresses acting on the surface of the sphere become [Eqs. (8–112)–(8–114)]

$$(\sigma_{\omega\omega})_{\omega=R} = -\left(p - p_\infty\right) = \frac{3}{2}\frac{\mu U_\infty}{R}\cos\varphi \tag{8–117a}$$

and

$$(\sigma_{\omega\varphi})_{\omega=R} = -\frac{3}{2}\frac{\mu U_\infty}{R}\sin\varphi \tag{8–117b}$$

The total drag on the sphere now can be calculated by integrating the z-component of the normal and shearing stresses over the surface of the sphere as follows:

$$\begin{aligned} D &= \int\int (p - p_\infty)\cos\varphi\, dA + \int\int \sigma_{\omega\varphi}\sin\varphi\, dA \\ &= 2\pi R^2 \int_\pi^0 \left(-\frac{3}{2}\frac{\mu U_\infty}{R}\cos\varphi\right)\cos\varphi\sin\varphi\, d\varphi \\ &\quad + 2\pi R^2 \int_\pi^0 \left(-\frac{3}{2}\mu\frac{U_\infty}{R}\sin\varphi\right)\sin^2\varphi\, d\varphi \\ &= 2\pi R\mu U_\infty + 4\pi R\mu U_\infty = 6\pi R\mu U_\infty \end{aligned} \tag{8–118}$$

The expression given in Eq. (8–118) is known as *Stokes' formula* for the drag of a sphere in which one-third of the value arises from the normal pressure forces and two-thirds from frictional forces. The coefficient of drag of the sphere at small Reynolds numbers is

$$C_D = \frac{D}{(\rho/2)U_\infty^2 \pi R^2} = \frac{24}{U_\infty(2R)/\nu} = \frac{24}{\mathrm{Re}} \tag{8–119}$$

A comparison between Stokes' drag coefficient in Eq. (8–119) and experimental results is shown in Fig. 8.15b. It is seen that Stokes' formula is valid only for cases when $\mathrm{Re} < 1$.

The stream function ψ for a uniform flow past a sphere can be calculated from Eq. (7–72) with the radial velocity given in Eq. (8–116a):

$$\begin{aligned} \psi &= \int (v_\omega)\omega^2 \sin\varphi\, d\varphi \\ &= U_\infty \int \left(1 - \frac{3}{2}\frac{R}{\omega} + \frac{1}{2}\frac{R^3}{\omega^3}\right)\omega^2 \cos\varphi\sin\varphi\, d\varphi \\ &= \frac{U_\infty}{2}\left(1 - \frac{3}{2}\frac{R}{\omega} + \frac{1}{2}\frac{R^3}{\omega^3}\right)\omega^2 \sin^2\varphi \end{aligned} \tag{8–120}$$

For the sake of clarity a system of reference has been chosen in which the undisturbed fluid is at rest and the sphere is moving through it. In this case a uniform flow $u = U_\infty$ is superimposed on the fluid, and the stream function for the uniform flow is given by Eq. (7–118):

$$\psi_{U_\infty} = \frac{1}{2} U_\infty \omega^2 \sin^2 \varphi \tag{7–118}$$

Hence the stream function for a sphere moving through a viscous fluid is the difference of Eqs. (7–118) and (8–120), i.e.,

$$\psi = -\frac{3}{4} U_\infty R\omega \left(1 - \frac{R^2}{3\omega^2}\right) \sin^2 \varphi \tag{8–121}$$

In Fig. 8.16 the streamlines for a sphere moving through a viscous fluid are shown. It is interesting to compare the stream function in Eq. (8–120) with the corresponding inviscid fluid stream function given in Eq. (7–167). The great difference between the two cases is due to the presence of viscous forces in the present problem.

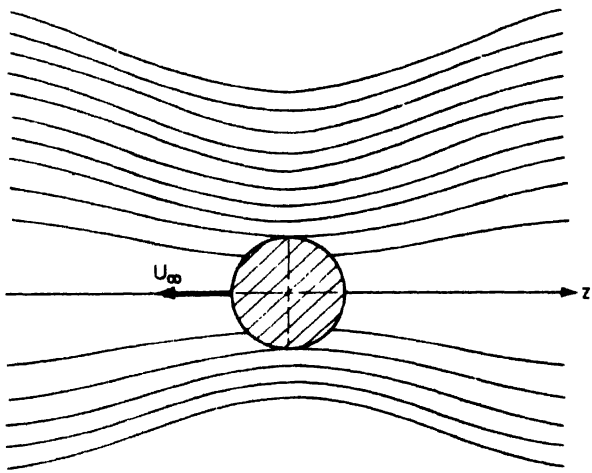


Figure 8.16

An improvement of Stokes' solution was made by Oseen in which the inertia-force terms were partly taken into consideration. Although the mathematical details are beyond the scope of this text, the improved formula for the coefficient of drag is

$$C_D = \frac{24}{\mathrm{Re}} \left(1 + \frac{3}{16} \mathrm{Re}\right) \tag{8–122}$$

Equation (8–122) is plotted in Fig. 8.15b together with experimental results and Stokes' formula. It is seen that Oseen's formula is good

up to Re = 5. The corresponding streamlines of Oseen's solution for a sphere moving in a viscous fluid are shown in Fig. 8.17.

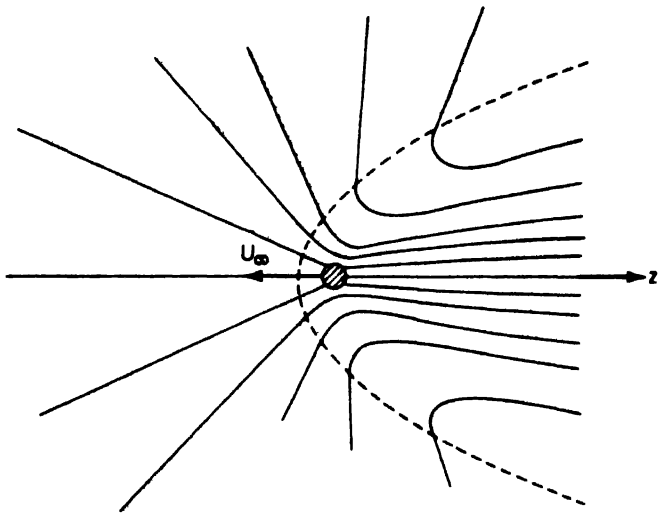


Figure 8.17

8.8. Unsteady Motion of a Flat Plate

In the previous four sections we have shown examples of exact solutions of the Navier-Stokes equations for steady flows. We now proceed to calculate an unsteady motion of a flat plate. The simplest flow of this class is that which results when the motion of the plate is started impulsively from rest. Consider an infinite plate in an infinite fluid initially at rest and then suddenly given a constant velocity U_∞ in the x-direction (Fig. 8.18a). Since the streamlines are parallel to the plate, the two components of velocity perpendicular to the plate are zero, and all the derivatives of u are zero except those in y and t. The Navier-Stokes equations [Eq. (5–19)] without any body force in this case become

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{8–123}$$

All three pressure gradients in Eq. (5–19) are zero because the plate is situated in an infinite fluid where the pressure is constant everywhere. Equation (8–123) is the same type of differential equation as

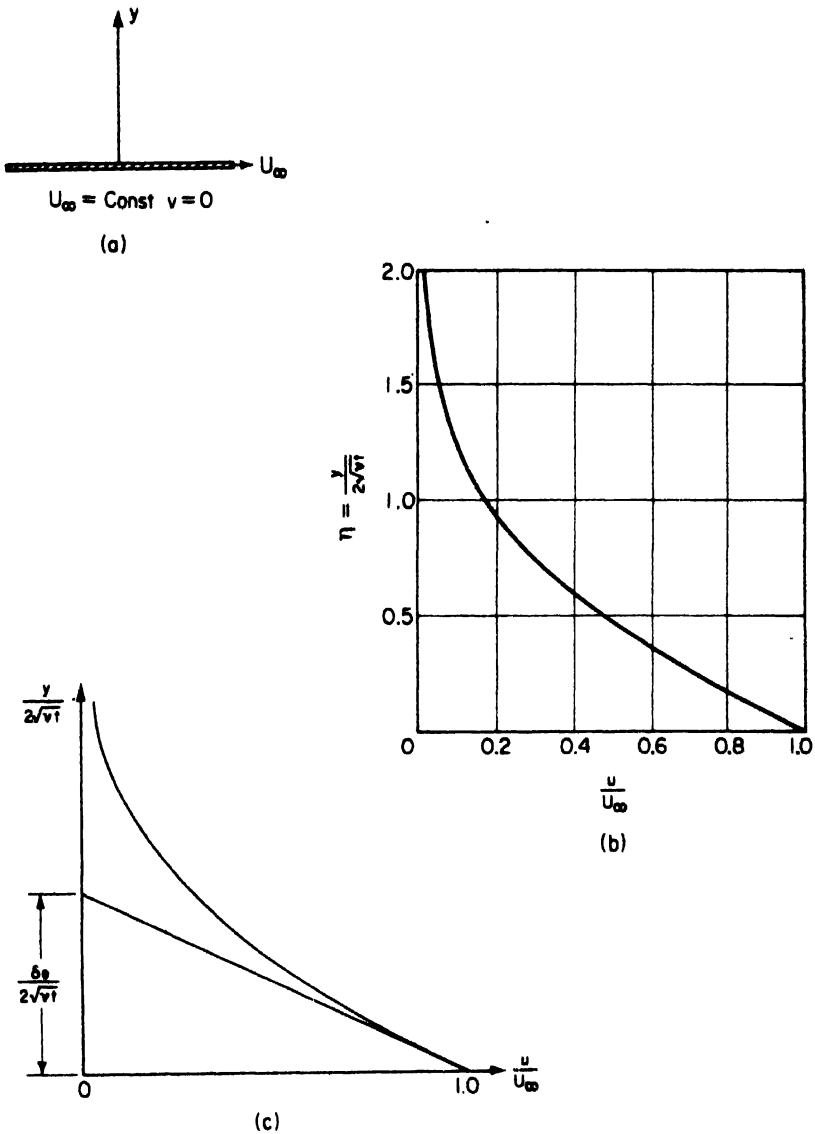


Figure 8.18

those of heat conduction, diffusion, etc. The initial and boundary conditions are

$$\left.\begin{array}{lll} u = 0 & \text{when} \quad t = 0 & \text{for all } y \\ u = U_\infty & \text{at} \quad y = 0 & \\ u = 0 & \text{at} \quad y = \infty & \end{array}\right\} \quad \text{when} \quad t > 0 \tag{8-124}$$

The partial differential equation, Eq. (8–123), can be transformed into an ordinary differential equation by the principle of similarity. The principle of similarity of the velocity profile can be written as

$$\frac{u}{U_\infty} = f(\eta) \tag{8-125}$$

and

$$\eta = \frac{1}{2} \nu^n y^\alpha t^\beta \tag{8-126}$$

where n, α, and β are numbers to be determined and the factor $\frac{1}{2}$ is inserted for convenience. Substituting Eqs. (8–125) and (8–126) into the partial differential equation, Eq. (8–123), one obtains

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = U_\infty \frac{\partial f}{\partial \eta}\left(\frac{\beta}{2} \nu^n y^\alpha t^{\beta-1}\right)$$

$$= U_\infty \beta t^{-1} \eta \frac{\partial f}{\partial \eta} \tag{8-127a}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = U_\infty \frac{\partial f}{\partial \eta}\left(\frac{\alpha}{2} \nu^n y^{\alpha-1} t^\beta\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial \eta}\left[U_\infty \frac{\partial f}{\partial \eta}\left(\frac{\alpha}{2} \nu^n y^{\alpha-1} t^\beta\right)\right]\frac{\partial \eta}{\partial y}$$

$$= U_\infty \left[\left(\frac{\alpha^2}{4} \nu^{2n} y^{2(\alpha-1)} t^{2\beta}\right)\frac{\partial^2 f}{\partial \eta^2} + \frac{\alpha}{2}(\alpha - 1)\nu^n y^{\alpha-2} t^\beta \frac{\partial f}{\partial \eta}\right] \tag{8-127b}$$

or

$$U_\infty \beta t^{-1} \eta \frac{\partial f}{\partial \eta} = U_\infty \left[\left(\frac{\alpha^2}{4} \nu^{2n+1} y^{2(\alpha-1)} t^{2\beta}\right)\frac{\partial^2 f}{\partial \eta^2}\right.$$

$$\left. + \frac{\alpha}{2}\left(\alpha - 1\right) \nu^{n+1} y^{\alpha-2} t^\beta \frac{\partial f}{\partial \eta}\right] \tag{8-128}$$

In order for Eq. (8–128) to become an ordinary differential equation, the power of t and y on both sides of the equation must be identical. Hence,

$$\alpha = 1, \qquad \beta = -\tfrac{1}{2}, \qquad n = -\tfrac{1}{2}$$

and Eq. (8–128) reduces to

$$\frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} = 0 \tag{8-129}$$

with the similarity parameter

$$\eta = \frac{y}{2\sqrt{\nu t}} \tag{8-130}$$

The solution of Eq. (8-129) is

$$f(\eta) = C_1 \int_0^\eta e^{-\eta^2}\, d\eta + C_2 \tag{8-131}$$

With the boundary conditions given in (8-124) and the similarity parameter in Eq. (8-130), the constants of integration C_1 and C_2 are determined as follows:

$$C_2 = 1; \qquad C_1 = -\frac{1}{\int_0^\infty e^{-\eta^2}\, d\eta} = -\frac{2}{\sqrt{\pi}}$$

The velocity distribution, Eq. (8-125), is therefore

$$\frac{u}{U_\infty} = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta^2}\, d\eta$$

$$= (1 - \operatorname{erf} \eta) \tag{8-132}$$

The integral in Eq. (8-132) is called the error function or the probability integral (erf η) which has been tabulated in many mathematical tables[3]. Selected values of the error function are given in Table 8.1 and the velocity distribution as given in Eq. (8-132) is plotted in Fig. 8.18b in terms of the similarity parameter η. It is seen from Fig. 8.18a that the distance y from the plate at $u = U_\infty$ extends asymptotically to infinity. In order to define a viscous layer* from the surface of the plate to a distance $y = \delta_e$ where the viscous stresses are negligible, a constant $(\tau_{yx})_{y=0}$ is assumed (Fig. 8.18c). This implies that

$$(\tau_{yx})_{y=0} = -\mu \left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu \frac{U_\infty}{\delta_e}$$

or

$$\frac{\delta_e}{2\sqrt{\nu t}} = -\frac{1}{(\partial f/\partial \eta)_{\eta=0}} = \frac{1}{[2/\sqrt{\pi}]\,[e^{-\eta^2}]_{\eta=0}} = \frac{\sqrt{\pi}}{2}$$

Therefore the thickness of the boundary layer is

$$\delta_e = \sqrt{\pi}\ \sqrt{\nu t} \tag{8-133}$$

We may also define the mean boundary layer thickness δ_m so that the total quantity of fluid is the same as the actual quantity (Fig. 8.19) which is expressed mathematically in Eq. (8-136). At any given time the total quantity of fluid moving above the moving plate is

*See Chap. Nine for a more comprehensive discussion.

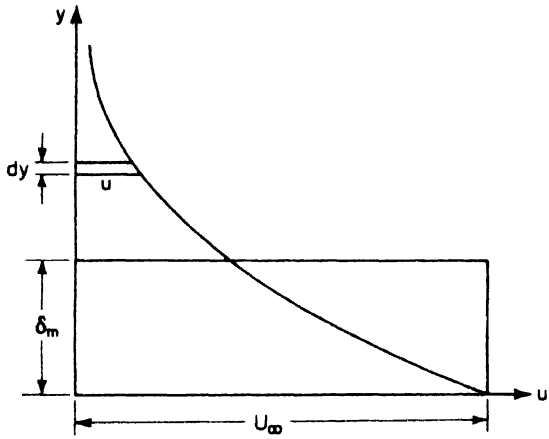


Figure 8.19

$$\int_0^\infty u\,dy \tag{8-134}$$

If $\delta_m U_\infty$ represents the actual quantity of fluid moving above the moving plate, then

$$U_\infty \delta_m = \int_0^\infty u\,dy \tag{8-135}$$

Solving for δ_m, we have

$$\delta_m = 2\sqrt{\nu t}\int_0^\infty f(\eta)\,d\eta = 2\sqrt{\nu t}\int_0^\infty [1 - \operatorname{erf}\eta]\,d\eta \tag{8-136}$$

The integral in Eq. (8-136) can be evaluated by integration by parts,

$$\int_0^\infty \operatorname{erf}\eta\,d\eta = \int_0^\infty \phi(\eta)\,d\eta = \lim_{m\to\infty}[\eta\phi(\eta)]_0^m - \int_0^\infty \eta\phi'(\eta)\,d\eta$$

but

$$\phi(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\eta^2}\,d\eta = \frac{2}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2}\right) = 1$$

$$\phi'(\eta) = \frac{2}{\sqrt{\pi}}\frac{d}{d\eta}\int_0^\eta e^{-\eta^2}\,d\eta = \frac{2}{\sqrt{\pi}}e^{-\eta^2}$$

Hence,

$$\begin{aligned}\int_0^\infty \phi(\eta)\,d\eta &= \lim_{m\to\infty}[m] - \frac{2}{\sqrt{\pi}}\int_0^\infty \eta e^{-\eta^2}\,d\eta \\ &= \lim_{m\to\infty}\left[m + \frac{1}{\sqrt{\pi}}(e^{-m^2} - 1)\right]\end{aligned} \tag{8-137}$$

Substituting Eq. (8–137) into Eq. (8–136), the mean boundary layer thickness is

$$\delta_m = 2\sqrt{\nu t} \lim_{m\to\infty} \left\{ m - \left[m + \frac{1}{\sqrt{\pi}}(e^{-m^2} - 1) \right] \right\}$$

$$= \frac{2}{\sqrt{\pi}} \sqrt{\nu t} \tag{8–138}$$

According to Eqs. (8–133) and (8–138) the value of δ is

$$\frac{2}{\sqrt{\pi}} \sqrt{\nu t} < \delta < \sqrt{\pi}\, \sqrt{\nu t}$$

Table 8.1. VALUES OF THE ERROR FUNCTION

η	erf η
0.0	0.0
0.01	0.01128
0.05	0.05637
0.1	0.11246
0.2	0.22270
0 3	0.32863
0.4	0.42839
0.5	0.52050
0.6	0.60386
0.7	0.67780
0.8	0.74210
0.9	0.79691
1.0	0.84270
1.1	0.88021
1.2	0 91031
1.3	0.93401
1.4	0.95229
1.5	0.96611
1.6	0.97635
1.7	0.98379
1.8	0.98909
1.9	0.99279
2.0	0.99532
2.1	0.99702
2.2	0.99814
2.3	0.99886
2.4	0.99931
2.5	0.99959
2.6	0.99976
2.7	0.99987
2.8	0.99992
2.9	0.99996
3.0	0.99998
∞	1.00000

The important conclusion of this result is that the thickness of the boundary layer is proportional to the square root of the kinematic viscosity and time.

Illustrative Example 8.5. Determine the relation between the boundary layer thickness δ and the velocity of propagation of δ perpendicular to the plate. Find the approximate magnitude of δ.

Solution. Let $\delta = c\sqrt{\nu t}$ where $(2/\sqrt{\pi}) < c < \sqrt{\pi}$. The velocity of propagation of δ perpendicular to the plate is

$$v_p = \frac{d\delta}{dt} = \frac{c}{2}\sqrt{\nu}\frac{1}{\sqrt{t}} = \frac{c^2}{2}\frac{\nu}{\delta} \tag{a}$$

Hence,

$$v_p \sim \frac{1}{\delta} \tag{b}$$

which is a general result for all transfer phenomena. If $\nu = 1.64 \times 10^{-4}$ ft^3/sec for air at 70 F is used, and $v_p = 1$ ft/sec is assumed, the thickness of the boundary layer is computed from Eq. (a) as the value

$$\delta = 1.64 \times 10^{-4} \text{ ft} = 1.366 \times 10^{-3} \text{ in.}$$

where $c^2 = 2$ is used.

Illustrative Example 8.6. An approximate solution for a plane surface which suddenly starts moving in its own plane with a constant velocity U_∞ is assumed to be

$$u = U_\infty e^{-ay/\delta} \tag{a}$$

where a is an arbitrary constant. Calculate the boundary layer thickness, δ.

Solution. The rate change of momentum per unit area in the flow direction in this case can be written as

$$\frac{d}{dt}\int_0^\infty \rho u \, dy \tag{b}$$

Substituting the velocity profile from Eq. (a) into Eq. (b), we obtain

$$\rho U_\infty \frac{d}{dt}\int_0^\infty e^{-ay/\delta}\, dy = \rho U_\infty \frac{d}{dt}\left[-\frac{\delta}{a}e^{-ay/\delta}\right]_0^\infty$$

$$= \rho \frac{U_\infty}{a}\frac{d\delta}{dt} \tag{c}$$

The expression given in Eq. (c) must be equal to the shearing force on the plate per unit area, i.e.,

$$\tau_0 = -\mu\left(\frac{\partial u}{\partial y}\right)_0 = a\mu\frac{U_\infty}{\delta} = \rho\frac{U_\infty}{a}\frac{d\delta}{dt} \tag{d}$$

Solving Eq. (d), we find

$$\frac{\delta^2}{2} = a^2\nu t \tag{e}$$

Hence the thickness of the boundary layer is

$$\delta = a\sqrt{2\nu t} \tag{f}$$

It is interesting to note that Eq. (f) is identical to Eq. (8–138) if $a = \sqrt{2/\pi}$ is assumed.

NOMENCLATURE FOR CHAPTER EIGHT

C_D	coefficient of drag, Eq. (8-119)
C_f	frictional coefficient, Eq. (8-36)
D	total drag on the sphere, Eq. (8-118)
f	Darcy's resistance coefficient of pipe flow, Eqs. (8-55), (8-56), and (8-57)
$f(\eta)$	nondimensional velocity parameter, Eq. (8-125)
F	Froude number, Eq. (8-3)
g	gravitational acceleration, Eq. (8-74)
h	distance between two parallel walls, Eqs. (8-17) and (8-77)
l	(1) typical linear dimension of the body, Eq. (8-1) (2) length of the pipe, Eq. (8-46)
m	mass of a single molecule, Eq. (8-8)
N	number of molecules per unit volume, Eq. (8-8)
p	pressure, Eq. (8-1)
$\mathbf{q}$	velocity vector, Eq. (8-1)
Q	flow rate, Eqs. (8-25) and (8-51)
R	(1) radius of the pipe, Eq. (8-43) (2) radius of the sphere, Eq. (8-108)
Re	Reynolds number, Eqs. (8-4) and (8-54)
r, θ, z	cylindrical coordinates
t	time, Eqs. (8-1) and (8-123)
T	total torque, Eq. (8-80)
u_{av}	average velocity, Eqs. (8-23) and (8-33)
u_{max}	maximum velocity, Eqs. (8-28) and (8-32)
U	velocity of the moving plate, Eq. (8-17)
U_∞	constant velocity of the infinite plate, § 8.8
u, v, w	velocity components in Cartesian coordinates
$v_{z_{av}}$	average velocity in pipe flow, Eq. (8-50)
$(v_z)_{max}$	velocity at the center line of the pipe, Eq. (8-49)
v_r, v_θ, v_z	velocity in cylindrical coordinates
$v_\omega, v_\varphi, v_\theta$	velocity in spherical coordinates

X_0	gravitational force per unit mass, Eq. (8-1)
x, y, z	Cartesian coordinates
α	nondimensional pressure gradient, Eq. (8-21)
δ	boundary layer thickness, Eqs. (8-133) and (8-138)
η	similarity parameter, Eq. (8-126)
λ	mean free path, Eq. (8-8)
μ	coefficient of viscosity, Eqs. (8-2) and (8-12)
ν	kinematic viscosity, Eq. (8-4)
ρ	density, Eq. (8-2)
τ_0	wall shearing stress, Eq. (8-100)
τ_{rz}	shearing stress in the z-direction, Eq. (8-46)
$\tau_{r\theta}$	shearing stress in the θ-direction, Eq. (8-71)
ψ	stream function, Eq. (8-120)
ω_1, ω_2	angular velocities of the inner and outer cylindrical walls, respectively, Eq. (8-68)
ω, φ, θ	spherical coordinates
$(\ )^*$	nondimensional quantity, Eq. (8-1)

PROBLEMS

8.1. If a body with a linear dimension of 1 ft is moving in a fluid with velocity q_∞, find the Froude number and the Reynolds number for the following cases: (a) when the fluid is air and has $q_\infty = 100$ ft/sec; (b) when the fluid is water and has $q_\infty = 5$ ft/sec.

8.2. Oil flows between two parallel plates (Fig. 8.3), one of which is at rest and the other moves with a velocity U. If the pressure is decreasing in the direction of the flow at a rate of 0.10 lbf/ft^2 per/foot, the dynamic viscosity is 10^{-3} lbf-sec/ft^2, the spacing of the plates is 2 in. and the volumetric flow Q per unit width is 0.15 ft^2/sec, what is the velocity U?

8.3. In Prob. 8.2, if the pressure is increasing in the direction of the flow at a rate of 0.10 lbf/ft^2 per foot, what is the velocity U?

8.4. Water at 70° F flows between two parallel plates (Fig. 8.3), one of which is fixed and the other moves with a velocity U. If the volumetric flow Q per unit width is zero, find the relation between U and dp/dx in terms of the dynamic viscosity μ, and the spacing of the plates, h. Calculate the pressure gradient for $U = \frac{1}{2}$ ft/sec and $h = \frac{1}{2}$ in.

8.5. Oil flows between two parallel plates (Fig. 8.3), one of which is at rest and the other moving with a velocity U. Find the relation between U and dp/dx for the following cases: (a) the wall shearing stress at the fixed plate is zero, and (b) the wall shearing stress at the moving plate is zero.

8.6. Calculate the pressure gradients and shearing stresses of the moving plate of (a) and the fixed plate of (b) in Prob. 8.5 if $U = 1.5$ ft/sec, $h = 1.5$ in., and $\mu = 10^{-3}$ lbf-sec/ft^2.

8.7. For plane Poiseuille flow verify that the shearing stress becomes zero at the middle plane and that

$$\frac{\partial \tau_{yx}}{\partial y} = \frac{dp}{dx}$$

8.8. Water flows between two very large parallel flat plates $\frac{1}{2}$ in. apart. If the average velocity of the water is $\frac{1}{2}$ ft/sec, what is the shearing stress at the lower plate, $\frac{1}{8}$ in. and $\frac{1}{4}$ in. from the lower plate?

8.9. Oil of dynamic viscosity 1.4×10^{-3} lbf-sec/ft^2 flows through a 1 in. diameter pipe with a mean velocity of 2 ft/sec. Calculate the pressure drop which occurs over a length of 125 ft of pipe.

8.10. The velocity along the center line of the Hagen-Poiseuille flow in a 4 in. diameter pipe is 5 ft/sec. If the viscosity of the fluid is 0.67 poises and its specific gravity is 0.92, calculate: (a) the volumetric flow in ft^3/sec; (b) the shearing stress of the fluid at the pipe wall; (c) the local frictional coefficient C_f.

8.11. A viscous fluid moves in a steady flow parallel to the axis in the annular space between two coaxial cylinders of radii r_1 and r_2. Show that the rate of volumetric flow is

$$Q = \frac{\pi r_1^4}{8\mu}\left(\frac{dp}{dx}\right)\left[(n^4 - 1) - \frac{(n^2 - 1)^2}{\ln (n)}\right]$$

where $n = r_2/r_1$.

8.12. If the volumetric flow of oil in Prob. 8.11 is 0.5 ft^3/sec and the dynamic viscosity of the oil is 1.4×10^{-3} lbf-sec/ft^2, calculate: (a) the maximum velocity and the average velocity in the annular space between two coaxial cylinders of radii $r_1 = 2$ in. and $r_2 = 3$ in.; (b) the shearing stresses at the walls of the inner and outer cylinders.

8.13. Oil is filled between two concentric rotating cylinders with radii 5 in. and $5\frac{1}{2}$ in. (see Fig. P. 8.13). Assuming $\mu = 0.005$ lbf-sec/ft^2, calculate the torque required to turn the inner cylinder at a speed of 5 rpm, while the outer cylinder is at rest. Neglect the resistance between the bottoms of the cylinders.

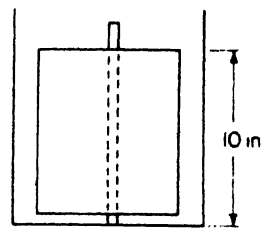


Figure P. 8.13

8.14. Verify that the expression for v_θ and p given in Eqs. (8-69) and (8-70), respectively, are solutions of Eqs. (8-64) and (8-65).

8.15. A rotational speed control device (see Fig. P. 8. 15) consists of a rotating circular flat plate in a bath of oil. The clearance between the plate and the housing is 0.02 in. and the dynamic viscosity is 0.8 poise. What torque is required to rotate the plate at a speed of 50 rpm?

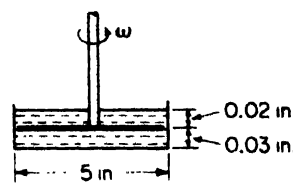


Figure P. 8.15

8.16. A 6 in. diameter shaft rotates at a speed of 500 rpm in a cylindrical sleeve 30 in. long with a 6.025 in. inside diameter. What

power is required to overcome the friction when the clearance contains oil, $\mu = 10^{-3}$ lbf-sec/ft^2.

8.17. The concentric cylinder viscometer (Fig. 8.11) has dimensions: $r_1 = 2.5$ in., $r_2 = 2.52$ in., $b = 0.012$ in., and $h = 6$ in. If a $\frac{1}{4}$ horsepower motor is required to revolve the outer cylinder at 100 rpm, what is the viscosity? Also, calculate the viscosity if the bottom gap is neglected.

8.18. Calculate the following shearing stresses from Prob. 8.17: (a) at the bottom of the inner cylinder; (b) at the circumferential surfaces of the concentric cylinders.

8.19. Figure 8.12b illustrates a slipper bearing. If $U = 2.5$ ft/sec, $\mu = 0.003$ lbf-sec/ft^2, $l/h_1 = 400$, $k = 2.0$, $h_0 = 0.001$ in., plot the pressure distribution in the gap as a function of x/l. Also, calculate: (a) the magnitude of the location of the maximum pressure in the lubricating fluid; (b) the total load that the bearing guide will support; and (c) the resistance force on the bearing guide.

8.20. Plot the velocity distribution of the lubricant between the block and bearing in Prob. 8.19 at the following locations: (a) $h = h_1$; (b) $h = h_2$; and (c) $h = h_0$.

8.21. Transform the velocity components and pressure given in Eqs. (8-111) and (8-112), respectively, into the following velocity components and pressure in a spherical coordinate system:

$$v_\omega = U_\infty \left[\frac{1}{2}\left(\frac{R}{\omega}\right)^3 - \frac{3}{2}\left(\frac{R}{\omega}\right) + 1 \right] \cos\varphi$$

$$v_\varphi = U_\infty \left[\frac{1}{4}\left(\frac{R}{\omega}\right)^3 + \frac{3}{4}\left(\frac{R}{\omega}\right) - 1 \right] \sin\varphi .$$

$$p = p_\infty - \frac{3}{2}\mu U_\infty \frac{R}{\omega^3} \cos\varphi$$

8.22. Verify that the velocity components and pressure in Prob. 8.21 satisfy the Navier-Stokes equations for axially symmetric creeping flow [Eqs. (5-78a) and (5-78b) without inertial and body forces and with $\partial(\ )/\partial\theta = 0$].

8.23. Plot the streamlines for steady flow around a sphere in both a viscous fluid ($\mathrm{Re} < 1$) and an inviscid fluid with the same U_∞.

8.24. Determine the largest possible diameter of a sphere, specific weight $\gamma = 10$ lbf/ft^3, which will settle in water at 50° F and obey Stokes' law. Calculate the settling velocity of the sphere.

8.25. A flat plate is suddenly moved to a velocity U_∞ in oil with $\rho = 1.785$ slugs/ft^3 and $\mu = 1.4 \times 10^{-3}$ lbf-sec/ft^2. Plot the velocity ratio u/U_∞ as a function of distance from the plate for $t = 0.01$ sec, 0.05 sec, and 0.1 sec.

8.26. Calculate the shearing stress at the plate in Prob. 8.25 for $U_\infty = 10$ ft/sec. The thickness of the boundary layer is given in both Eqs. (8-133) and (8-138).

9

THE LAMINAR BOUNDARY LAYER

In Chap. Eight we have shown a small number of special cases of flow problems (§ 8.3, § 8.4, § 8.5, and § 8.8) for which exact solutions of the Navier-Stokes equations can be found. In those cases, the fundamental equations of a viscous fluid happened to be linear because of the simple configurations of the flow patterns and the assumption of incompressible fluid. The solutions obtained in the cases mentioned above are valid for all practical values of viscosity. We have also discussed a case of the approximate solution of the Navier-Stokes equations for very small Reynolds numbers (§ 8.7). In this problem the frictional forces are very large in comparison with the inertial forces and the fundamental equations of a viscous fluid are again linearized, this time by neglecting the convective acceleration. In general, there is no available method yet for finding the solution of the Navier-Stokes equations which is valid for all values of viscosity.

In the present chapter the cases of flow problems for very large Reynolds numbers (very small viscosities of the fluids and moderate or high velocities and dimensions of the bodies) will be discussed. For a steady flow of an incompressible fluid without body forces the Navier-Stokes equation in nondimensional vector notation is given in § 8.1 as follows:

$$(\mathbf{q}^* \cdot \nabla^*)\mathbf{q}^* = -\left(\frac{p_\infty}{\rho q_\infty^2}\right)\nabla^* p^* + \frac{1}{\mathrm{Re}}\nabla^{*2}\mathbf{q} \tag{8-2}$$

where $\mathrm{Re} = q_\infty l/\nu$ is the Reynolds number and the nondimensional quantities denoted by asterisks are defined in Eq. (8-1). If the fluid velocity is increased (for very small viscosity of the fluid) then the Reynolds number becomes progressively larger and there will be a velocity beyond which the second term on the right-hand side of Eq. (8-2) becomes negligibly small relative to the term on the left-hand side of Eq. (8-2). In other words, the inertial force becomes much greater than the viscous force. Under this condition, Eq. (8-2) reduces to Euler's equation without body forces (second-order differential equations reduce to first-order ones). The question now arises: Can the solutions for incompressible inviscid flows be regarded as exact solutions of the Navier-Stokes equations for very large Reynolds numbers? The following simple analogue may exhibit the mathematical structure of the phenomenon which would serve as an answer to the question above.

Let us consider the second-order differential equation

$$\epsilon \frac{d^2 f}{dy^2} + \frac{df}{dy} + f = 0 \tag{9-1}$$

where $f(y)$ is defined throughout the interval $0 \leq y \leq 1$ and the constant coefficient $\epsilon \ll 1$. The boundary conditions which the solution of Eq. (9-1) is to satisfy may be prescribed:

$$f(0) = 0 \tag{9-2a}$$

$$f(1) = 1 \tag{9-2b}$$

The general solution of Eq. (9-1) which satisfies the boundary conditions (9-2) is

$$f = \frac{1}{1 - e^{1-1/\epsilon}} (e^{1-y} - e^{1-y/\epsilon}) \tag{9-3}$$

In analogy with the Navier-Stokes equations with very large Reynolds numbers (or very small viscosity), we consider the case of negligibly small ϵ in Eq. (9-1). In this case Eq. (9-1) reduces to the first-order differential equation

$$\frac{df}{dy} + f = 0 \tag{9-4}$$

The solution of Eq. (9-4) can be easily obtained in the form

$$f = e^{1-y} \tag{9-5}$$

It is seen that the solution in Eq. (9-5) can only satisfy the boundary condition $f(1) = 1$ but the condition $f(0) = 0$ cannot be satisfied. This phenomenon is evident as the order of the second-order differential equation is lowered to the first-order differential equation by assuming $\epsilon = 0$. The number of boundary conditions which the solution of the

reduced differential equation must satisfy is also reduced by one.

In Fig. 9.1 a graphical representation of the solution of Eq. (9–3) is shown. For the case of $e = 0$ the value of $f(y)$, which is represented by the first term in Eq. (9–3), decreases continuously from $f = e$ at $y = 0$ to $f = 1$ at $y = 1$. This curve also represents the solution of Eq. (9–4). However, for any finite value of very small ϵ, the function $f(y)$ first increases rapidly from $f = 0$ at $y = 0$ to a maximum value at a small value of y, then decreases continuously along the curve for $e = 0$. This indicates that (for small values of ϵ) the complete solution of Eq. (9–3) is important only near the vicinity of $y = 0$.

In § 7.8b the boundary condition which the solution of an inviscid potential flow must satisfy is given, namely the vanishing of the normal

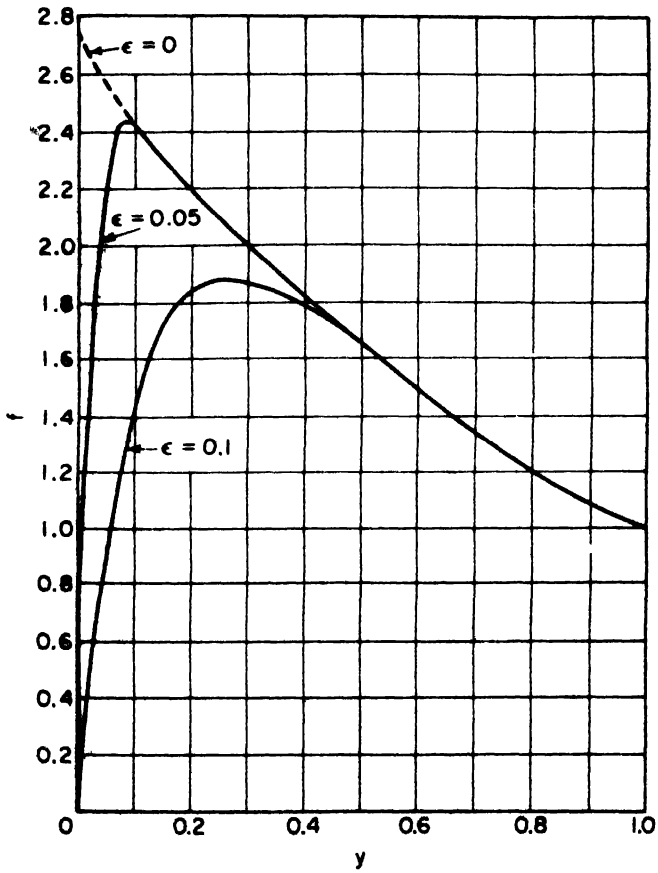


Figure 9.1

velocity at a fixed wall. But the boundary condition which the solution of viscous flow must satisfy in addition to the vanishing normal velocity at a wall is the *no-slip* condition (zero relative tangential velocity at the wall). The boundary conditions for viscous flow are given in § 8.7. In comparing the example above with the solution of the Navier-Stokes equation, it can be deduced that the solution of viscous flow corresponds to the solution given by Eq. (9–3), and the solution of inviscid potential flow resembles the solution given by Eq. (9–5). The conclusion which may be drawn from this analogy is that for very large Reynolds numbers the viscous effects are important only in the immediate vicinity of the boundary of a solid. This immediate region is generally referred to as a *boundary layer* into which the effect of viscosity is confined and in which that effect is predominant. The flow in the region outside the boundary layer where the main stream prevails can be regarded as inviscid. It must be borne in mind that there is no actual sharp region of division between the boundary layer and the main stream. The definition of boundary layer thickness is rather an arbitrary one which will be discussed in § 9.3.

In § 9.1 the properties of the Navier-Stokes equations are discussed. Based on the similarity of the vorticity transport equation and the energy equation, the concept of boundary layer is explained.

In § 9.2 the boundary layer equations in two-dimensional flow are obtained from the Navier-Stokes equations. Limitation of the application of these equations is discussed.

In § 9.3 the exact solution of the boundary layer along a flat plate is presented. The shearing stress at the wall and the thickness of the boundary layer are calculated.

In § 9.4 a simple case of the boundary layer on a surface with pressure gradient is discussed.

In § 9.5 von Karman momentum integral theorems for a boundary layer are discussed, and forms of the momentum integral relation are derived.

In § 9.6 the solution of the boundary layer equations by the momentum integral method is discussed and the limitation of the method is explained. A simplified solution of the momentum integral equation is also given.

In § 9.7 the application of the momentum integral equation to the case of axially symmetrical flow is discussed.

In § 9.8 the mathematical criterion, physical examples, and the method of prediction of boundary layer separation are presented.

In § 9.9 methods of boundary layer control are discussed, and separation prevention by boundary layer suction is presented.

9.1. *Properties of the Navier-Stokes Equations—Boundary Layer Concept*

The mathematical example which we have discussed in the preceding section clearly indicates that the solution of the potential flow cannot satisfy both the boundary conditions for a viscous flow. The analogy further indicates that the solution of the Navier-Stokes equations for very large Reynolds numbers are important only in the immediate vicinity of a solid wall. Hence, beyond the immediate vicinity of a solid wall the solution of the Navier-Stokes equations is replaced by the solution of Euler's equations. Now we shall illustrate some properties of the Navier-Stokes equations which will lead to the basic concept of boundary layer theory.

If the curl of the Navier-Stokes equations [Eq. (5–19)] is taken and the body force is neglected, we have

$$\frac{D\boldsymbol{\Omega}}{Dt} - (\boldsymbol{\Omega} \cdot \nabla)\mathbf{q} = \nu\nabla^2\boldsymbol{\Omega} \tag{9–6}$$

where $\boldsymbol{\Omega} = \nabla \times \mathbf{q}$ [Eq. (3–7)]. Equation (9–6) is referred to as the vorticity transport equation. The term on the left-hand side of Eq. (9–6) signifies the local and convective variation of the vortex strength, and the term on the right-hand side, the rate of dissipation of vorticity through friction. In the case of two-dimensional flow in the x-y plane the vorticity transport equation becomes

$$\frac{D\Omega_z}{Dt} = \nu\nabla^2\Omega_z \tag{9–7}$$

where $\Omega_z = \partial v/\partial x - \partial u/\partial y$. Equation (9–7) together with the equation of continuity, Eq. (5–9b), forms a system of two equations for the two unknowns u and v. In nondimensional form Eq. (9–7) can be written as

$$\frac{D\Omega_z^*}{Dt^*} = \frac{1}{\mathrm{Re}}\nabla^{*2}\Omega_z^* \tag{9–8}$$

where $\Omega_z^* = \Omega_z/\Omega_0$ and the other nondimensional quantities are given in Eq. (8–1).

If the frictional heat generated in the fluid is neglected the energy equation, Eq. (5–34), of an incompressible fluid is given by

$$\rho C_\tau \frac{D\theta}{Dt} = k\nabla^2\theta \tag{9–9}$$

where $\theta = T - T_\infty$. In nondimensional form Eq. (9–9) becomes

$$\frac{D\theta^*}{Dt^*} = \frac{\kappa}{\mathrm{Re}\,\mathrm{Pr}} \nabla^{*2}\theta^* \tag{9-10}$$

where $\theta^* = (T - T_\infty)/(T_0 - T_\infty)$ and $\kappa = C_p/C_v$. Equation (9-9) may be used to describe the temperature distribution around a heated body at a temperature T_0 immersed in a fluid stream with a temperature T_∞ (Fig. 9.2). However, from the experience of heat conduction, we know that if the surrounding fluid is moving slowly around a heated body (fluid at rest would be the limiting case), the heat would be transferred from the body outwards in all directions, far into the fluid. On the other hand, if the fluid is flowing at high velocities (Re is large) only a relatively thin layer of the fluid near the body and a narrow wake (Fig. 9.2) will be influenced by the heated body.

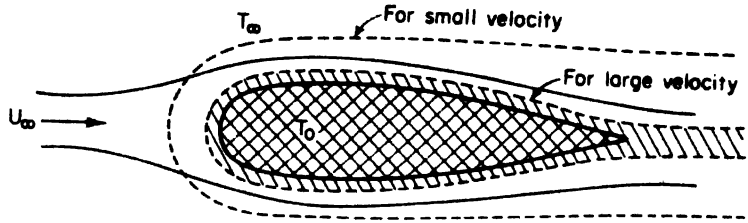


Figure 9.2

Now let us compare the energy equation, Eq. (9-10), with the vorticity transport equation, Eq. (9-8). These two differential equations are exactly the same except the constant in Eq. (9-10) differs by a factor of κ/Pr from that of Eq. (9-8). The values of κ/Pr for most gases are in the order of magnitude of unity. According to the temperature distribution phenomenon mentioned above, we can describe in the same manner the vorticity which is represented by Eq. (9-8). For small Reynolds numbers the vorticity distribution is noticeable at a large distance from the body (see § 8.7), whereas for large Reynolds numbers the vorticity distribution is limited to the immediate neighborhood of the body and in a wake behind the body. Thus, for large Reynolds numbers, one expects the following solution of Navier-Stokes equations:

(a) In the region of a thin boundary layer near the vicinity of a wall where $\Omega \neq 0$, viscous forces are of the same order of magnitude as inertial forces. The fluid velocity varies from zero at the wall to the value given by potential flow at the outer edge of the boundary layer..

(b) In the region outside of this thin boundary layer where $\Omega = 0$, viscous forces can be completely neglected in comparison with the inertial forces; hence only potential flow exists.

The concept of a boundary layer which divides the field of flow into the potential flow and the essentially viscous flow permits the simplification of the Navier-Stokes equations to such an extent that it becomes possible to treat many important practical problems.

9.2. *The Boundary Layer Equations in Two-Dimensional Flow*

The viscosity of water, air, and many other fluids encountered in engineering practice is very small, or the Reynolds number is very large. From the discussion in the preceding section we have deduced that for a flow with very large Reynolds numbers the effects of viscosity are confined to a very thin layer in the immediate vicinity of a solid boundary. This led Prandtl* in 1904 to introduce the concept of a boundary layer and analyze the flow in the boundary layer. Subsequently, the boundary layer equations have been well investigated for many engineering problems, and the results play a very important role in the fluid dynamics of viscous fluids. Since no completely mathematical theory on the boundary layer has been developed, the following treatment is rather qualitative and must be based ultimately on direct experiment and observation. The mathematical procedure for reducing the Navier-Stokes equations to the boundary layer equations is that of finding the first-order approximation of the former equations when the Reynolds number is very large.

Let us consider the flow around a wedge profile submerged in a fluid of very small viscosity (Fig. 9.3). At the leading stagnation point O, the thickness of the boundary layer is zero and it grows slowly toward the rear of the wedge. The pattern of streamlines and the velocity distribution deviate only slightly from those in potential flow with the exception of the immediate vicinity of the wall. Within a very thin boundary layer of thickness δ a large velocity gradient exists, i.e., the velocity increases from zero at the wall to the value of potential flow at the edge of the boundary layer. If the x-axis of the coordinate system coincides with the wall of the wedge and the y-axis is perpendicular to it, the Navier-Stokes equations, Eq. (5–19), without body force for two-dimensional flow are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{9-11a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{9-11b}$$

*NACA Tech. Memo. 452 (1928).

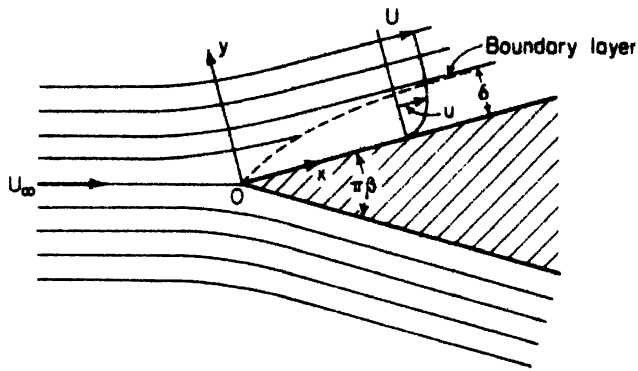


Figure 9.3

The continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9-11c}$$

According to the exact solution of the unsteady motion of a flat plate (§ 8.8), we found that the thickness of the boundary layer δ is proportional to the square root of the kinematic viscosity ν which is indeed very small. For this reason $\delta \ll x$ except near the stagnation point O where the boundary layer begins. In order to compare the order of magnitude of the individual terms in Eqs. (9–11) it is more advantageous to put Eqs. (9–11) in nondimensional form by letting

$$x^* = \frac{x}{l}, \qquad y^* = \frac{y}{\delta}, \qquad u^* = \frac{u}{U}, \qquad v^* = \frac{v}{V}$$
$$t^* = \frac{t}{l/U}, \qquad p^* = \frac{p}{p_\infty} \tag{9-12}$$

where l, δ, U, V, and p_∞ are certain reference values of the corresponding quantities x, y, u, v, and p. The nondimensional quantities are all of the order of unity. From the continuity equation, Eq. (9–11c), it follows that

$$\frac{U}{l}\frac{\partial u^*}{\partial x^*} + \frac{V}{\delta}\frac{\partial v^*}{\partial y^*} = 0 \tag{9-13a}$$

Integrating the equation above we obtain, after making use of the condition $(v^*)_{y^*=1} = 1$,

$$\frac{V}{U} = -\frac{\delta}{l}\int_0^1 \frac{\partial u^*}{\partial x^*}\, dy^* \tag{9-13b}$$

Since the integral in Eq. (9–13b) is of the order of unity, the velocity ratio V/U is of the order of δ/l. Hence, $V \ll U$. The nondimensional form of Eq. (9–11a) is obtained by inserting the nondimensional quantities from Eq. (9–12) into Eq. (9–11a),

$$\frac{U^2}{l}\frac{\partial u^*}{\partial t^*} + \frac{U^2}{l}u^*\frac{\partial u^*}{\partial x^*} + \frac{VU}{\delta}v^*\frac{\partial u^*}{\partial y^*} = -\frac{p_\infty}{\rho l}\frac{\partial p^*}{\partial x^*} + \frac{\nu U}{l^2}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{l^2}{\delta^2}\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

or

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + \frac{V}{U}\frac{l}{\delta}v^*\frac{\partial u^*}{\partial y^*} = -\frac{p_\infty}{\rho U^2}\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{l^2}{\delta^2}\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

$$1 \qquad 1 \qquad \delta\frac{1}{\delta} \qquad 1 \qquad\qquad \delta^2 \quad 1 \qquad \frac{1}{\delta^2} \tag{9–14}$$

Similarly, the nondimensional form of Eq. (9–11b) is

$$\frac{V}{U}\frac{\partial v^*}{\partial t^*} + \frac{V}{U}u^*\frac{\partial v^*}{\partial x^*} + \frac{V^2}{U^2}\frac{l}{\delta}v^*\frac{\partial v^*}{\partial y^*} =$$

$$\delta \qquad \delta \qquad\qquad \delta^2\ \frac{1}{\delta}$$

$$-\frac{p_\infty}{\rho U^2}\frac{\partial p^*}{\partial y^*}\frac{l}{\delta} + \frac{1}{\mathrm{Re}}\frac{V}{U}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{l^2}{\delta^2}\frac{\partial^2 v^*}{\partial y^{*2}}\right) \tag{9–15}$$

$$\delta^2 \quad \delta \quad 1 \qquad \frac{1}{\delta^2}$$

The order of magnitude of various terms in Eqs. (9–14) and (9–15) is indicated under the respective equation. Since the Reynolds number is inversely proportional to ν it is of the order $1/\delta^2$. This is reasonable because the last viscous-force term in Eq. (9–14) is of the same order of magnitude as the inertial term. By neglecting the terms of the order of δ and smaller from Eqs. (9–14) and (9–15) and reverting to dimensional variables, we obtain the equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{9–16a}$$

$$\frac{\partial p}{\partial y} = 0 \tag{9–16b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9–16c}$$

Equations (9–16) are generally known as Prandtl's boundary layer equations with boundary conditions

$$\begin{aligned} y = 0: &\qquad u = v = 0 \\ y = \infty: &\qquad u = U(x, t) \end{aligned} \tag{9–17}$$

According to Eq. (9–16b) the pressure distribution is a function of x only, i.e., for a given x, p is constant through the boundary layer. In other words, the pressure throughout the boundary layer has the same value of p as in the potential flow. Hence the boundary layer equations have only two variables, u and v, instead of the former three variables (u, v, p) in the Navier-Stokes equations. This is a great simplification in the solution of the differential equations. Now, let U be the velocity in the potential flow (just outside the boundary layer), then the Euler equation reduced from Eq. (9–16a) [or Eq. (6-5b)] becomes (since, $\partial u/\partial y = v = 0$ at $y = \delta$)

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{dp}{dx} \tag{9–18}$$

Introducing the expression above for dp/dx into Eqs. (9–16), we have

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{9–19a}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9–19b}$$

Although the equations above are derived for rectilinear flow, they hold for curved flow if the radius of curvature is large compared to the boundary layer thickness. The boundary layer for axially symmetric flow will be discussed in § 9.7.

The integration of Eqs. (9–19) can be simplified if we can reduce the number of variables by introducing the stream function:

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \tag{9–20}$$

The continuity equation, Eq. (9–19b), is satisfied automatically by Eq. (9–20). In terms of the variable ψ the boundary layer equation, Eq. (9–19a), becomes

$$\frac{\partial^2 \psi}{\partial t\,\partial y} + \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x\,\partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu\frac{\partial^3 \psi}{\partial y^3} + \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} \tag{9–21}$$

The boundary conditions are

$$\begin{aligned} y = 0: &\qquad \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0 \\ y = \infty: &\qquad \frac{\partial \psi}{\partial y} = U(x, t) \end{aligned} \tag{9–22}$$

An exact discussion of Eq. (9–21) was first given by Blasius in 1908 for the case of a flat plate (U = Constant) and for steady motion ($\partial/\partial t = 0$).

9.3. The Boundary Layer Along a Flat Plate

a. The Blasius Solution

Consider the flow of an incompressible viscous fluid past a thin flat plate which is placed in the direction of a uniform velocity U_∞ (Fig. 9.4). The plate is of infinite length. Hence the problem is one of two-dimensional motion which can be analyzed by using the Prandtl boundary layer equations. Let the origin of the coordinates be at the leading edge of the plate, the x-axis be the direction of the uniform stream, and the y-axis, normal to the plate. The Prandtl boundary layer equations in the case under consideration are

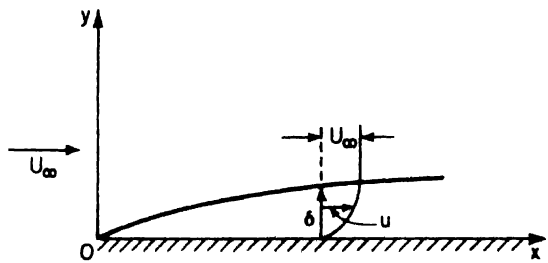


Figure 9.4.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{9-23a}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9-23b}$$

In addition to Eq. (9-23) the boundary conditions

$$\begin{aligned} y = 0: &\qquad u = v = 0 \\ y = \infty: &\qquad u = U_\infty \end{aligned} \tag{9-24}$$

are to be satisfied by $u(x, y)$ and $v(x, y)$.

The characteristic parameters in these problems are U_∞, ν, x, y, i.e., our flow problem is defined completely by these parameters. In accordance with the procedure of the law of similarity, the velocity profile may be assumed as

$$\frac{u}{U_\infty} = F(y, x, \nu, U_\infty) = F(\eta) \tag{9-25}$$

According to the exact solution of the unsteady motion of a flat plate (§ 8.8), we found that

$$\delta \sim \sqrt{\nu t} \sim \sqrt{\frac{\nu x}{U_\infty}} \tag{9-26}$$

The time t is interpreted here as the time required for a fluid particle to travel a distance x with velocity U_∞. Hence the nondimensional distance parameter may be expressed by

$$\eta = \frac{y}{\delta} = \frac{y}{\sqrt{\nu x/U_\infty}} \tag{9-27}$$

which is in agreement with Eq. (9–25).

The stream function can now be obtained in terms of the velocity function of Eq. (9–25):

$$\psi = \int u\, dy = \frac{U_\infty}{\sqrt{U_\infty/\nu x}} \int F(\eta)\, d\eta = \sqrt{U_\infty \nu x}\, f(\eta) \tag{9-28}$$

The velocity components and their derivatives (differentiations with respect to η are denoted by primes) are

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = U_\infty f'(\eta) \tag{9-29a}$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}} f(\eta) - \sqrt{U_\infty \nu x}\left(-\frac{\eta}{2x}\right) f'(\eta)$$

$$= \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}} [\eta f'(\eta) - f(\eta)] \tag{9-29b}$$

$$\frac{\partial u}{\partial x} = \frac{\partial^2 \psi}{\partial x\, \partial y} = -\frac{U_\infty}{2}\frac{\eta}{x} f''(\eta) \tag{9-29c}$$

$$\frac{\partial u}{\partial y} = U_\infty \sqrt{\frac{U_\infty}{\nu x}} f''(\eta) \tag{9-29d}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{\nu x} f'''(\eta) \tag{9-29e}$$

where

$$\frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{\nu x}}, \qquad \frac{\partial \eta}{\partial x} = -\frac{1}{2}\frac{y}{x}\sqrt{\frac{U_\infty}{\nu x}} = -\frac{1}{2}\frac{\eta}{x}$$

Substituting Eqs. (9–29) into Eq. (9–23a), we have

$$-\frac{U_\infty^2}{2}\frac{\eta}{x} f' f'' + \frac{U_\infty^2}{2x}[\eta f' - f] f'' = \frac{U_\infty^2}{x} f''' \tag{9-30a}$$

After simplification, Eq. (9–30) becomes

$$2\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} = 0 \tag{9-30b}$$

The boundary conditions from Eq. (9–24) in terms of f and η are

$$\begin{aligned} \eta = 0: \qquad & f = 0, \qquad \frac{df}{d\eta} = 0 \\ \eta = \infty: \qquad & \frac{df}{d\eta} = 1 \end{aligned} \tag{9–31}$$

Equation (9–30b) is a third-order nonlinear differential equation and no closed form solution has been found. Blasius in 1908 obtained the solution in the form of power series expansion about $\eta = 0$.

$$f = A_0 + A_1\eta + \frac{A_2}{2!}\eta^2 + \frac{A_3}{3!}\eta^3 + \cdots \tag{9–32a}$$

$$f' = A_1 + A_2\eta + \frac{A_3}{2!}\eta^2 + \frac{A_4}{3!}\eta^3 + \cdots \tag{9–32b}$$

$$f'' = A_2 + A_3\eta + \frac{A_4}{2!}\eta^2 + \frac{A_5}{3!}\eta^3 + \cdots \tag{9–32c}$$

$$f''' = A_3 + A_4\eta + \frac{A_5}{2!}\eta^2 + \frac{A_6}{3!}\eta^3 + \cdots \tag{9–32d}$$

Applying the boundary conditions [Eq. (9–31)] at $\eta = 0$ to Eqs. (9–32), we obtain

$$A_0 = 0; \qquad A_1 = 0 \tag{9–33a}$$

Substituting Eqs. (9–32) for f, f'', and f''' into Eq. (9–30b), we find

$$2A_3 + (2A_4)\eta + (A_2^2 + 2A_5)\frac{\eta^2}{2!} + (4A_2A_3 + 2A_6)\frac{\eta^3}{3!} + \cdots = 0$$

In order for the equation above to be true the coefficients of the various powers of η must be identically equal to zero. Hence,

$$\begin{aligned} A_3 &= A_4 = A_6 = A_7 = 0 \\ A_5 &= -\frac{A_2^2}{2} \\ A_8 &= \tfrac{11}{4}A_2^3 \end{aligned} \tag{9–33b}$$

Substituting Eqs. (9–33a) and (9–33b) into Eq. (9–32a) the series for f in terms of η and A_2 is obtained:

$$f = \frac{A_2}{2!}\eta^2 - \frac{1}{2}\frac{A_2^2}{5!}\eta^5 + \frac{1}{4}\frac{11A_2^3}{8!}\eta^8 - \frac{1}{8}\frac{375A_2^4}{11!}\eta^{11} + \frac{1}{16}\frac{27{,}897}{14!}A_2^5\eta^{14} - \cdots \tag{9–34}$$

Equation (9–34) satisfies the two boundary conditions at $\eta = 0$, and the constant A_2 will be determined from the boundary condition at $\eta = \infty$ [Eq. (9–31)].

Now Eq. (9–34) can be rewritten in the form

$$f = A_2^{1/3}\left[\frac{(A_2^{1/3}\eta)^2}{2!} - \frac{1}{2}\frac{(A_2^{1/3}\eta)^5}{5!} + \frac{1}{4}\frac{11(A_2^{1/3}\eta)^8}{8!} - \frac{1}{8}\frac{375(A_2^{1/3}\eta)^{11}}{11!} + \cdots\right]$$
$$= A_2^{1/3}F(A_2^{1/3}\eta) \tag{9-35}$$

Applying the boundary condition [Eq. (9-31)] at $\eta = \infty$ to the equation above, we have

$$\lim_{\eta\to\infty}[A_2^{2/3}F'(A_2^{1/3}\eta)] = f'(\infty) = 1$$

or

$$A_2 = \left[\frac{1}{\lim_{\eta\to\infty} F'(\eta)}\right]^{3/2} \tag{9-36}$$

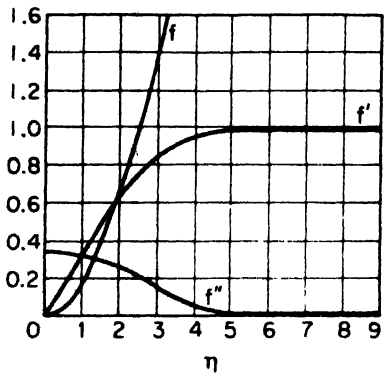


Figure 9.5

The value of A_2 can be determined numerically from the equation above to any desired approximation. Howarth[1] found that $A_2 = 0.33206$. With this value of A_2 the functions f, f', and f'' are plotted in Fig. 9.5. The velocity components u and v inside the boundary layer can be readily computed from Eqs. (9-29a) and (9-29b) with the calculated values of f and f'. Figure 9.6 shows excellent agreement between the Blasius

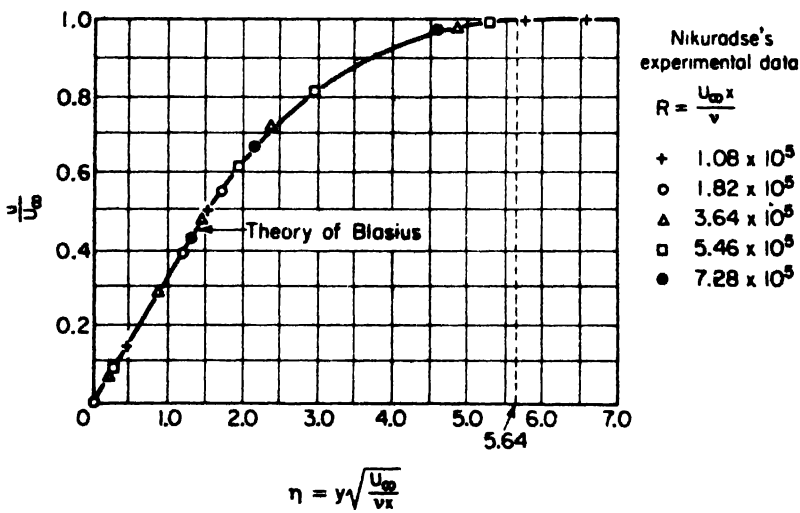


Figure 9.6

solution and the experiments by Nikuradse[2] for the velocity profile. Hence, it can be said that the Prandtl boundary layer equations have been verified by experiments.

b. Shearing Stress and Boundary Layer Thickness

The shearing stress on the surface of the plate can be easily calculated from the results of the Blasius solution. From Eqs. (1-7) and (9-29a), we have

$$\tau_0 = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\mu U_\infty f''(0)}{\sqrt{\nu x/U_\infty}},$$

$$= \mu \frac{U_\infty A_2}{\sqrt{\nu x/U_\infty}} = \frac{0.332}{\sqrt{\mathrm{Re}_x}} \rho U_\infty^2 \tag{9-37}$$

where $\mathrm{Re}_x = U_\infty x/\nu$. The local skin-friction coefficient or frictional drag coefficient is

$$C_f = \frac{\tau_0}{\frac{1}{2}\rho U_\infty^2} = \frac{0.664}{\sqrt{\mathrm{Re}_x}} \tag{9-38}$$

The total frictional force per unit width for one side of the plate of length l is given by

$$F = \int_0^l \tau_0 \, dx = 0.664 \rho U_\infty^2 \sqrt{\frac{\nu l}{U_\infty}} \tag{9-39}$$

Equation (9-39) indicates that the frictional force is proportional to the $\frac{3}{2}$ power of the free-stream velocity. The average skin-friction coefficient in this case is

$$C_F = \frac{F}{\frac{1}{2}\rho U_\infty^2 l} = \frac{0.664 \rho U_\infty^2 \sqrt{\nu l/U_\infty}}{\frac{1}{2}\rho U_\infty^2 l} = \frac{1.328}{\sqrt{\mathrm{Re}_l}} \tag{9-40}$$

where $\mathrm{Re}_l = U_\infty l/\nu$. In Fig. 9.7 a comparison between theoretical and experimental[3] results for the local skin-friction coefficient is shown. Again excellent agreement between theory and experiment is obtained.

According to the boundary condition [Eq. (9-24)], the velocity in the boundary layer does not reach the value of free-stream velocity until $y \to \infty$. Hence, this theory does not give an exact prediction of the boundary layer thickness. However, at a certain finite value of η the velocity in the boundary layer asymptotically blends into the free-stream velocity. If an arbitrary limit of the boundary layer at $u = 0.9975$ is used, the thickness of the boundary layer can be found from Fig. 9.6 to be

$$\delta = 5.64 \sqrt{\frac{\nu x}{U_\infty}} \tag{9-41}$$

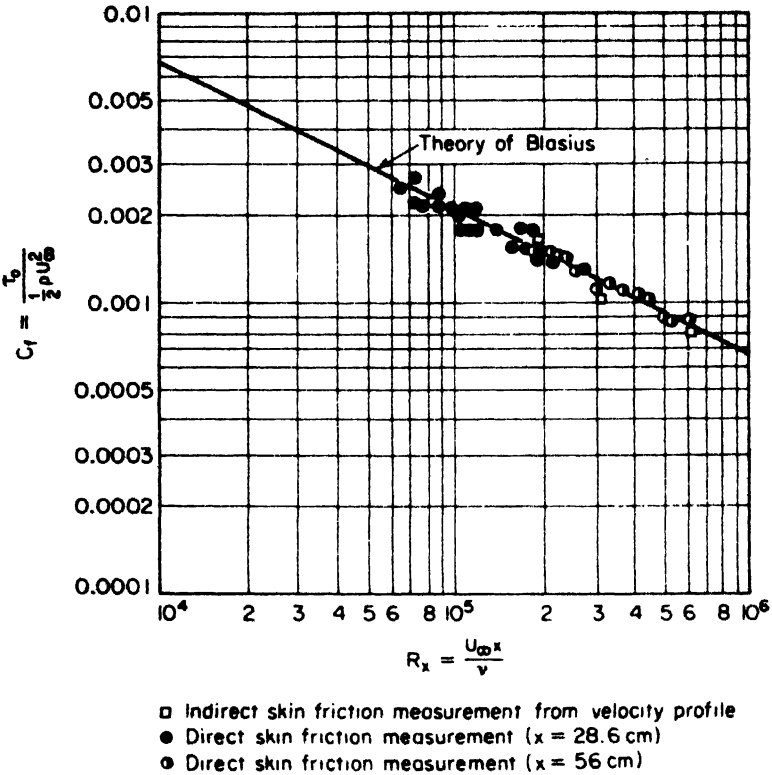


Fig. 9.7. From H. Schlichting, *Boundary Layer Theory*. New York: McGraw-Hill, Inc., 1960.

Since the definition above of the boundary layer thickness is somewhat arbitrary, a more physically meaningful definition of the thickness δ is introduced as

$$U_\infty \delta^* = \int_0^\infty (U_\infty - u)\, dy \tag{9-42a}$$

The right-hand side of Eq. (9-42a) signifies the decrease in total flow rate caused by the action of the friction and the left-hand side represents the potential flow that has been displaced from the wall (Fig. 9.8). Hence the distance by which the free stream is displaced outwards due to the decrease in velocity in the boundary layer is

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) dy \tag{9-42b}$$

where δ^* is called the displacement thickness in a boundary layer. Substituting the expressions for u/U_∞ and η from Eqs. (9-29a) and

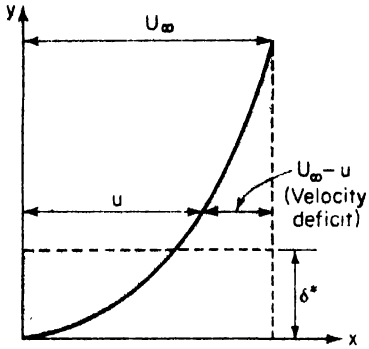


Figure 9.8

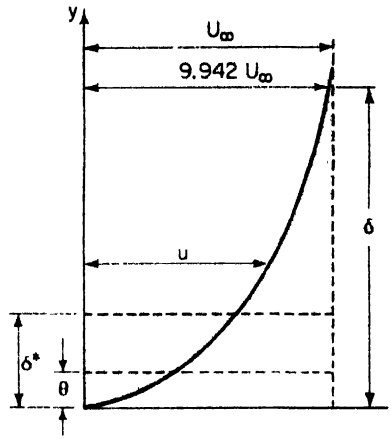


Figure 9.9

(9–27) into Eq. (9–42b), we obtain

$$\delta^* = \sqrt{\frac{\nu x}{U_\infty}} \int_0^\infty (1 - f')\, d\eta = \sqrt{\frac{\nu x}{U_\infty}} \lim_{\eta \to \infty} [\eta - f(\eta)]$$

$$= 1.7208 \sqrt{\frac{\nu x}{U_\infty}} \tag{9–42c}$$

where the numerical value of $f(\eta)$ is obtained from the numerical solution of Eq. (9–30b) by Howarth.[1]

In analogy to the displacement thickness a momentum thickness may be defined in accordance with the momentum law. This is established by equating the loss of momentum flow as a consequence of the wall friction in the boundary layer (right-hand side of the following equation) to the momentum flow in the absence of the boundary layer. Thus

$$\rho \theta U_\infty^2 = \rho \int_0^\infty u(U_\infty - u)\, dy \tag{9–43a}$$

or the momentum thickness is defined as

$$\theta = \int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy \tag{9–43b}$$

With the expression of u/U_∞ and η in Eqs. (9–29a) and (9–27) we can evaluate numerically the value of θ for a flat plate as follows:

$$\theta = \sqrt{\frac{\nu x}{U_\infty}} \int_0^\infty f'(1 - f')\, d\eta = 0.664 \sqrt{\frac{\nu x}{U_\infty}} \tag{9–43c}$$

Figure 9.9 shows a comparison among various thicknesses of a boundary layer as discussed in this section.

Illustrative Example 9.1. Using five terms of the series for f in Eq. (9–34), compare the approximate velocity profile with the exact solution. Also calculate the local skin-friction coefficient, the displacement thickness, and the momentum thickness from the approximate expression for f.

Solution. Differentiating Eq. (9–34) with respect to η and substituting $A_2 = 0.33206$ into the resulting equation, we obtain

$$\frac{u}{U_\infty} = f'(\eta) = 0.332\eta - 0.00230\eta^4 + 1.998 \times 10^{-5}\eta^7 - 1.571 \times 10^{-7}\eta^{10} + 1.130 \times 10^{-9}\eta^{13} \qquad \text{(a)}$$

This approximate velocity profile together with the exact velocity profile is shown in Fig. 9.A. For $u/U_\infty = 0.976$, $\eta \doteq 4$ is obtained from Eq. (a).

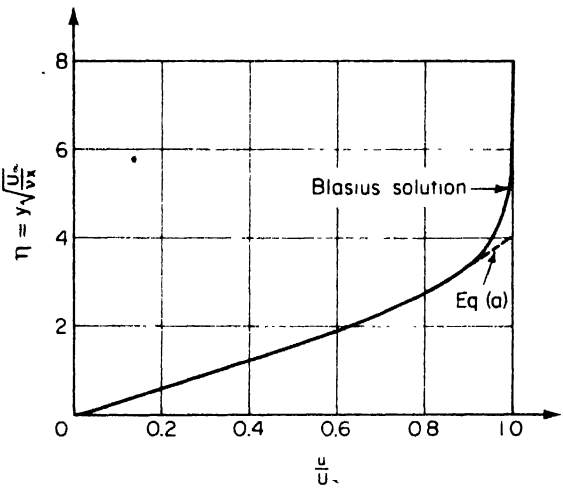


Figure 9.A

The shearing stress on the surface of the plate is given by

$$\tau_0 = \mu \frac{U_\infty}{\sqrt{\nu x/U_\infty}} f''(0) = \frac{0.332}{\sqrt{\mathrm{Re}_x}} \rho U_\infty \qquad \text{(b)}$$

where the value for $f''(0)$ is obtained from Eq. (a). The local skin friction coefficient is

$$C_f = \frac{\tau_0}{\frac{1}{2}\rho U_\infty^2} = \frac{0.664}{\sqrt{\mathrm{Re}_x}} \qquad \text{(c)}$$

which is identical with the result obtained from the exact solution.

The displacement thickness may be obtained by substituting Eq. (a) into Eq. (9–42c) as follows:

$$\delta^* = \sqrt{\frac{\nu x}{U_\infty}} \int_0^4 (1 - f')\, d\eta = 1.69\sqrt{\frac{\nu x}{U_\infty}} \tag{d}$$

The upper limit of the integral corresponds to $f' = 1.0$.

In a like manner the momentum thickness is obtained by substituting Eq. (a) into Eq. (9–43c),

$$\theta = \sqrt{\frac{\nu x}{U_\infty}} \int_0^4 f'(1 - f')\, d\eta = 0.62\sqrt{\frac{\nu x}{U_\infty}} \tag{e}$$

9.4. *Boundary Layer on a Surface with Pressure Gradient*

We have just discussed the boundary layer along a flat plate (zero pressure gradient) based on the similar velocity profile. Now, we are going to find the similar solution of the two-dimensional boundary layer equations if the velocity outside the boundary layer is assumed to be proportional to a power of distance along the wall. This is the case for flow past a wedge (Fig. 9.3) and the velocity of the potential flow is given by

$$U(x) = U_1 x^m \tag{9–44}$$

where U_1 is a constant, the coordinate x is measured from the stagnation point O and the wedge angle is denoted by $\pi\beta$ (Fig. 9.3). The fundamental equation of the boundary layer in this case is

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x\, \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = U\frac{\partial U}{\partial x} + \nu\frac{\partial^3 \psi}{\partial y^3} \tag{9–45}$$

with boundary conditions

$$\begin{aligned} y = 0: &\qquad \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0 \\ y = \infty: &\qquad \frac{\partial \psi}{\partial y} = U(x) \end{aligned} \tag{9–46}$$

The equation and boundary conditions above are reduced from Eqs. (9–21) and (9–22), respectively, for the case of steady flow.

The similar solution of Eq. (9–45) can be obtained if the following transformation [see Eq. (9–27)] is made:

$$\eta_1 = yC\sqrt{\frac{U}{\nu x}} = yC\sqrt{\frac{U_1}{\nu}}\, x^{(m-1)/2} \tag{9–47}$$

where the arbitrary constant C is to be determined. The stream function in this transformation is obtained by integrating the continuity equation, i.e.,

$$\psi = \int u\,dy = \frac{U}{C\sqrt{\frac{U_1}{\nu}}\,x^{(m-1)/2}} \int F(\eta_1)\,d\eta_1$$

$$= \frac{1}{C}\sqrt{\nu U_1}\; x^{(m+1)/2} f(\eta_1) \tag{9-48}$$

The derivatives of ψ with respect to x and y are transformed as follows:

$$\begin{aligned}
\frac{\partial \psi}{\partial y} &= \frac{\partial \psi}{\partial \eta_1}\frac{\partial \eta_1}{\partial y} = Uf' \\
\frac{\partial \psi}{\partial x} &= \left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial \eta_1}\frac{\partial \eta_1}{\partial x}\right) = C\sqrt{\frac{\nu U}{x}}\left[f + \frac{C^2 - 1}{C^2}\eta_1 f'\right] \\
\frac{\partial^2 \psi}{\partial x \partial y} &= (2C^2 - 1)\frac{U}{x}\left[f' + \frac{C^2 - 1}{2C^2 - 1}\eta_1 f''\right] \\
\frac{\partial^2 \psi}{\partial y^2} &= CU\sqrt{\frac{U}{\nu x}}\,f'' \\
\frac{\partial^3 \psi}{\partial y^3} &= U\left(C^2\frac{U}{\nu x}\right) f'''
\end{aligned} \tag{9-49}$$

With the relations in Eq. (9-49), Eq. (9-45) transforms to

$$\begin{aligned}
Uf'\left[(2C^2 - 1)\frac{U}{x}\left(f' + \frac{C^2 - 1}{2C^2 - 1}\eta_1 f''\right)\right] - \left[C\sqrt{\frac{\nu U}{x}}\left(f + \frac{C^2 - 1}{C^2}\eta_1 f'\right)\right] \\
\times \left[CU\sqrt{\frac{U}{\nu x}}\,f''\right] = \nu U\left(C^2\frac{U}{\nu x}\,f'''\right) + m\frac{U^2}{x}
\end{aligned}$$

$$f''' + ff'' - \left(\frac{2C^2 - 1}{C^2}\right) f'^2 + \frac{m}{C^2} = 0 \tag{9-50}$$

The coefficients of the last two terms can be equal if

$$2C^2 - 1 = m$$

or

$$C = \sqrt{\frac{1 + m}{2}} \tag{9-51}$$

Hence, the differential equation for $f(\eta_1)$:

$$\frac{d^3 f}{d\eta_1^3} + f\frac{d^2 f}{d\eta_1^2} - \frac{2m}{m + 1}\left[\left(\frac{df}{d\eta_1}\right)^2 - 1\right] = 0 \tag{9-52}$$

with the boundary conditions

$$\begin{aligned}
&\eta_1 = 0: \qquad f = 0, \qquad \frac{df}{d\eta_1} = 0 \\
&\eta_1 = \infty: \qquad \frac{df}{d\eta_1} = 1
\end{aligned} \tag{9-53}$$

It is interesting to note that for $m = 0$, Eq. (9-52) reduces to Eq. (9-30b) which is the case of the flow over a flat plate. For the case of a flat plate the velocity of the potential flow given in Eq. (9-44) becomes a constant, and the wedge angle $\pi\beta$ (Fig. 9.3) becomes zero. When $m = 1$, Eq. (9-52) reduces to the case of flow in the vicinity of a stagnation point[4] in a plane with wedge angle $\pi\beta = \pi$ or $\beta = 1$. In this case the velocity of the potential flow is linearly proportional to the distance along the wall. From the results of the analogy above we can relate the coefficient of the third term in Eq. (9-52) to the wedge angle $\pi\beta$, i.e.,

$$\beta = \frac{2m}{1+m} \tag{9-54}[5]$$

The solution of Eq. (9-52) was made by Hartree[6], and the result is shown in Fig. 9.10. When $m > 0$ ($\beta > 0$), the flow is accelerated and the velocity profiles have no point of inflection. On the other hand, if $m < 0$ ($\beta < 0$) the flow is decelerated (with adverse pressure gradient) and a point of inflection occurs in the velocity profile. As m reaches a value of -0.091 ($\beta = -0.199$) the separation of flow from the wall occurs (§ 8.3a and § 9.8).

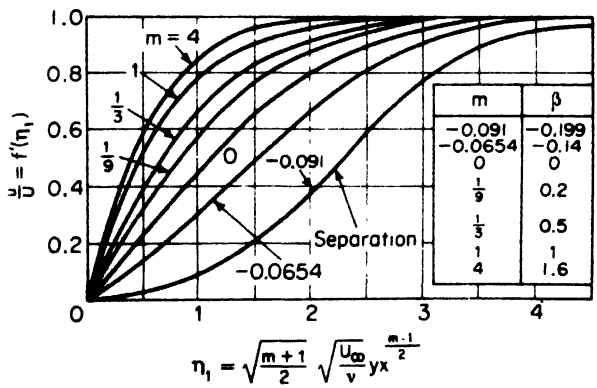


Figure 9.10

9.5. Momentum Integral Theorems for the Boundary Layer

As previously discussed, the solution of the steady boundary layer equations, Eqs. (9-16a)–(9-16c) is in general very difficult. The solutions obtained in § 9.3 and § 9.4 represent only very special cases. For engineering problems, it is often acceptable if an approximate solution

can be obtained. One of the most useful methods is the von Kármán-Pohlhausen method based on the integral theorem. The basic concept of this method is that the solutions satisfy the differential equations only on the average. In other words, it is not anticipated that the solution satisfies the boundary layer differential equations at every point (x,y) but the momentum integral equation and the boundary conditions must be satisfied. The momentum integral equation is obtained by integrating the boundary layer equations with respect to y or by the momentum law.

a. The von Kármán Integral Relation

The boundary layer equations which were discussed in § 9.2 are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{9-19a}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9-19b}$$

Integrating Eq. (9-19a) with respect to y from $y = 0$ to $y = \delta(x)$, the outer edge of the boundary layer, we have

$$\frac{\partial}{\partial t}\int_0^\delta u\,dy + \int_0^\delta u\frac{\partial u}{\partial x}dy + \int_0^\delta v\frac{\partial u}{\partial y}dy = -\frac{1}{\rho}\int_0^\delta \frac{dp}{dx}dy + \frac{\mu}{\rho}\int_0^\delta \frac{\partial^2 u}{\partial y^2}dy \tag{9-55}$$

The third term on the left-hand side of Eq. (9-55) can be integrated by parts :

$$\int_0^\delta v\frac{\partial u}{\partial y}dy = \int_0^\delta \frac{\partial(uv)}{\partial y}dy - \int_0^\delta u\frac{\partial v}{\partial y}dy$$

$$= \Big[uv\Big]_0^\delta - \int_0^\delta u\frac{\partial v}{\partial y}dy = U\int_0^\delta \frac{\partial v}{\partial y}dy - \int_0^\delta u\frac{\partial v}{\partial y}dy \tag{9-56a}$$

With the [aid of the continuity equation [Eq. (9-19b)], Eq. (9-56a) becomes

$$\int_0^\delta v\frac{\partial u}{\partial y}dy = -U\int_0^\delta \frac{\partial u}{\partial x}dy + \int_0^\delta u\frac{\partial u}{\partial x}dy \tag{9-56b}$$

Substituting Eq. (9-56b) into Eq. (9-55) and making use of the fact that $\tau = \mu(\partial u/\partial y)$ is zero at $y = \delta$, we have

$$\frac{\partial}{\partial t}\int_0^\delta u\,dy + \int_0^\delta \frac{\partial u^2}{\partial x}dy - U\int_0^\delta \frac{\partial u}{\partial x}dy = -\frac{1}{\rho}\frac{dp}{dx}\delta - \frac{\tau_0}{\rho} \tag{9-57}$$

where

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

According to the Leibnitz rule the second and third items of Eq. (9-57) are

$$\int_0^\delta \frac{\partial u^2}{\partial x}\,dy = \frac{d}{dx}\int_0^\delta u^2\,dy - U^2\frac{d\delta}{dx}$$

$$U\int_0^\delta \frac{\partial u}{\partial x}\,dy = U\frac{d}{dx}\int_0^\delta u\,dy - U^2\frac{d\delta}{dx}$$

Hence Eq. (9-57) reduces to

$$\frac{\partial}{\partial t}\int_0^\delta u\,dy + \frac{d}{dx}\int_0^\delta u^2\,dy - U\frac{d}{dx}\int_0^\delta u\,dy = -\frac{1}{\rho}\frac{dp}{dx}\delta - \frac{\tau_0}{\rho} \qquad (9\text{-}58)$$

Equation (9-58) is one form of the von Kármán integral relation; it is also called the momentum integral equation of the boundary layer.

b. von Kármán Integral Relation by Momentum Law

The von Kármán momentum integral equation of the bounday layer represents the relation between the overall rate of flux of momentum across a section of the boundary and the surface forces due to the wall shearing stress and the pressure gradient. Hence, Eq. (9-58) can be derived from the momentum theorem of fluid mechanics (§ 7.6).

Consider an element of the boundary layer, *ABCDA* of unit length perpendicular to the x-y plane as shown in Fig. 9.11. Let $AD = \Delta X$ be the small length of the element in the x-direction and δ the thickness of the boundary layer at distance x from the leading edge of the plate. For the purpose of integration, it may be assumed that

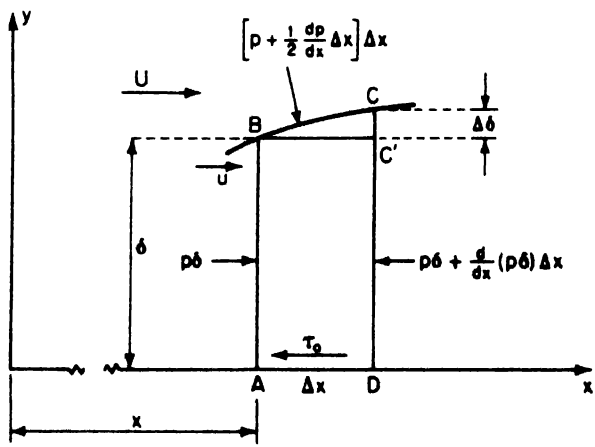


Figure 9.18

the velocity of the boundary layer flow at the outer edge of the boundary layer is the same as that of the potential flow, i. e., $u = U$ at $y = \delta$. Although the surface illustrated in Fig. 9.11 is flat, the resulting expression would apply to surfaces with slight curvature.

The rate of mass flow across AB into the element is

$$\int_0^\delta \rho u \, dy$$

and the corresponding rate of mass flow across DC out of the element is

$$\int_0^\delta \rho u \, dy + \frac{d}{dx}\left[\int_0^\delta \rho u \, dy\right] \Delta x$$

Hence the net rate of mass flow across AB and DC out of the element is

$$\frac{d}{dx}\left[\int_0^\delta \rho u \, dy\right] \Delta x \tag{9-59a}$$

Since there is no flow across the surface of the plate AD, the rate of mass flow out of the element across BC according to the continuity equation must be equal to

$$-\frac{d}{dx}\left[\int_0^\delta \rho u \, dy\right] \Delta x \tag{9-59b}$$

In the same manner the net rate of change of momentum in the x-direction across AB and DC of the element is obtained as

$$\frac{d}{dx}\left[\int_0^\delta \rho u^2 \, dy\right] \Delta x \tag{9-60a}$$

The rate of change of momentum in the x-direction across BC out of the element is

$$-U\frac{d}{dx}\left[\int_0^\delta \rho u \, dy\right] \Delta x \tag{9-60b}$$

The time rate of increase of momentum within the element is

$$\frac{\partial}{\partial t}\int_0^\delta \rho u \, dy \, \Delta x \tag{9-60c}$$

The force in the x-direction on the element due to the pressure, after neglecting terms of order $\Delta x \, \Delta\delta$, is

$$p\delta - \left[p\delta + \frac{d}{dx}(p\delta)\,\Delta x\right] + \left[p + \frac{1}{2}\frac{dp}{dx}\Delta x\right]\Delta x \frac{d\delta}{dx} = -\delta\frac{dp}{dx}\Delta x \tag{9-61a}$$

and that due to the shearing stress at the wall is

$$- \tau_0 \Delta x \tag{9-61b}$$

According to the momentum law, we can equate expressions (9-60a), (9-60b), and (9-60c) to expressions (9-61a) and (9-61b) as follows :

$$\frac{\partial}{\partial t}\int_0^\delta \rho u \, dy \Delta x + \frac{d}{dx}\int_0^\delta \rho u^2 \, dy \Delta x - U\frac{d}{dx}\int_0^\delta \rho u \, dy \Delta x = -\delta \frac{dp}{dx}\Delta x - \tau_0 \Delta x \tag{9-62}$$

After dividing Eq. (9-62) by $\rho \Delta x$, we obtain the von Kármán integral relation which is identical with Eq. (9-58):

$$\frac{\partial}{\partial t}\int_0^\delta u \, dy + \frac{d}{dx}\int_0^\delta u^2 dy - U\frac{d}{dx}\int_0^\delta u \, dy = -\frac{1}{\rho}\frac{dp}{dx}\delta - \frac{\tau_0}{\rho} \tag{9-58}$$

c. Other Forms of the von Kármán Integral Relation

It is often convenient to have the integral relation in a form in terms of the displacement and momentum thicknesses. Rearranging Eq. (9-58) and making use of Eq. (9-18), we have for steady flow

$$\frac{d}{dx}\int_0^\delta u^2 \, dy - U\frac{d}{dx}\int_0^\delta u \, dy - \frac{dU}{dx}U\delta = -\frac{\tau_0}{\rho}$$

or

$$\frac{d}{dx}\int_0^\delta u^2 \, dy - \frac{d}{dx}\int_0^\delta Uu \, dy + \frac{dU}{dx}\int_0^\delta u \, dy - \frac{dU}{dx}\int_0^\delta U \, dy = -\frac{\tau_0}{\rho}$$

or

$$\frac{d}{dx}\int_0^\infty u(U-u)dy + \frac{dU}{dx}\int_0^\infty (U-u)dy = \frac{\tau_0}{\rho} \tag{9-63}$$

The form given in Eq. (9-63) permits us to take the limit $\delta \to \infty$; hence, we have the alternative form,

$$\frac{d}{dx}\int_0^\infty u(U-u)\,dy + \frac{dU}{dx}\int_0^\infty (U-u)\,dy = \frac{\tau_0}{\rho} \tag{9-64}$$

If the displacement and momentum thicknesses are introduced from Eqs. (9-42b) and (9-43b) into Eq. (9-64), we obtain

$$\frac{d}{dx}(U^2\theta) + U\frac{dU}{dx}\delta^* = \frac{\tau_0}{\rho}$$

or

$$\frac{d\theta}{dx} + (2\theta + \delta^*)\frac{1}{U}\frac{dU}{dx} = \frac{\tau_0}{\rho U^2} \tag{9-65}$$

where the momentum and displacement thicknesses take the form,

$$\theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$
$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy \tag{9-66}$$

Equation (9–65) gives an ordinary differential equation for the boundary layer thickness. In the calculation of the boundary layer flow based on Eq. (9–65), it is necessary to find a suitable expression for the velocity function in the boundary layer. This velocity function must satisfy the no-slip condition at the wall and the conditions at the point where the potential flow solution exists. A detailed discussion of the method is given in the succeeding section.

***Illustrative Example* 9.2.** Based on the von Kármán integral relation, determine the local frictional coefficient, C_f, for flow over a flat plate.

Solution. For a flat plate the momentum integral equation [Eq. (9–65)] reduces to

$$\frac{d\theta}{dx} = \frac{\tau_0}{\rho U_\infty^2} \tag{a}$$

The momentum thickness [Eq. (9–66)] can be written in the form

$$\frac{\theta}{\delta} = \int_0^1 \left(1 - \frac{u}{U_\infty}\right)\frac{u}{U_\infty} d\eta = K_1 \tag{b}$$

where $\eta = y/\delta$. Substituting Eq. (b) into Eq. (a) we obtain

$$K_1 \frac{d\delta}{dx} = \frac{\tau_0}{\rho U_\infty^2} \tag{c}$$

where the shearing stress on the surface of the plate, τ_0, may be expressed in the form

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_0 = \mu \frac{U_\infty}{\delta}\left[\frac{\partial}{\partial \eta}\left(\frac{u}{U_\infty}\right)\right]_0 = K_2 \mu \frac{U_\infty}{\delta} \tag{d}$$

Combining Eqs. (c) and (d), after integrating, we have

$$\delta = \sqrt{2\frac{K_2}{K_1}\left(\frac{\nu x}{U_\infty}\right)} \tag{e}$$

Substituting Eq. (e) into Eq. (d), the local frictional coefficient is obtained :

$$C_f = \frac{\tau_0}{\frac{1}{2}\rho U_\infty^2} = \frac{K_2 \mu U_\infty}{\frac{1}{2}\rho U_\infty^2 \delta} = \frac{\sqrt{2K_1K_2}}{\sqrt{\mathrm{Re}_x}} \tag{f}$$

where

$$K_1 = \int_0^1 \left[1 - \left(\frac{u}{U_\infty}\right)\right]\frac{u}{U_\infty} d\eta$$

$$K_2 = \left[\frac{\partial}{\partial \eta}\left(\frac{u}{U_\infty}\right)\right]_{\eta=0} \quad \text{and} \quad \mathrm{Re}_x = \frac{U_\infty x}{\nu}$$

9.6. *Application of the Momentum Integral Equation to Boundary Layers—von Kármán-Pohlhausen Method*

The momentum integral equation discussed in the preceding section may be used to treat the general boundary layer flow with a pressure gradient. At the suggestion of von Kármán, Pohlhausen carried out the calculation of the boundary layer flow based on Eq. (9-65).

The exact solution of the boundary layer equation (§ 9.3) reveals that the velocity distribution in the boundary layer is proportional to the ratio y/δ. In addition, the velocity distribution must satisfy the boundary condition that $y = 0$: $u = 0$, $v = 0$, and $y = \delta$: $u = U$ and $\partial u/\partial y = 0$. In § 9.4 it has been shown that a point of inflection occurs in the velocity profile if the flow is decelerated. Furthermore, the separation of flow at the wall begins when the velocity gradient at the wall is equal to zero. With this information Pohlhausen introduced a polynomial of the fourth degree for the velocity function in terms of the nondimensional distance from the wall η as follows.

$$\frac{u}{U} = f(\eta) = a\eta + b\eta^2 + c\eta^3 + d\eta^4 \tag{9-67}$$

in the range $0 \leq \eta \leq 1$, where $\eta = y/\delta$. The constants a, b, c, d are to be determined from the following boundary conditions:

$$y = 0: \quad u = 0, \quad v = 0; \quad \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{dp}{dx} = -\frac{U}{\nu}\frac{dU}{dx} \tag{9-68a}$$

$$y = \delta: \quad u = U; \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 0 \tag{9-68b}$$

The first two conditions in Eq. (9-68a) are identically satisfied by the velocity function of Eq. (9-67). These and the first condition in Eq. (9-68b) constitute the three boundary conditions which are satisfied by all exact solutions. The second condition in Eq. (9-68b) is introduced in order that the velocity profile satisfies the continuous condition at $y = \delta$ where the solution of potential flow exists. The third condition in Eq. (9-68a) is obtained from the Prandtl boundary layer equation [Eq. (9-16a)] at $y = 0$. The significance of this condition can be seen from the illustration shown in Fig. 9.12. For an accelerated flow ($dp/dx < 0$ and $dU/dx > 0$) we have $(\partial^2 u/\partial y^2)_0 < 0$ and also $\partial^2 u/\partial y^2 < 0$ over the entire thickness of the boundary layer (Fig. 9.12a). On the

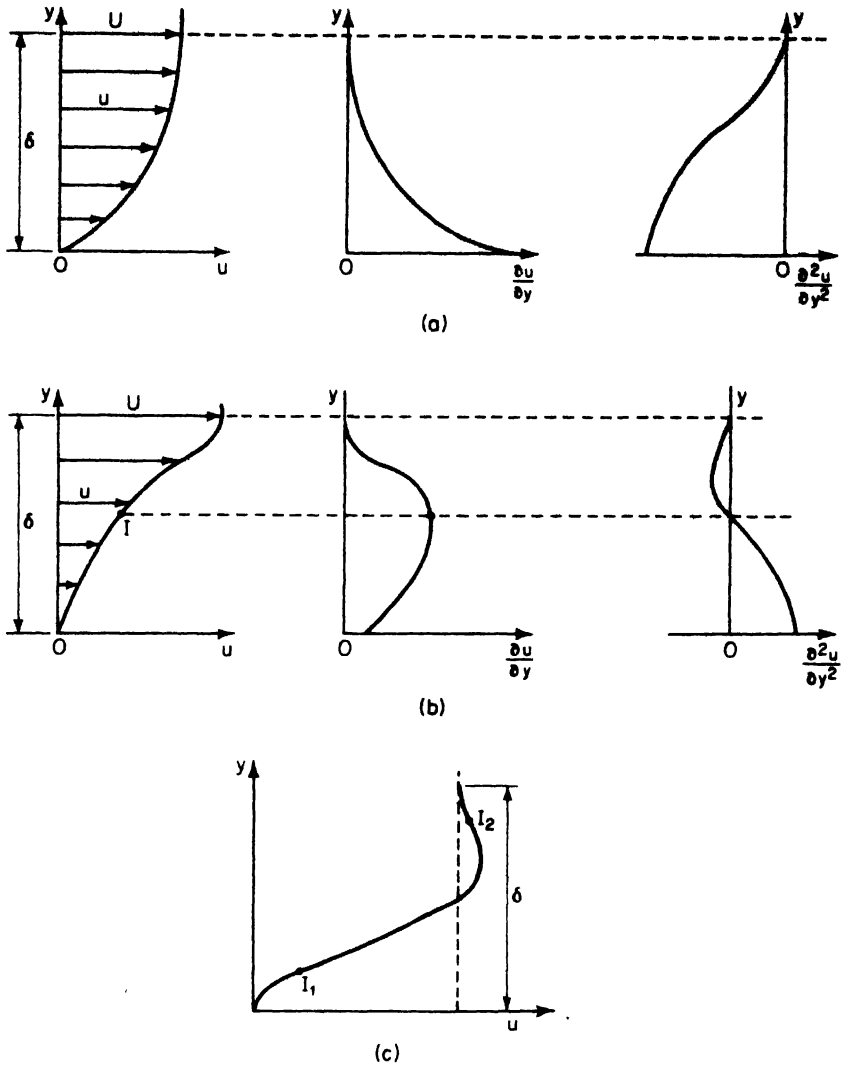


Figure 9.12

other hand, if the flow is decelerated ($dp/dx > 0$ and $dU/dx < 0$) the curvature of the velocity at the wall is positive, i.e., $(\partial^2 u/\partial y^2)_0 > 0$ (Fig. 9.12b); but due to the nature of the velocity profile, $\partial^2 u/\partial y^2 < 0$ must occur within the boundary layer thickness. For this reason there must exist a point for which $\partial^2 u/\partial y^2 = 0$. This point is called a point of inflection of the velocity profile in the boundary layer (point I in

Fig. 9.12b). The third condition in Eq. (9-68b) is introduced to guard against velocity profiles of the type shown in Fig. 9.12c. From Eq. (9-67) we have for $\partial^2 u/\partial y^2 = 0$,

$$\frac{\partial^2 u}{\partial y^2} = 2b + 6c\eta + 12d\eta^2 = 0 \tag{9-69}$$

which contains two roots, i.e., two inflection points in the boundary layer. One of the inflection points, I_1 is located near the wall, and the second inflection point, I_2, occurs at the upper region in the boundary layer. The presence of the inflection point I_2 would result in values of $u/U > 1$ in the boundary layer, which is physically inadmissible. In order to avoid this type of velocity profile we can remove the inflection point I_2 out of the boundary layer by imposing the condition $\partial^2 u/\partial y^2 = 0$ at $y = \delta$. Hence, according to Eq. (9-69), we cannot have a third inflection point within the boundary layer. Therefore, only velocity profiles of the types shown in Figs. 9.12a and 9.12b will occur in the velocity function of Eq. (9-67). Finally, the four constants in Eq. (9-67) are obtained from the boundary conditions [Eq. (9-68)] as follows:

$$\begin{aligned}
&\eta = 0: \quad \frac{\partial^2 u}{\partial y^2} = -\frac{U}{\nu}\frac{dU}{dx} \quad \text{or} \quad b = -\frac{\delta^2}{2\nu}\frac{dU}{dx} = -\frac{\Lambda}{2} \\
&\eta = 1: \quad \frac{u}{U} = 1 \quad \text{or} \quad a - \frac{\Lambda}{2} + c + d = 1 \\
&\eta = 1: \quad \frac{\partial u}{\partial y} = 0 \quad \text{or} \quad a - \Lambda + 3c + 4d = 0 \\
&\eta = 1: \quad \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{or} \quad -\Lambda + 6c + 12d = 0
\end{aligned} \tag{9-70a}$$

The simultaneous solutions of Eq. (9-70a) yield

$$a = 2 + \frac{\Lambda}{6}; \qquad b = -\frac{\Lambda}{2}; \qquad c = -2 + \frac{\Lambda}{2}; \qquad d = 1 - \frac{\Lambda}{6} \tag{9-70b}$$

and, hence, the expression for the velocity profile which satisfies all the boundary conditions in Eq. (9-68) is

$$\frac{u}{U} = f(\eta) = F(\eta) + \Lambda G(\eta) \tag{9-71}$$

where

$$\begin{aligned}
F(\eta) &= 2\eta - 2\eta^3 + \eta^4 \\
G(\eta) &= \frac{1}{6}(\eta - 3\eta^2 + 3\eta^3 - \eta^4) = \frac{\eta}{6}(1 - \eta)^3 \\
\Lambda &= \frac{\delta^2}{\nu}\frac{dU}{dx}
\end{aligned} \tag{9-71a}$$

The velocity profiles expressed in Eq. (9–71) constitute a one-parameter family of curves with a form parameter Λ which depends mainly on the pressure gradient of the flow. The limitation of the values of Λ can be examined in the following manner.

From Eq. (9–71) the slope and curvature of the velocity in the boundary layer are obtained:

$$\frac{df}{d\eta} = (2 - 6\eta^2 + 4\eta^3) + \frac{\Lambda}{6}\left[(1-\eta)^2(1-4\eta)\right]$$

$$\frac{d^2f}{d\eta^2} = (1-\eta)\left[12\left(\frac{\Lambda}{6} - 1\right)\eta - \Lambda\right]$$

Now, we have the velocity gradient at $\eta = 0$,

$$\left(\frac{df}{d\eta}\right)_0 = 2 + \frac{\Lambda}{6}$$

Hence the velocity profile at separation with $(df/d\eta)_0 = 0$ occurs for $\Lambda = -12$. This is the lower limit of Λ. The upper limit of Λ can be determined from the condition of zero curvature of the velocity, i.e., $d^2f/d\eta^2 = 0$ gives $\eta = (\Lambda/12)[(\Lambda/6) - 1]$. It is seen that for $\Lambda \leq 12$, $\eta \geq 1.0$ and for $\Lambda > 12$, $\eta < 1.0$. Hence for $\Lambda > 12$, the inflection point occurs within $\eta = 1.0$, that is, the velocity profile in the boundary layer becomes greater than the velocity in the potential flow (see Fig. 9.13). This is not correct physically; therefore, the form parameter is limited to the range

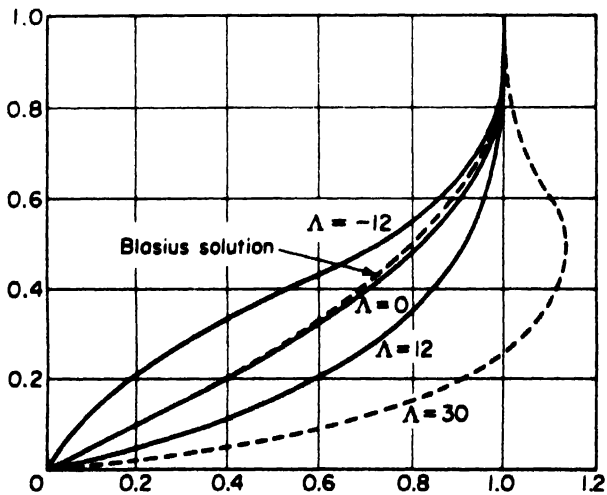


Figure 9.13

$$-12 \leq \Lambda \leq 12$$

The velocity profiles for various values of Λ are shown in Fig. 9.13. For $\Lambda = 0$, the velocity profile corresponds to the Blasius solution (§ 9.3).

Now we are in position to calculate the thickness of the boundary layer from Eq. (9-65) with the aid of the assumed velocity profile in Eq. (9-71). First, we must calculate the displacement thickness and the momentum thickness from Eq. (9-66) together with Eq. (9-71)

$$\frac{\delta^*}{\delta} = \int_0^1 \Big[1 - F(\eta) - \Lambda G(\eta)\Big]\, d\eta$$

$$= \frac{3}{10} - \frac{\Lambda}{120} \tag{9-72a}$$

$$\frac{\theta}{\delta} = \int_0^1 \Big[F(\eta) + \Lambda G(\eta)\Big]\Big[1 - F(\eta) - \Lambda G(\eta)\Big]\, d\eta$$

$$= \frac{37}{315} - \frac{\Lambda}{945} - \frac{\Lambda^2}{9072} \tag{9-72b}$$

The momentum integral equation [Eq. (9-65)] after multiplying by $U\theta/\nu$ becomes

$$\frac{U}{\nu}\,\theta\,\frac{d\theta}{dx} + \Big(2 + \frac{\delta^*}{\theta}\Big)\frac{U'\theta^2}{\nu} = \frac{\tau_0}{\mu}\,\frac{\theta}{U} \tag{9-73}$$

where $U' = dU/dx = \Lambda\nu/\delta^2$. Equation (9-73) can be simplified if the following parameters are introduced :

$$z = \frac{\theta^2}{\nu} \tag{9-74a}$$

$$\kappa = zU' = \frac{\theta^2}{\nu}\,U' = \Big(\frac{\theta}{\delta}\Big)^2 \Lambda \tag{9-74b}$$

With the aid of Eq. (9-72), we have

$$\kappa = \Big(\frac{\theta}{\delta}\Big)^2 \Lambda = \Big(\frac{37}{315} - \frac{\Lambda}{945} - \frac{\Lambda^2}{9072}\Big)^2 \Lambda \tag{9-74c}$$

$$f_1(\kappa) = \frac{\delta^*}{\theta} = \frac{3/10 - \Lambda/120}{37/315 - \Lambda/945 - \Lambda^2/9072} \tag{9-74d}$$

$$f_2(\kappa) = \frac{\tau_0}{\mu}\,\frac{\theta}{U} = \Big(\frac{12 + \Lambda}{6}\Big)\Big(\frac{37}{315} - \frac{\Lambda}{945} - \frac{\Lambda^2}{9072}\Big) \tag{9-74e}$$

where

$$\tau_0 = \mu\Big(\frac{\partial u}{\partial y}\Big)_0 = \frac{\mu U}{\delta}\Big(2 + \frac{\Lambda}{6}\Big)$$

Substituting z, κ, $f_1(\kappa)$, and $f_2(\kappa)$ from Eqs. (9-74a) - (9-74e) into Eq. (9-73) together with $(\theta/\nu)(d\theta/dx) = \frac{1}{2}(dz/dx)$, the momentum equation finally becomes

$$\frac{dz}{dx} = \frac{F(\kappa)}{U} \tag{9-75a}$$

where

$$F(\kappa) = 2f_2(\kappa) - 2[2 + f_1(\kappa)]\kappa, \qquad \kappa = zU' \tag{9-75b}$$

Equation (9-75a) is a nonlinear differential equation of the first order for z. In order to obtain a solution from Eq. (9-75a), one must start it at the stagnation point $x = 0$ where $U = 0$. Since dz/dx cannot be infinite at $x = 0$, $F(\kappa)$ must be equal to zero at the stagnation point. This yields (see small print below) the initial value of Λ at the stagnation point

$$\Lambda_0 = 7.052 \tag{9-76a}$$

and the initial value and initial slope of the integral curve

$$z_0 = \frac{\kappa_0}{U'_0} = \frac{0.077}{U'_0} \tag{9-76b}$$

$$\left(\frac{dz}{dx}\right)_0 = -0.0652\left(\frac{U''}{U'}\right)_0 \tag{9-76c}$$

The initial values of Λ_0, z_0, and $(dz/dx)_0$ can be calculated as follows:

(i) *The Value of* Λ_0

Since $(dz/dx)_0 \neq \infty$, $[F(\kappa)]_0 = 0$ at $x = 0$, we have

$$2\left(\frac{12 + \Lambda_0}{6}\right)\left(\frac{37}{315} - \frac{\Lambda_0}{945} - \frac{\Lambda_0^2}{9072}\right) - 2\left[2 + \frac{3/10 - \Lambda_0/120}{37/315 - \Lambda_0/945 - \Lambda_0^2/9072}\right]\left(\frac{37}{315} - \frac{\Lambda_0}{945} - \frac{\Lambda_0^2}{9072}\right)\Lambda = 0 \tag{9-77a}$$

or

$$\Lambda_0^3 + 147.4\Lambda_0^2 - 1670.4\Lambda_0 + 9072 = 0 \tag{9-77b}$$

Equation (9-77b) yields the solutions

$$\Lambda_0 = 7.052, \quad \Lambda_0 = 17.75 \text{ and } \Lambda_0 = 70 \tag{9-76a}$$

but the limitation to the range of Λ is $-12 < \Lambda < 12$. Hence, $\Lambda_0 = 7.052$ is the only solution of Eq. (9-77b) for this case.

(ii) *The Values of* z_0 *and* $(dz/dx)_0$

In order to evaluate the value of $(dz/dx)_0$ in Eq. (9-75) we must use L'Hospital's rule to determine the indeterminate form

$$\left(\frac{dz}{dx}\right)_0 = \frac{[F(\kappa)]_0}{[U]_0} = \frac{0}{0} \tag{9-77c}$$

The differentiation of the numerator of the term on the right-hand side of Eq. (9-75) with respect to x gives

$$\frac{d}{dx}[F(\kappa)] = \frac{d}{dx}[2f_2(\kappa) - 4\kappa - 2\kappa f_1(\kappa)] = \left[2\frac{df_2(\kappa)}{d\Lambda}\frac{d\Lambda}{d\kappa} - 4 - 2f_1(\kappa) - 2\kappa\frac{df_1(\kappa)}{d\Lambda}\frac{d\Lambda}{d\kappa}\right]\frac{d\kappa}{dx} \tag{9-77d}$$

The terms on the right-hand side of Eq. (9-77d) are calculated as follows: From Eq. (9-74c) we have

$$\kappa_0 = \left(\frac{37}{315} - \frac{\Lambda_0}{945} - \frac{\Lambda_0^2}{9072}\right)^2 \Lambda_0$$

$$= (0.1045)^2(7.052) = 0.077 \tag{a}$$

and

$$\left(\frac{d\kappa}{d\Lambda}\right)_0 = \Lambda_0\left[2\left(\frac{37}{315} - \frac{\Lambda_0}{945} - \frac{\Lambda_0^2}{9072}\right)\left(-\frac{1}{945} - \frac{2\Lambda_0}{9072}\right)\right] + \left(\frac{37}{315} - \frac{\Lambda_0}{945} - \frac{\Lambda_0^2}{9072}\right)^2$$

$$= (7.052)[2(0.1045)(-0.002612)] + (0.1045)^2 = 0.00707 \tag{b}$$

With the value of $\kappa_0 = 0.077$, the initial value of z is obtained from Eq. (9-74b),

$$z_0 = \frac{\kappa_0}{U_0'} = \frac{0.077}{U_0'} \tag{9-76b}$$

and upon the differentiation of Eq. (9-74b) with respect to x,

$$\left(\frac{d\kappa}{dz}\right)_0 = U_0'\left(\frac{dz}{dx}\right)_0 + 0.077\,\frac{U_0''}{U_0} \tag{c}$$

The values of $[f_1(\kappa)]_0$, $[df_1/d\Lambda]_0$, and $[df_2/d\Lambda]_0$ are obtained from Eq. (9-74d) and (9-74e), respectively,

$$[f_1(\kappa)]_0 = \left(\frac{3}{10} - \frac{\Lambda_0}{120}\right)\left(\frac{37}{315} - \frac{\Lambda_0}{945} - \frac{\Lambda_0^2}{9072}\right)^{-1}$$

$$= \frac{0.2412}{0.1045} = 2.305$$

$$\left|\frac{df_1(\kappa)}{d\Lambda}\right|_0 = \frac{\left(\frac{37}{315} - \frac{\Lambda_0}{945} - \frac{\Lambda_0^2}{9072}\right)\left(-\frac{1}{120}\right) + \left(\frac{3}{10} - \frac{\Lambda_0}{120}\right)\left(\frac{1}{945} + \frac{2\Lambda_0}{9072}\right)}{\left(\frac{37}{315} - \frac{\Lambda_0}{945} - \frac{\Lambda_0^2}{9072}\right)^2}$$

$$= \frac{0.00087 + 0.00063}{0.01092} = 0.022 \tag{d}$$

and

$$f_2(\kappa) = \left(\frac{12 + \Lambda}{6}\right)\left(\frac{37}{315} - \frac{\Lambda}{945} - \frac{\Lambda^2}{9072}\right)$$

$$\left|\frac{df_2(\kappa)}{d\Lambda}\right|_0 = \frac{1}{6}\left|\left(\frac{37}{315} - \frac{\Lambda_0}{945} - \frac{\Lambda_0^2}{9072}\right) - (12 + \Lambda_0)\left(\frac{1}{945} + \frac{2\Lambda_0}{9072}\right)\right|$$

$$= \frac{1}{6}[(0.1045) - 19.052(0.002612)] = 0.00915 \tag{e}$$

Substituting Eqs. (a), (b), (c), (d), and (e) into Eq. (9-77d), we have

$$\left|\frac{dF(\kappa)}{dx}\right|_0 = \left|2(0.00915)\left(\frac{1}{0.00707}\right) - 2(2.305) - 4\right.$$

$$\left. - 2(0.077)(-0.022)\left(\frac{1}{0.00707}\right)\right]\left|U_0'\left(\frac{dz}{dx}\right)_0 + 0.077\,\frac{U_0''}{U_0'}\right|$$

$$= -5.545\left|U_0'\left(\frac{dz}{dx}\right)_0 + 0.077\frac{U_0''}{U_0'}\right|$$

Hence, Eq. (9-77c) becomes

$$\left(\frac{dz}{dx}\right)_0 = \frac{[dF(\kappa)/dx]_0}{(dU/dx)_0} = -5.545\left[\left(\frac{dz}{dx}\right)_0 + 0.077\frac{U_0''}{(U_0')^2}\right]$$

Rearranging the equation above, finally we obtain

$$\left(\frac{dz}{dx}\right)_0 = -\frac{0.077}{1.1805}\frac{U_0''}{(U_0')^2} = -0.0652\frac{U_0''}{(U_0')^2} \tag{9-76c}$$

Illustrative Example 9.3. In order to show the effectiveness of the Pohlhausen method let us calculate the boundary layer over a flat plate and compare the results with the Blasius solution in § 9.3. Since the potential flow velocity in the case of the boundary layer over a flat plate is constant, i.e., $U = U_\infty$, the velocity gradient and the pressure gradient are zero in the region outside the boundary layer. Hence, from Eqs. (9-71a), (9-74c), (9-74d), and (9-74e), respectively, we have

$$\Lambda = \frac{\delta^2}{\nu}\frac{dU}{dx} = 0$$

$$\kappa = \left(\frac{\theta}{\delta}\right)\Lambda = 0$$

$$f_1(\kappa) = \frac{\delta^*}{\theta} = \frac{945}{370} \tag{9-77a}$$

$$f_2(\kappa) = \frac{74}{315}$$

$$F(\kappa) = 2f_2(\kappa) - 2\,[2 + f_1(\kappa)]\kappa = 0.4698 = F(0)$$

Substituting the values for $F(\kappa)$ above into Eq. (9-75) yields

$$\frac{dz}{dx} = \frac{d\left(\frac{\theta^2}{\nu}\right)}{dx} = \frac{0.4698}{U_\infty} \tag{9-77b}$$

The solution of the equation above with the initial values $\theta = 0$ and $x = 0$ gives the momentum thickness

$$\theta = 0.685\sqrt{\frac{\nu x}{U_\infty}} \tag{9-77c}$$

and the displacement thickness

$$\delta^* = \theta f_1(\kappa) = \frac{945}{370}\left(0.685\sqrt{\frac{\nu x}{U_\infty}}\right)$$

$$= 1.75\sqrt{\frac{\nu x}{U_\infty}} \tag{9-77d}$$

The shearing stress at the wall is obtained from Eq. (9-74e):

$$\tau_0 = \mu \frac{U_\infty}{\theta} f_2(\kappa)$$

$$= \left(\frac{74}{315}\right)\left(\frac{1}{0.685}\right) \mu U_\infty \sqrt{\frac{U_\infty}{\nu x}} = 0.343 \frac{\rho U_\infty^2}{\sqrt{\mathrm{Re}_x}} \qquad (9\text{-}77e)$$

where $\mathrm{Re}_x = U_\infty x/\nu$.

The corresponding results of θ, δ^*, and τ_0 given by the Blasius solution are

$$\theta = 0.664 \sqrt{\frac{\nu x}{U_\infty}} \qquad (9\text{-}43c)$$

$$\delta^* = 1.7208 \sqrt{\frac{\nu x}{U_\infty}} \qquad (9\text{-}42c)$$

$$\tau_0 = 0.332 \frac{\rho U_\infty^2}{\sqrt{\mathrm{Re}_x}} \qquad (9\text{-}37)$$

It is seen that the agreement between the approximate solution of Pohlhausen and the exact solution of Blasius is quite satisfactory at least for the case of the boundary layer along a flat plate. In general, the Pohlhausen method is satisfactory for flows with decreasing pressure or even for the case of pressure increasing sufficiently gradually that separation is not approached. It breaks down in the neighborhood of a separation point and fails completely in predicting the exact location of the separation point unless the increase of pressure is very rapid as in the case of flow past a circular cylinder.

Illustrative Example 9.4. Simplified Solution of the Momentum Integral Equation. Figure 9.14 shows a plot of $F(\kappa)$ [Eq. (9–75)] vs. κ. It can be seen from the curve that the function $F(\kappa)$ can be approximated closely by a straight line

$$F(\kappa) = -a\kappa + b \qquad (9\text{-}78)$$

Substituting Eq. (9–78) into Eq. (9–75) and remembering that $\kappa = zU'$, we have

$$\frac{dz}{dx} + \mathrm{a} \frac{U'}{U} z = \frac{\mathrm{b}}{U} \qquad (9\text{-}79)$$

The solution of Eq. (9–79) with initial conditions $\theta = 0$ and $x = 0$ gives

$$z = \frac{b}{U^a} \int_0^x U^{a-1}\, dz \qquad (9\text{-}80a)$$

The momentum thickness is obtained with the aid of Eq. (9–74a),

$$\theta^2 = \frac{b\nu}{U^a} \int_0^x U^{a-1}\, dx \qquad (9\text{-}80b)$$

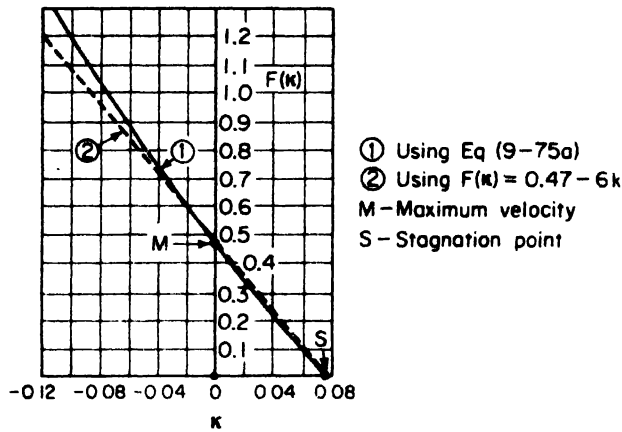


Fig. 9.14. H. Schlichting, "Boundary Layer Theory." New York: McGraw-Hill, 1960

Holstein and Bohlen[7] found the values for a and b to be: $a = 6$ and $b = 0.47$. With these values of a and b, the function $F(\kappa)$ given in Eq. (9–78) is very close to Pohlhausen's solution, Eq. (9–75a), in the region between the stagnation point ($\Lambda = 7.052$) and the point of maximum velocity ($\Lambda = 0$). Thwaites[8] gave the values of $a = 6$ and $b = 0.45$. Equation (9–80b) has proved to be a very useful technique in the solution of a variety of problems.

Illustrative Example 9.5. Plot and compare the velocity profiles calculated from the simplified solution of the momentum integral equation and the exact solution for the following cases: (a) flow past a flat plate, (b) flow past a wedge.

Solution. (a) According to Eq. (9–80b) the momentum thickness is given by

$$\theta^2 = \frac{0.47\nu}{U^6} \int_0^x U^5 \, dx \tag{a}$$

For flow past a flat plate the potential flow is $U = U_\infty = \text{Constant}$; hence Eq. (a) gives

$$\frac{\theta^2 U}{\nu} = 0.47x \tag{b}$$

or

$$\theta = 0.686 \sqrt{\frac{\nu x}{U_\infty}} \tag{c}$$

From Eq. (9–72b) we have

$$\delta = \left(\frac{315}{37}\right)\theta = 5.83\sqrt{\frac{\nu x}{U_\infty}} \tag{d}$$

The velocity profile is obtained from Eq. (9–71) in which $\Lambda = 0$ for this case:

$$\frac{u}{U} = 2\eta - 2\eta^3 + \eta^4 \tag{e}$$

where $\eta = y/\delta$. The abscissa in Fig. 9.B is given by ($m = 0$):

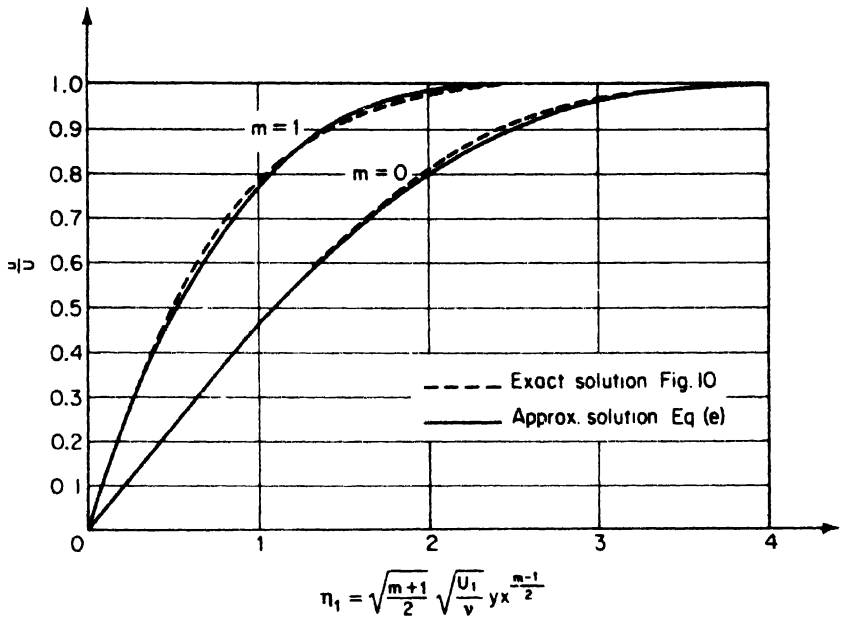


Figure 9.B

$$\eta_1 = y\sqrt{\frac{m+1}{2}}\sqrt{\frac{U_1}{\nu}}\, x^{(m-1)/2} = \sqrt{\frac{1}{2}\left(\frac{U_1}{\nu x}\right)}\, y$$

$$= 4.12\left(\frac{y}{\delta}\right) = 4.12\eta \tag{f}$$

(b) For flow past a wedge the potential flow is given by

$$U = U_1 x \tag{g}$$

Substituting Eq (g). into Eq. (a) we find

$$\frac{\theta^2}{\nu} U_1 = \frac{0.47}{x^6}\int_0^x x_1^5\, dx_1 = \frac{0.47}{6} = 0.0783 \tag{h}$$

The form parameter Λ is

$$\Lambda = \frac{\delta^2}{\nu}\frac{dU}{dx} = \frac{\delta^2}{\nu}U_1 \tag{i}$$

Combining Eqs. (h) and (i) we have

$$\left(\frac{\theta}{\delta}\right)^2 = \frac{0.0783}{\Lambda}, \qquad \Lambda \neq 0 \tag{j}$$

Substituting the expression for θ/δ from Eq. (9–72b) into Eq. (j), the result is

$$\frac{0.0783}{\Lambda} = \left(\frac{37}{315} - \frac{\Lambda}{945} - \frac{\Lambda^2}{9072}\right)^2 \tag{k}$$

The meaningful solution for Eq. (k) gives $\Lambda = 7.25$. With this value of Λ the velocity profile for flow past a wedge is obtained,

$$\frac{u}{U} = 2\eta - 2\eta^3 + \eta^4 + (7.25)\frac{1}{6}\eta(1-\eta)^3 \tag{l}$$

The expression for η_1 in this case is ($m = 1$)

$$\eta_1 = y\sqrt{\frac{U_1}{\nu}} = y\sqrt{\frac{\Lambda}{\delta^2}} = \sqrt{\Lambda}\,\eta \tag{m}$$

Figure 9.B shows the plot of u/U vs. η_1 for flows past both a flat plate and a wedge. The agreement between the simplified solution and the exact solution (Fig. 9.10) is quite good.

9.7. Boundary Layer for Axially Symmetrical Flow

In the preceding sections we have considered only boundary layers for two-dimensional cases in which the two velocity components depend on two space coordinates. The general three-dimensional case of a boundary layer in which the three velocity components depend on all three coordinates involves enormous mathematical difficulties which are beyond the scope of this text. However, the mathematical difficulties encountered in the three-dimensional boundary layer flow do not appear in the study of axially symmetrical boundary layers. In this case the flows are symmetrical with respect to the axis of revolution (Fig. 9.15a).

For a steady flow of incompressible fluids without a body force the Navier-Stokes equations and the continuity equation are obtained from Eqs. (5–19) and (5–9b), respectively,

$$(\mathbf{q}\cdot\nabla)\mathbf{q} = \frac{\nabla p}{\rho} + \nu\nabla^2\mathbf{q} \tag{9–81}$$

$$\nabla\cdot\mathbf{q} = 0 \tag{5–9b}$$

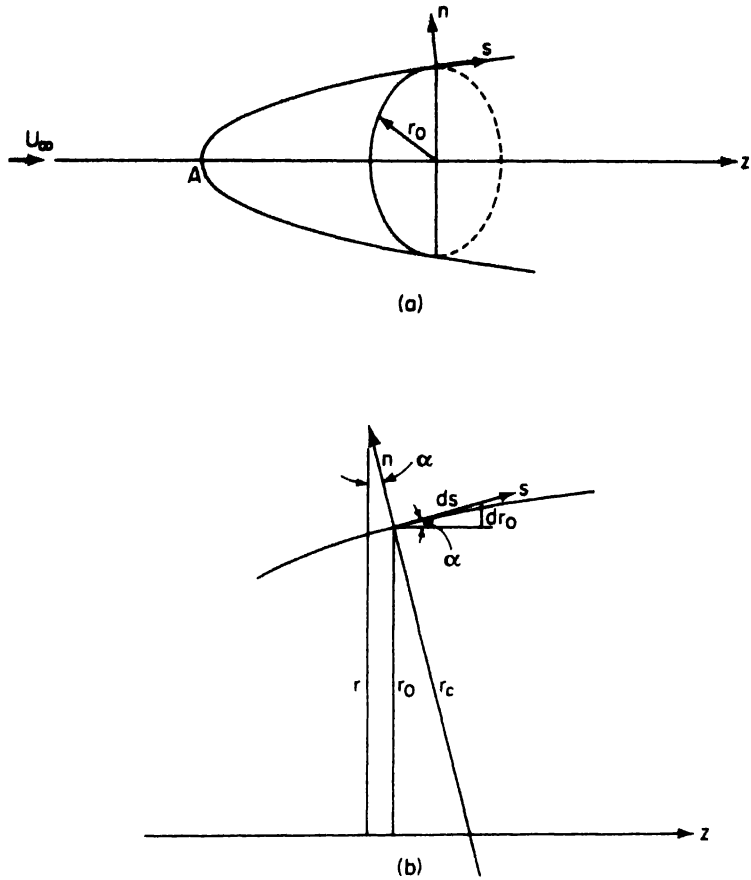


Figure 9.15

In the present case (Fig. 9.15a) a curvilinear coordinate system is introduced with s measured along the boundary surface in the direction of flow (a meridian) from the stagnation point A, n denoting the coordinate at right angles to the surface. The contour of the body of revolution is specified by the radius $r_0(s)$ of the sections taken at right angles to the axis of symmetry (z-axis), and the distance of a point outside the body of revolution from the z-axis is denoted by $r(n, s)$. If the boundary layer thickness δ is assumed to be small compared to the radius of curvature of the boundary, r_c (Fig. 9.15b), then the element $dsdn$ is assumed to be rectangular so that the line element becomes

$$dl^2 = ds^2 + r^2\,d\theta^2 + dn^2 \tag{9-82}$$

The velocity components parallel to the s-axis and n-axis are denoted by u and v, respectively. With the aid of the vector-curvilinear coordinates transformation (Appendix A) and the following relations:

$$h_1 = 1; \qquad h_2 = r; \qquad h_3 = 1$$

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial s}; \qquad \frac{\partial}{\partial x_2} = \frac{\partial}{\partial \theta} = 0; \qquad \frac{\partial}{\partial x_3} = \frac{\partial}{\partial n}$$

we have

$$(\mathbf{q}\cdot\nabla)\mathbf{q} = \mathbf{i}_s\left[\left(u\frac{\partial u}{\partial s} + v\frac{\partial v}{\partial s}\right) + v\left(\frac{\partial u}{\partial n} - \frac{\partial v}{\partial s}\right)\right] + \mathbf{i}_n\left[\left(u\frac{\partial u}{\partial n} + v\frac{\partial v}{\partial n}\right) - u\left(\frac{\partial u}{\partial n} - \frac{\partial v}{\partial s}\right)\right]$$

$$= \mathbf{i}_s\left(u\frac{\partial u}{\partial s} + v\frac{\partial u}{\partial n}\right) + \mathbf{i}_n\left(u\frac{\partial v}{\partial s} + v\frac{\partial v}{\partial n}\right)$$

$$\nabla p = \mathbf{i}_s\frac{\partial p}{\partial s} + \mathbf{i}_n\frac{\partial p}{\partial n}$$

$$\nabla^2\mathbf{q} = \mathbf{i}_s\left[\frac{1}{r}\frac{\partial}{\partial n}\left\{r\left(\frac{\partial u}{\partial n} - \frac{\partial v}{\partial s}\right)\right\}\right] - \mathbf{i}_n\left[\frac{1}{r}\frac{\partial}{\partial s}\left\{r\left(\frac{\partial u}{\partial n} - \frac{\partial v}{\partial s}\right)\right\}\right]$$

$$= \mathbf{i}_s\left[\frac{1}{r}\frac{\partial r}{\partial n}\left(\frac{\partial u}{\partial n} - \frac{\partial v}{\partial s}\right) + \frac{\partial^2 u}{\partial n^2} - \frac{\partial^2 v}{\partial n\,\partial s}\right] - \mathbf{i}_n\left[\frac{1}{r}\frac{\partial r}{\partial s}\left(\frac{\partial u}{\partial n} - \frac{\partial v}{\partial s}\right) + \frac{\partial^2 u}{\partial s\,\partial n} - \frac{\partial^2 v}{\partial s^2}\right]$$

$$\nabla\cdot\mathbf{q} = \frac{1}{r}\left[\frac{\partial}{\partial s}(ru) + \frac{\partial}{\partial n}(rv)\right]$$

Substituting the expressions above into Eqs. (9-81) and (5-9b), the Navier-Stokes equations and the continuity equation in the new coordinate system become

$$u\frac{\partial u}{\partial s} + v\frac{\partial u}{\partial n} = -\frac{1}{\rho}\frac{\partial p}{\partial s} + \nu\left[\frac{\partial^2 u}{\partial n^2} - \frac{\partial^2 v}{\partial n\,\partial s} + \frac{1}{r}\left(\frac{\partial u}{\partial n} - \frac{\partial v}{\partial s}\right)\frac{\partial r}{\partial n}\right] \tag{9-83a}$$

$$\frac{u^2}{r_c} + u\frac{\partial v}{\partial s} + v\frac{\partial v}{\partial n} = -\frac{1}{\rho}\frac{\partial p}{\partial n} + \nu\left[\frac{\partial^2 v}{\partial s^2} - \frac{\partial^2 u}{\partial s\,\partial n} + \frac{1}{r}\left(\frac{\partial v}{\partial s} - \frac{\partial u}{\partial n}\right)\frac{\partial r}{\partial s}\right] \tag{9-83b}$$

$$\frac{\partial}{\partial s}(ru) + \frac{\partial}{\partial n}(rv) = 0 \tag{9-83c}$$

where u^2/r_c in Eq. (9-83b) is the centrifugal force per unit mass. Using the method employed in § 9.2 by assuming $v \ll u$ and $n \ll s$, the first-order approximation of Eqs. (9-83) for large Reynolds numbers can be written[9]

$$u\frac{\partial u}{\partial s}+v\frac{\partial u}{\partial n}=-\frac{1}{\rho}\frac{\partial p}{\partial s}+\nu\left[\frac{\partial^2 u}{\partial n^2}+\frac{1}{r}\frac{\partial u}{\partial n}\frac{\partial r}{\partial n}\right] \tag{9-84a}$$

$$-\frac{\partial p}{\partial n}=\frac{u^2}{r_c} \tag{9-84b}$$

$$\frac{\partial(ru)}{\partial s}+\frac{\partial(rv)}{\partial n}=0 \tag{9-84c}$$

Equations (9–84) are the approximate boundary layer equations for an axially symmetrical flow. If the radius of curvature of the boundary, r_c, is large compared with the thickness of the boundary layer, Eqs. (9–84) reduce to

$$u\frac{\partial u}{\partial s}+v\frac{\partial u}{\partial n}=-\frac{\partial p}{\partial s}+\nu\frac{\partial^2 u}{\partial n} \tag{9-85a}$$

$$\frac{\partial p}{\partial n}=0 \tag{9-85b}$$

$$\frac{\partial(r_0 u)}{\partial s}+\frac{\partial(r_0 v)}{\partial n}=0 \tag{9-85c}$$

where $r_0=r-n\cos\alpha=r-n\sqrt{1+(dr_0/ds)^2}\doteq r$. It is seen that the only difference between Eqs. (9–85) and the corresponding equations [see Eqs. (9–16)] for a two-dimensional curved surface is in the continuity equation.

Mangler (ZAMM **28**, 1948, p. 97) suggested that the system of Eqs. (9–85) can be transformed into a system of the corresponding equations for two dimensional flow by the following transformations:

$$\bar{s}=\int_0^s\frac{r_0^2(s)}{L^2}\,ds,\qquad \bar{n}=\frac{r_0(s)}{L}n$$

$$\bar{\psi}(\bar{s},\bar{n})=\frac{\psi(s,n)}{L},\qquad \bar{p}(\bar{s})=p(s) \tag{9-86a}$$

where L is a fixed reference length.

According to Eqs. (7–70), the steam function ψ is defined by

$$r_0 u=\frac{\partial\psi}{\partial n},\qquad -r_0 v=\frac{\partial\psi}{\partial s} \tag{9-86b}$$

The velocity components $\bar{u}$ and $\bar{v}$ are given by

$$\bar{u}=\frac{\partial\bar{\psi}}{\partial\bar{n}},\qquad -\bar{v}=\frac{\partial\bar{\psi}}{\partial\bar{s}} \tag{9-86c}$$

In terms of the barred variables, we have

$$u=\frac{1}{r_0}\frac{\partial\psi}{\partial n}=\frac{L}{r_0}\frac{\partial\bar{\psi}}{\partial\bar{n}}\frac{\partial\bar{n}}{\partial n}=\bar{u} \tag{9-87d}$$

$$v=-\frac{1}{r_0}\frac{\partial\psi}{\partial s}=-\frac{L}{r_0}\left[\frac{\partial\bar{\psi}}{\partial\bar{s}}\frac{\partial\bar{s}}{\partial s}+\frac{\partial\bar{\psi}}{\partial\bar{n}}\frac{\partial\bar{n}}{\partial\bar{s}}\right]$$

$$=\frac{r_0}{L}\bar{v}-\frac{r_0'}{r_0}n\bar{u} \tag{9-87e}$$

$$u\frac{\partial u}{\partial s}+v\frac{\partial u}{\partial n}=\bar{u}\left[\frac{\partial \bar{u}}{\partial \bar{s}}\frac{\partial \bar{s}}{\partial s}+\frac{\partial \bar{u}}{\partial \bar{n}}\frac{\partial \bar{n}}{\partial s}\right]$$

$$+\left[\frac{r_0}{L^2}\bar{r}-\frac{r_0'}{r_0}n\bar{u}\right]\frac{\partial \bar{u}}{\partial \bar{n}}\frac{\partial \bar{n}}{\partial n}=\frac{r_0^2}{L^2}\left[\bar{u}\frac{\partial \bar{u}}{\partial \bar{s}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{n}}\right] \tag{9-87f}$$

$$\frac{\partial p}{\partial s}=\frac{\partial \bar{p}}{\partial \bar{s}}\frac{\partial \bar{s}}{\partial s}=\frac{r_0^2}{L^2}\frac{\partial \bar{p}}{\partial \bar{s}} \tag{9-87g}$$

$$\frac{\partial p}{\partial s}=\frac{r_0}{L}\frac{\partial \bar{p}}{\partial \bar{n}} \tag{9-87h}$$

$$\nu\frac{\partial^2 u}{\partial n^2}=\nu\frac{\partial^2 \bar{u}}{\partial \bar{n}^2}\left(\frac{\partial \bar{n}}{\partial n}\right)^2=\nu\frac{r_0^2}{L^2}\frac{\partial^2 \bar{u}}{\partial \bar{n}^2} \tag{9-87i}$$

$$\frac{\partial}{\partial \bar{s}}(r_0u)+\frac{\partial}{\partial n}(r_0v)=\left[r_0'\bar{u}+r_0\frac{\partial \bar{u}}{\partial \bar{s}}\frac{\partial \bar{s}}{\partial s}+r_0\frac{\partial \bar{u}}{\partial \bar{n}}\frac{\partial \bar{n}}{\partial \bar{s}}\right]$$

$$+\left[\frac{r_0^2}{L}\frac{\partial \bar{r}}{\partial \bar{n}}\frac{\partial \bar{n}}{\partial \bar{n}}-r_0'\bar{u}-r_0'n\frac{\partial \bar{u}}{\partial \bar{n}}\frac{\partial \bar{n}}{\partial n}\right]$$

$$=\frac{r_0^3}{L^2}\left[\frac{\partial \bar{u}}{\partial \bar{s}}+\frac{\partial \bar{r}}{\partial \bar{n}}\right] \tag{9-87j}$$

Substituting Eqs. (9-87f) — (9-87j) into Eqs. (9-85), we have

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{s}}+\bar{r}\frac{\partial \bar{u}}{\partial \bar{n}}=-\frac{\partial \bar{p}}{\partial \bar{s}}+\nu\frac{\partial^2 \bar{u}}{\partial \bar{n}^2} \tag{9-88a}$$

$$\frac{\partial \bar{p}}{\partial \bar{n}}=0 \tag{9-88b}$$

$$\frac{\partial \bar{u}}{\partial \bar{s}}+\frac{\partial \bar{r}}{\partial \bar{n}}=0 \tag{9-88c}$$

Equation (9-88) is a system of boundary layer equations which is identical to that of two-dimensional flow. The approximate solution of Eqs. (9-88) by the momentum integral method is given in §9.5 and §9.6.

9.8. *Separation of Boundary Layer Flow*

a. Mathematical Criterion

In the general Couette flow (§ 8. 3) we have discussed that the separation of flow at the wall occurs when the pressure gradient in the flow direction reaches a certain positive value. The influence of the adverse pressure gradient in this case surpasses the effect of the dragging action of the viscous force. In the case of boundary layer flow and its separation from the wall, the phenomenon may be explained by applying the Prandtl boundary layer equation [Eq. (9-16a)] both outside the boundary layer and at the wall. Outside the boundary layer, Eq. (9-16a) applied to steady flow is simply

$$U\frac{dU}{dx}=-\frac{1}{\rho}\frac{dp}{dx}$$

The expression above which is the differential form of the Bernoulli equation (neglecting body forces) reveals that the pressure distribution

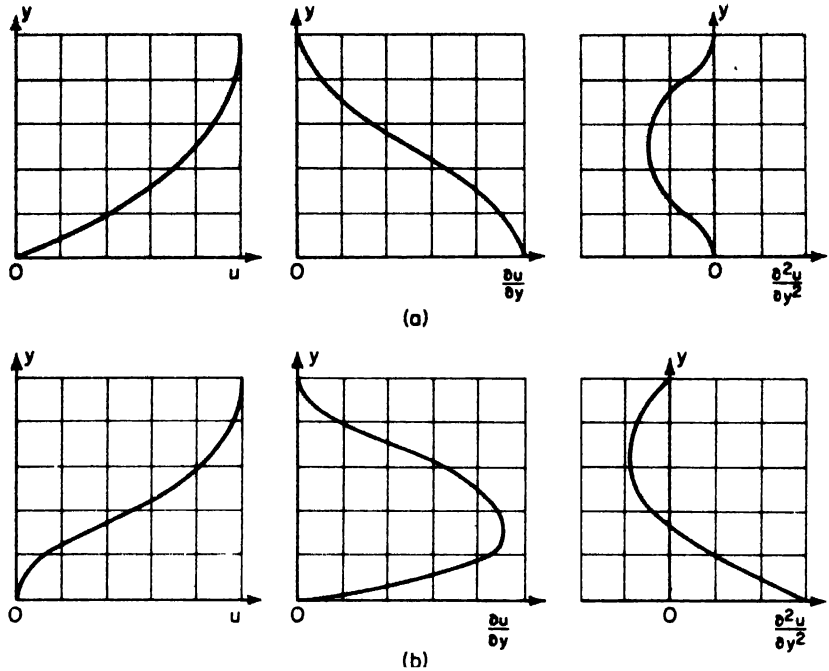


Figure 9.16

is determined by the nature of the flow outside the boundary layer. Equation (9–16b) indicates that this pressure is transmitted without change through the boundary layer to the surface.

At the wall, where $y = 0$, we have $u = 0$, $v = 0$, and Eq. (9–16a) becomes

$$\mu \left(\frac{d^2u}{dy^2}\right)_0 = \frac{dp}{dx} \tag{9–68a}$$

With reference to § 9.6 and Fig. 9.16a, it is seen that in the case of a zero pressure gradient in the flow, Eq. (9–68a) gives $(\partial^2u/\partial y^2)_0 = 0$, and hence, the velocity gradient $\partial u/\partial y$ decreases steadily from a positive value at the wall to zero at the outer edge of the boundary layer. The velocity profile must therefore have a steadily decreasing form, i.e., it must be without an inflection point. If the flow is accelerated, i.e., $dp/dx < 0$, Eq. (9–68a) gives $(\partial^2u/\partial y^2)_0 < 0$ and this negative value of $(\partial^2u/\partial y^2)_0$ (Fig. 9.12a) at $y = 0$ increases steadily to zero at $y = \delta$. The velocity gradient $\partial u/\partial y$ again decreases steadily from a positive value at the wall to zero at $y = \delta$, and the velocity profile has a form very similar to the case of zero pressure gradient except it is fuller throughout. If, however, $dp/dx > 0$, i. e., the flow is decelerated, it follows from Eq. (9–68a) that $(\partial^2u/\partial y^2)_0$ will be a positive quantity. In order to have a positive value of $\partial^2u/\partial y^2$ at $y = 0$, the slope of the

velocity gradient, $\partial u/\partial y$ at $y = 0$ must be positive. But the boundary condition requires $\partial u/\partial y = 0$ at $y = \delta$. Therefore, the slope of the velocity gradient must change signs from positive to negative in the boundary layer which results in a point of inflection of the velocity profile in the boundary layer. This discussion above is illustrated in Fig. 9.12b. It can be seen from Fig. 9.12 that the slope of the velocity profile (at the wall) under an adverse pressure gradient is much smaller than that of the case with zero pressure gradient.

As the adverse pressure gradient increases further the velocity profile may become increasingly distorted until the velocity gradient at the wall $(\partial u/\partial y)_0$ is zero, as shown in Fig. 9.16b. At this point separation of flow from the wall begins. Still further downstream from the point of separation there will be a back-flow region as the velocity component u is negative near the wall.

It may be mentioned here that the type of velocity profile shown in Fig. 9.12b is inherently unstable, and it frequently happens that the transition to turbulent flow in the boundary layer will take place before laminar separation can occur. Under these circumstances the turbulent boundary layer will be maintained and separation of flow from the wall will be delayed (see Chap. Ten).

b. Physical Examples

For an example of a boundary layer flow with pressure gradient, consider a long circular cylinder with its axis perpendicular to the direction of incoming flow. The problem is essentially two dimensional. If the fluid is assumed to be nonviscous, the pattern of the streamlines is shown in Fig. 7. 36. There will be a stagnation point on the upstream side of the cylinder intersecting the x-axis (the axis parallel to the incoming flow). The velocity of the flow along the cylinder increases continuously from the forward stagnation point until the vertical axis (the axis perpendicular to the incoming flow) is reached, beyond which the flow velocity should continuously decrease in an exactly symmetrical manner until the rear stagnation point (see § 7.18b) is reached. According to the Bernoulli equation the pressure gradient is negative in the front half of the cylinder and positive in the rear half of the cylinder. The velocity and pressure changes become less marked for flow further away from the cylinder, and at large distances they are imperceptible.

The presence of viscosity in a fluid, however, complicates the pattern of flow due to the dissipative effect of the viscous force. As mentioned in § 9.8a the fluid pressure in the boundary layer is the same as that for the corresponding location outside it. Consequently, the pressure distribution in the boundary layer is essentially that for an inviscid fluid. The flow pattern for the viscous fluid past the upstream portion

of the cylinder is essentially the same as that indicated in Fig. 7.36 for the inviscid fluid except in the region within the boundary layer. The stagnation point in this case is pushed upstream from the surface of the cylinder along the x-axis by an amount identical to thickness of the boundary layer at that point. Since the effect of the viscosity in the boundary layer is to oppose the fluid flow, the energy of the fluid particle near the top (or bottom) of the front portion of the cylinder is now lower than for that corresponding to inviscid flow. Consequently, the fluid particle motion is stopped by the adverse pressure gradient at points on the surface where the vertical axis is reached (experiments indicate the location of the points is about 9

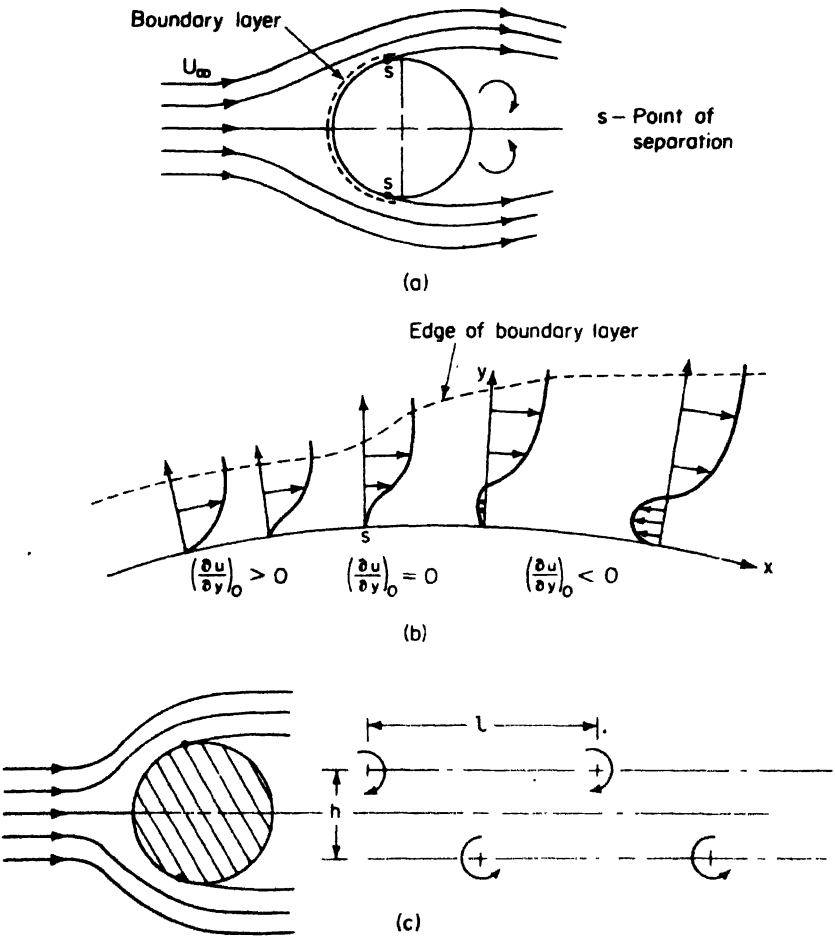


Figure 9.17

before the vertical axis is reached). Thus, the fluid particles are forced outwards from the wall, separating from it. In general, the fluid particles behind the point of separation follow the pressure gradient and move against the direction of the main flow. This flow phenomenon explains the association of the occurrence of separation with the appearance of vorticity in the fluid. Under this circumstance the boundary layer thickens considerably downstream, and the field of flow behind the cylinder changes completely in the wake. The fluid separation and the wake about a circular cylinder are shown in Fig. 9.17a, and the development of velocity distribution in the boundary layer within the region of separation is illustrated in Fig. 9.17b.

Experiments have shown that vortices formed in the separated region of the boundary layer will be shed from the cylinder and carried downstream. Upper and lower vortices may break away in an alternating fashion from the cylinder. At a greater distance downstream from the cylinder a regular pattern of two rows of vortices will be observed, rotating alternately counterclockwise and clockwise, respectively. This system of vortices is known as the *Kármán vortex street* and is shown in Fig. 9.17c. If l is the distance between the centers of consecutive vortices with the same direction of rotation and h the distance between the two lines of vortex centers, von Kármán showed that, for stability of the vortex arrangement, the ratio $h/l = 0.281$. The alternate breaking away of these vortices results in a periodic transverse force on the cylinder. If the natural frequency of the cylinder in the transverse direction is in resonance with this forced transverse oscillation, severe vibration of the cylinder can take place. The aerodynamic instability of suspension bridges and the vibration of telephone and transmission lines at certain wind velocities are familiar examples.

The separation phenomenon just described is not particular to circular cylinders alone; it is observed also in divergent-channel flow (Fig. 9.18), in flow past a step (Fig. 9.18b), and in flow around an airfoil at a high angle of attack (Fig. 9.18c).

c. Prediction of Boundary Layer Separation

The following discussion is limited only to two-dimensional laminar boundary layers. In the analysis of flow over a solid boundary with pressure gradient (e.g., a circular cylinder) two distinct regions must be examined separately: (i) the flow up to the point of separation and (ii) the separated region which contains trailing vortices. In the region up to the vicinity of separation the solutions of the boundary layer equations discussed in §9.4 and §9.6 can be applied with

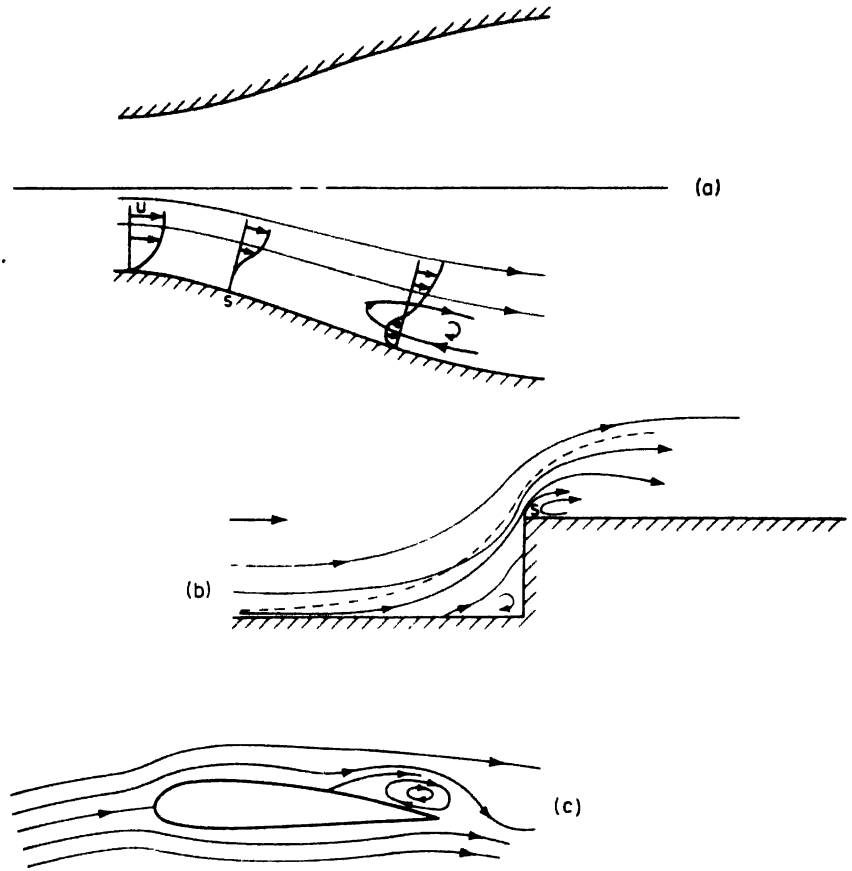


Figure 9.18

satisfactory results. However, in the separated region both methods given in § 9.4 and §9.6 do not apply. This is evident because the pressure distribution of a separated flow is usually quite different from that of an unseparated flow about the same body. Furthermore, all the assumptions made in the derivation of boundary layer equations break down in the separated region. It is clear that the Navier-Stokes equations must be used to calculate the fluid motion in the wake behind a body; but no satisfactory theory for obtaining this except for certain special cases has been found and experimental solution is the only resort at the present time.

The main difficulty in predicting the separation point of the laminar boundary layer lies in the determination of the pressure distribution for a body; this pressure distribution is not known a priori and is quite different from the calculated value based on potential flow

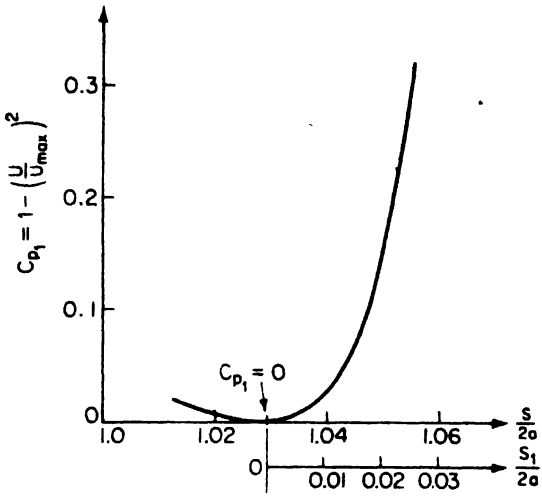


Figure 9.D

Fig. 9.D as follows :

$$A = 5.096, \qquad B = -736.754$$
$$C = 5{,}334.37, \qquad D = -540{,}582.79 \tag{d}$$

Upon the substitution of Eq. (c) for C_{p_1} and its derivative with respect to $s_1/2a$ into Eq. (b), an algebraic equation in $s_1/2a$ is obtained. The solution of this equation, which we are interested in, is found to be

$$\left(\frac{s_1}{2a}\right)_{\text{separation}} = 0.0179$$

Hence, the point of separation of the elliptical airfoil under consideration is

$$\left(\frac{s}{2a}\right) \quad 1.0469$$

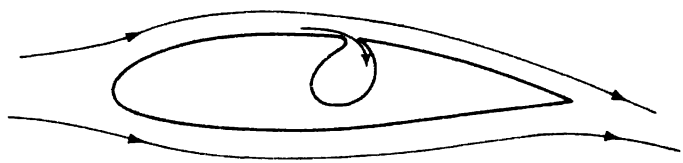
measured from the front stagnation point.

9.9. *Boundary Layer Control*

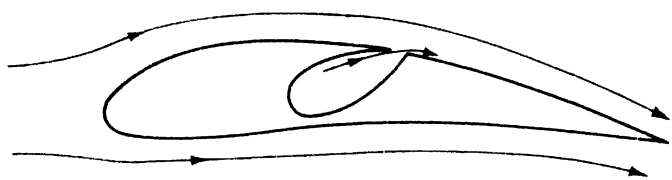
a. Methods of Boundary Layer Control

In the preceding section we have discussed separation of the boundary layer which causes a large vortex field behind a body and

consequently high drag. It is logical now to ask whether separation of the boundary layer can be prevented or, more generally, can the growth of the boundary layer be controlled ? Indeed, according to the physical concept of the boundary layer, it should be possible to delay or even prevent separation by removing the decelerated fluid particles caused by the adverse pressure gradient in the region where separation is likely to develop (see Fig. 9.19a). This possibility was first pointed out by L. Prandtl in his paper published as early as 1904. By applying suction through a small slit on the upper rear portion of a circular cylinder, Prandtl showed in his experiments that the flow adheres to the cylinder over a considerably larger portion of its surface instead of separating at 81° from the stagnation point as it does in the case when no suction is applied. The total drag is reduced considerably, and simultaneously a large lifting force in the direction perpendicular to the free stream is induced.



(a)



(b)

Figure 9.19

An alternative means of controlling the boundary layer is the supplying of additional energy to the boundary layer by injecting fluid parallel to the surface (Fig. 9.19b), thus enabling the boundary layer to proceed further against an adverse pressure gradient. The ideal method of prevention of separation would be a means to eliminate the boundary layer. This can be accomplished in principle by having a solid wall moving with the stream. The logical way to obtain such a result is to place a rotating circular cylinder with its axis at a right

angle to the flow (see § 7. 19). It is clear that on the upper surface of the cylinder where the cylinder moves (angular velocity) in the same direction as the flow, no separation is possible. However, incomplete separation is developed on the lower side of the cylinder. The resultant flow field does not vary appreciably from the potential flow theory (§ 7. 19). This example was first applied by Prandtl to illustrate the boundary layer theory, but it encounters great practical complications as far as engineering applications are concerned. Hence, the two main methods of boundary layer control are suction of the boundary layer into the surface and blowing air into the boundary layer. The case of blowing air into the boundary layer is connected mainly with high lift devices* and transpiration cooling† which are beyond the scope this text.

b. Separation Prevention by Boundary Layer Suction

In order to show the effectiveness of separation prevention by boundary layer suction, consider a laminar boundary layer on a porous flat plate with continuous suction. The use of a porous wall is a logical extension of a series of slots by increasing the number of slots infinitely and making their width infinitesimally small. This analysis is due to Griffith and Meredith[13] who derived the asymptotic velocity profile for a laminar boundary layer on a porous flat plate with suction. The condition for the asymptotic case is that the gradients of the velocity profiles with respect to x can be neglected as compared with the gradients with respect to y. Hence, the boundary layer equations [Eqs. (9-23a) and (9-23b)] reduce to

$$v\frac{du}{dy} = \nu\frac{d^2u}{dy^2} \tag{9-90a}$$

$$\frac{dv}{dy} = 0 \tag{9-90b}$$

with the boundary conditions

$$\begin{aligned} y &= 0: \quad u = 0 \qquad v = -v_w \\ y &= \delta: \quad u = U_\infty \end{aligned} \tag{9-91}$$

According to Eqs. (9-90b) and (9-91), we have

$$v = -v_w = \text{Constant} \tag{9-92}$$

Hence, Eq. (9-90a) becomes

$$\frac{(d^2u/dy^2)}{(du/dy)} = -\frac{v_w}{\nu} \tag{9-93}$$

*Yuan, S. W., and Th. von Kármán, "Rotor Control", U.S. Patent 2, 925, 129 (1960).

†Reference 56.

Solving Eq. (9–93) with proper boundary conditions [Eq. (9–91)], we obtain

$$\frac{u}{U_\infty} = 1 - e^{-v_w y/\nu} \tag{9–94}$$

It follows that the displacement thickness and momentum thickness, are, respectively,

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) dy = \frac{\nu}{v_w} \tag{9–95}$$

and

$$\theta = \int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy = \frac{\nu}{2v_w} \tag{9–96}$$

The velocity distribution in the boundary layer in terms of the momentum thickness can be written

$$\frac{u}{U_\infty} = 1 - e^{-y/2\theta} \tag{9–97}$$

A comparison between the asymptotic suction velocity profile and the corresponding profile without suction (Blasius profile in § 9. 3a) is shown in Fig. 9.20. For a given value of y/θ the asymptotic suction velocity profile is fuller then the Blasius profile. The tendency is more pronounced towards the wall. Therefore, by means of suction of the boundary layer through a porous wall (or slots) the velocity profile is made more stable. The stability of the velocity profile is not only favorable for the prevention of separation but also for delaying the transition from laminar to turbulent boundary layer.

The calculation of separation prevention by boundary layer suction for bodies of arbitrary shape may be made by approximate methods based on the Kármán–Pohlhausen momentum integral equation (§ 9.6). If we take into consideration the boundary condition that at $y = 0$, $v = -v_w(x)$, Eq. (9–56b) may be written

$$\int_0^\delta v \frac{\partial u}{\partial y} = -Uv_w - U\int_0^\delta \frac{\partial u}{\partial x} dy + \int_0^\delta u \frac{\partial u}{\partial x} dy \tag{9–98}$$

Following the same procedure as in § 9.5a, the momentum integral equation of the boundary layer with suction is obtained:

$$\frac{d}{dx}\int_0^\delta u(U-u)dy + \frac{dU}{dx}\int_0^\delta (U-u)dy + Uv_w = \frac{\tau_0}{\rho} \tag{9–99}$$

The additional term $v_w U$, as compared with Eq. (9–63), represents the change of momentum due to suction at the wall. Equation (9–99) can be written in terms of the momentum thickness and displacement thickness as

$$U^2\frac{d\theta}{dx} + \left(2 + \frac{\delta^*}{\theta}\right)\theta U\frac{dU}{dx} + Uv_w = \frac{\tau_0}{\rho} \tag{9–100}$$

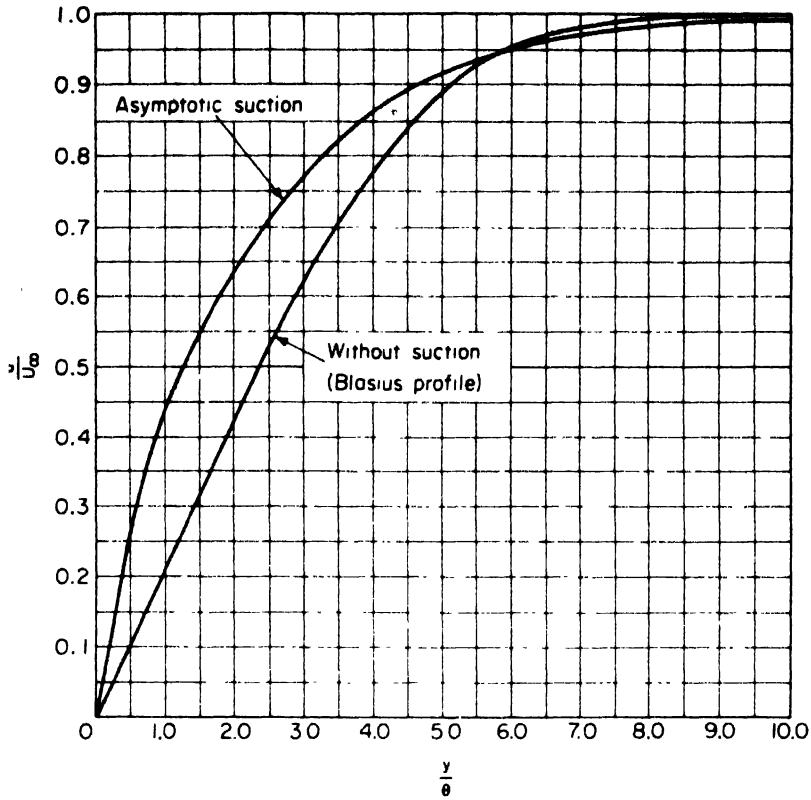


Figure 9.20

where θ and δ^* are given in Eq. (9-66).

There are several possibilities in the solution of the nonlinear differential equation, Eq. (9-100): (i) to calculate the suction-velocity distribution along the wall for an arbitrarily assumed distribution of boundary layer thickness, (ii) for a given suction-velocity distribution along the wall, the variation of boundary layer thickness can be determined, and (iii) a desired shape of velocity profile in the boundary layer can be maintained by means of suction, and hence separation prevented.

Prandtl employed Eq. (9-100) to make an approximate estimate of the suction velocity required to prevent separation. In order to simplify the calculations the Pohlhausen velocity profile for $\Lambda = -12$ is maintained by suction along the whole length of the wall. For $\Lambda = -12$, Eq. (9-71a) becomes

$$\frac{u}{U} = 6\eta^2 - 8\eta^3 + 3\eta^4 \tag{9-101}$$

The slope of this velocity profile is equal to zero at the wall. The displacement and momentum thicknesses as calculated from Eq. (9-66) with the velocity profile of Eq. (9-101) are, respectively,

$$\delta^* = \delta \int_0^1 (1 - 6\eta^2 + 8\eta^3 - 3\eta^4) d\eta = \frac{2}{5}\delta$$

and

$$\theta = \delta \int_0^1 (6\eta^2 - 8\eta^3 + 3\eta^4)(1 - 6\eta^2 + 8\eta^3 + 3\eta^4)\, d\eta = \frac{4}{35}\delta$$

With the values above for δ^* and θ and the assumption that the thickness of the boundary layer is constant, Eq. (9-100) becomes

$$v_w = \frac{22}{35}\delta \frac{dU}{dx} \tag{9-102}$$

According to the boundary condition at the wall Eq. (9-19a) takes the form

$$-v_w \left(\frac{\partial u}{\partial y}\right)_0 = U \frac{dU}{dx} + \nu \left(\frac{\partial^2 u}{\partial y^2}\right)_0 \tag{9-103}$$

The term on the left-hand side of Eq. (9-103) is equal to zero, hence

$$\delta = -\frac{12\nu}{dU/dx} \tag{9-104}$$

Substituting this value of δ into Eq. (9-102), we have

$$\frac{v_w}{U_\infty} = \frac{44\sqrt{3}}{35} \sqrt{\frac{\nu}{U_\infty l}} \left| \frac{d(U/U_\infty)}{d(x/l)} \right| \tag{9-105}$$

The suction velocity given in Eq. (9-105) is the magnitude just sufficient to prevent separation all along the wall. In order to illustrate the numerical amount of v_w a flow past a circular cylinder of radius R is considered. The velocity in the potential flow for this case is given by Eq. (7-153) and the velocity gradient at the rear stagnation point is

$$\frac{dU}{ds} = -\frac{2U_\infty}{R}$$

where s is the arc length. The suction velocity required to prevent separation is given by

$$\frac{v_w}{U_\infty} = \frac{44\sqrt{3}}{35} \frac{\sqrt{2}}{\sqrt{U_\infty R/\nu}} = \frac{3.08}{\sqrt{U_\infty R/\nu}}$$

It is seen that for a Reynolds number equal to 10^4 the velocity of

suction required to prevent separation is about 1 per cent of the free-stream velocity.

NOMENCLATURE FOR CHAPTER NINE

C_F	average skin-friction coefficient, Eq. (9-40)
C_f	local skin-friction coefficient, Eq. (9-38)
F	frictional force per unit width, Eq. (9-39)
$F(\eta)$	nondimensional velocity profile, Eq. (9-25)
$f(\eta)$	(1) nondimensional stream function, Eq. (9-28) (2) nondimensional velocity profile, Eq. (9-67)
l	length of the plate, Eq. (9-39)
m	power of x in potential flow velocity, Eq. (9-44)
p	pressure, Eq. (9-11a)
P_r	Prandtl number, Eq. (9-10)
$\mathbf{q}$	velocity vector, Eq. (9-6)
r	distance of a point outside of the body of revolution from z-axis, Fig. 9.15
Re, Re_x	Reynolds number, Eqs. (9-10) and (9-38)
s, n	coordinates, tangential and normal to the surface of the body of revolution, Eq. (9-82) and Fig. 9.15
U_∞	free-stream velocity, Eq. (9-25)
$U(x)$	velocity of the potential flow, Eq. (9-44)
u, v, w	velocity in Cartesian coordinates
v_w	suction velocity at the wall
x, y, z	Cartesian coordinates
z	momentum thickness parameter, Eq. (9-74a)
δ	boundary layer thickness, Eq. (9-12)
δ^*	displacement thickness in the boundary layer, Eq. (9-42b)
η	nondimensional distance parameter, Eqs. (9-27) and (9-67)
η_1	nondimensional distance parameter, Eq. (9-47)
θ	momentum thickness in the boundary layer, Eq. (9-43b)
κ	(1) ratio of specific heats, Eq. (9-10) (2) a variable parameter, Eq. (9-74b)
Λ	form parameter, Eq. (9-71a)
ν	kinemetric viscosity, Eq. (9-6)
τ_0	wall shearing stress, Eq. (9-39)
ψ	stream function, Eq. (9-20)
$\boldsymbol{\Omega}$	vorticity vector, Eq. (9-6)
$(\ )^*$	nondimensional quantity, Eq. (9-12)

PROBLEMS

9.1. Equation (9-3) is a solution of the differential equation, Eq. (9-1), for $\epsilon \ll 1$. Find the exact solution of Eq. (9-1) for any ϵ and compare it with Eq. (9-3) for $\epsilon \ll 1$.

9.2. Following the same procedure of § 9.2, derive the boundary layer equations for two-dimensional flow in the cylindrical coordinates from Eqs. (5-54a) and (5-54b).

9.3. Using the first three terms of the series for f in Eq. (9-34) compare the approximate velocity profile with the exact solution. Also calculate the local skin-friction coefficient, the displacement thickness, and the momentum thickness from the approximate expression for f.

9.4. Based on the momentum integral equation calculate the local frictional coefficient, C_f, for the following velocity profiles on a flat plate:

(a) $\frac{u}{U_\infty} = \eta$

(b) $\frac{u}{U_\infty} = \frac{3}{2}\eta - \frac{1}{2}\eta^3$

(c) $\frac{u}{U_\infty} = 2\eta - 2\eta^3 + \eta^4$

9.5. Calculate $(\delta^*/x)\sqrt{\mathrm{Re}_x}$ and $(\theta/x)\sqrt{\mathrm{Re}_x}$ for each of the three velocity profiles in Prob. 9.4.

9.6. Using Eq. (9-75), obtain the expression for δ^*, θ, and τ_0, for the following velocity profiles:

(a) $\frac{u}{U} = 2\eta - \eta^5$

(b) $\frac{u}{U} = \sin\left(\frac{\pi}{2}\eta\right)$

9.7. If a polynomial of the fifth degree is assumed for the velocity profile in the boundary layer, determine the five constants from the appropriate boundary conditions.

9.8. A one-parameter family for the velocity distribution in the boundary layer may be expressed by

$$\frac{u}{U} = 1 - e^{-\eta} + \mathrm{k}\left(1 - e^{-\eta} - \sin\frac{\pi}{6}\eta\right), \qquad 0 \leq \eta \leq 3$$

$$\frac{u}{U} = 1 - e^{-\eta} - \mathrm{k}e^{-\eta}, \qquad 3 \leq \eta$$

where $\eta = y/\delta$. Show that the velocity profile above satisfies the following boundary conditions:

(a) $y = 0$: $u = 0$, and $v = 0$

(b) $y = \infty$: $u = U$, and $\partial u/\partial y = \partial^2 u/\partial y^2 = 0$

and determine k from an appropriate boundary condition.

9.9. Obtain the momentum and displacement thicknesses and the wall shearing stress from the velocity profile in Prob. 9.8 and compare them with the Blasius exact solution.

9.10. Plot the following velocity profiles for a flat plate:
(a) Blasius solution (b) Prob. 9.6
(c) Pohlhausen velocity profile (d) Prob. 9.8

9.11. From the momentum equation for the boundary layer, Eq. (9-65), show that the expression for the frictional drag coefficient is given by

$$C_f\sqrt{\mathrm{Re}_l} = \frac{2}{\sqrt{z_1}}\left(\frac{u}{U_\infty}\right)\left[\frac{1}{2}\left(\frac{u}{U_\infty}\right)\frac{dz_1}{d(x/l)} + \left(2 + \frac{\delta^*}{\theta}\right)z_1\frac{d}{d(x/l)}\left(\frac{u}{U_\infty}\right)\right]$$

where $C_f = \tau_0/\frac{1}{2}\rho U_\infty^2$, $\mathrm{Re}_l = U_\infty l/\nu$, and $z_1 = \theta^2 U_\infty/\nu l$.

9.12. Using the velocity profile in the boundary layer given in Prob. 9.8 show that the differential equation for the momentum thickness, Eq. (9-65), can be reduced to the following form:

$$\frac{dz}{dx} = \frac{2}{U}[g(\Lambda)][f\Lambda)]$$

where

$$g(\Lambda) = \frac{\theta}{\delta} = 0.4099 + 0.1137\Lambda - 0.02358\Lambda^2$$

$$f(\Lambda) = 0.5236 - 1.433\Lambda - 0.1373\Lambda^2 + 0.04716\Lambda^3$$

$$z = \frac{\theta^2}{\nu}, \qquad \text{and} \qquad \Lambda = \frac{\delta^2}{\nu}\frac{dU}{dx}.$$

9.13. Using the simplified solution of the momentum integral equation calculate the wall shearing stresses for the following cases:
(a) flow past a flat plate,
(b) flow past a wedge.

9.14. The potential flow velocity for flow past a circular cylinder is given by $u/U_\infty = 2\sin\theta$. Determine the point of separation by using Eq. (9-89).

9.15. Determine the required suction velocity ratios, v_w/U_∞ in order that a velocity profile in the boundary layer with $\Lambda = -6$ and $\Lambda = 0$, respectively, can be maintained.

10

INTRODUCTION TO TURBULENT FLOW

The flow of fluids through conduits or past bodies is rarely as simple as that described in the preceding two chapters. In general, a complete laminar flow in the vicinity of a solid wall is a rather exceptional case such as the flow of lubricating oil, of water through pipes of small cross section, and the flow of very viscous liquids. On the other hand, whenever we have the opportunity to observe the natural state of motion, we find the motion with irregular velocity fluctuations such as the flow of water in rivers or the motion of air near the surface of the earth. This type of motion is called turbulent.

The first systematic experimental investigations of turbulent flow were made by Osborne Reynolds.[1] In his classical experiments, Reynolds used glass tubes with flowing water from a reservoir and observed the flow pattern by injecting a thin stream of dye into the main stream (Fig. 10.1). If the velocity of the water is sufficiently low, the colored filament of dye remains straight and parallel to the walls of the tube which indicates that the flow is steady and orderly in behavior. However, if the flow velocity is increased beyond a certain value, the colored filament begins to oscillate and finally loses its identity and diffuses through the tube.

The first type of flow is clearly laminar, which has been discussed in Chap. Eight. The fluid appears to move with parallel layers, one layer of fluid sliding over the other. The second type of flow is called

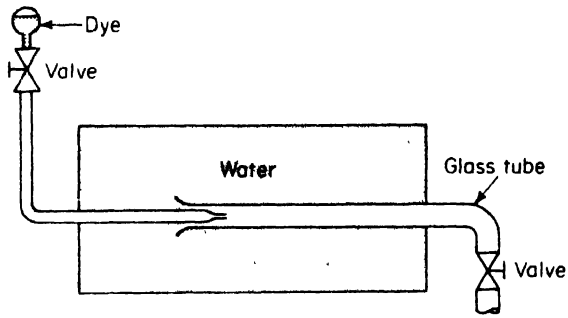


Figure 10.1

turbulent which is characterized by rapid and continuous mixing of the fluid in a rather chaotic manner throughout the tube. Since the Reynolds number is defined as the ratio of inertial force to frictional force, the frictional forces tend to constrain the motion of the fluid in parallel layers when the Reynolds number is low. On the other hand when the Reynolds number is large, the inertial forces, which dominate in the flow, tend to diffuse the fluid particles, hence causing the mixing of the fluid.

The discussion so far has hinged upon only the physical phenomenon of the disorderliness and random irregularities of turbulent flow; yet we do not have a clear-cut definition of *turbulence*. This is rather difficult. In his book *Turbulence*, Hinze[2] suggests, "Turbulent fluid motion is an irregular condition of flow in which the various quantities show a random variation with time and space coordinates, so that statistically distinct average values can be discerned." In §3.1 we have shown that an unsteady flow in one space reference (an observer at a fixed location) can often be transferred to another space reference (an observer moving with the fluid) where the flow becomes steady. Furthermore, if a motion is irregular in space alone, it becomes a steady flow. Hence, it is not sufficient to define turbulent flow as irregular merely in time alone or in space alone.

Actually, the discrete molecules which make up the fluid are incessantly bombarded from all sides so that even in laminar flow there always exist random irregularities. Since we are interested in continuum theory, the random irregularities of molecules are unrelated to the average motion of fluid particles. On the other hand, the random irregularities of turbulent flow are on a much larger scale and are related to the bulk motion of the fluid. In fact, these fluctuating values of the velocity can be measured by a fine measuring instrument such as the hot-wire anemometer.[3]

According to the definition suggested by Taylor and von Kármán,[4] turbulence can be generated by fluid flow past solid surfaces or by the

flow of layers of fluids at different velocities past or over one another. The definition above indicates that there are two distinct types of turbulence: (1) Turbulence generated by the viscous effect due to the presence of a solid wall is designated by *wall turbulence*; (2) Turbulence, in the absence of a wall, generated by the flow of layers of fluids at different velocities is called *free turbulence*. Turbulent flow through conduits and past bodies are examples of wall turbulence, and turbulent jet mixing regions and wakes fall into the category of free turbulence.

The necessity of studying turbulence and its relation to practical problems needs little explanation. For the flow in a pipe, the pressure drop for turbulent flow may be hundreds of times greater than the value computed by the laminar flow theory at the same Reynolds number. There are many practical problems such as boundary layer, friction, and the heat transfer and diffusion of fluids which cannot be calculated correctly without the consideration of turbulence. In this chapter we will discuss only elementary topics of wall turbulence. The reader should refer to advanced texts for a more comprehensive treatment.

In §10.1 the origin of turbulence based on instability of the laminar flow theory is briefly discussed. Factors affecting transition from laminar to turbulent flow are also mentioned.

In §10.2 Reynolds modification of the Navier-Stokes equations for turbulent flow is derived. The significance of Reynolds stresses is explained.

In §10.3 semiempirical theories of turbulence are discussed. These include both Prandtl's mixing length theory and the von Kármán similarity hypothesis.

In §10.4 the expression for the universal velocity profile near a wall is derived from the Reynolds equations with the aid of Prandtl's mixing length theory.

In §10.5 the Blasius formula for the frictional resistance of smooth pipes and the Blasius $\frac{1}{7}$th power law for the velocity distribution based on empirical data are given. The universal velocity distribution and the universal resistance law for smooth pipes are derived. The logarithmic velocity distribution and the universal resistance formula for fully rough pipes are also discussed.

In §10.6 the average skin-friction coefficient of a smooth flat plate is obtained for the turbulent boundary layer flow. The reduced skin-friction coefficient which accounts for the laminar flow at the front portion of the plate is also presented.

10.1. *The Origin of Turbulence*

In the previous discussion we have mentioned that Reynolds used a dye experiment to investigate the circumstances of the transition

from laminar to turbulent flow. Based on his experimental results Reynolds concluded that transition from laminar to turbulent flow in pipes always occurred at nearly the same Reynolds number, $v_{z_{av}}(2R)/\nu$ [see Eq. (8–54)]. The approximate value of the critical Reynolds number, Re_{cr} at which the laminar regime breaks down was established to be the order of 2×10^3. Later, with Reynolds' apparatus, Ekman[5] was able to maintain laminar flow up to a critical Reynolds number of 4×10^4 when the testing conditions were made extremely free from disturbances. This suggested that the upper limit of the critical Reynolds number depends very strongly on the initial disturbance as it increases with the decrease of the disturbance in the flow. In spite of the uncertainty of the upper limit of the critical Reynolds number, there exists a lower limit for a critical Reynolds number below which the flow always remains laminar. For flow through a circular pipe with smooth walls this lower critical Reynolds number is established as being approximately 2×10^3. This led one to believe that laminar flow was stable for an infinitesimal disturbance, and transition occurred as a result of an external disturbance of finite magnitude.

The critical Reynolds number which we have just discussed has considerable practical significance in connection with the origin of turbulence. It defines a certain definite limit below which all the initial disturbances in the flow will be damped out, and the laminar flow becomes stable. The theoretical problem of the stability of laminar flow[6] has been investigated by many scientists and mathematicians since Reynolds' time. These investigations are based on the method of small disturbances. In other words, they are based on the assumption that laminar flows are influenced by certain small disturbances.

The problem of a two-dimensional laminar boundary layer of a flat plate under constant pressure was successfully investigated by Tollmien[7] and Schlichting.[8] The theory consists of a time- and space-dependent disturbance which is superimposed on the main flow. If the disturbances decay with time, the main flow is considered to be stable; however, there exists the possibility of transition to a turbulent flow when the disturbances increase with time. The mathematical discussion of Tollmien's analysis is too complicated and involved to be presented here. The result of the analysis for the prescribed similar velocity profile of Blasius can be represented graphically in an $\alpha\delta^*$ vs. $U_\infty\delta^*/\nu$ diagram where $\lambda = 2\pi/\alpha$ is the wavelength of the disturbance and δ^* the displacement thickness. In Fig. 10.2 the curve of neutral stability is shown. It can be seen that instability of the flow exists when the Reynolds number $U_\infty\delta^*/\nu$ exceeds the critical value of 420. This is the point of instability for the laminar boundary layer of a flat plate.

In a series of remarkable experiments performed in a special low-

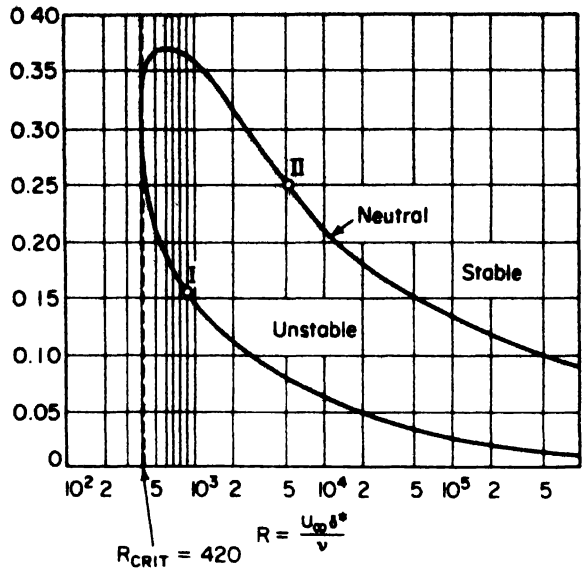


Fig 10.2. H Schlichting, "Boundary Layer Theory," New York: McGraw-Hill, 1960

turbulence type wind tunnel, Schubauer and Skramstad[9] obtained good agreement between the experimental data and Tollmien's[7] theoretical predictions. On the other hand, if the free-stream turbulence is the controlling factor, the transition to turbulence may originate from momentary separation produced by fluctuating pressure gradients accompanying the free-stream turbulence. This explanation of the origin of turbulence was advanced by Sir Geoffrey Taylor[10] and verified by experiments.

In addition to the free-stream turbulence which has an adverse effect on the stability of laminar flow, other factors[11] affecting transition from laminar flow to a turbulent flow are: (a) The influence of the pressure gradient on transition is demonstrated by Schubauer and Skramstad's[9] experimental results. For accelerated flow ($dp/dx < 0$ and $dU/dx > 0$) the critical Reynolds number, Re_{cr} increases, whereas for retarded flow ($dp/dx > 0$ and $dU/dx < 0$) Re_{cr} decreases. The latter case was also explained in Chap. Nine. (b) The effect of roughness of the wall on transition is to decrease the critical Reynolds number. This is clear because roughness would be equivalent to additional disturbances in the laminar stream. (c) Curvature has negligible effect on convex surfaces (the radius of curvature is negative), but the critical Reynolds number was found to decrease as the radius of curvature of a concave surface decreases. (d) Suction of the boundary layer at the wall has a very powerful influence on stabilizing the laminar flow; but, injection

will tend to decrease the critical Reynolds number. (e) For a heated wall when $T_w > T_\infty$, the transfer of heat from the wall to the gas tends to destabilize the laminar boundary layer, whereas the heat transfer from the gas to the wall ($T_w < T_\infty$) increases the critical Reynolds number.

10.2. Reynolds Modification of the Navier-Stokes Equations for Turbulent Flow

According to the preceding discussion we see that the physical nature of turbulence and its development is far from understandable; however, it can be studied both experimentally and theoretically as a statistical phenomenon. By this approach it is hoped that some important aspects of the structure of turbulent flows might be revealed. Topics on statistical approaches to turbulence are beyond the scope of this text.

a. Mean Values and Fluctuations

In the mathematical description of turbulent flow it is convenient to assume that the motion consists of a mean flow and a fluctuation about the mean value (Fig. 10.3). Let u, v, w be the velocity compo-

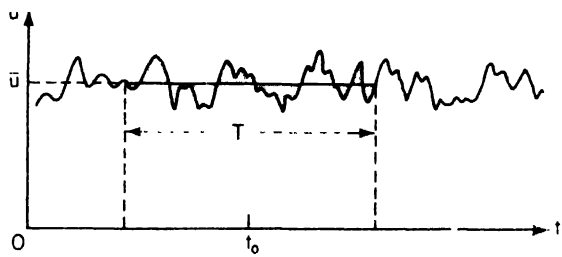


Figure 10.3

nents relative to Cartesian axes (x, y, z); however, they are also a function of time. The mean velocity components of the flow may be taken either with respect to time at a fixed point (temporal mean) or with respect to one of the coordinates at a given instant of time (spatial mean). By definition, the temporal mean of any quantity $q(x, y, z, t)$ at a time t_0 will be denoted by $\bar{q}$,

$$\frac{1}{T}\int_{t_0-T/2}^{t_0+T/2} q\,dt \tag{10-1}$$

where T is the period of sampling which must be sufficiently large to include an adequate number of fluctuations. When $\bar{q}$ is independent of

T and t_0, T can be taken to be infinite. It follows that the mean velocity components (Fig. 10.3) can be expressed by the relations

$$\bar{u} = \frac{1}{T}\int_{t_0-T/2}^{t_0+T/2} u\,dt; \qquad \bar{v} = \frac{1}{T}\int_{t_0-T/2}^{t_0+T/2} v\,dt; \qquad \bar{w} = \frac{1}{T}\int_{t_0-T/2}^{t_0+T/2} w\,dt \tag{10-2}$$

where u, v, and w are independent of T (quasi-steady flow). The fluctuating velocity component is defined as the difference between the total velocity component at any instant and the mean velocity component. Thus

$$u'(t) = u(t) - \bar{u}; \qquad v'(t) = v(t) - \bar{v}; \qquad w'(t) = w(t) - \bar{w};$$
$$p'(t) = p(t) - \bar{p} \tag{10-3}$$

According to Eqs. (10–2) and (10–3) the temporal mean value of u' is

$$\overline{u'} = \frac{1}{T}\int_{t_0-T/2}^{t_0+T/2} u'\,dt = \frac{1}{T}\int_{t_0-T/2}^{t_0+T/2} u\,dt - \frac{\bar{u}}{T}\int_{t_0-T/2}^{t_0+T/2} dt$$

or

$$\overline{u'} = \bar{u} - \bar{u} = 0 \tag{10-4}$$

Similarly,

$$\overline{v'} = \overline{w'} = 0 \tag{10-5}$$

Let us proceed to calculate the time average of the following quantities which will be found useful in the next section. By Eqs. (10–2) and (10–3) we find that

$$u^2 = (\bar{u} + u')^2 = \bar{u}\bar{u} + 2\bar{u}u' + u'^2 \tag{10-6}$$

and

$$\text{averaging } (u^2) = \bar{u}\bar{u} + 2\overline{\bar{u}u'} + \overline{u'^2}$$

but

$$\overline{\bar{u}u'} = \frac{1}{T}\int_{t_0-T/2}^{t_0+T/2} \bar{u}u'\,dt = \frac{\bar{u}}{T}\int_{t_0-T/2}^{t_0+T/2} u'\,dt = 0 \tag{10-7}$$

hence,

$$\text{averaging } (u^2) = \bar{u}\bar{u} + \overline{u'^2} \tag{10-8}$$

Similarly,

$$\text{averaging } (uv) = \overline{(\bar{u} + u')(\bar{v} + v')} = \bar{u}\bar{v} + \overline{\bar{u}v'} + \overline{\bar{v}u'} + \overline{u'v'}$$
$$= \bar{u}\bar{v} + \overline{u'v'} \tag{10-9a}$$

$$\text{averaging } (uw) = \bar{u}\bar{w} + \overline{u'w'} \tag{10-9b}$$

$$\text{averaging } (vw) = \bar{v}\bar{w} + \overline{v'w'} \tag{10-9c}$$

Furthermore, if η denotes any one of the variables x, y, z, t, then we have the following rules of average:

$$\overline{\frac{\partial u}{\partial \eta}} = \frac{\partial \bar{u}}{\partial \eta} \quad \text{and} \quad \overline{\int u \, d\eta} = \int \bar{u} \, d\eta \tag{10-10}$$

Illustrative Example 10.1. Show that

$$\overline{\left(\frac{\partial F}{\partial t}\right)} = \frac{\partial \bar{F}}{\partial t} \tag{a}$$

Solution. Consider $F = F(x, y, z, t)$. According to Eq. (10-2) we can write

$$\overline{\left(\frac{\partial F}{\partial t}\right)} = \frac{1}{T}\int_{t_0-T/2}^{t_0+T/2} \frac{\partial F}{\partial t}\, dt = \frac{1}{T}\left[F(x, y, z, t)\right]_{t_0-T/2}^{t_0+T/2}$$

$$= \frac{1}{T}\left[F\left(x, y, z, t_0 + \frac{T}{2}\right) - F\left(x, y, z, t_0 - \frac{T}{2}\right)\right] \tag{b}$$

or

$$\bar{F} = \frac{1}{T}\int_{t_0-T/2}^{t_0+T/2} F\, dt \tag{c}$$

Differentiating Eq. (c) with respect to t we obtain

$$\frac{\partial \bar{F}}{\partial t} = \frac{1}{T}\frac{\partial}{\partial t}\int_{t_0-T/2}^{t_0+T/2} F\, dt = \frac{1}{T}\left[F(x, y, z, t)\right]_{t_0-T/2}^{t_0+T/2}$$

$$= \frac{1}{T}\left[F\left(x, y, z, t_0 + \frac{T}{2}\right) - F\left(x, y, z, t_0 - \frac{T}{2}\right)\right] \tag{d}$$

Since the right-hand sides of Eqs. (b) and (d) are identical we have

$$\overline{\left(\frac{\partial F}{\partial t}\right)} = \frac{\partial \bar{F}}{\partial t}$$

b. Reynolds Equations and Reynolds Stresses

If the equation of continuity for an incompressible flow is multiplied by ρu,

$$\rho u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \tag{5-9b}$$

and substituted into the equation of motion in the x-direction without a body force [Eq. (5-15)] we find

$$\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(\sigma_{xx} - \rho u^2) + \frac{\partial}{\partial y}(\sigma_{yx} - \rho uv) + \frac{\partial}{\partial z}(\sigma_{zx} - \rho uw) \tag{10-11a}$$

Similar equations for motion in the y and z-directions are

$$\rho \frac{\partial v}{\partial t} = \frac{\partial}{\partial x}(\sigma_{xy} - \rho vu) + \frac{\partial}{\partial y}(\sigma_{yy} - \rho v^2) + \frac{\partial}{\partial z}(\sigma_{zy} - \rho vw) \tag{10-11b}$$

$$\rho\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}(\sigma_{xz} - \rho wu) + \frac{\partial}{\partial y}(\sigma_{yz} - \rho wv) + \frac{\partial}{\partial z}(\sigma_{zz} - \rho w^2) \qquad (10\text{–}11c)$$

Equations (10–11) are simply another form for the equations of motion. The term ρu^2 in Eq. (10–11a) is the momentum leaving in the x-direction per unit surface per unit time; hence $(\partial/\partial x)\rho u^2$ represents the change of momentum in the x-direction per unit volume per unit time which is equivalent to a normal force per unit volume acting in the x-direction. The term ρuv represents the product of the mass leaving in the x-direction per unit surface per unit time and the velocity in the y-direction. This rate of transport of momentum per unit area is equivalent to a shearing stress in the x-direction (see §8.2). Similarly the terms ρv^2 and ρw^2 are equivalent to normal stresses in the y and z-directions, respectively, and ρvw and ρwu are equivalent to shearing stresses in the y and z-directions, respectively.

If we now average every term in Eqs. (5–9b) and (10–11) by using the rules given in Eqs. (10–8), (10–9), and (10–10), we have, respectively, the continuity equation

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \qquad (10\text{–}12)$$

and the equations of motion

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z}\right) = \frac{\partial}{\partial x}(\bar{\sigma}_{xx} - \rho\overline{u'^2}) + \frac{\partial}{\partial y}(\bar{\sigma}_{xy} - \rho\overline{u'v'}) + \frac{\partial}{\partial z}(\bar{\sigma}_{zx} - \rho\overline{u'w'}) \qquad (10\text{–}13a)$$

$$\rho\left(\bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + \bar{w}\frac{\partial \bar{v}}{\partial z}\right) = \frac{\partial}{\partial x}(\bar{\sigma}_{xy} - \rho\overline{u'v'}) + \frac{\partial}{\partial y}(\bar{\sigma}_{yy} - \rho\overline{v'^2}) + \frac{\partial}{\partial z}(\bar{\sigma}_{yz} - \rho\overline{v'w'}) \qquad (10\text{–}13b)$$

$$\rho\left(\bar{u}\frac{\partial \bar{w}}{\partial x} + \bar{v}\frac{\partial \bar{w}}{\partial y} + \bar{w}\frac{\partial \bar{w}}{\partial z}\right) = \frac{\partial}{\partial x}(\bar{\sigma}_{zx} - \rho\overline{u'w'}) + \frac{\partial}{\partial y}(\bar{\sigma}_{yz} - \rho\overline{v'w'}) + \frac{\partial}{\partial z}(\bar{\sigma}_{zz} - \rho\overline{w'^2}) \qquad (10\text{–}13c)$$

where $\partial\bar{u}/\partial t = 0$ for quasi-steady flows and

$$\bar{\sigma}_{xx} = -\bar{p} + 2\mu\frac{\partial \bar{u}}{\partial x}$$

$$\bar{\sigma}_{yy} = -\bar{p} + 2\mu\frac{\partial \bar{v}}{\partial y}$$

$$\bar{\sigma}_{zz} = -\bar{p} + 2\mu\frac{\partial \bar{w}}{\partial z}$$

$$\bar{\sigma}_{xy} = \bar{\sigma}_{yx} = \mu\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right)$$

$$\bar{\sigma}_{yz} = \bar{\sigma}_{zy} = \mu\left(\frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial y}\right)$$

$$\bar{\sigma}_{zx} = \bar{\sigma}_{xz} = \mu\left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\right) \tag{10-14}$$

Equations (10–12)-(10–13c) are usually referred to as the Reynolds equations. If these equations for the mean flow are compared with the Navier-Stokes equations [Eqs. (5–15)] for the instantaneous flow it is noticed that additional terms $-\rho\partial(\overline{u'^2})/\partial x, -\rho\partial(\overline{u'v'})/\partial y, \ldots, -\rho\,\partial(\overline{w'^2})/\partial z$ appear in the Reynolds equations. These additional stresses $-\rho\overline{u'^2}$, $-\rho\overline{u'v'}, \ldots, -\rho\overline{w'^2}$ are called the Reynolds stresses.

In order to distinguish the Reynolds stresses and the viscous stresses we shall rewrite Eq. (10–13a) in the form

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z}\right) = -\frac{\partial \bar{p}}{\partial x} + \mu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right) - \left[\frac{\partial}{\partial x}(\rho\overline{u'^2}) + \frac{\partial}{\partial y}(\rho\overline{u'v'}) + \frac{\partial}{\partial z}(\rho\overline{u'w'})\right] \tag{10-15}$$

It is seen that the second term on the right-hand side of Eq. (10–15) represents the viscous stresses in terms of the mean velocity gradient. The last three terms on the right-hand side of Eq. (10–15) represent the Reynolds stresses or the apparent stresses due to the exchange of momentum in the turbulent mixing process.

The physical significance of the Reynolds stress may be explained separately as follows: The normal stress $-\rho\overline{u'^2}$ represents the additional momentum leaving in the x-direction per unit surface per unit time due to the fluctuating velocity component u'. This is equivalent to an additional dynamic pressure in the x-direction. The shearing stress $-\rho\overline{u'v'}$ can be interpreted with the aid of the illustrated example which we have discussed in §8.2. Figure 10.4 shows the two open freight cars (A and B) loaded with gravel moving in the same direction on parallel

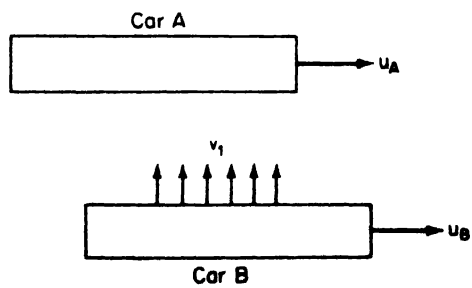


Figure 10.4

tracks at velocities u_A and u_B, respectively. Assuming that the gravel is thrown evenly and continuously across a surface S from car B to car A with a uniform velocity v_1, let m be the mass of the gravel transported per unit time from car B to car A across S. If u_1 is the difference between u_B and u_A, the momentum change at car A in the direction of motion is $\dot{m}u_1$ which is equivalent to a tangential force acting on car A in the direction of motion. Thus:

$$\text{Shearing stress} = \frac{\text{Tangential force}}{S} = \frac{\dot{m}u_1}{S} = \rho v_1 u_1 \qquad (10\text{–}16)$$

In a similar manner we can explain that in fluids the shearing stress is the result of the rate of momentum transport of fluid masses due to their random motion. In §8.2 we have discussed the small scale random movements of molecules in a gas. The analogy is now extended to the random movement of small discrete lumps or particles of fluid. Let the dotted line A-A in Fig. 10.5 represent a surface moving with velocity

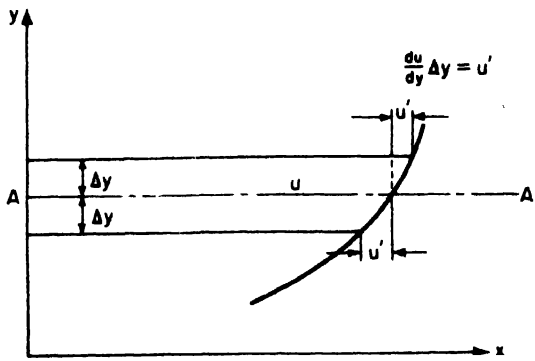


Figure 10.5

u. Then the flow velocities at layers a small distance Δy above and below A-A are $u + u'$ and $u - u'$, respectively. Due to the molar random movements of fluid elements in turbulent motion (molecular movements in laminar motion) there is a fluctuating velocity component v' of fluid elements in the y-direction. The fluid element now at the layer above moves down with a velocity $-v'$ and a mean rate of transport of momentum per unit area $-\rho\overline{v'u'}$ results. Similarly, the fluid element at the layer below moves up with a velocity v' and again a mean rate of transport of momentum per unit area $-\rho\overline{v'u'}$ results. It is seen that a positive $\rho v'$ is associated with the negative u' and vice versa. The instantaneous rate of momentum transfer per unit area is positive and is therefore equal to $-\rho u'v'$. Finally we can equate the mean rate of momentum transfer per unit area to a turbulent shearing stress, i.e.,

$$(\sigma_{yx})_t = -\rho\overline{u'v'} \qquad (10\text{–}17)$$

In view of the random process of the turbulent fluctuations it is therefore to be expected that the problem may be approached by statistical theory. This theory permits us to investigate the correlation between the two components of the fluctuating velocities u' and v' at a particular point. The correlation coefficient between u' and v' at a point is, by definition,

$$\bar{R}(u', v') = \frac{-\overline{u'v'}}{\sqrt{\overline{u'^2}}\sqrt{\overline{v'^2}}} \tag{10-18}$$

When there is no correlation between u' and v', $\overline{u'v'} = 0$ and $\bar{R} = 0$. For complete correlation between u' and v', $\overline{u'v'} = \overline{u'^2} = \overline{v'^2}$ and $\bar{R} = 1$. In the turbulent boundary layer, $\bar{R}$ is found experimentally to have a nearly constant value between 0.45 and 0.55.

Illustrative Example 10.2. Illustrate the equivalence of the exchange of momentum and tangential force.

Solution. Figure 10.4 shows the two open freight cars moving in the same direction on parallel tracks at velocities u_A and u_B, respectively. If $u_1 = 30$ ft/sec is the difference between u_B and u_A, and the amount of the gravel transported per second from car B to car A is 2,000 lbf, the equivalent tangential force acting on car A in the direction of motion is

$$F_t = \dot{m}u_1 = \frac{1}{g}(2000)(30) = 1863 \text{ lbf}$$

Illustrative Example 10.3. Illustrate the significance of the Reynolds stress $\rho\overline{v_z'v_r'}$ for turbulent flow through a smooth pipe.

Solution. For turbulent flow in a circular pipe the Reynolds stress $\rho\overline{v_z'v_r'}$ can be obtained by integrating Eq. (a) in Prob. 10.3 as

$$\rho\overline{v_z'v_r'} = \mu\frac{\partial\bar{v}_z}{\partial_r} - \frac{r}{2}\frac{\partial\bar{p}}{\partial z} \tag{a}$$

According to Darcy's[12] measurement the velocity distribution for a smooth pipe can be written as

$$\frac{(\bar{v}_z)_{\max} - (\bar{v}_z)}{\sqrt{\dfrac{\tau_w}{\rho}}} = 5.08\left(\frac{r}{R}\right)^{1/2} \tag{b}$$

where r is the radial distance measured from the pipe wall. The pressure gradient is given by Eq. (10–58) as

$$\frac{\partial\bar{p}}{\partial z} = -\frac{2\tau_w}{R} \tag{c}$$

Substituting Eqs. (b) and (c) into Eq. (a) we obtain

$$\frac{\overline{v_z' v_r'}}{\tau_w/\rho} = 3.38\frac{\nu}{Rv_*}\sqrt{\frac{r}{R}} + \frac{r}{R} \tag{d}$$

Since Darcy's formula is not valid near the wall, the first term on the right-hand side of Eq. (d) (due to viscous effects) is negligible in comparison with the second term. Therefore, the value of $\overline{v_z' v_r'}$ is proportional to r, and the resistance is almost entirely determined by the turbulent stress.

10.3. Semiempirical Theories of Turbulence

For an incompressible (quasi-steady) turbulent flow Eqs. (10–12) and (10–13) constitute the basic equations for the calculation of the time-averages of the magnitudes which describe the flow. Unfortunately the four Reynolds equations mentioned above contain six unknown Reynolds stresses in addition to variables p, u, v, w. The exact relation between the Reynolds stresses and the mean flow quantities is not yet known. While turbulence has been studied statistically, the theory is still rather meager for the application to problems of engineering interest. On the other hand, several semiempirical theories (used as alternative approaches to turbulence) have been developed and found to yield useful results in many problems. Several of these theories will be discussed here.

a. Prandtl's Mixing Length Theory

In analogy with the coefficient of viscosity in laminar flow [Eq. (1–7)], Boussinesq introduced a mixing coefficient $\rho\mathcal{E}$ for Reynolds stress in turbulent flow by putting

$$(\sigma_{yx})_t = \rho\mathcal{E}\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right) \tag{10–19a}$$

where $\mathcal{E}$ is called the apparent kinematic viscosity which is much greater than the corresponding kinematic viscosity ν for laminar flow.

In order to interpret the significance of $\mathcal{E}$ it is necessary to connect it with the mean flow. Consider a two-dimensional parallel flow in which the velocity varies only in the direction perpendicular to the flow. If the direction of the flow is taken to be the x-axis, we have

$$\bar{u} = \bar{u}(y), \qquad \bar{v} = 0, \qquad \bar{w} = 0$$

In the present case the shearing stress expressed in Eq. (10–19a) becomes

$$(\sigma_{yx})_t = \rho\mathcal{E}\frac{d\bar{u}}{dy} = -\rho\overline{u'v'} \tag{10–19b}$$

In §8.2 we have discussed the mechanism of momentum transfer by molecular motion. According to the kinetic theory of gases, corresponding to Eq. (10–19b), we have

$$(\sigma_{yx})_l \sim \rho\lambda c \frac{du}{dy} \tag{8–9}$$

hence for laminar flow

$$\nu \sim \lambda c \tag{8–10}$$

where λ is the mean free path and c is the root-mean-square molecular velocity. In analogy with the relation given in Eq. (8–10) it is reasonable to consider the following relation for turbulent flow,

$$\varepsilon = l v_* \tag{10–20}$$

where v_* is the root-mean-square value for the turbulent velocity components (also called frictional velocity) and l is a unique length which characterizes the local intensity of the turbulent mixing at any level. Besides the fact that l is much larger in magnitude that λ, it may also be a function of position, the mean velocity, etc.

Prandtl assumed that there is complete correlation between the fluctuating velocity components u' and v'. Hence, we obtain the root-mean-square value from Eqs. (10–19b) and (10–18) in the form

$$v_*^2 = \frac{(\sigma_{ux})_l}{\rho} = -\overline{u'v'} = \sqrt{\overline{u'^2}}\sqrt{\overline{v'^2}} \tag{10–21}$$

Combining Eq. (10–21) with Eqs. (10-19b) and (10–20) we have

$$(\sigma_{yx})_l = \rho v_*^2 = \rho v_* l \frac{d\bar{u}}{dy} \tag{10–22}$$

from which

$$v_* = l \frac{d\bar{u}}{dy} \tag{10–23}$$

With this value of v_* the Reynolds stress is obtained

$$(\sigma_{yx})_l = \rho l^2 \left(\frac{d\bar{u}}{dy}\right)^2 \tag{10–24a}$$

Since the Reynolds stress depends on the sign of the mean velocity gradient, Eq. (10–24a) can be written

$$(\sigma_{yx})_l = \rho l^2 \left|\frac{d\bar{u}}{dy}\right| \frac{d\bar{u}}{dy} \tag{10–24b}$$

where $|d\bar{u}/dy|$ is the absolute value of the mean velocity gradient and $d\bar{u}/dy$ is used with its actual sign. The length l in Eq. (10–24b) is known as the Prandtl mixing length. For $\bar{R} < 1.0$ [Eq. (10–18)], we have

$\overline{|v'|} = c\overline{|u'|}$ and $l =$ Constant $\times\, l_1$. Therefore, Eq. (10-24b) is valid as long as $\overline{|v'|} \sim \overline{|u'|}$.

The advantage of Eq. (10-24b) over Eq. (10-19b) is evident since l is much less complicated to interpret than the apparent kinematic viscosity ε. Prandtl argued that the length l must be proportional to the distance y from the wall as the mixing movements of u' and v' must reduce to zero at a solid wall. The simplest relation between l and y is given by

$$l = ky \tag{10-25}$$

where k is a proportionality constant which must be determined from experiments.

b. von Kármán Similarity Hypothesis

Prandtl's hypothesis of the mixing length expression makes possible an analytic treatment of the flow along a solid wall. Such a treatment must be restricted to flow of shallow layers since the value of l is of the same order of magnitude as the distance above the reference surface. An additional important relationship for the mixing length l based on similarity considerations has been developed by von Kármán.

Consider a two-dimensional mean motion parallel to the x-axis, such that $\bar{u} = \bar{u}(y)$ and $\bar{v} = 0$. If a system of coordinates moving with the mean motion is introduced, the motion becomes steady with respect to such a system. von Karmán's basic hypothesis is that in such a system the fluctuations at all such points differ only in length and time scales. In other words, the mixing length l is a function of the distribution of the mean motion in the turbulent flow but only indirectly related to the distance y from a solid wall. With reference to the moving coordinate system the velocity components u and v, after expanding the mean flow in a Taylor series in the neighborhood of point P, may be written (see Fig. 10.6)

$$u = \left(\frac{\partial \bar{u}}{\partial y}\right)_P y + \left(\frac{\partial^2 \bar{u}}{\partial y^2}\right)_P \frac{y^2}{2} + \cdots + u' \tag{10-26a}$$

$$v = v' \tag{10-26b}$$

We now introduce the quantities ξ, η, u_1', v_1' for the amplitudes of the disturbance and l and A for the length scale and velocity scale, respectively. According to the similarity hypothesis, we have the relations

$$\begin{aligned} x &= l\xi, & y &= l\eta \\ u' &= Au_1', & v' &= Av_1' \end{aligned} \tag{10-27}$$

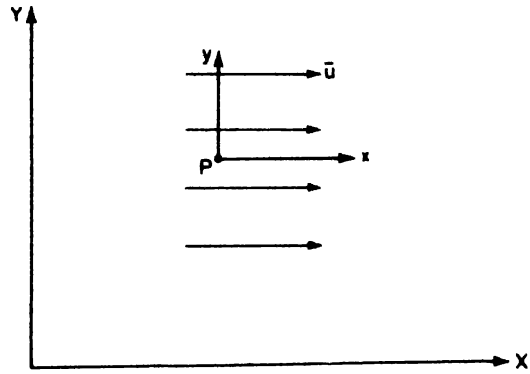


Figure 10.6

Substituting the relations in Eq. (10–27) into Eqs. (10–26), the results yield

$$u = \left(\frac{\partial \bar{u}}{\partial y}\right)_P l\eta + \left(\frac{\partial^2 \bar{u}}{\partial y^2}\right)_P \frac{l^2\eta^2}{2} + \cdots + Au_1' \tag{10–28a}$$

$$v = Av_1' \tag{10–28b}$$

Since u_1' and v_1' are to be functions of ξ and η only, the similarity conditions as deduced from Eqs. (10–28) require

$$l\left(\frac{\partial \bar{u}}{\partial y}\right)_P \sim l^2\left(\frac{\partial^2 \bar{u}}{\partial y^2}\right)_P \sim A$$

or

$$l \sim \frac{(d\bar{u}/dy)_P}{(d^2\bar{u}/dy^2)_P} \qquad \text{and} \qquad A \sim l\left(\frac{d\bar{u}}{dy}\right)_P$$

von Kármán suggested that the mixing length takes the form

$$l = k\left|\frac{d\bar{u}/dy}{d^2\bar{u}/dy^2}\right| \tag{10–29}$$

where k is an empirical constant.

Equation (10–29) indicates that the mixing length l is a local function and depends only on the velocity distribution in the neighborhood of a particular point. The turbulent shearing stress in this case is

$$(\sigma_{yx})_t = \rho k^2 \frac{(d\bar{u}/dy)^4}{(d^2\bar{u}/dy^2)^2} \tag{10–30}$$

10.4. *The Universal Velocity Profile Near a Wall*

Consider a steady two-dimensional mean motion in the vicinity of a solid wall. If the mean velocity $\bar{u}$ is in the x-direction, such that

$\bar{u}=\bar{u}(y)$, $\bar{v} = \bar{w} = 0$, the Reynolds equations [Eqs. (10–12) and (10–13)] reduce to the single equation

$$\frac{\partial}{\partial y}\left[\mu\frac{\partial \bar{u}}{\partial y} + (\sigma_{yx})_t\right] = \frac{\partial \bar{p}}{\partial x} \tag{10–31}$$

In a turbulent flow along a wall it was suggested by Nikuradse, von Kármán, and others that the flow should be divided into three zones. The first zone is a thin layer in the immediate vicinity of the wall in which the viscous stress predominates while the Reynolds stress is insignificant. This region is the so-called *laminar sublayer.* The zone immediately above the laminar sublayer is called the *buffer zone* in which the Reynolds stress and the viscous stress are of the same order of magnitude. Still farther from the wall we have the *turbulent zone* in which the viscous stresses may be completely neglected in comparison with the Reynolds stresses. For convenience, the flow is usually divided into two zones, the viscous layer and the turbulent layer. For flows with zero pressure gradient in the x-direction and sufficiently large Reynolds numbers, the viscous layer is very thin and Eq. (10–31) may be written

$$(\sigma_{yx})_t = (\sigma_{yx})_{t_o} = \text{Constant} \tag{10–32}$$

where the subscript o indicates the intersections of the turbulent and viscous zones (buffer zone is neglected here). This implies the Reynolds stress is constant in the flow field. Substituting Eqs. (10–21) and (10–24a) into Eq. (10–32), we have

$$l^2\left(\frac{d\bar{u}}{dy}\right)^2 = v_*^2 = \frac{(\sigma_{yx})_{t_o}}{\rho} \tag{10–33}$$

If $l = ky$ from Eq. (10–25) is used as the expression for mixing length, Eq. (10–33) becomes

$$\frac{1}{v_*}\frac{d\bar{u}}{dy} = \frac{1}{ky} \tag{10–34}$$

On integration this gives

$$\frac{\bar{u}}{v_*} = \frac{1}{k}(\ln y + \text{Constant}) \tag{10–35}$$

The constant of integration in Eq. (10–35) can be determined from an appropriate boundary condition. The no-slip condition at the wall cannot be satisfied with a finite constant of integration. This is expected as Eq. (10–35) is valid only in the turbulent region. The appropriate condition for the present problem should be that $\bar{u} = 0$ at a certain distance $y = y_0$ from the wall. Hence Eq. (10–35) becomes

$$\frac{\bar{u}}{v_*} = \frac{1}{k}(\ln y - \ln y_0) \tag{10–36}$$

The distance y_0 is of the order of magnitude of the thickness of the viscous layer. Equating the Reynolds stress and the viscous stress at $y = y_0$, we have

$$(\sigma_{yx})_{t_0} = \rho v_*^2 = \mu\left(\frac{d\bar{u}}{dy}\right)_{y=y_0}$$

or

$$v_* = \frac{\nu}{v_*}\left(\frac{d\bar{u}}{dy}\right)_{y=0} \sim y_0\left(\frac{d\bar{u}}{dy}\right)_{y=0} \tag{10-37}$$

from which

$$y_0 = \beta\frac{\nu}{v_*} \tag{10-38}$$

where β is a nondimensional constant. Substituting Eq. (10–38) into Eq. (10–36), we finally obtain

$$\frac{\bar{u}}{v_*} = \frac{1}{k}(\ln\frac{v_* y}{\nu} - \ln\beta) \tag{10-39}$$

where k and β are to be determined experimentally. Equation (10–39) is the so-called universal velocity profile or the logarithmic velocity distribution law. The same result can also be obtained by using von Kármán's similarity hypothesis.

Based on Nikuradse's and Reichardt's experimental data[13] on velocity distribution the value of 0.40 for k and 0.111 for β have been suggested. With these values for the two constants, k and β, Eq. (10–39) may be written

$$\frac{\bar{u}}{v_*} = 2.5\ln\frac{v_* y}{\nu} + 5.5 \tag{10-40a}$$

or

$$\frac{\bar{u}}{v_*} = 5.75\log_{10}\frac{v_* y}{\nu} + 5.5 \tag{10-40b}$$

Excellent agreement between the experimental measurements of velocity profiles and the corresponding calculated result from Eq. (10–40b) is obtained. This comparison is shown in a semilogarithmic plot with nondimensional parameters $\bar{u}/v_*$ and $v_* y/\nu$ as coordinates (Fig. 10.7).

As mentioned previously the universal velocity profile [Eq. (10–40)] is not valid in the region near the wall where the viscous shear is predominate in the flow. It is clearly indicated in Fig. 10.7 that there is a progressive divergence between the velocity profile from Eq. (10–40b) and the experimental data for values of $v_* y/\nu$ less than 70. In the laminar sublayer ($\sigma_{yx} = \tau_w$ = Constant) the linear velocity distribution is obtained as follows:

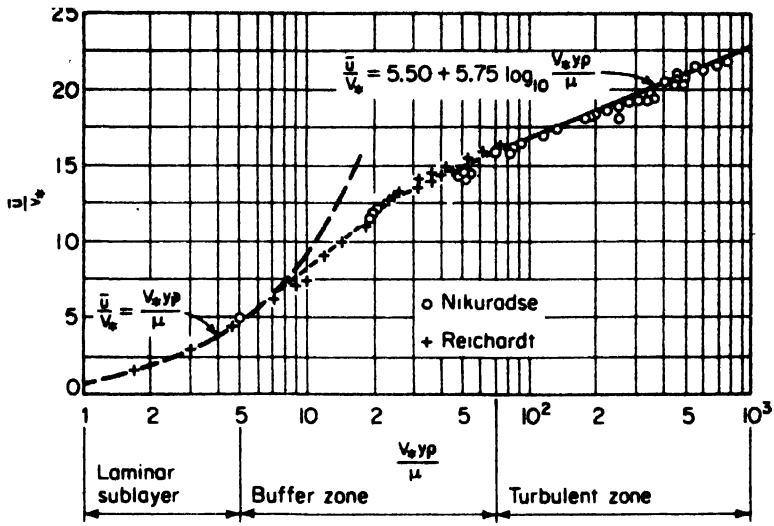


Figure 10.7

$$\tau_w = \mu \frac{d\bar{u}}{dy} = \rho v_*^2$$

or

$$\frac{\bar{u}}{v_*} = \frac{v_* y}{\nu} \tag{10-41}$$

for $v_* y/\nu < 5.0$. von Kármán[14] suggested a logarithmic equation for the buffer zone, $5 < v_* y/\nu < 70$, as

$$\frac{\bar{u}}{v_*} = 5.0 \ln \frac{v_* y}{\nu} - 3.05 \tag{10-42a}$$

or

$$\frac{\bar{u}}{v_*} = 11.5 \log_{10} \frac{v_* y}{\nu} - 3.05 \tag{10-42b}$$

The calculated velocity profiles from Eqs. (10–41) and (10–42b) are shown in Fig. 10.7 which are in good agreement with the measured values.

10.5. *Turbulent Flow in Pipes*

a. Empirical Relations for Smooth Pipes

The fully developed laminar flow through a long straight pipe of circular cross section with axial symmetry has been discussed in §8.4a.

The solution of the Navier-Stokes equations and the continuity equation gives the laminar velocity distribution in the axial direction. The theoretically determined wall frictional coefficient is in excellent agreement with the experimental data (Fig. 8.7). As we explained previously, the solution of the Reynolds equations for turbulent flow is not yet known. Therefore, the velocity distribution and the frictional coefficient in this case can only be obtained empirically or semiempirically.

Let z be the direction of flow along the axis of the pipe and r denote the radial direction measured outwards from the z-axis. According to the equilibrium condition we can write the relation between the shearing stress and the pressure difference for a fully developed turbulent pipe flow as (Fig. 10.8)

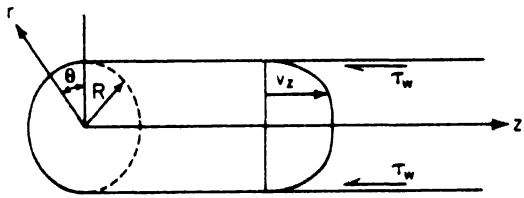


Figure 10.8

$$\sigma_{rz} = \left(\frac{p_1 - p_2}{L}\right)\frac{r}{2} = \tau_{rz} \tag{10-43}$$

Equation (10-43) was originally derived for laminar flow through pipes [Eq. (8-46)]; however, in the present case σ_{rz} represents the sum of laminar and turbulent shearing stresses as shown in Eq. (10-31) and $\bar{p}$ is replaced by p as a matter of simplicity. The shearing stress at the wall can be directly obtained from Eq. (10-43) in the form

$$\tau_w = \left(\frac{p_1 - p_2}{L}\right)\frac{R}{2} \tag{10-44}$$

Since the value on the right-hand side of Eq. (10-44) can be determined directly by measuring the pressure gradient along the pipe, the wall shearing stress is experimentally determined. According to Eqs. (8-54) and (8-57) the coefficient of resistance may be expressed in the form

$$\tau_w = \frac{f}{8}\rho(\bar{v}_z)_{av}^2 = \frac{f}{8}\rho\bar{u}_{av}^2 \tag{10-45}$$

where $(\bar{v}_z)_{av} = \bar{u}_{av}$ is the average turbulent mean velocity in the pipe which is defined as

$$\bar{u}_{av} = \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R \bar{v}_z\, r\, dr\, d\theta \tag{10-46}$$

In the case of laminar flow through a pipe the coefficient of resistance f is given by

$$f = \frac{64}{\mathrm{Re}} \tag{8-54}$$

where Re is based on the average velocity of the laminar flow through pipes. Based on Saph and Schoder's[15] extensive experimental results, Blasius[16] formulated the following empirical expression for the coefficient of friction f for turbulent flow

$$f = \frac{0.3164}{(\mathrm{Re})^{1/4}} \tag{10-47}$$

where $\mathrm{Re} = \bar{u}_{av} D/\nu$. This expression for the frictional resistance of smooth pipes is known as the Blasius formula. It is valid only in the range of Reynolds numbers $4 \times 10^3 \leq \mathrm{Re} \leq 10^5$. For Reynolds numbers larger than 10^5 the universal resistance law for smooth pipes has been found to be in excellent agreement with the experimental data (§ 10.5c).

The shearing stress at the wall can be obtained by substituting f from Eq. (10-47) into Eq. (10-45):

$$\tau_w = 0.03955\rho(\bar{u}_{av})^{7/4}(\nu)^{1/4}(D)^{-1/4} \tag{10-48}$$

In terms of the radius R instead of the diameter D, Eq. (10-48) becomes

$$\tau_w = 0.03325\rho(\bar{u}_{av})^{7/4}(\nu)^{1/4}(R)^{-1/4} = \rho v_*^2 = \rho(v_*)^{7/4}(v_*)^{1/4} \tag{10-49}$$

where $v_* = \sqrt{\tau_w/\rho}$. Rearranging Eq. (10-49), we obtain

$$\frac{\bar{u}_{av}}{v_*} = 6.99\left(\frac{v_* R}{\nu}\right)^{1/7} \tag{10-50}$$

According to the measured results of Nikuradse for turbulent pipe flow in the range of Reynolds numbers $\mathrm{Re} \leq 10^5$, the ratio of $\bar{u}_{av}/\bar{u}_{max}$ is approximately 0.8. Equation (10-50) may be written in the form

$$\frac{\bar{u}_{max}}{v_*} = 8.74\left(\frac{v_* R}{\nu}\right)^{1/7} \tag{10-51}$$

It is reasonable to assume that Eq. (10-51) is valid not only for the center line (z-axis) but also for any distance r' ($r' = R - r$) from the wall. Hence, we obtain

$$\frac{\bar{u}}{v_*} = 8.74\left[\frac{v_*(R-r)}{\nu}\right]^{1/7} = 8.74\left(\frac{v_* r'}{\nu}\right)^{1/7} \tag{10-52}$$

The velocity ratio

$$\frac{\bar{u}}{\bar{u}_{max}} = \frac{\bar{v}_z}{(\bar{v}_z)_{max}} = \left(\frac{R-r}{R}\right)^{1/7} = \left(\frac{r'}{R}\right)^{1/7} \tag{10-53}$$

is the so-called Blasius $\frac{1}{7}$th power law for the velocity distribution in turbulent flow.

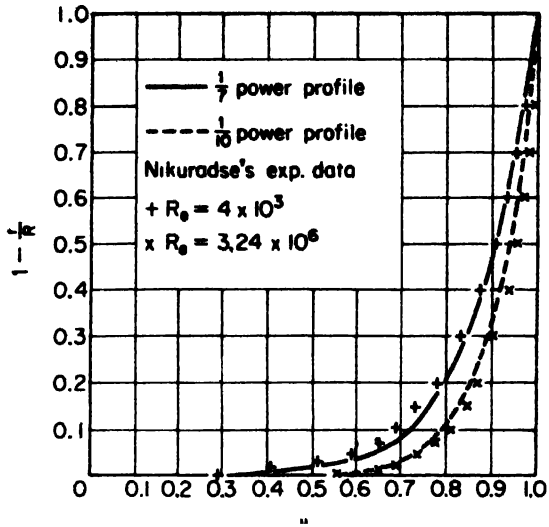


Figure 10.9

In Fig. 10.9, the Blasius formula, Eq. (10–53), is used for comparison with experimental results of Nikuradse.[17] It is seen that the $\frac{1}{7}$th power velocity profile is in good agreement with the experimental data for Reynolds numbers equal to 4×10^3; however, it deviates considerably in relation to $\mathrm{Re} = 3.24 \times 10^6$. Since the experimental results indicate that the velocity profile becomes fuller as the Reynolds number increases, a $\frac{1}{10}$th power velocity profile has been found to be in good agreement with the experimental data for $\mathrm{Re} = 3.24 \times 10^6$. The above fact suggests that for very large Reynolds numbers there exists a universal velocity profile in the form of a logarithmic expression. This will be discussed in the succeeding section.

b. Universal Velocity Distribution

Let z be the direction of the flow along the axis of the pipe and r denote the radial direction measured outwards from the z-axis. The mean velocity components, $\bar{v}_\theta$ and $\bar{v}_r$, in the radial and tangential directions, respectively, are zero. Under these flow conditions the equation of continuity [Eq. (5–41) for mean velocities] reduces to

$$\frac{\partial \bar{v}_z}{\partial z} = 0 \tag{10–54}$$

or

$$\bar{v}_z = \bar{v}_z(r) = u(r) \tag{10–55}$$

The mean velocity component $\bar{v}_z$ will be replaced by u as a matter of simplicity. The equation of motion for a fully developed turbulent flow, without body forces, through a pipe of circular cross section with axial symmetry can be reduced from Eqs. (5-48) in the form:

$$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial \theta} = 0$$

$$\frac{\partial p}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) \tag{10-56}$$

where σ_{rz} is the sum of the Reynolds stress and the viscous stress instead of the viscous stress alone in Eq. (5-48). The negative sign is introduced in Eq. (10-56) because the direction of r is measured from the center of the pipe. Since in the present case we have shown that $u = u(r)$ and $p = p(z)$, Eq. (10-56) can be integrated with the result (for simplicity, $\sigma_{rz} = \tau$)

$$r\sigma_{rz} = r\tau = -\frac{r^2}{2}\left(\frac{dp}{dz}\right) + C \tag{10-57}$$

where the constant of integration C is determined from the boundary conditions that

$$r = 0, \qquad \tau = 0, \qquad C = 0$$

At the wall, we have

$$\tau_w = -\frac{R}{2}\frac{dp}{dz} \tag{10-58}$$

Combining Eqs. (10-57) and (10-58) the linear shearing-stress distribution is obtained:

$$\frac{\tau}{\tau_w} = \frac{r}{R} \tag{10-59}$$

(i) *Prandtl's Law.* If the Reynolds stress τ from Eq. (10-24b) and the mixing length from Eq. (10-25) with k_1 replacing k are employed, Eq. (10-59) becomes

$$\tau = \rho k_1^2 (R - r)^2 \left|\frac{du}{dr}\right| \frac{du}{dr} = \frac{r}{R}\tau_w \tag{10-60a}$$

Equation (10-60a) is not valid in the region very near to the wall where viscous stress cannot be neglected. Since the right-hand side of Eq. (10-60a) always remains positive we can rewrite Eq. (10-60a), after using the relation of v_* and τ from Eq. (10-21),

$$k_1^2 (R - r)^2 \left(\frac{du}{dr}\right)^2 = \frac{v_*^2 r}{R}$$

or

$$\frac{du}{dr} = \frac{1}{k_1} \frac{v_*}{\sqrt{R}} \frac{\sqrt{r}}{(R-r)} \tag{10-60b}$$

and by integration

$$\frac{u}{v_*} = -\frac{1}{k_1}\left[\ln\left(\frac{1+\sqrt{r/R}}{1-\sqrt{r/R}}\right) - 2\sqrt{\frac{r}{R}}\right] + C_1 \tag{10-61}$$

The constant of integration C_1 can be found from the condition $u = u_{max}$ at $r = 0$. Therefore,

$$C_1 = \frac{u_{max}}{v_*}$$

and hence, Eq. (10-61) becomes

$$\frac{u_{max} - u}{v_*} = \frac{1}{k_1}\left[\ln\left(\frac{1+\sqrt{r/R}}{1-\sqrt{r/R}}\right) - 2\sqrt{\frac{r}{R}}\right] \tag{10-62}$$

This is the universal velocity distribution in turbulent pipe flow due to Prandtl. Since the velocity distribution in Eq. (10-62) is a function of the radius of the pipe and an empirical constant k_1, it holds for all Reynolds numbers and all wall roughnesses in the turbulent flow.

In Fig. 10.10 the result calculated from Eq. (10-62) is compared with Nikuradse's experimental data ($Re = 3.24 \times 10^6$) in smooth pipes by selecting k_1 to give agreement at $r/R = 0.7$ (approximate location of the average velocity). The value of k_1 is found to be 0.23. The agree-

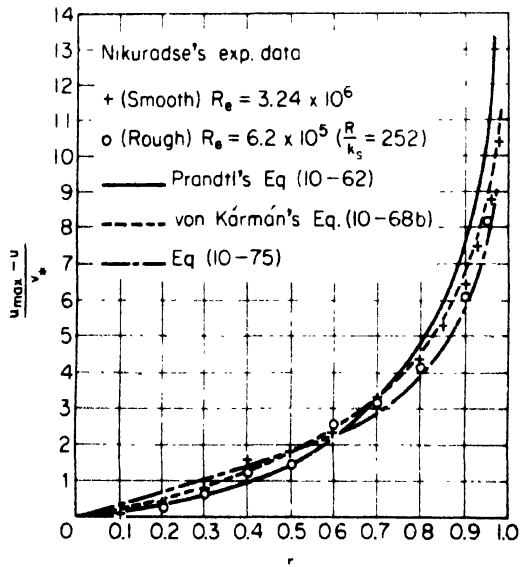


Figure 10.10

ment between Prandtl's formula and the experimental results is quite satisfactory in the central region of the pipe but as expected, discrepancies become more obvious as the wall is approached.

(ii) *von Kármán's Law.* To determine a universal velocity distribution expression based on von Kármán's similarity hypothesis, the expression for turbulent shearing stress given in Eq. (10–30) may be used for τ of Eq. (10–59). The combination of these two equations leads to

$$\tau_w \frac{r}{R} = \rho v_*^2 \frac{r}{R} = \rho k_2^2 \frac{(du/dr)^4}{(d^2u/dr^2)^2} \tag{10–63}$$

Taking the square root, we obtain

$$\frac{d^2u/dr^2}{(du/dr)^2} = \pm \frac{k_2\sqrt{R}}{v_*} \frac{1}{\sqrt{r}} \tag{10–64}$$

Integrating,

$$-\left(\frac{du}{dr}\right)^{-1} = \pm \frac{2k_2}{v_*}\sqrt{R}\sqrt{r} + C_2 \tag{10–65}$$

The constant of integration is obtained from the condition when $r \to R$, $(du/dr)^{-1} \to 0$. Therefore

$$C_2 = \mp \frac{2k_2R}{v_*}$$

With this value for C_2, Eq. (10-65) becomes

$$\frac{du}{dr} = -\frac{v_*}{2k_2\sqrt{R}(\sqrt{R} - \sqrt{r})} \tag{10–66}$$

The negative sign is used here because du/dr is negative. The integration of Eq. (10-66) yields

$$u = -\frac{v_*}{k_2}\left[\left(1 - \sqrt{\frac{r}{R}}\right) - \ln(\sqrt{R} - \sqrt{r})\right] + C_3 \tag{10–67}$$

where C_3 may be evaluated by the boundary condition at the center of the pipe: $u = u_{max}$ at $r = 0$. Therefore,

$$C_3 = \frac{v_*}{k_2}(1 - \ln\sqrt{R}) + u_{max}$$

The universal velocity distribution expression due to von Kármán is obtained by substituting the value of C_3 into Eq. (10–67):

$$u = u_{max} + \frac{v_*}{k_2}\left[\ln\left(1 - \sqrt{\frac{r}{R}}\right) + \sqrt{\frac{r}{R}}\right] \tag{10–68a}$$

or

$$\frac{u_{max} - u}{v_*} = -\frac{1}{k_2}\left[\ln\left(1 - \sqrt{\frac{r}{R}}\right) + \sqrt{\frac{r}{R}}\right] \tag{10–68b}$$

Equation (10–68b) also holds for all Reynolds numbers and all wall roughnesses in the turbulent flow since it contains only r/R and the empirical constant k_2.

von Kármán's universal velocity distribution is seen to be in very good agreement with the experimental data if $k_2 = 0.30$ is selected at $r/R = 0.7$. This comparison is shown in Fig. 10.10.

It is interesting to show that in the vicinity of the pipe wall, i. e., $r \to R$, Eqs. (10–60a) and (10–63) will yield the same result. For $r \to R$ Eq. (10–63) may be written

$$\frac{(du/dr)^4}{(d^2u/dr^2)^2} = \frac{v_*^2}{k^2} \tag{10–69}$$

Taking the square root and integrating,

$$-\left(\frac{du}{dr}\right)^{-1} = \pm\frac{k}{v_*}r + C_4 \tag{10–70}$$

The boundary condition, $du/dr \to \infty$ at $r \to R$, gives

$$C_4 = \mp\frac{k}{v_*}R$$

Hence,

$$-\frac{du}{dr} = \frac{v_*}{k}\frac{1}{(R-r)} \tag{10–71}$$

Integrating,

$$u = \frac{v_*}{k}\ln(R-r) + C_5 \tag{10–72}$$

Similarly, in the vicinity of the wall Eq. (10–60a) may be written

$$(R-r)^2\left(\frac{du}{dr}\right)^2 = \frac{v_*^2}{k^2} \tag{10–73}$$

Taking the square root,

$$-\frac{du}{dr} = \frac{v_*}{k}\frac{1}{(R-r)}$$

Integrating,

$$u = \frac{v_*}{k}\ln(R-r) + C_6 \tag{10–74}$$

This velocity distribution is the same as Eq. (10–72), and both of them are identical to Eq. (10–40a) which is the universal velocity profile in turbulent flow near a wall. Equation (10–74) may be written in the form

$$\frac{u_{max} - u}{v_*} = \frac{1}{k}\ln\left(\frac{R}{R-r}\right) \tag{10–75}$$

where the constant C_5 is determined from the condition at the center of the pipe where $u = u_{max}$. A comparison of this velocity distribution ($k = 0.4$) with experimental data is given in Fig. 10.10.

For flow in the immediate vicinity of the pipe wall where the viscous effect exists, both Prandtl's and von Kármán's velocity distribution expressions are not valid. In this case the laminar sublayer approximation which has been discussed in §10.4 must be applied.

c. *Universal Resistance Law for Smooth Pipes*

In §10.5 we have discussed the empirical expression for the coefficient of friction which was formulated by Blasius from rather limited experimental data. The Blasius resistance formula [Eq. (10–47)] is valid only in the range of Reynolds numbers $4 \times 10^3 \leq \mathrm{Re} \leq 10^5$. Since the Blasius resistance formula is a function of u_{av}, R, and ν, it is reasonable to believe that a universal resistance law may be derived from the average value of the logarithmic velocity distribution. According to Eqs. (10–46) and (10–74) the average velocity distribution of turbulent pipe flow may be obtained,

$$u_{av} = \frac{2v_*}{R^2}\int_0^R \frac{u}{v_*} r\,dr = \frac{2v_*}{R^2}\int_0^R \left[\frac{1}{k}\ln\frac{(R-r)}{\nu}v_* + C_1\right] r\,dr$$

$$u_{av} = v_*\left[\frac{1}{k}\ln\frac{Rv_*}{\nu} + C_2\right] \tag{10–76}$$

The relation between the coefficient of resistance and the average velocity is obtained from Eq. (10–45) in the form

$$f = 8\left(\frac{v_*}{u_{av}}\right)^2 \tag{10–77}$$

Substituting the average velocity from Eq. (10–76) into Eq. (10–77), we have

$$f = \frac{1}{[(1/k)\ln \mathrm{Re}\sqrt{f} - (1/k)\ln 4\sqrt{2} + C_2]^2} \tag{10–78}$$

where $Rv_*/\nu = \mathrm{Re}(\sqrt{f}/4\sqrt{2})$ and $\mathrm{Re} = 2u_{av}R/\nu$. Since logarithms to the base 10 are preferable in engineering practice, Eq. (10–78) may be written in the form

$$\frac{1}{\sqrt{f}} = A\log_{10}\mathrm{Re}\sqrt{f} + C_3 \tag{10–79}$$

where A and C_3 are constants. Equation (10–79) gives a straight line relation between $1/\sqrt{f}$ and $\log_{10}\mathrm{Re}\sqrt{f}$ which agrees very well with the experimental results of Nikuradse. It can be seen from Fig. 10.11 that the straight line passing through the experimental points may be expressed by the equation

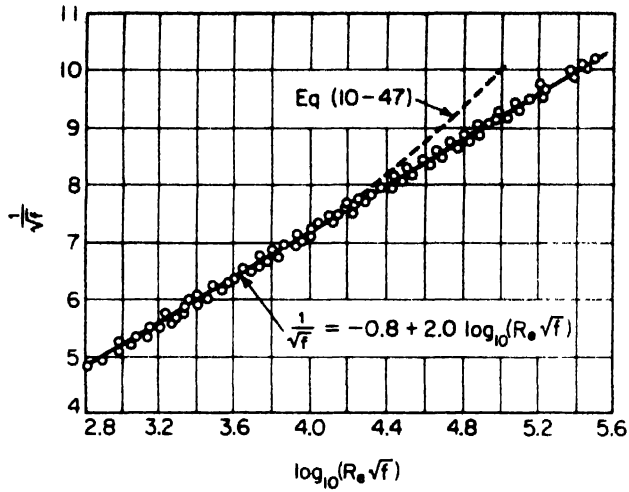


Figure 10.11

$$\frac{1}{\sqrt{f}} = 2.0 \log_{10} \mathrm{Re}\sqrt{f} - 0.8 \tag{10-80}$$

This is the Prandtl universal resistance law for a smooth pipe which is valid for practically all Reynolds numbers. It is evident that the constants in Eqs. (10–79) are $A = 2.0$ and $C_3 = -0.8$. The value of $A =$ corresponds to a value of $k = 0.408$ which is quite close to the value 2.0 of 0.4 previously determined in Eq. (10–40a). It is interesting to note that the universal resistance law agrees quite well with the Blasius resistance formula [Eq. (10–47)] up to $\mathrm{Re} = 10^5$. This comparison is depicted in Fig. 10.11.

d. Flow in Rough Pipes

The results we have obtained so far are limited only to turbulent flow in pipes with smooth surfaces. In engineering practice, most pipes being used, such as cement pipes, castiron pipes, and chemically corroded metal pipes, cannot be considered smooth. Although the universal velocity distribution obtained in §10.5b is also valid for the central region of the turbulent flow in rough pipes (see Fig. 10.10), the flow conditions in the immediate vicinity of the wall depend on the relative magnitude of the thickness of the laminar sublayer and the roughness of the wall.

In determining the effect of wall roughness on the flow and resistance of rough pipes it will be necessary to take into consideration the varied geometrical forms of roughness as well as the manner of their distribu-

tion. This makes the treatment of the roughness problem very difficult. According to Nikuradse's results[18] from his extensive work on sand-grain roughness in pipes, sand-grain roughness has been adopted as a standard in studies of resistance of rough walls. It is assumed that mean roughness elements consist of sand grains (or similar particles) of nearly uniform size but not necessarily regular shape, packed closely on a plain surface. In this way the significant dimensions used to describe the wall roughness then reduce to a linear dimension alone, the mean height of the roughness element, k_s. For other types of roughness, it is convenient to correlate any given roughness with its equivalent sand-grain roughness. Experimentally determined values of equivalent sand roughness for a large number of roughnesses arranged in a regular fashion are given by Schlichting.[5]

In Eq. (10–38) we have found that the thickness of the laminar sublayer is proportional to the ratio ν/v_*. Now we can use this ratio to define a roughness Reynolds number

$$\frac{v_* k_s}{\nu}$$

which serves as the criterion to characterize the roughness. It has been found experimentally that, if $v_* k_s/\nu < 4$, the surface may be considered hydraulically (or aerodynamically) smooth; if $v_* k_s/\nu > 100$, the surface may be considered fully rough. For $4 < v_* k_s/\nu < 100$, the flow condition involves both the roughness and viscous effects. Since the thickness of the laminar sublayer has been chosen for $v_* y/\nu = 5.0$, it is seen that the mean height of the roughness element is smaller than the laminar sublayer of the flow when the surface is considered as hydraulically smooth.

According to Eq. (10–74) the universal velocity distribution is given by

$$\frac{u}{v_*} = \frac{1}{k'} \ln r' + C_5 \tag{10-74}$$

where r' is the radial distance measured from the pipe wall. The constant of integration C_5 can be determined from the boundary condition that $u \to 0$ at $r' \to Ck_s$ where C is an arbitrary constant. Hence, Eq. (10–74) becomes

$$\begin{aligned} \frac{u}{v_*} &= \frac{1}{k'}(\ln r' - \ln Ck_s) \\ &= \frac{1}{k'} \ln \frac{r'}{k_s} + C_6 \end{aligned} \tag{10-81}$$

or

$$\frac{u}{v_*} = \frac{2.3026}{k'} \log_{10} \frac{r'}{k_s} + C_6 \tag{10-82}$$

where C_6 is a constant to be determined experimentally From Nikuradse's experimental results of sand-grain roughness, for a fully rough wall, the logarithmic velocity profile expression [Eq. (10-82)] is found to agree with measured data for $k' = 0.4$ and $C_6 = 8.5$ (see Fig. 10.12). Thus Eq. (10-82) becomes

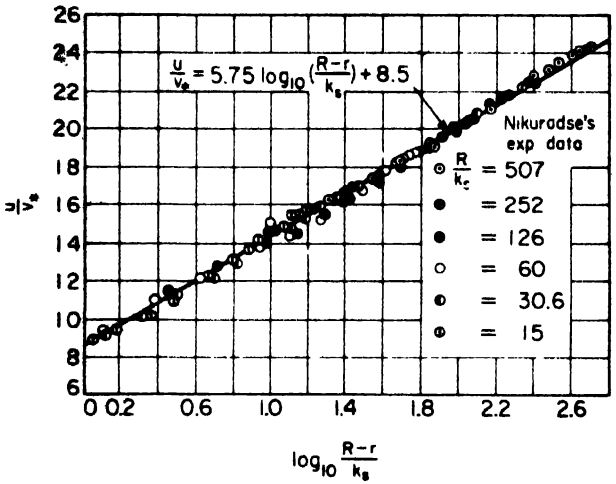


Figure 10.12

$$\frac{u}{v_*} = 5.75 \log_{10} \left(\frac{R-r}{k_s}\right) + 8.5 \tag{10-83}$$

This is the velocity distribution expression for turbulent flow in fully rough pipes.

In the case that the wall of the pipe is moderately rough, i. e., $4 < v_* k_s/\nu < 100$, the value of C_6 is no longer a constant but becomes a function of the roughness Reynolds number $v_* k_s/\nu$. According to Fig. 10.13 the relation between C_6 and $v_* k_s/\nu$ may be approximated by the following expressions:

$$\begin{aligned} C_6 &= 5.5 + 5.75 \log_{10} \frac{v_* k_s}{\nu} && \left(\frac{v_* k_s}{\nu} \leqq 7.08\right) \\ &= 9.58 && \left(7.08 \leqq \frac{v_* k_s}{\nu} \leqq 14.1\right) \\ &= 11.50 - 1.65 \log_{10} \frac{v_* k_s}{\nu} && \left(14.1 \leqq \frac{v_* k_s}{\nu} \leqq 70.8\right) \\ &= 8.5 && \left(70.8 \leqq \frac{v_* k_s}{\nu}\right) \end{aligned} \tag{10-84}$$

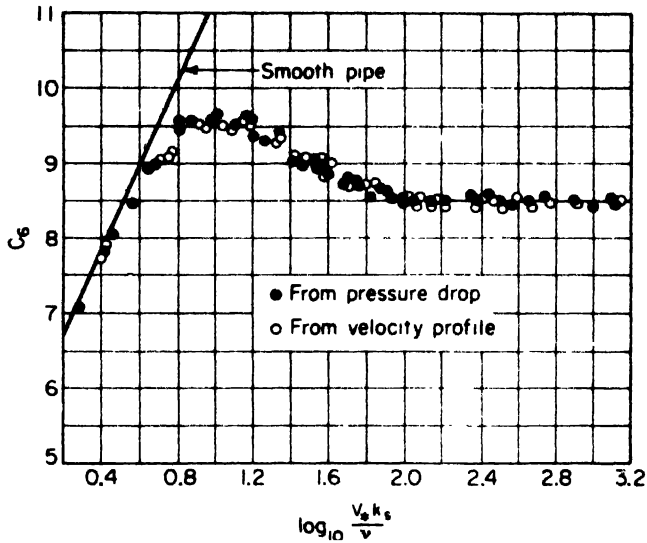


Fig 10.13. H. Schlichting, "Boundary Layer Theory," New York: McGraw-Hill, 1960

For turbulent flow in fully rough pipes the universal resistance law can be derived in a similar manner as that of §10.5c. The average velocity distribution in this case is obtained with the aid of Eq. (10–81) as follows:

$$\frac{u_{av}}{v_*} = \frac{2}{R^2}\int_0^R \frac{u}{v_*} r\,dr = \frac{2k_s^2}{R^2}\int_0^R \left[\frac{1}{k}\ln\frac{r'}{k_s} + C_6\right]\frac{(R-r')}{k_s}\,d\left(\frac{r'}{R}\right)$$

$$= A_1 \ln\frac{R}{k_s} + B_1 \qquad (10\text{–}85)$$

where A_1 and B_1 are two arbitrary constants to be determined experimentally. The resistance coefficient f for a fully rough pipe is obtained by combining Eqs. (10-85) and (10-77),

$$f = \frac{8}{[A_1 \ln (R/k_s) + B_1]^2} \qquad (10\text{–}86)$$

or

$$\frac{1}{\sqrt{f}} = A_2 \log_{10}\frac{R}{k_s} + B_2 \qquad (10\text{–}87a)$$

where A_2 and B_2 are constants to be determined. In a comparison with Nikuradse's experimental data on turbulent flow in fully rough pipes (Fig. 10.14), the values of $A_2 = 2.0$ and $B_2 = 1.74$ are obtained. Hence,

$$\frac{1}{\sqrt{f}} = 2.0 \log_{10}\frac{R}{k_s} + 1.74 \qquad (10\text{–}87b)$$

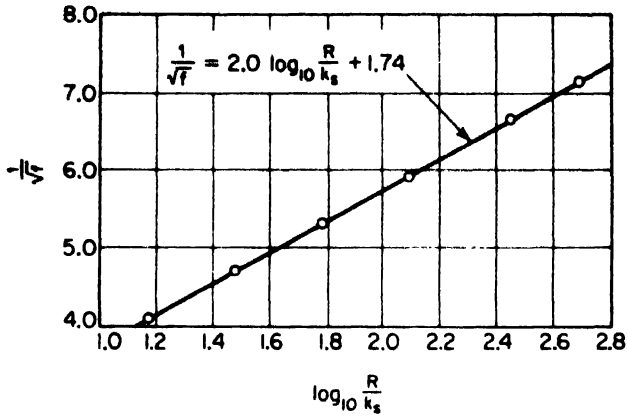


Figure 10.14

This is the universal resistance formula for turbulent flow in fully rough pipes. It can be seen from Fig. 10.14 that the experimental data lie very close to a straight line in a plot of $1/\sqrt{f}$ vs. $\log_{10} R/k_s$. If the radius R of the circular pipe is replaced by the hydraulic radius which is defined by

$$R_h = \frac{\text{Cross-sectional area}}{\text{Wetted perimeter}}$$

Eq. (10–87b) may also be used for pipes with noncircular cross-sectional area.

Illustrative Example 10.4 In the solution of practical pipe-flow problems, we need the following two relations:

$$Q = \bar{u}_{av} \left(\frac{\pi D^2}{4} \right) \tag{a}$$

and

$$h_f = \frac{p_1 - p_2}{\rho g} = f \cdot \left(\frac{L}{D} \right) \cdot \frac{\bar{u}_{av}^2}{2g} \tag{b}$$

Equation (a) relates the volume flow rate Q to the average velocity $\bar{u}_{av}$ and the pipe diameter D. Equation (b), which can be obtained from Eqs. (10–44) and (10–45), relates the frictional loss h_f to coefficient of resistance f, the average velocity, and the pipe length-diameter ratio. Equation (b) is called Darcy's equation for friction loss.

As an example let us assume a 20 in. diameter (inside) horizontal smooth pipe line transporting 300,000 gal/hr of water at 60°F. What horsepower per mile would be required to maintain the flow?

Solution.

$$\bar{u}_{av} = \frac{Q}{(\pi D^2/4)} = \frac{4(300{,}000)(0.1337/3600)}{\pi(20/12)^2} = 5.11 \text{ ft/sec}$$

Since $\nu = 1.21 \times 10^{-5}$ ft^2/sec for water at 60°F, we have

$$\text{Re} = \frac{\bar{u}_{av}D}{\nu} = \frac{(5.11)(20/12)}{1.21 \times 10^{-5}} = 704{,}000$$

According to Eq. (10.80),

$$f = [2.0 \log_{10} \text{Re}\sqrt{f} - 0.8]^{-2} = 0.0123$$

According to Darcy's equation for frictional loss,

$$\frac{h_f}{L} = \frac{f}{D} \cdot \frac{\bar{u}_{av}^2}{2g} = \frac{0.0123}{(20/12)} \cdot \frac{(5.11)^2}{(2)(32.2)} = 0.00298$$

The required horsepower is then

$$\left(\frac{h_f}{L}\right) \cdot \rho g \frac{(L)(Q)}{550} = \frac{(0.00298)(62.4)(5280)(3 \times 10^5)}{550}\left(\frac{0.1337}{3600}\right)$$

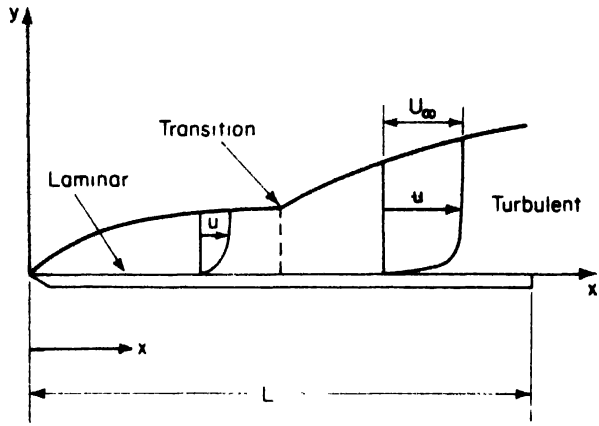
$$= 19.9 \text{ hp/mi}$$

10.6. *Turbulent Boundary Layer Over a Smooth Flat Plate*

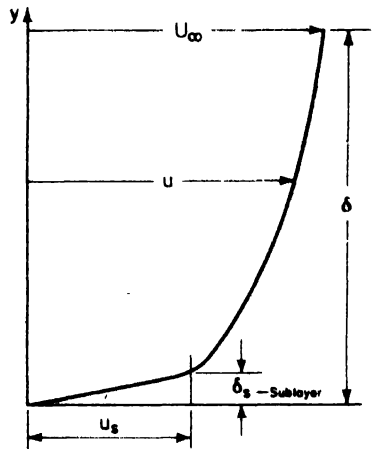
In §9.3 and §9.6a we have discussed the laminar boundary layer flow on a flat plate. The transition from laminar to turbulent flow has been discussed briefly in §10.1. In reality the boundary layer along a flat plate (or an arbitrary body) is not usually laminar or turbulent over the entire plate, but it generally begins with laminar flow over the forward portion of the plate and becomes turbulent beyond (Fig. 10.15a). The position of the point of transition depends on the intensity of turbulence in the free flow, etc. (§10.1). In the present discussion we shall first calculate the skin friction of a fully developed turbulent boundary layer and subsequently the calculation of the total skin friction including the effect of the laminar initial length will be made.

a. Fully Turbulent Boundary Layer

We shall assume that the boundary layer is turbulent from the leading edge ($x = 0$) of a flat plate with respect to a system of coordinates as shown in Fig. 10.15a. The momentum equation [Eq. (9–63)] may also be applied to the turbulent boundary layer. For a turbulent



(a)



(b)

Figure 10.15

boundary layer flow on a flat plate, Eq. (9–63) may be written in the form

$$\frac{d}{dx}\int_0^\delta u(U_\infty - u)\,dy = \frac{\tau_w}{\rho} \tag{10–88}$$

The shearing stress at the wall, τ_w, in Eq. (10–88) actually consists of both the turbulent stress and the viscous stress. Since we do not have an accurate knowledge of the thickness and velocity distribution in the

laminar sublayer, we shall limit our analysis to the turbulent zone (Fig. 10.15b).

According to Prandtl[19] we can assume that the velocity distribution within a turbulent boundary layer of a flat plate is essentially the same as that in a large diameter pipe. If we replace the maximum velocity at the center of the pipe by the velocity U_∞ at the outer edge of the boundary layer and the radius R by the thickness of the layer, Eq. (10–53) can therefore be written as

$$\frac{u}{U_\infty} = \left(\frac{y}{\delta}\right)^{1/7} \tag{10–89}$$

where $\delta = \delta(x)$ is the thickness of the boundary layer. The Blasius $\frac{1}{7}$th power law for the velocity distribution given in Eq. (10–89) is valid for Reynolds numbers $U_\infty L/\nu < 10^6$ where L is the length of the plate. For the turbulent shearing stress τ_w the empirical formula for pipe flow may be used, i. e.,

$$\tau_w = 0.03955\,\rho u_{av}^2 \left(\frac{u_{av} D}{\nu}\right)^{-1/4} \tag{10–48}$$

Since the velocity distribution for turbulent pipe flow corresponding to Eq. (10–89) is given by

$$\frac{u}{u_{max}} = \left(\frac{r'}{D/2}\right)^{1/7} \tag{10–53}$$

and $u_{av} = 0.817\,u_{max}$, the expression for the shearing stress in the turbulent boundary layer can be obtained from Eq. (10–48) in the form

$$\tau_w = 0.0233\,\rho U_\infty^2 \left(\frac{U_\infty \delta}{\nu}\right)^{-1/4} \tag{10–90}$$

Substituting Eqs. (10–89) and (10–90) into Eq. (10–88) gives

$$U_\infty^2 \frac{d}{dx}\left\{\delta \int_0^1 \left(\frac{y}{\delta}\right)^{1/7}\left[1 - \left(\frac{y}{\delta}\right)^{1/7}\right] d\left(\frac{y}{\delta}\right)\right\} = 0.0233\,U_\infty^2 \left(\frac{U_\infty \delta}{\nu}\right)^{-1/4} \tag{10–91}$$

from which a differential equation for the thickness of the turbulent boundary layer is obtained:

$$\frac{d\delta}{dx} = 0.239\left(\frac{\nu}{U_\infty \delta}\right)^{1/4} \tag{10–92}$$

Integrating between 0 and x yields

$$\frac{4}{5}\delta^{5/4} = 0.239\left(\frac{\nu}{U_\infty}\right)^{1/4} x$$

or

$$\frac{\delta}{x} = 0.379\left(\frac{\nu}{U_\infty x}\right)^{1/5} = \frac{0.379}{(\mathrm{Re}_x)^{1/5}} \tag{10–93}$$

where the constant of integration becomes zero since $x = 0$ when $\delta = 0$. The result shown in Eq. (10–93) indicates that the thickness of the turbulent boundary layer on a flat plate increases with $x^{4/5}$, whereas the thickness of the laminar boundary layer increases with $x^{1/2}$ [Eq. (9–41)] Hence, the thickness of the turbulent boundary layer grows at a more rapid rate along the length of the plate than the corresponding laminar boundary layer. Introducing the expression for δ from Eq. (10–93) into Eq. (10–90) the turbulent shearing stress takes the form

$$\tau_w = 0.0295\,\rho U_\infty^2 (\mathrm{Re}_x)^{-1/5} \tag{10–94}$$

and the total frictional force per unit width for one side of the plate of length L is given by

$$F = \int_0^L \tau_w\,dx = 0.0368\,\rho U_\infty^2 (\mathrm{Re}_L)^{-1/5} \tag{9–39}$$

The average skin-friction coefficient in this case is

$$C_F = \frac{F}{\frac{1}{2}\rho U_\infty^2 L} = 0.074(\mathrm{Re}_L)^{-1/5} \tag{10–95}$$

Equation (10–95) is valid for a range of $5 \times 10^5 < \mathrm{Re}_L < 10^7$. For larger Reynolds numbers in the turbulent boundary layer over a flat plate one can replace the $\frac{1}{7}$th power law for the velocity distribution by the logarithmic law [Eq. (10-40a)]. Prandtl[19] made the calculation of skin-friction coefficient by substituting the logarithmic law for the velocity distribution [Eq. (10–39)] into the momentum equation, Eq. (10–88), and found

$$C_F = \frac{0.455}{[\log_{10} \mathrm{Re}_L]^{2.58}} \tag{10–96}$$

Equation (10-96) is valid for Re_L from 10^6 up to 10^9.

b. Boundary Layers in the Transition Range

If the boundary layer along a flat plate begins with laminar flow over the forward portion of the plate and becomes turbulent beyond (Fig. 10.15a), the total frictional force on the plate decreases. Following Prandtl[19] this reduced frictional force can be estimated for any known transition Reynolds number. According to Eq. (9–40) the skin friction of the laminar boundary layer flow over a distance x from the leading edge of the plate is

$$F_{\mathrm{lam}} = \frac{1.328}{(\mathrm{Re}_x)_{tr}^{1/2}} \left(\frac{x}{2}\rho U_\infty^2\right) \tag{10–97}$$

The skin friction of the turbulent boundary layer flow over the rear portion of the plate (from a distance x o L) is

$$F_{\text{turb}} = \left[\frac{(0.074)L}{(\text{Re}_L)^{1/5}} - \frac{(0.074)x}{(\text{Re}_x)_{cr}^{1/5}}\right]\left(\frac{1}{2}\rho U_\infty^2\right) \tag{10-98}$$

Hence, the total frictional force on the entire plate is the sum of Eqs. (10–98) and (10–97) which yields

$$F_{\text{total}} = \left[\frac{(1.328)x}{(\text{Re}_x)_{cr}^{1/2}} + \frac{(0.074)L}{(\text{Re}_L)_{cr}^{1/5}} - \frac{(0.074)x}{(\text{Re}_x)_{cr}^{1/5}}\right]\left(\frac{1}{2}\rho U_\infty^2\right) \tag{10-99}$$

But the ratio

$$\frac{x}{L} = \frac{(\text{Re}_x)_{cr}}{\text{Re}_L}$$

Hence Eq. (10-99) may be written as

$$F_{\text{total}} = \left[\frac{1.328}{\text{Re}_L}(\text{Re}_x)_{cr}^{1/2} + \frac{0.074}{(\text{Re}_L)^{1/5}} - \frac{0.074}{\text{Re}_L}(\text{Re}_x)_{cr}^{1/5}\right]\left(\frac{1}{2}\rho U_\infty^2 L\right) \tag{10-100}$$

and the total skin-friction coefficient becomes

$$C_{F_{\text{total}}} = \frac{0.074}{(\text{Re}_L)^{1/5}} - \frac{1}{\text{Re}_L}[0.074(\text{Re}_x)_{cr}^{1/5} - 1.328(\text{Re}_x)_{cr}^{1/2}] \tag{10-101}$$

Taking the transition Reynolds number to be 5×10^5 $[(\text{Re}_x)_{cr} = 5 \times 10^5]$, the coefficient of the reduced skin friction which accounts for the laminar region at the front portion of the plate becomes

$$C_f = \frac{0.074}{(\text{Re}_L)^{1/5}} - \frac{1700}{\text{Re}_L} \tag{10-102}$$

This equation is valid for the range $5 \times 10^5 \leq \text{Re}_L \leq 10^7$.

Figure 10.16 gives the relation between C_F and the Reynolds number on the basis of the analysis above. Line 1 corresponds to the

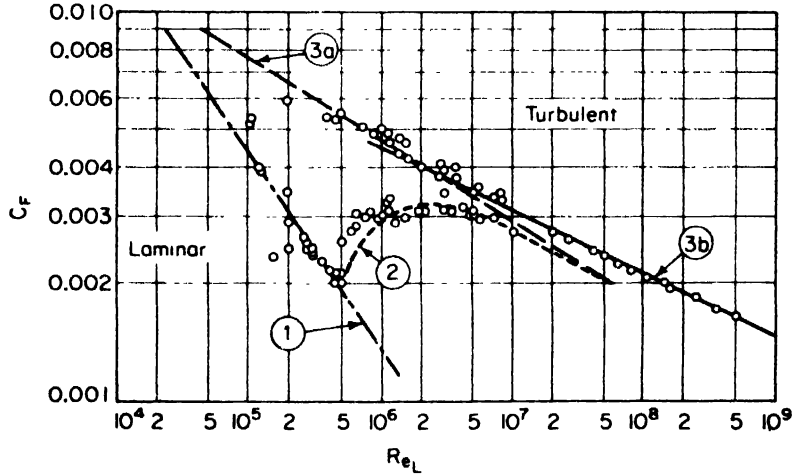


Fig. 10.16. H. Schlichting, "Boundary Layer Theory." New York: McGraw-Hill, 1960

laminar region [Eq. (9–40)], line 2 to the transition region [Eq. (10–102)], and lines 3a and 3b to the turbulent region [Eqs. (10–95) and (10–96)]. Good agreement between theory and measurement is also depicted in this figure.

Illustrative Example 10.5 Estimate the frictional drag force on a plate 10 ft long and 20 ft wide which is placed in a wind tunnel at a velocity of 150 ft/sec under the following assumptions: (a) The boundary layer is to remain laminar over the entire surface of the plate, (b) The transition occurs $(\mathrm{Re}_x)_{cr} = 5 \times 10^5$, (c) The boundary layer is turbulent over the entire smooth surface of the plate, and (d) The boundary layer is turbulent over the entire rough surface with $U_\infty k_s/\nu = 10^4$.

Solution. Let us assume that the interference between the plate and tunnel wall is not taken into consideration, and the air has a value of

$$\nu = 1.58 \times 10^{-4}\ \mathrm{ft^2/sec}.$$

(a) The Reynolds number is

$$\mathrm{Re}_L = \frac{U_\infty L}{\nu} = \frac{(150)(10)}{1.58 \times 10^{-4}} = 9.5 \times 10^6$$

From Eq. (9–40) we obtain the frictional coefficient for the laminar boundary layer as

$$C_F = \frac{1.32}{\sqrt{\mathrm{Re}_L}} = \frac{1.32}{\sqrt{9.5 \times 10^6}} = 0.000428$$

The total frictional drag force (both sides) is

$$F = 2[\tfrac{1}{2} C_F \rho U_\infty^2 A] = (0.000428)(0.002378)(150)^2(200)$$
$$= 4.58\ \mathrm{lbf}$$

(b) If the transition occurs at $(\mathrm{Re}_x)_{cr} = 5 \times 10^5$, the frictional coefficient which accounts for the laminar region at the front portion of the plate is given by Eq. (10–102) as

$$C_F = \frac{0.074}{(\mathrm{Re}_L)^{1/5}} - \frac{1700}{\mathrm{Re}_L}$$
$$= \frac{0.074}{(9.5 \times 10^6)^{0.2}} - \frac{1700}{9.5 \times 10^6} = 0.0028$$

The total frictional drag force is

$$F = (0.0028)(0.002378)(150)^2(200) = 30\ \mathrm{lbf}$$

(c) The frictional coefficient for the turbulent boundary layer over a smooth plate is given by Eq. (10–95) as

$$C_F = \frac{0.074}{(\mathrm{Re}_L)^{1/5}} = 0.00298$$

and Eq. (10–96) gives

$$C_F = \frac{0.455}{(\log_{10} \mathrm{Re}_L)^{2.58}} = 0.0030$$

The total frictional drag force in this case is almost equal to that obtained in (b).

(d) In Reference 10.8 (p. 553) the frictional coefficient for the turbulent boundary layer over a rough plate is

$$C_F = \left[1.89 + 1.62 \log_{10}\left(\frac{L}{k_s}\right)\right]^{-2.5}$$

but

$$\frac{L}{k_s} = \frac{U_\infty L}{\nu}\left(\frac{\nu}{U_\infty k_s}\right) = (9.5 \times 10^6)(10^{-4}) = 9.5 \times 10^2$$

Hence,

$$C_F = [1.89 + 1.62 \log_{10}(9.5 \times 10^2)]^{-2.5} = 0.00861$$

The frictional drag force is

$$F = (0.00861)(0.002378)(150)^2(200) = 92 \text{ lbf}$$

NOMENCLATURE FOR CHAPTER TEN

C_F	average skin-friction coefficient of a flat plate, Eq. (10-95)
f	coefficient of friction for pipe flow, Eq. (10-47)
k	proportionality constant of mixing length, Eq. (10-25)
k'	proportionality constant of mixing length (rough pipe), Eq. (10-74)
k_s	mean height of the roughness element, §10.5d
l	Prandtl mixing length, Eq. (10-20)
r	radial direction measured from the center line of a pipe, Eq. (10-43)
R	radius of a pipe, Eq. (10-44)
Re	Reynolds number, Eq. (10-47)
Re_{cr}	critical Reynolds number, §10.1
$\bar{R}(u', v')$	correlation coefficient between u' and v' at a point, Eq. (10-18)
R_h	hydraulic radius, §10.5d
T	period of sampling, Eq. (10-2)
t_0	a time, Eq. (10-2)
U_∞	free-stream velocity, §10.1
$u_{\max}$	maximum velocity, Eqs. (10-62) and (10-68b)
$\bar{u}_{av}$	average turbulent mean velocity, Eq. (10-46)
u, v, w	velocity components in Cartesian coordinates

u', v', w'	fluctuating velocity components in Cartesian coordinates, Eq. (10-3)
$\bar{u}, \bar{v}, \bar{w}$	mean velocity components in Cartesian coordinates, Eq. (10-2)
$\bar{v}_z = u$	mean velocity component in z-direction of a pipe flow (the overbar was dropped later for convenience), Eq. (10-45)
v_*	frictional velocity, Eq. (10-20)
x, y, z	Cartesian coordinates
δ	boundary layer thickness of a flat plate, Eq. (10-93)
δ^*	displacement thickness, §10.1
ε	apparent kinematic (or eddy) viscosity, Eqs. (10-19a) and (10-20)
ρ	density, Eq. (10-11)
$\rho\overline{u'v'}, \ldots$	Reynolds stresses, Eq. (10-15)
ν	kinematic viscosity, §10.1
$\bar{\sigma}_{xx}, \bar{\sigma}_{yy}, \bar{\sigma}_{zz}$	average normal stresses in Cartesian coordinates, Eq. (10-13)
$\bar{\sigma}_{xy}, \bar{\sigma}_{yz}, \bar{\sigma}_{zx}$	average shearing stresses in Cartesian coordinates, Eq. (10-13)
$(\sigma_{yx})_t$	turbulent shearing stress in x-direction, Eq. (10-17)
$\sigma_{rz} = \tau_{rz} = \tau$	shearing stress of pipe flow in z-direction (both laminar and turbulent), Eqs. (10-43) and (10-57)
τ_w	wall shearing stress, Eq. (10-44)

PROBLEMS

10.1. Verify that

$$\overline{\int_{t-T/2}^{t+T/2} u\, d\eta} = \int_{t-T/2}^{t+T/2} \bar{u}\, d\eta$$

10.2. For compressible flows the fluctuating density is defined by

$$\rho' = \rho - \bar{\rho}$$

With the definition of the fluctuating velocity components [Eq. (10-3)] show that the continuity equation [Eq. (5-9a)] can be written as

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x}(\bar{\rho}\bar{u} + \overline{\rho' u'}) + \frac{\partial}{\partial y}(\bar{\rho}\bar{v} + \overline{\rho' v'}) + \frac{\partial}{\partial z}(\bar{\rho}\bar{w} + \overline{\rho' w'}) = 0$$

10.3. For an incompressible laminar flow in a circular pipe the equations of motion are given by

$$\frac{1}{r}\mu \frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) = \frac{\partial p}{\partial z} \quad \text{and} \quad \frac{\partial p}{\partial r} = 0$$

Show that for turbulent flow in a circular pipe the equations of motion can be written as

$$\frac{1}{r}\mu \frac{\partial}{\partial r}\left(r \frac{\partial \bar{v}_z}{\partial r}\right) - \frac{1}{r}\rho \frac{\partial}{\partial r}[r(\overline{v_z' v_r'})] = \frac{\partial \bar{p}}{\partial z} \tag{a}$$

$$\frac{1}{r}\rho\frac{\partial}{\partial r}[r(\overline{v_r'v_r'})] - \frac{1}{r}\rho\overline{v_\theta'v_\theta'} = -\frac{\partial\bar{p}}{\partial r} \tag{b}$$

where $p = \bar{p}$, $v_r' = v_r - \bar{v}_r$, $v_z' = v_z - \bar{v}_z$, and $v_\theta' = v_\theta - \bar{v}_\theta$.

10.4. Using von Kármán's similarity hypothesis derive the universal velocity profile near a wall [Eq. (10-39)].

10.5. For a two-dimensional flow near the boundary of a flat plate, Eq. (10-31) reduces to

$$\mu\frac{\partial\bar{u}}{\partial y} + (\sigma_{yx})_t = \tau_w$$

If $(\tau_{yx})_t = \rho k\bar{u}^2$ is an approximation for the turbulent stress, find the velocity profile.

10.6. Compare the velocity distributions for a pipe flow by plotting (a) Poiseuille's formula, and (b) $u/u_{\max} = (1-r/R)^n$ when $n = \frac{1}{7}$ and $\frac{1}{9}$.

10.7. A 2 in. diameter (inside) horizontal smooth pipe is used to transport oil having $\nu = 1.1 \times 10^{-3}$ ft^2/sec. What horsepower per 1000 ft would be required to maintain the flow when:

(a) The flow rate of the oil is 3,000 gal/hr?

(b) The flow rate of the oil is 30,000 gal/hr?

10.8. Oil having $\nu = 1.2 \times 10^{-1}$ ft^2/sec is transported through a horizontal smooth pipe line ($\frac{1}{2}$ mile long) at a rate of 75,000 gal/hr. Determine the horsepower required to maintain the flow at a Reynolds number of 35,000.

10.9. Oil flows through a horizontal smooth pipe line with maximum velocity of 7 ft/sec. If the diameter of the pipe is 10 in. and $\nu = 1.2 \times 10^{-4}$ ft^2/sec, find the horsepower required per mile to maintain the flow at the velocity given above.

10.10. The universal velocity distribution in turbulent pipe flow due to Prandtl is given by Eq. (10-62). Show that the expression for the average velocity is given by

$$\frac{u_{\text{av}}}{u_{\max}} = 1 - \frac{2}{3k_1}\left(\frac{v_*}{u_{\max}}\right)$$

10.11. Using Nikuradse's experimental data (Fig. 10.10) determine the value of k_1 in Eq. (10-62) by choosing $r/R = 0.75$ so that agreement between experimental data and the result from Eq. (10-62) is obtained at this location.

10.12. If the distance $2h$ between two parallel walls of a rectangular channel is taken to be much smaller than the width, show that the universal velocity distribution in a turbulent channel flow can be obtained by using the Prandtl mixing length theory in the form

$$\frac{u}{u_{\max}} = 1 - \frac{1}{k_a}\left(\frac{v_*}{u_{\max}}\right)\left[\ln\left(\frac{1+\sqrt{y/h}}{1-\sqrt{y/h}}\right) - 2\sqrt{\frac{y}{h}}\right]$$

where y is the distance measured from the center line of the channel and k_a is an empirical constant.

10.13. Find the average velocity of the turbulent flow in a channel (see Prob. 10.11) in terms of the maximum velocity and the frictional velocity v_*.

10.14. Using von Kármán's similarity law show that the universal velocity distribution in a turbulent channel flow is given by

$$\frac{u_{max} - u}{v_*} = -\frac{1}{k_b}\left[\ln\left(1 - \sqrt{\frac{y}{h}}\right) + \sqrt{\frac{y}{h}}\right]$$

where y, h are defined in Prob. 10.12, and k_b is an empirical constant.

10.15. From Fig. 10.10 determine the value of k_2 in Eq. (10-68b). The value of $(u_{max} - u)/v_*$ at $r/R = 0.75$ may be used.

10.16. Determine the average velocity of the turbulent flow in a pipe with the velocity distribution given in Eq. (10-68b).

10.17. Water at 60°F is transported through a horizontal smooth pipe line $\frac{1}{2}$ mile long, with maximum velocity of 10 ft/sec. If the Reynolds number is 10^6, find the diameter of the pipe and the horsepower required to maintain the flow. Prandtl's universal velocity distribution may be used.

10.18. Using von Kármán's universal velocity distribution find the diameter of the pipe and horsepower required for Prob. 10.17.

10.19. Calculate the Reynolds stress in the form $\overline{v_x' v_r'}/v_*^2$ [see Eq. (a) in illustrative example 10.3] for turbulent flow in fully rough pipes. Discuss the significance of the Reynolds stress.

10.20. Show that the velocity distribution for turbulent flow in rough pipes can be expressed as

$$\frac{u_{max} - u}{u_{av}} = -2.03\sqrt{f}\log_{10}\left(1 - \frac{r}{R}\right)$$

and plot $(u_{max} - u)/u_{av}$ vs. r/R for the roughness Reynolds number $v_* k_s/\nu = 100$ and $\mathrm{Re} = 10^6$.

10.21. Water at 60°F is transported through a horizontal rough pipe line $(v_* k_s/\nu = 100)$, $\frac{1}{2}$ mile long, with maximum velocity of 10 ft/sec. If the Reynolds number is 10^6, find the diameter of the pipe and horsepower required to maintain the flow. Compare the present results with those obtained for the smooth pipe and the moderately rough pipe $(v_* k_s/\nu = 25)$

10.22. If the velocity distribution in the turbulent boundary layer on a flat plate is given as

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/n}$$

determine the ratio θ/δ and δ^*/δ in terms of n.

For $n = 7$, plot θ/x vs. Re_x in the range of Re_x from 10^3 to 6×10^6. The transition Reynolds number is assumed to occur at $(\mathrm{Re}_x)_{cr} = 10^5$.

10.23. Estimate the power required to move a flat plate, 30 ft long and 10 ft wide, in oil ($\rho = 1.785$ slugs/ft^3 and $\mu = 1.4 \times 10^{-3}$ lbf-sec/ft^2) at 25 ft/sec under the following assumptions:

(a) The boundary layer is to remain laminar over the entire surface of the plate.
(b) The transition occurs at $(\mathrm{Re}_L)_{cr} = 3 \times 10^5$.
(c) The boundary layer is turbulent over the entire smooth surface of the plate.
(d) The boundary layer is turbulent over the entire rough surface with $U_\infty k_s/\nu = 5 \times 10^3$ (see illustrative example 10.5).

11

INVISCID COMPRESSIBLE FLOW

Elementary physics has taught us that liquids do not change their volume appreciably under pressure and temperature changes and they can be treated as incompressible fluids for all practical purposes. On the other hand, the density of gases and vapors varies with both pressure and temperature and hence they are compressible. For a perfect gas the relation among density, pressure, and temperature is given by the equation of state in Appendix B.1.

In a large number of practical applications, the gas density varies very little if the pressure differences produced by the flow regime are small. Under this condition, the flow of gas around bodies or inside conduits may be studied with sufficient accuracy by assuming the gas density to be constant. The basic concepts of the flow of this type were discussed in Chaps. Six and Seven for an inviscid fluid and Chaps. Eight through Ten for a viscous fluid.

When the variations of pressure produced by the flow regime are large, the assumption of incompressible flow is no longer valid as the corresponding changes in density must be considered. Experience shows that compressions and expansions of the gas involve work done on and by the gas, respectively, and this results in changes in temperature of the gas. Hence the laws of mechanics alone are insufficient for a complete description of the physical quantities involved. In addition

thermodynamics is necessary to completely determine these quantities. This branch of science is sometimes called *aerothermodynamics*. A review of the basic concepts of thermodynamics is presented in Appendix B.

In Chap. Nine we have learned that the influence of the fluid viscosity is limited to a thin boundary layer. Therefore, for a very large group of gas dynamic problems, such as the pressure distribution over a body in a compressible flow, we may neglect the viscosity of the fluid and still obtain a very good approximation to the real physical situation. This simplification greatly reduces the difficulties as well as the labor involved for the solution of the fundamental differential equations of a compressible fluid. In this chapter our discussions are confined to inviscid compressible fluids.

In § 11.1 the significance of the four important parameters in viscous compressible fluids are discussed, based on the similarity of flows.

In § 11.2 the equation of continuity for a multidimensional compressible fluid is reduced from the general form of the equation in Chap. Five. The continuity equation for flow along a narrow stream tube is also derived, based on the law of conservation of mass.

In § 11.3 the equation of motion for an irrotational flow is given. The equations of motion along a streamline are reduced from the equations of motion in cylindrical coordinates (§ 5.5b).

In § 11.4 the energy equation is found from the simplification of the general form in Chap. Five.

In § 11.5 the pressure equation or Bernoulli's equation is derived from the equation of motion. The case for isentropic flow is also discussed.

In § 11.6 the circulation theorem of Kelvin for compressible fluids is discussed and derived.

In § 11.7 the propagation of a pressure change (sound velocity) through both an elastic solid and a fluid are discussed. Nonstationary waves are also discussed, based on the linearized equations of continuity and motion, and the isentropic relation.

In § 11.8 the propagation of a plane wave of high intensity is discussed and the formation of a shock wave is also illustrated and discussed.

In § 11.9 the physical significance of the Mach number, Mach line and Mach cone is illustrated and discussed.

In § 11.10 Bernoulli's equation is expressed in terms of the stagnation quantities or maximum velocity. The relation of local velocity and the local Mach number is also given.

In § 11.11 the critical velocity is discussed, and the thermodynamic quantities at the critical condition are derived.

In § 11.12 the pressure, density, and temperature ratios are given as a function of the Mach number.

In § 11.13 the pressure coefficient for the case of small disturbances is derived.

In § 11.14 the isentropic relation for flow through tubes, the mass flow through a converging nozzle, and the flow through a convergent-divergent nozzle are discussed.

In § 11.15 the fundamental equations for flow through a constant-area tube with heat addition are derived. The relationships of the thermodynamic properties for a Rayleigh process are derived from the basic equations in terms of the Mach number.

In § 11.16 the general theorems and equations for finite motion of a normal shock wave are discussed. Rankine-Hugoniot relations for a normal shock wave are derived and the entropy increase across the shock is shown.

In § 11.17 the general theorems and equations for finite motion of an oblique shock wave are discussed. Relations between the wave and the deflection angles and the Mach number are derived and the formation of detached shock waves are discussed.

In § 11.18 graphical representation of the shock wave relations, the shock polar, is discussed and illustrated.

In § 11.19 compression and expansion by turning in supersonic flow are discussed. The Prandtl-Meyer function is derived.

In § 11.20 applications of the shock-expansion theory are presented. These include the loss of pressure head through a shock wave, flow in a de Laval nozzle, and the occurrence of supersonic wave drag.

A. FUNDAMENTAL EQUATIONS OF THE FLOW OF COMPRESSIBLE INVISCID FLUIDS

In the study of flows of an incompressible inviscid fluid we need only to find the velocity and pressure at every point in space. When we study flows of a compressible fluid, we have to consider all components of velocity together with the pressure, density, and temperature of the fluid. In order to determine these quantities, one vector and three scalar, we must have one vector equation and three scalar equations, i.e., equation of motion, equation of continuity, equation of state for a perfect gas, and the energy equation. The equation of state for a perfect gas is given in § 5.4 as well as in Appendix B.1.

11.1. Controlling Parameters in Compressible Flow

Before proceeding to the discussion of the dynamics of compressible fluids, it is important to obtain an orientation of the subject by considering the controlling parameters involved in the problem. These parameters are expressed in terms of fluid properties which characterize the physical situation under consideration. In § 8.1 we have deduced Reynolds' similarity principle from the Navier-Stokes equation and obtained the controlling parameter in viscous incompressible flow. In the same manner as in § 8.1, controlling parameters in a viscous compressible flow can be deduced from the energy equation. For a viscous compressible flow the energy equation is given [Eq. (5–32)] as follows:

$$\rho\frac{D(C_pT)}{Dt} = \frac{Dp}{Dt} + \nabla\cdot(k\nabla T) + \mu\left[-\frac{2}{3}(\nabla\cdot\mathbf{q})^2 + \cdots\right] \tag{11-1}$$

The last term on the right-hand side of Eq. (11–1) is the dissipation function [Eq. (5–23b)].

Let the quantity with subscript ∞ denote the characteristic reference magnitude of the corresponding quantity at infinity, and l and $t_0 = l/q_\infty$ a linear dimension of the body and time scale, respectively. The nondimensional quantities may now be introduced

$$\begin{aligned} t^* &= \frac{tq_\infty}{l}, & \nabla^* &= l\nabla, & \mathbf{q}^* &= \frac{\mathbf{q}}{q_\infty} \\ \rho^* &= \frac{\rho}{\rho_\infty}, & p^* &= \frac{p}{\rho_\infty q_\infty^2}, & T^* &= \frac{T}{T_\infty} \\ \mu^* &= \frac{\mu}{\mu_\infty}, & k^* &= \frac{k}{k_\infty}, & C_p^* &= \frac{C_p}{C_{p_\infty}} \end{aligned} \tag{11-2}$$

Substituting the nondimensional quantities above into Eq. (11–1), the nondimensional form of the energy equation is obtained:

$$\begin{aligned} \rho^*\frac{D(C_p^*T^*)}{Dt^*} = {} & \left(\frac{q_\infty^2}{C_{p_\infty}T_\infty}\right)\frac{Dp^*}{Dt^*} + \left(\frac{\mu_\infty}{\rho_\infty q_\infty l}\right)\left(\frac{k_\infty}{C_{p_\infty}\mu_\infty}\right)\nabla^*\cdot(k^*\nabla^*T^*) \\ & + \left(\frac{\mu_\infty}{\rho_\infty q_\infty l}\right)\left(\frac{q_\infty^2}{C_{p_\infty}T_\infty}\right)\mu^*\left[-\frac{2}{3}(\nabla^*\cdot\mathbf{q}^*)^2 + \cdots\right] \end{aligned} \tag{11-3}$$

The coefficients of the terms on the right-hand side of Eq. (11–3) may be interpreted individually. In the first group the nondimensional quantity can be expressed as

$$\frac{q_\infty^2}{C_{p_\infty}T_\infty} = \frac{\kappa Rq_\infty^2}{C_{p_\infty}\kappa RT_\infty} = (\kappa-1)\frac{q_\infty^2}{a_\infty^2} = (\kappa-1)\mathrm{M}_\infty^2$$

This gives the Mach number and the ratio of the specific heats. The Mach number is defined as the ratio of the free-stream velocity to the corresponding velocity of sound as

$$M_\infty = \frac{q_\infty}{a_\infty} \tag{11-4}$$

Hence our first parameter, the Mach number, is a measure of the compressibility of the fluid. The second parameter in a compressible fluid is the ratio of the specific heats of the fluid at constant pressure to that at constant volume:

$$\kappa = \frac{C_p}{C_\tau} \tag{11-5}$$

which is a measure of the relative internal complexity of the molecules of the fluid.

In the second group the nondimensional quantities can be written as

$$\left(\frac{\mu_\infty}{\rho_\infty q_\infty l}\right)\left(\frac{k_\infty}{C_{p_\infty}\mu_\infty}\right) = \frac{1}{\text{RePr}}$$

where the third parameter Re is the already familiar Reynolds number. The following nondimensional parameter

$$\text{Pr} = \frac{\mu_\infty C_{p_\infty}}{k_\infty} = \frac{\mu_\infty/\rho_\infty}{k_\infty/\rho_\infty C_{p_\infty}} = \frac{\text{Kinematic viscosity}}{\text{Thermal diffusivity}} \tag{11-6}$$

is called the Prandtl number, which is our fourth parameter for a compressible fluid. The Prandtl number is a measure of the relative importance of heat conduction and viscosity of a fluid. In the third group the nondimensional quantities yield no parameters other than the four parameters mentioned above.

The significance of these four parameters can be seen in the requirement of dynamical similarity of flows of a viscous compressible fluid. In § 8.1 it was pointed out that in order to have flows around two bodies be completely similar, not only must the bodies be geometrically similar, but also the four parameters (Mach number, ratio of specific heats, Reynolds number, and Prandtl number) in the two flow patterns must be equal in magnitude. For flows of an inviscid compressible fluid we do not need to be concerned about the Reynolds number and the Prandtl number; however, the Mach number and the ratio of specific heats are important in this case.

11.2. *Equation of Continuity*

a. The Multidimensional Flow

For a compressible fluid the equation of continuity is given in Eq. (5-7) as follows:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q}) = 0 \tag{5-7}$$

Equation (5–7) may be written in terms of the material derivative as

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{q}) = 0 \tag{5-8}$$

In Cartesian coordinates Eq. (5–7) may be written

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{5-9a}$$

For two-dimensional or three-dimensional axially symmetrical steady flows [Eq. (5–40)] the equation of continuity may be written

$$\frac{\partial(\rho q_1)}{\partial x_1} + \frac{\partial(\rho q_2)}{\partial x_2} + n\left(\frac{\rho q_2}{x_2}\right) = 0 \tag{11-7a}$$

where $n = 0$, $q_1 = u$, $q_2 = v$, $x_1 = x$, and $x_2 = y$ for the two-dimensional case, and $n = 1$, $q_1 = v_z$, $q_2 = v_r$, $x_1 = z$, and $x_2 = r$ for the axially symmetrical case. The forms of the equation of continuity in other coordinate systems are given in § 5.5a and § 5.6a.

From Eq. (11-7a) a stream function ψ can be defined so that

$$\rho q_1 x_2^n = \frac{\partial \psi}{\partial x_2}, \qquad \rho q_2 x_2^n = -\frac{\partial \psi}{\partial x_1} \tag{11-7b}$$

It can easily be shown that Eq. (11-7a) is automatically satisfied by the stream function ψ.

b. Flow Along a Narrow Stream Tube

In § 6.1 we discussed the continuity equation for flow along a narrow stream tube in the case of steady incompressible flow. For an unsteady flow the stream tube changes shape with each succeeding instant of time. Figure 11.1 shows a stream tube with the mass of fluid, $\rho A\, ds$, moving in a unit time through the two normal cross sections A and $A + dA$ which are separated by a distance ds. The mathematical expression for the law of conservation of mass in this case can be written

$$\frac{D}{Dt}(\rho A\, ds) = 0 \tag{11-8}$$

Expanding the substantial derivative, Eq. (11–8), it becomes

$$\frac{\partial}{\partial t}(\rho A\, ds) + V\frac{\partial}{\partial s}(\rho A)\, ds = 0 \tag{11-9}$$

where $V = ds/dt$ is the velocity along the mean streamline (V is assumed constant over A). Equation (11–9) can be written in the form

$$\frac{\partial(\rho A)}{\partial t} ds + \rho A\, dV + V\frac{\partial}{\partial s}(\rho A)\, ds = 0 \tag{11-10}$$

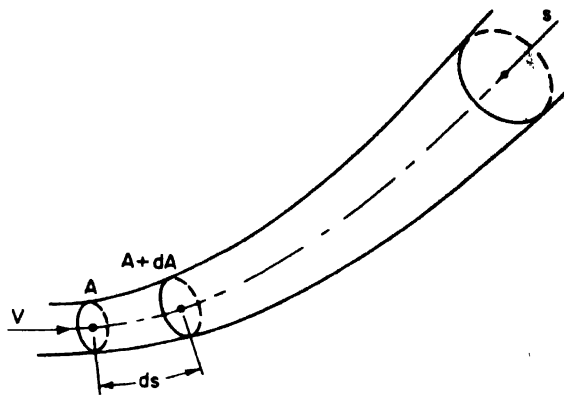


Figure 11.1

Hence, the equation of continuity is

$$\frac{\partial(\rho A)}{\partial t} + \frac{\partial(\rho V A)}{\partial s} = 0 \tag{11-11}$$

For steady flow of gas through a stream tube (or a pipe), the continuity equation reduces to

$$\rho V A = \text{Constant} \tag{11-12}$$

It should be noted that the equations above may be applied for flow along a stream tube with a large cross-sectional area if average values of velocities and densities at each cross section are used.

11.3. Equations of Motion

a. The Multidimensional Flow

The general vector equation defining the conservation of momentum for both inviscid compressible and incompressible flows is given by Eq. (5-20) as

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla)\mathbf{q} = \mathbf{X} - \frac{\nabla p}{\rho} \tag{5-20}$$

or in Cartesian notations [Eq. (7-2)],

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X_x - \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{7-2a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = X_y - \frac{1}{\rho}\frac{\partial p}{\partial y} \tag{7-2b}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = X_z - \frac{1}{\rho}\frac{\partial} \tag{7-2c}$$

For gases, the external force is usually very small, therefore **X** can be neglected in Eq. (5–20). The expressions of the equations of motion in other systems of coordinates are given in § 5.5b and § 5.6b.

In order to introduce the velocity potential ϕ in the equation of motion for irrotational flow, the convective term in Eq. (5–20) is expanded [Eq. (7–21)], and an alternative form of the equation of motion is obtained.

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla\left(\frac{q^2}{2}\right) + \boldsymbol{\Omega} \times \mathbf{q} = \mathbf{X} - \frac{\nabla p}{\rho} \tag{11-13}$$

For an irrotational flow, $\boldsymbol{\Omega} \times \mathbf{q}$ becomes zero in Eq. (11–13). If we assume that the body force derives from a potential ϕ_f, then Eq. (11–13) becomes

$$\nabla\left(\frac{\partial \phi}{\partial t} + \frac{q^2}{2} - \phi_f\right) = -\frac{\nabla p}{\rho} \tag{11-14}$$

For a steady potential flow Eq. (11–14) reduces to

$$\nabla\left(\frac{q^2}{2} - \phi_f\right) = -\frac{\nabla p}{\rho} \tag{11-15}$$

or in differential form,

$$q\,dq - d\phi_f + \frac{dp}{\rho} = 0 \tag{11-16}$$

b. Equations of Motion Along a Streamline

Consider a streamline passing through a point O, the origin of a coordinate system with the s-axis tangent to the streamline, n-axis normal to the streamline and z perpendicular to the s–n plane (Fig. 11.2). The equations of motion along these axes (which are called intrinsic equations) can be obtained from Eqs. (5–52*a*), (5–52b), and (5–52c), respectively, and are written:

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s} = X_s - \frac{1}{\rho}\frac{\partial p}{\partial s} \tag{11-17a}$$

$$\frac{V^2}{R} = X_n - \frac{1}{\rho}\frac{\partial p}{\partial n} \tag{11-17b}$$

$$0 = X_z - \frac{1}{\rho}\frac{\partial p}{\partial z} \tag{11-17c}$$

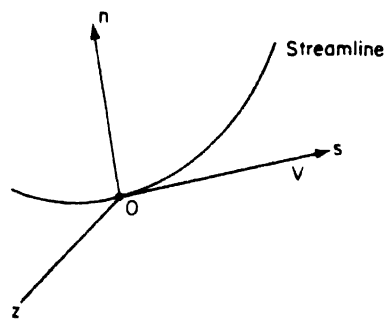


Figure 11.2

where V is the velocity in the s-direction, R the radius of curvature at point O, and X_s, X_n, and X_z are the external forces in the s, n, and z-directions, respectively.

For an irrotational flow Eq. (11-17a) becomes

$$\frac{\partial}{\partial s}\left(\frac{\partial \phi}{\partial t} + \frac{V^2}{2} + \phi_f\right) = -\frac{1}{\rho}\frac{\partial p}{\partial s} \tag{11-18}$$

Equation. (11-18) can also be obtained directly from Eq. (11-14).

11.4. The Energy Equation

The energy equation based on the law of conservation of energy is given in Eq. (5-26). For an inviscid, adiabatic flow with negligible body forces, Eq. (5-26) reduces to

$$\frac{Dh}{Dt} = \frac{1}{\rho}\frac{Dp}{Dt} = \frac{1}{\rho}\frac{\partial p}{\partial t} + \mathbf{q} \cdot \frac{\nabla p}{\rho} \tag{5-36}$$

Scalar multiplication of each term of Eq. (5-20) by $\mathbf{q}$ yields, after neglecting the body force,

$$\frac{1}{2}\frac{Dq^2}{Dt} = -\mathbf{q} \cdot \frac{\nabla p}{\rho} \tag{11-19}$$

Substituting Eq. (11-19) into Eq. (5-36) gives the energy equation in the form:

$$\frac{D}{Dt}\left(h + \frac{q^2}{2}\right) = \frac{1}{\rho}\frac{\partial p}{\partial t} \tag{11-20}$$

Equation (11-20) shows that a change in the quantity $h_0 = h + q^2/2$, the total enthalpy, is due to a nonsteady effect of the pressure.

For a steady flow the energy equation is

$$d\left(h + \frac{q^2}{2}\right) = 0$$

or

$$h + \frac{q^2}{2} = C_p T + \frac{q^2}{2} = \text{Constant} = h_0 = C_p T_0 \tag{11-21}$$

where T_0 is the absolute temperature at the points where the fluid is at rest.

11.5. The Pressure Equation or Bernoulli's Equation

In the preceding section the energy equation has been derived, based on the thermodynamic law. Similar relations can be obtained by utilizing the equation of motion and the results will show impor-

tant physical significance. Scalar multiplication of $\nabla p/\rho$ by the displacement $d\mathbf{r}$ results in the following relation:

$$d\mathbf{r} \cdot \frac{\nabla p}{\rho} = \frac{dp}{\rho} = d\int \frac{dp}{\rho} = d\mathbf{r} \cdot \nabla \int \frac{dp}{\rho}$$

The substitution of this relation into Eq. (11-13), the equation of motion with negligible body forces, yields

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla\left(\int \frac{dp}{\rho} + \frac{q^2}{2}\right) = \mathbf{q} \times \boldsymbol{\Omega} \tag{11-22}$$

Equation (11-22) can be written for steady flow, after scalar multiplication by $d\mathbf{r}$,

$$d\left(\int \frac{dp}{\rho} + \frac{q^2}{2}\right) = (\mathbf{q} \times \boldsymbol{\Omega}) \cdot d\mathbf{r} \tag{11-23}$$

The term on the right-hand side of Eq. (11-23) is zero because $(\mathbf{q} \times \boldsymbol{\Omega})$ is perpendicular to $d\mathbf{r}$. Integrating Eq. (11-23), we have

$$\int \frac{dp}{\rho} + \frac{q^2}{2} = C \tag{11-24}$$

where the constant C remains unchanged along a given streamline but it may vary from one streamline to another.

According to the entropy relation (Appendix B) we have

$$dQ = T\,dS - dh - \frac{dp}{\rho} = d\left(h - \int \frac{dp}{\rho}\right) \tag{11-25}$$

Combining Eqs. (11-25) and (11-23), we obtain

$$d\left(h + \frac{q^2}{2}\right) - T\,dS = (\mathbf{q} \times \boldsymbol{\Omega}) \cdot d\mathbf{r} \tag{11-26}$$

But the first term on the left-hand side of Eq. (11-26) is equal to zero [see Eq. (11-21)], so Eq. (11-26) reduces to

$$T\,dS = (\boldsymbol{\Omega} \times \mathbf{q}) \cdot d\mathbf{r} \tag{11-27}$$

Since the term $(\boldsymbol{\Omega} \times \mathbf{q}) \cdot d\mathbf{r}$ vanishes along a streamline, it follows that entropy is constant along a given streamline. However, entropy may vary from one streamline to another.

If the flow is irrotational over the entire space, it follows from Eq. (11-27) that entropy is constant, hence the flow is isentropic, i.e.,

$$p = \text{Constant}\ \rho^{\kappa} \tag{B-50}$$

where $\kappa = C_p/C_\tau$. For an isentropic flow, Eq. (11-22), with $\mathbf{q} = \nabla\phi$, becomes after integration

$$\frac{\partial \phi}{\partial t} + \frac{\kappa}{\kappa - 1}\frac{p}{\rho} + \frac{q^2}{2} = \text{Constant} \tag{11-28}$$

Equation (11-28) is called the pressure equation for an unsteady flow or the Bernoulli-Lagrange equation. For a steady irrotational flow both

Eqs. (11-28) and (11-24) reduce to

$$\frac{\kappa}{\kappa-1}\frac{p}{\rho}+\frac{q^2}{2}=\text{Constant}=\frac{\kappa}{\kappa-1}\frac{p_0}{\rho_0} \tag{11-29}$$

where p_0 and ρ_0 are the pressure and density, respectively, at points where the fluid is at rest. Equation. (11-29) is generally referred to as the pressure equation or Bernoulli equation for compressible fluids. If the fluid is steady and incompressible, Eq. (11-24) reduces to Eq. (6-10a).

11.6. Kelvin's Theorem

In § 7.3c we have proved for an ideal homogeneous fluid (inviscid, incompressible) that the circulation is independent of time along a closed fluid line provided the external forces are conservative and are derived from a single-valued potential. In an inviscid compressible fluid the constancy of circulation is true only for barotropic fluids (pressure is a function of density only) and isothermal fluids. The proof of Kelvin's theorem for the compressible fluids mentioned above can be made by integrating Eq. (7-15) along a closed contour C as follows:

$$\frac{D\Gamma}{Dt}=\left[-\phi_f+\frac{q^2}{2}\right]_C-\oint_C\frac{dp}{\rho} \tag{11-30}$$

If, as assumed, ϕ_f and $\mathbf{q}$ are single-valued functions of $\mathbf{r}$, then the first term on the right-hand of Eq. (11-30) is equal to zero. Hence

$$\frac{D\Gamma}{Dt}=-\oint_C\frac{dp}{\rho} \tag{11-31a}$$

However, for a barotropic fluid we have, from Stokes' theorem (Appendix A),

$$\begin{aligned}\oint_C\frac{dp}{\rho}&=\oint_C d\mathbf{r}\cdot\frac{\nabla p}{\rho}=\oint_C\nabla f\cdot d\mathbf{r}\\&=\oint_A\mathbf{n}\cdot[\nabla\times\nabla f]\,dA=0\end{aligned} \tag{11-31b}$$

For an isothermal fluid, we may apply the thermodynamic relations of Eq. (11-25) to obtain

$$\oint_C\frac{dp}{\rho}=\oint_C(dh-T\,dS)=[h]_C+T[S]_C=0 \tag{11-31c}$$

In Eq. (11-31c) h and S are single-valued functions of $\mathbf{r}$. Hence the rate change of circulation along a closed curve [expressed on the left-hand side of Eq. (11-31a)] is equal to zero for both barotropic

[$p = p(\rho)$] and isothermal fluids. Equation. (11-31c) also shows that

$$\oint_c \frac{dp}{\rho} = 0$$

for an isentropic fluid. This result is expected as an isentropic process is a special case of the barotropic process.

B. PROPAGATION OF MOTION IN COMPRESSIBLE FLUIDS

11.7. Propagation of Pressure Change: Sound Velocity

a. Stationary Waves

The term *velocity of sound* is frequently confused with the velocity of fluid particles by readers having an insufficient comprehension of wave motion. For this reason it is believed to be very helpful if an example of the propagation of waves in elastic solid media is examined prior to the discussion of the phenomenon of propagation of pressure change in fluids.

Consider an elastic prismatical bar under the action of a suddenly applied compressive stress at the left end of the bar (Fig. 11.3). If the stress is uniformly distributed, initially a uniform compression of an infinitely thin layer at the left end of the bar will be produced. This compression is not transmitted at once to the far end of the bar but will travel along the bar with a certain velocity a. After a time interval t, the portion of the bar of length at will be compressed while the remaining portion will be undisturbed. The velocity a is called the velocity of wave propagation which is related to the compressibility of the medium.

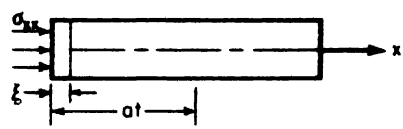


Figure 11.3

If ξ is designated as the small displacement of the bar due to the compressive stress σ_{xx}, the unit contraction (negative elongation) in the compressive zone is ξ/at. According to Hooke's law (for isotropic material) we have

$$\frac{\xi}{at} = \frac{\sigma_{xx}}{E} \tag{11-32}$$

where E is the modulus of elasticity of the material in compression. The velocity of particles in the compressed zone as determined from

Eq. (11-32) is

$$u = \frac{\xi}{t} = \frac{a\sigma_{xx}}{E} \tag{11-33}$$

Now let A be the cross-sectional area of the bar and ρ the mass per unit volume. In accordance with the momentum law the impulse of the compressive force must be equal to the momentum produced in the compressed zone, i.e.,

$$(\sigma_{xx}A)t = (\rho Aat)u$$

or

$$a = \frac{\sigma_{xx}}{\rho u} \tag{11-34}$$

Using the relation in Eq. (11-33) we find

$$a = \sqrt{\frac{E}{\rho}} \tag{11-35}$$

It is seen that the velocity of wave propagation is independent of the compressive force; however, the velocity of particles is directly proportional to the compressive stress.

In the case of the propagation of a pressure change in fluids, the phenomenon can be explained in a manner similar to that of the elastic solid media. Let us examine a constant-area duct containing gas initially at rest with a piston which may slide without friction in the duct as shown in Fig. 11.4a. The pressure p and the density ρ of the gas are initially uniform throughout the duct. If a small impulse is given to the piston, the gas layer immediately adjacent to the piston will experience an infinitesimal pressure increase dp. After a time interval t, the small pressure wave has progressed a distance at. During this same interval of time the gas particles have moved a distance $(dV)t$ from the initial position. Here again a is the velocity of the pressure wave propagation and dV the velocity of gas particles.

If the coordinate system is so chosen that it moves with the pressure wave, then the wave appears to be stationary with reference to the new coordinate system. Now the gas to the right of the pressure wave moves toward the left (toward the wave) with a velocity a and the gas on the left of the wave moves with a velocity $a - dV$ (Fig. 11.4b). Recalling the continuity equation we can write

$$a\rho = (\rho + d\rho)(a - dV) \tag{11-36a}$$

Dropping the second-order term in Eq. (11-36a) the velocitv of the particles is obtained as

$$dV = a\frac{d\rho}{\rho} \tag{11-36b}$$

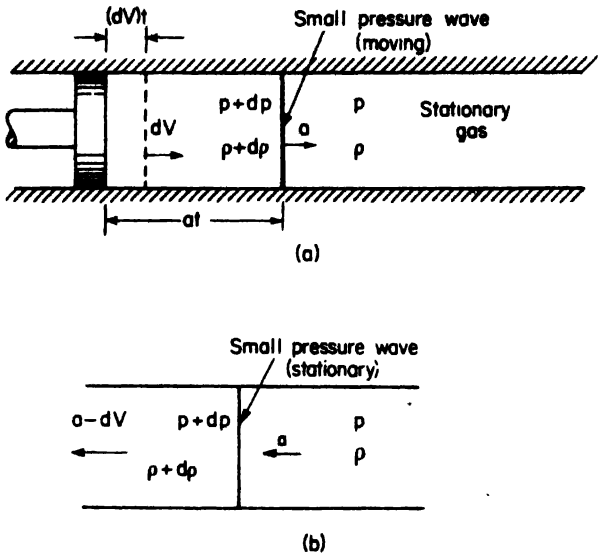


Figure 11.4

The momentum equation may be written, after neglecting the frictional and body forces, as

$$(dp)A = \rho a A[a - (a - dV)] \tag{11-37a}$$

where A is the cross-sectional area of the duct. Simplifying, we find

$$a = \frac{dp}{\rho dV} \tag{11-37b}$$

From Eq. (11-36b) the expression for dV when substituted into Eq. (11-37b) yields

$$= \sqrt{\frac{dp}{d\rho}} \tag{11-38a}$$

which is the velocity of propagation of the pressure wave (or sound wave) in the fluid. In terms of the bulk modulus of elasticity K [see Eq. (1-17)] Eq. (11-38a) becomes

$$= \sqrt{\frac{K}{\rho}} \tag{11-38b}$$

If K is replaced by E for an elastic solid media, Eq. (11-38b) becomes identical to Eq. (11-35).

According to Eq. (11-21) the energy equation may be written for the gas on the two sides of the wave front as

$$C_pT + \frac{a^2}{2} = C_p(T + dT) + \frac{(a - dV)^2}{2}$$

which reduces to, after neglecting high-order terms,

$$C_p\,dT = a\,dV \tag{11-39}$$

For a perfect gas, we have

$$C_p\,dT = C_p\,d\left(\frac{p}{R\rho}\right) = \frac{\kappa}{\kappa - 1}d\left(\frac{p}{\rho}\right) \tag{11-40}$$

Combination of Eqs. (11-40) and (11-37b) with Eq. (11-39) results in

$$\frac{\kappa}{\kappa - 1}d\left(\frac{p}{\rho}\right) = \frac{dp}{\rho} \tag{11-41}$$

The relation given in Eq. (11-41) is true only if the change of state is isentropic. Consequently, for a small pressure wave, the change of state can be considered isentropic. It is therefore more proper to use the partial differential notation instead of Eq. (11-38a), namely,

$$a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_S} \tag{11-38c}$$

where the subscript S indicates the partial derivative at constant entropy, i.e., isentropic process. For a perfect gas the relation between pressure and density in an isentropic process is given by

$$p = \text{Constant}\ \rho^{\kappa}$$

Putting this relation into Eq. (11-38c) we obtain

$$a = \sqrt{\kappa\left(\frac{p}{\rho}\right)} \tag{11-42}$$

It is seen that the finiteness of the velocity of sound is due to the compressibility of the gas. For an incompressible fluid the velocity of sound would be infinite because there would be no adjustment in volume possible; and therefore, any pressure change would be immediately transmitted and felt throughout the field of gas.

For historical interest we should mention Newton[1] who was the first man to calculate the propagation of pressure in air. Based on the assumed isothermal process in a perfect gas, he found the speed of propagation to be equal to the square root of the ratio of the pressure change to the corresponding density change involved in the process, i.e.,

$$a = \sqrt{\frac{p}{\rho}}$$

Since thermodynamics was not known as a science at Newton's time,

the 17% difference between his theory and experiment was never justified.

Almost a century later Marquis de Laplace[2] corrected Newton's calculation. The basic difference between Laplace's theory and Newton's lies in the fact that the propagation of pressure takes place in an adiabatic process. Since the compressions taking place in the propagation of pressure produce a very small temperature gradient (see §11.7b), it is not possible for heat due to compression to be transferred to the surrounding region. Laplace's correction for the sound velocity based on the adiabatic process modified Newton's formula by multiplying it by a factor $\sqrt{\kappa}$. This is in agreement with Eq. (11–42) which indicates that a sound wave propagates isentropically.

b. Nonstationary Waves

In the preceding discussion the pressure wave was assumed to be stationary in order for the process of wave propagation to be illustrated more explicitly. Actually the pressure change, due to disturbances, is gradually transmitted into undisturbed space. The disturbed and undisturbed spaces are separated by a surface known as a wave surface which moves with time. In order to describe this nonstationary surface wave the general form of the equation of continuity and the equation of motion given in Eqs. (5–7) and (5–20) must be used.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q}) = 0 \tag{5-7}$$

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla)\mathbf{q} = -\frac{\nabla p}{\rho} \tag{5-20a}$$

where the body forces in Eq. (5–20) are assumed to be negligible for the present discussion. The relation between p and ρ will be assumed to be that given by the isentropic relation.

Let p_∞ and ρ_∞ be the pressure and density when the gas is at rest and let the following quantities be defined as

$$p = p_\infty + p_1, \qquad \rho = \rho_\infty + \rho_1, \qquad \mathbf{q} = \mathbf{q}_1$$

In the case of a small disturbance, we impose the conditions that $p_1 \ll p_\infty$, $\rho_1 \ll \rho_\infty$, and the magnitude of velocity $\mathbf{q}_1$ is small compared to the velocity of sound. Furthermore, we assume that all the directional derivatives of the velocity components, pressure, and density are small. According to our hypothesis, the term $(\mathbf{q} \cdot \nabla)\mathbf{q}$ in Eq. (5–20a) can be neglected and the term

$$\frac{\nabla p}{\rho} = \frac{\nabla p_1}{\rho_\infty + \rho_1} = \frac{1}{\rho_\infty}\left(1 - \frac{\rho_1}{\rho_\infty} + \cdots\right)\nabla p_1 \doteq \frac{\nabla p_1}{\rho_\infty}$$

Hence the equation of motion may be reduced from Eq. (5-20a) as follows:

$$\frac{\partial \mathbf{q}_1}{\partial t} + \frac{\nabla p_1}{\rho_\infty} = 0 \tag{11-43}$$

Similarly, Eq. (5-7) may be written

$$\frac{\partial \rho_1}{\partial t} + (\mathbf{q}_1 \cdot \nabla)\rho_1 + (\rho_\infty + \rho_1)(\nabla \cdot \mathbf{q}_1) = 0$$

Neglecting the second-order terms, we have

$$\frac{\partial \rho_1}{\partial t} + \rho_\infty(\nabla \cdot \mathbf{q}_1) = 0 \tag{11-44}$$

In terms of the pressure, Eq. (11-44) becomes

$$\frac{\partial p_1}{\partial t} + a_\infty^2 \rho_\infty(\nabla \cdot \mathbf{q}_1) = 0 \tag{11-45}$$

where

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \left(\frac{\partial p_1}{\partial \rho_1}\right)_s = a^2 = \left(a_\infty - \frac{\kappa - 1}{2} q_1\right)^2 \doteq a_\infty^2$$

Taking the divergence of Eq. (11-43) and the derivative with respect to time of Eq. (11-45), Eqs. (11-43) and (11-45) become, respectively,

$$\frac{\partial(\nabla \cdot \mathbf{q}_1)}{\partial t} + \frac{\nabla^2 p_1}{\rho_\infty} = 0 \tag{11-46a}$$

and

$$\frac{\partial^2 p_1}{\partial t^2} + a_\infty^2 \rho_\infty \frac{\partial}{\partial t}(\nabla \cdot \mathbf{q}_1) = 0 \tag{11-46b}$$

Combining Eqs. (11-46a) and (11-46b), we have

$$\frac{\partial^2 p_1}{\partial t^2} - a_\infty^2 \nabla^2 p_1 = 0 \tag{11-47}$$

Equation (11-47) is a linear partial differential equation called the *wave equation.* The same type of equation represents the electromagnetic waves which propagate with the speed of light. Hence we see that the pressure wave propagates with the speed of sound. In special cases the wave equations of the same form represent the vibrating membrane and vibrating string when the disturbances are two and one-dimensional, respectively.

In a similar manner $\mathbf{q}_1$ and p_1 can be eliminated from Eqs. (11-43) and (11-44) and Eqs. (11-43) and (11-45), respectively, to yield

$$\frac{\partial^2 \rho_1}{\partial t^2} - a_\infty^2 \nabla^2 \rho_1 = 0 \tag{11-48}$$

and

$$\frac{\partial^2 \mathbf{q}_1}{\partial t^2} - a_\infty^2 \nabla^2 \mathbf{q}_1 = 0 \tag{11-49}$$

It is seen that Eqs. (11–47), (11–48), and (11–49) are the same type of equation, called the wave equation. Hence both the density change and the velocity increment follow the same wave pattern as the propagation of the pressure change.

Solutions of the three-dimensional wave equation are in general extremely complicated and our study will therefore be confined to the problem of propagation of plane disturbances of low intensity. From Eq. (11–47) the one-dimensional wave equation is obtained as

$$\frac{\partial^2 p_1}{\partial t^2} - a_\infty^2 \frac{\partial^2 p_1}{\partial x^2} = 0 \tag{11-50}$$

The general solution of the wave equation for a relative pressure change is expressed as

$$\frac{p_1}{p_\infty} = f_1(\xi) + g_1(\eta) = f_1(x - a_\infty t) + g_1(x + a_\infty t) \tag{11-51}$$

where f_1 and g_1 are arbitrary functions. To investigate the nature of this solution, let us first consider the family of solutions given by the arbitrary function $f_1(\xi)$ with $g_1(\eta) = 0$. The relative pressure change in this case at time $t = 0$ is

$$\frac{p_1}{p_\infty} = f_1(x) \tag{11-52a}$$

At an interval of time t_1 later, we have

$$\frac{p_1}{p_\infty} = f_1(x - a_\infty t_1) \tag{11-52b}$$

It can easily be shown that the relative pressure change in Eq. (11–52a) is the same as in Eq. (11–52b). In other words both of these two equations reduce to

$$\frac{p_1}{p_\infty} = f_1(x_0) \tag{11-53}$$

when $x = x_0$ at $t = 0$ is substituted in Eq. (11–52a) and $x = x_0 + a_\infty t_1$ at $t = t_1$ is substituted in Eq. (11–52b). Equation (11–53) is true for any interval of time $t_2, t_3, t_4, \ldots, t_n$. Hence we conclude that the disturbance of pressure change is propagated to the right with velocity a_∞ (Fig. 11.5). Similarly the solution $g_1(\eta)$ represents a small relative pressure change which propagates to the left with velocity a_∞.

If the lines of constant p_1/p_∞ are plotted on an x–t plane (Fig. 11.6), a family of straight lines with constant value of slope, either $+a_\infty$ or $-a_\infty$, is obtained. The positive slope is for the rightward propagating wave and the negative slope is for the leftward propagating

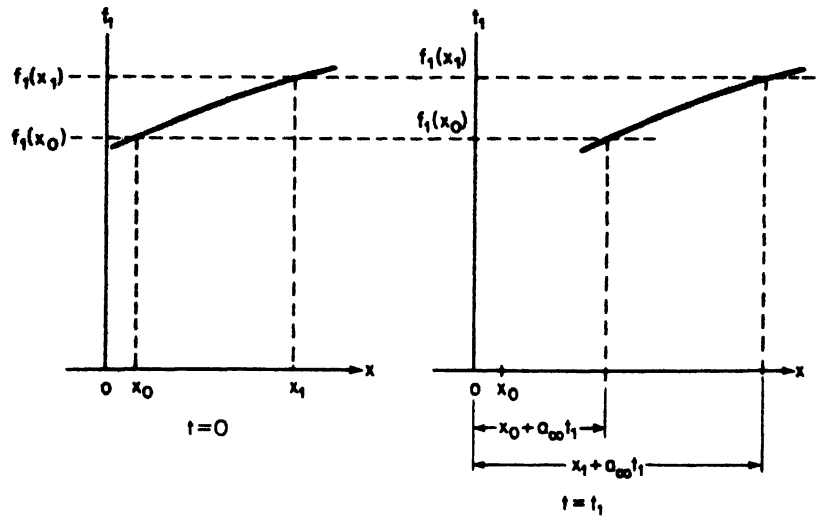


Figure 11.5

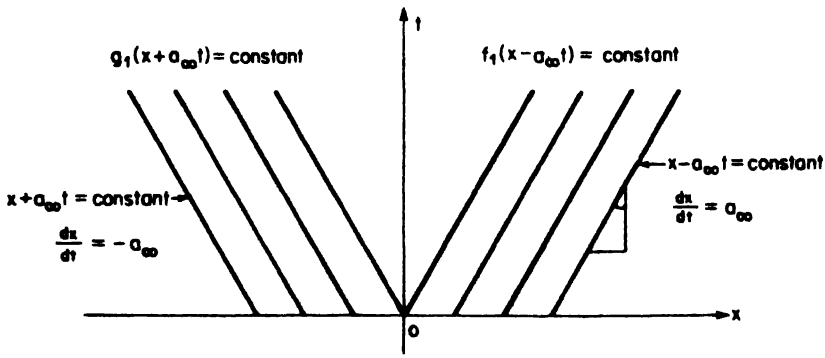


Figure 11.6

wave. These lines of constant $f_1(\xi)$ and $g_1(\eta)$ are called the *characteristics* of the wave equation [Eq. (11–50)].

Similarly the one-dimensional wave equations for the density change and the velocity increment are obtained from Eqs. (11–48) and (11–49), respectively, and they are written as follows:

$$\frac{\partial^2 \rho_1}{\partial t^2} - a_\infty^2 \frac{\partial^2 \rho_1}{\partial x^2} = 0 \tag{11–54}$$

and

$$\frac{\partial^2 u_1}{\partial t^2} - a_\infty^2 \frac{\partial^2 u_1}{\partial x^2} = 0 \tag{11-55}$$

The solutions for ρ_1/ρ_∞ and u_1 may be written in terms of two arbitrary functions.

$$\frac{\rho_1}{\rho_\infty} = f_2(x - a_\infty t) + g_2(x + a_\infty t) \tag{11-56}$$

and

$$u_1 = f_3(x - a_\infty t) + g_3(x + a_\infty t) \tag{11-57}$$

The relations f_1 and f_3, and g_1 and g_3 can be found by substituting p_1/p_∞ and u_1 from Eqs. (11-51) and (11-57), respectively, into Eqs. (11-43) and (11-45). The results are

$$-\rho_\infty a_\infty f_3' + \rho_\infty a_\infty g_3' + p_\infty f_1' + p_\infty g_1' = 0 \tag{11-58a}$$

and

$$\rho_\infty a_\infty^2 f_3' + \rho_\infty a_\infty^2 g_3' - p_\infty a_\infty f_1' + p_\infty a_\infty g_1' = 0 \tag{11-58b}$$

Multiplying Eq. (11-58a) by a_∞ and subtracting it from Eq. (11-58b) yields

$$p_\infty a_\infty f_1' = \rho_\infty a_\infty^2 f_3'$$

which is satisfied if

$$f_1 = \frac{\rho_\infty a_\infty}{p_\infty} f_3 \tag{11-59a}$$

and similarly,

$$g_1 = -\frac{\rho_\infty a_\infty}{p_\infty} g_3 \tag{11-59b}$$

In a like manner we obtain the relations f_2 and f_3, and g_2 and g_3 from Eqs. (11-56), (11-57), (11-43), and (11-44),

$$f_3 = a_\infty f_2 \tag{11-59c}$$

$$g_3 = -a_\infty g_2 \tag{11-59d}$$

So far we have made no statement regarding the nature of the flow in the discussion above. If the flow is assumed to be irrotational we can rewrite Eq. (11-55) in terms of a velocity potential ϕ as follows:

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 \phi}{\partial t^2}\right) - a_\infty^2 \frac{\partial}{\partial x}\left(\frac{\partial^2 \phi}{\partial x^2}\right) = 0$$

After integration, we have

$$\frac{\partial^2 \phi}{\partial t^2} - a_\infty^2 \frac{\partial^2 \phi}{\partial x^2} = 0 \tag{11-60}$$

where the constant of integration which is a function of time is

included in ϕ. The solution of Eq. (11-60) is similar to Eq. (11-55),

$$\phi = F(x - a_\infty t) + G(x + a_\infty t) \tag{11-61}$$

The discussion above is confined only to the propagation of plane disturbances of low intensity. In the case of propagation of spherical and cylindrical disturbances of low intensity the respective wave equation can also be obtained from Eqs. (11-47)-(11-49).

Illustrative Example 11.1. Illustrate the propagation of an acoustic wave.

Solution. If we choose a sine function for $f_1(x - a_\infty t)$ in Eq. (11-52b), the relative pressure change in this case becomes

$$\frac{p_1}{p_\infty} = 0.01 \sin (x \pm a_\infty t) \tag{a}$$

It can easily be verified that Eq. (a) is a solution of Eq. (11-50).

The significance of Eq. (a) can be investigated first by considering the solution

$$\frac{p_1}{p_\infty} = 0.01 \sin (x - a_\infty t) \tag{b}$$

At time $t = 0$, the pressure change is

$$\frac{p_1}{p_\infty} = 0.01 \sin x \tag{c}$$

This pressure function which is represented by a sine curve is shown in Fig. 11.A(a). At time $t = 10^{-3}$ sec (assume $a_\infty = 400\pi$ ft/sec), Eq. (b) becomes

$$\frac{p_1}{p_\infty} = 0.01 \sin \left(x - \frac{2}{5}\pi\right) \tag{d}$$

If we replace $x - \frac{2}{5}\pi$ by x_1, Eq. (d) becomes

$$\frac{p_1}{p_\infty} = 0.01 \sin x_1 \tag{e}$$

It is seen that the pressure change in Eqs. (c) and (e) are identical except for a shifting of a distance $a_\infty t_1 = \frac{2}{5}\pi$ along the x-axis which is shown in Fig. 11.A(b). Similarly, Fig. 11.A(c) depicts the pressure change at time $t = 2 \times 10^{-3}$ sec. Thus it is evident that the wave represented by Eq. (b) propagates without distortion at a velocity a_∞ to the right, and hence it is called a rightward propagating wave.

The lines in the x-t plane with slope $dx/dt = a_\infty$ (Fig. 11.A) which trace the progress of the waves are called the characteristics of the wave equation, Eq. (11-50). In a like manner the solution

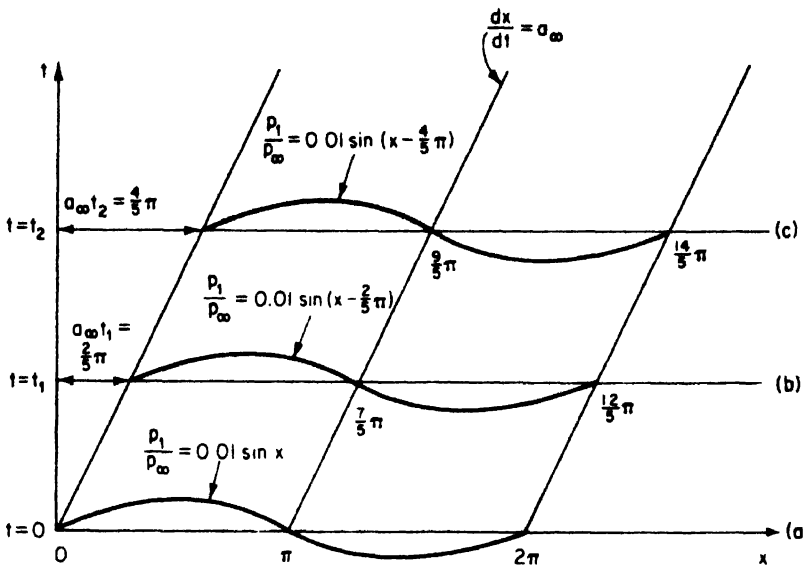


Figure 11.A

$$\frac{p_1}{p} \quad 0.01 \sin (x + a_\infty t)$$

may be interpreted as a pressure change propagation, without distortion, at a velocity a_∞ to the left, and hence it is called a leftward propagating wave. The characteristics of the wave equation in this case are the lines with slope $dx/dt = -a_\infty$.

11.8. *Propagation of a Plane Wave of High Intensity—Formation of Shock Waves*

Based on the small disturbance theory, we have just obtained the linearized unsteady one-dimensional equations of continuity and momentum. The solutions of the linearized equations show that the sound wave propagates in a fluid with constant velocity and its simple wav shape remains unchanged. If the basic assumptions that the changes of the thermodynamic properties from their undisturbed values are very small are considered invalid, then the wave propagates in a fluid with a velocity which changes from point to point and its simple wave shape becomes distorted as it propagates. Under these conditions of finite disturbances the complete nonlinear equations [Eqs. (5–7)

and (5–20)] must be used for the complete description. In the case of one-dimensional motion, these nonlinear equations form a system of quasi-linear partial differential equations of the hyperbolic type to which the method of characteristics can be applied. The reader may refer to an advanced text in compressible flow for the treatment of the method of characteristics.

In order to find a simple analytic solution of the one-dimensional form of Eqs. (5–7) and (5–20) we may use the results of the preceding section in which the density is considered a function of velocity only, i.e.,

$$\rho = \rho(u) \tag{11–62}$$

With the relation of Eq. (11–62) the continuity and momentum equations in one-dimensional flow can be written [see Eqs. (5–7) and (5–20a)], respectively,

$$\frac{d\rho}{du}\frac{\partial u}{\partial t} + u\frac{d\rho}{du}\frac{\partial u}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \tag{11–63a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{dp}{d\rho}\frac{d\rho}{du}\frac{\partial u}{\partial x} = 0 \tag{11–64}$$

Equation (11–63a), after dividing by $d\rho/du$, may be written

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\rho}{d\rho/du}\frac{\partial u}{\partial x} = 0 \tag{11–63b}$$

Combining Eqs. (11–64) and (11–63b), we have

$$\frac{du}{d\rho} = \pm\frac{1}{\rho}\sqrt{\frac{dp}{d\rho}} = \pm\frac{a}{\rho} \tag{11–65}$$

Using the isentropic relation $a^2 = dp/d\rho = \kappa(p/\rho)$ and integrating Eq. (11–65) we obtain

$$u = \pm\int_{\rho_\infty}^{\rho}\sqrt{\frac{dp}{d\rho}}\frac{d\rho}{\rho} = \pm\frac{2}{\kappa - 1}[a - a_\infty] \tag{11–66}$$

Thus the relation between the local speed of sound and the particle velocity is obtained:

$$a = a_\infty \pm \frac{\kappa - 1}{2}u \tag{11–67}$$

where $(\ )_\infty$ represents the respective quantity in the undisturbed region in which the gas is at rest.

Now substituting Eq. (11–65) into Eqs. (11–63a) and (11–64) we obtain, respectively,

$$\frac{\partial \bar{p}}{\partial t} + (u \pm a)\frac{\partial \bar{p}}{\partial x} = 0 \tag{11–68}$$

and

$$\frac{\partial u}{\partial t} + (u \pm a)\frac{\partial u}{\partial x} = 0 \tag{11-69}$$

where $\bar{\rho} = \rho/\rho_\infty - 1$.

For constant characteristics, $u \pm a$, the solutions of Eqs. (11–68) and (11–69) are, respectively,

$$\bar{\rho} = F_1[x - (u \pm a)t] \tag{11-70}$$

$$u = F_2[x - (u \pm a)t] \tag{11-71}$$

where F_1 and F_2 are arbitrary functions. Equations (11–70) and (11–71) mean that the variation of $\bar{\rho}$ and u with respect to a point moving with velocity $u + a$ or $u - a$ is zero. In other words, they show that the disturbance is propagated with an instantaneous velocity, $u + a$ or $u - a$. To investigate the physical nature of this solution let us consider one part of the solution given in Eq. (11–70),

$$\bar{\rho} = F_1[x - (u + a)t] \tag{11-72a}$$

with $a = a_\infty + [(\kappa - 1)/2]u$ from Eq. (11–67); then we have

$$\bar{\rho} = F_1\left[x - \left(a_\infty + \frac{\kappa + 1}{2}u\right)t\right] = F_1(\xi) \tag{11-72b}$$

which shows the disturbance is propagated with an instantaneous velocity $a_\infty + [(\kappa - 1)/2]u$. In order to make it easier to comprehend this phenomenon, the initial density distribution, when $t = 0$, is chosen to resemble a sine wave function, i.e., at $x = 0$, $x = x_1$, and $x = x_2$; we have values for $\bar{\rho}$ obtained from Eq. (11–72b), respectively, as

$$\bar{\rho} = F_1(0) = 0 \tag{11-73a}$$

$$\bar{\rho} = F_1(x_1) = \bar{\rho}_1 \tag{11-73b}$$

$$\bar{\rho} = F_1(x_2) = \bar{\rho}_2 = 0 \tag{11-73c}$$

This initial variation of the density function is shown in Fig. 11.7a.

It can easily be seen that the values of the density distribution given in Eqs. (11–73) can be maintained at a later time t_1 if the corresponding locations ($x = 0$, $x = x_1$, and $x = x_2$) on the x-axis are shifted accordingly:

$$x = a_\infty t_1 \tag{11-74a}$$

$$x = x_1 + a_\infty t_1 + \frac{\kappa + 1}{2}ut_1 \tag{11-74b}$$

$$x = x_2 + a_\infty t_1 \tag{11-74c}$$

If the increment of the velocity u is negligibly small in comparison with the speed a_∞, we have exactly the case of the sound wave which was discussed in the preceding section. In this case the curve $\bar{\rho} = F_1(\xi)$ described by Eq. (11–72b) does not change its shape as the disturbance

propagates. From Eq. (11-74b), it is seen that the same situation would exist if $\kappa = -1$; however, this is not true for any real gas.

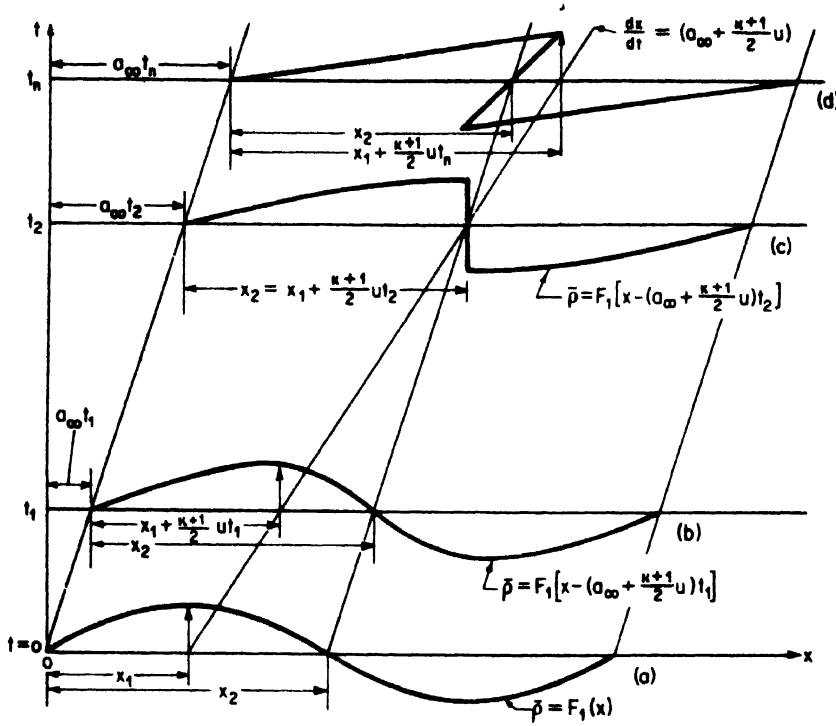


Figure 11.7

On the other hand if the variation of density is not small, the value of $\bar{\rho} = \bar{\rho}_1$ in Eq. (11-74b) will no longer occur at $x_1 + a_\infty t_1$, but will appear further by a distance $[(\kappa + 1)/2]ut_1$. However, the points of zero $\bar{\rho}$ remain at the same positions [see Eqs. (11-74a) and (11-74c)] at all times. Hence the curve $\bar{\rho} = F_1(\xi)$ changes its shape as the disturbance propagates. These distortions of the density distribution are shown in Figs. 11.7b, 11.7c, and 11.7d. If the value of $\bar{\rho}$ at a certain later time is sufficiently large, the value of $\bar{\rho}_1$ tends to move beyond the point $x_2 + a_\infty t_n$ which is shown in Fig. 11.7d. The density distribution in this case is physically impossible as these are three different values for $\bar{\rho}$ at one given value of x ($x = x_2 + a_\infty t_n$). In fact, for longitudinal waves such as those under consideration the velocity and density must be single-valued functions of position. Actually, the progressive wave

does not pass beyond the ordinate of the point $x_2 + a_\infty t_n$ where the value of the density decreases suddenly and the vertical slope $d\bar{\rho}/dx$ reaches infinity (see Fig. 11.7c). It is clear that a surface of discontinuity would be formed because a progressive disturbance of finite amplitude cannot last indefinitely. Across this surface of discontinuity which is known as a compressive wave or shock wave, the velocity and all the thermodynamic properties change suddenly and the gas no longer follows the isentropic process. A discussion on shock waves will be given in § 11.16.

11.9. *Mach Number, Mach Line, and Mach Cone*

In the previous sections we have discussed that a small pressure disturbance (or pressure impulse) in a stationary compressible fluid propagates with the speed of sound uniformly in all directions. Hence it may be stated that the disturbances propagate through spherical wave surfaces. Figure 11.8a shows a plane view of the physical situation. For an incompressible fluid we assume that $a \to \infty$; hence this case applies not only to the stationary fluid but also to the very slowly moving fluid where $V/a \to 0$.

Let us now consider a small projectile (or any shape of a small body) moving from right to left in a stationary fluid at a velocity smaller than the velocity of sound as illustrated in Fig. 11.8b. As the projectile moves, pressure disturbances are generated which propagate in all directions at the velocity of sound; hence the resulting propagation of the disturbance is no longer symmetrical. It is seen from Fig. 11.8b that the propagation velocity of the disturbances is faster in the downstream direction and slower in the upstream direction. Let us assume that the small projectile during a time interval of $3(\Delta t)$ moves from position O_1 to position O_4, a distance of $3V\Delta t$. In this same interval of time, $3\Delta t$, the initial spherical wave originating at position O_1 would have propagated to a radius $3a\Delta t$. The wave originating at position O_2 propagates to a radius $2a\Delta t$ when the projectile reaches a distance $3V\Delta t$ and so on. It is seen that if $V < a$ the pressure disturbance precedes the projectile and will eventually reach the entire fluid space; hence we have the case which defines subsonic flow.

When the velocity of the projectile is equal to the velocity of sound, the pressure waves of the disturbance can neither precede nor exceed the projectile. In this case, the propagation of the pressure waves is restricted to the half space bounded by a plane perpendicular to the direction of the moving projectile. This is shown in Fig. 11.8c which characterizes transonic flow.

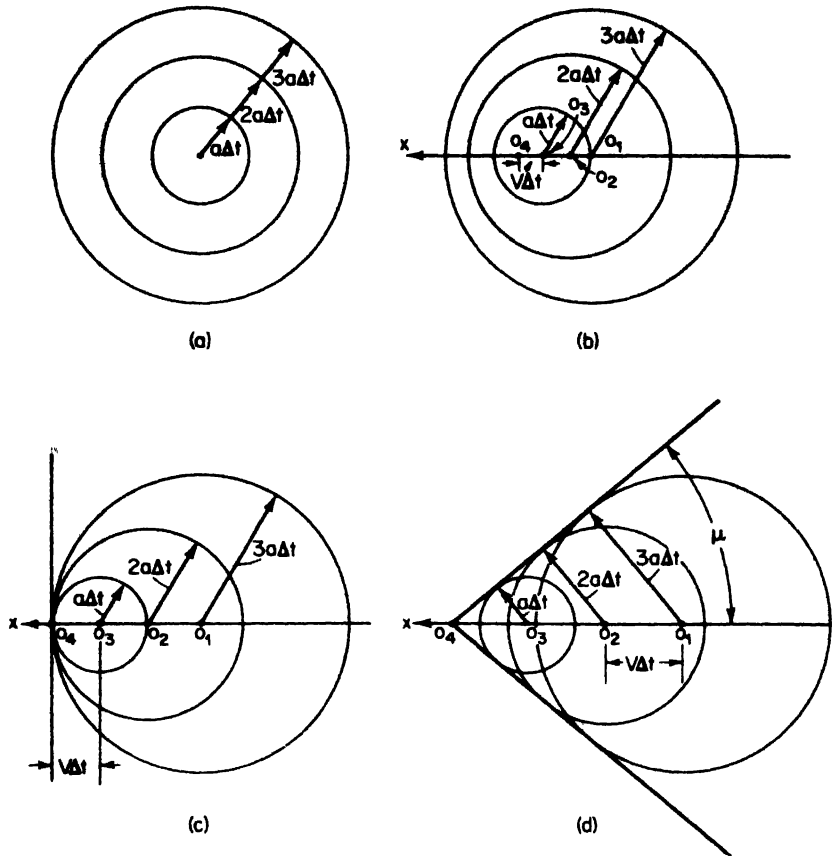


Figure 11.8

In the case when the velocity of the projectile exceeds the sound velocity, the pressure impulse (produced by the projectile) then travels faster than the spherical waves can be generated. By referring to Fig. 11.8d we see that after a time $3\Delta t$ the projectile originating at position O_1 would travel a distance of $3V\Delta t$ to position O_4 while the spherical wave propagates to a radius $3a\Delta t$ which is much less than $3V\Delta t$. It can also be seen that the projectile at position O_4 is outside the spherical waves originating at O_2 and O_3 after the interval of time $2\Delta t$ and Δt, respectively. Hence the effect of the pressure disturbance is restricted to the interior of the envelope of these spheres which is a cone whose vertex coincides with the source of disturbance (the assumed small projectile) and whose semivertex angle is given by the relation

$$\sin \mu = \frac{a}{V} \tag{11-75}$$

This angle is called the *Mach angle* and the resultant cone, the *Mach cone* (Fig. 11.8d). In two-dimensional flow this cone reduces to a pair of intersecting lines each of which is called a *Mach line* or a *Mach wave*. The ratio of the flow velocity to the velocity of sound, i.e.,

$$M = \frac{V}{a} = \frac{1}{\sin \mu} \tag{11-76}$$

is designated as the *Mach number*, named after the German scientist, Ernst Mach.

The Mach cone divides the fluid into two regions: the region inside the cone which is disturbed by the motion of the projectile, and the region outside the cone which remains undisturbed. The former region was referred to by von Karman as the zone of action and the latter as the zone of silence. This phenomenon characterizes the nature of supersonic flow. If an airplane is flying at a subsonic speed, the noise created by it can be heard long before the plane reaches directly overhead; however, the situation of a supersonic airplane is entirely different as you cannot hear the noise until the plane is far beyond you.

C. ISENTROPIC FLOW RELATIONS IN TERMS OF THE SONIC VELOCITY AND THE MACH NUMBER

11.10. Alternative Forms of Bernoulli's Equation

In § 11.1 we discussed the importance of the Mach number in compressible fluids and in the preceding section we have shown the distinctive character of subsonic, transonic, and supersonic flows. It is therefore necessary and important to express Bernoulli's equation and to determine later the variation of the physical characteristics in terms of the Mach number.

The Bernoulli equation given in Eq. (11-29),

$$\frac{1}{2} q^2 + \frac{\kappa}{\kappa - 1} \frac{p}{\rho} = \frac{\kappa}{\kappa - 1} \frac{p_0}{\rho_0} \tag{11-29}$$

can be used to obtain the velocity and pressure relation. After substituting the isentropic relation $\rho/\rho_0 = (p/p_0)^{1/\kappa}$ into Eq. (11-29), we have

$$q^2 = \frac{2\kappa}{\kappa - 1} \frac{p_0}{\rho_0} \left[1 - \left(\frac{p}{p_0} \right)^{(\kappa-1)/\kappa} \right] = \frac{2a_0^2}{\kappa - 1} \left[1 - \left(\frac{p}{p_0} \right)^{(\kappa-1)/\kappa} \right] \tag{11-77}$$

Now consider a flow system which consists of an infinite reservoir and a convergent nozzle as shown in Fig. 11.9. The quantities p_0 and ρ_0 are designated as the pressure and density under the reservoir conditions. If the local pressure at the exit is equal to the reservoir pressure, the local velocity at the exit is zero. As the local pressure at the exit decreases, the velocity q increases and it reaches a maximum value when the pressure approaches zero. Denoting the local velocity corresponding to the absolute vacuum ($P_{exit} = 0$) by q_{max}, then Eq. (11–77) becomes

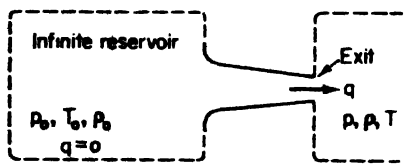


Figure 11.9.

$$q_{max}^2 = \frac{2\kappa}{\kappa - 1}\frac{p_0}{\rho_0} = \frac{2\kappa RT_0}{\kappa - 1} = \frac{2a_0^2}{\kappa - 1} \tag{11–78}$$

where a_0 is the velocity of sound based on reservoir conditions. It must be realized that Eq. (11–78) represents an ideal flow system while in actual practice the velocity q_{max} cannot be reached because not all the thermal energy of the system can be converted into mechanical energy. In terms of the maximum velocity, Eq. (11–77) can be written

$$q^2 = q_{max}^2 - \frac{2\kappa}{\kappa - 1}\frac{p}{\rho} = q_{max}^2\left[1 - \left(\frac{p}{p_0}\right)^{(\kappa-1)/\kappa}\right] \tag{11–79}$$

Since the velocity of sound and the Mach number are the fundamental parameters in inviscid compressible fluids, it is useful to express the Bernoulli equation in terms of these parameters. By applying the relation $a^2 = \kappa p/\rho$ [Eq. (11–42)], the Bernoulli equation takes the form

$$\frac{1}{2}q^2 + \frac{a^2}{\kappa - 1} = \frac{a_0^2}{\kappa - 1} = \frac{1}{2}q_{max}^2 = \frac{\kappa}{\kappa - 1}RT_0 = \text{Constant} \tag{11–80}$$

Dividing Eq. (11–80) by q^2, a relation between the local velocity and the local Mach number is obtained:

$$\frac{1}{M^2} = \frac{\kappa - 1}{2}\left(\frac{q_{max}^2}{q^2} - 1\right) \tag{11–81}$$

11.11. Critical Conditions (Sonic Conditions)

According to Eq. (11–80) we see that the local velocity of sound decreases as the local velocity increases and vice versa. At any given

instant when the local velocity and the local velocity of sound are equal then the velocity q is said to have attained a critical value,

$$q = a = a_c$$

where a_c is called the critical velocity of sound or simply the critical velocity. If q and a are replaced by a_c, Eq. (11–80) gives the relation

$$a_c^2 = \frac{2}{\kappa + 1} a_0^2 = \frac{2\kappa}{\kappa + 1} RT_0 \tag{11–82}$$

It is seen that the critical velocity depends on the static conditions at the reservoir as well as the physical characteristics of the gas. For air ($\kappa = 1.405$), $a_c = 0.915\, a_0$.

The pressure, density, and temperature corresponding to the critical velocity are known as critical pressure, critical density, and critical temperature, respectively. The value of the critical pressure can be obtained from Eqs. (11–77) and (11–82) as follows:

$$a_c^2 = \frac{2a_0^2}{\kappa - 1}\left[1 - \left(\frac{p_c}{p_0}\right)^{(\kappa-1)/\kappa}\right] = \frac{2a_0^2}{\kappa + 1}$$

Hence,

$$\frac{p_c}{p_0} = \left(\frac{2}{\kappa + 1}\right)^{\kappa/(\kappa-1)} \tag{11–83}$$

The critical density and critical temperature can be obtained from Eq. (11–83) with the aid of an isentropic relation and the equation of state for a perfect gas, and they are written as follows:

$$\left(\frac{\rho_c}{\rho_0}\right) = \left(\frac{p_c}{p_0}\right)^{1/\kappa} = \left(\frac{2}{\kappa + 1}\right)^{1/(\kappa-1)} \tag{11–84}$$

$$\left(\frac{T_c}{T_0}\right) = \left(\frac{p_c}{p_0}\right)\left(\frac{\rho_0}{\rho_c}\right) = \left(\frac{p_c}{p_0}\right)^{(\kappa-1)/\kappa} = \frac{2}{\kappa + 1} \tag{11–85}$$

For air ($\kappa = 1.405$), Eqs. (11–83), (11–84), and (11–85) give, respectively,

$$\frac{p_c}{p_0} = 0.528, \qquad \frac{\rho_c}{\rho_0} = 0.634, \qquad \frac{T_c}{T_0} = 0.833 \tag{11–86}$$

11.12. *Pressure, Density, and Temperature in Terms of the Mach Number*

The importance of the Mach number in the analysis of compressible flow can now be demonstrated by its introduction into the expressions for temperature. Dividing Eq. (11–80) by $a^2/(\kappa - 1)$, we have

$$\frac{\kappa - 1}{2}\left(\frac{q}{a}\right)^2 + 1 = \left(\frac{a_0}{a}\right)^2 = \frac{T_0}{T} \tag{11–87a}$$

from which the variation of temperature as a function of the Mach number is obtained:

$$\frac{T_0}{T} = 1 + \frac{\kappa - 1}{2} M^2 \tag{11-87b}$$

Equation (11-87b) may be used to determine the free-stream temperature of the gas if values are known for the stagnation temperature T_0, the ratio of specific heats, and the free-stream Mach number. In connection with the free-stream temperature one must realize that it cannot be measured by placing a thermometer in the stream. This restriction exists because the velocity at the wall of a thermometer is equal to zero (see Chap. Nine, Boundary Layer Theory) and the temperature at the wall is approximately equal to the stagnation temperature T_0.

It is both useful and necessary to define the Mach number corresponding to stagnation conditions by

$$M_0 = \frac{q}{a_0} \tag{11-88}$$

where a_0 is the velocity of sound in a fluid at rest. Multiplying Eq. (11-87a) by $(a/a_0)^2$ we have the variation of temperature as a function of M_0

$$\frac{T}{T_0} = 1 - \frac{\kappa - 1}{2}\left(\frac{q}{a_0}\right)^2 = 1 - \frac{\kappa - 1}{2} M_0^2 \tag{11-89}$$

The variation of pressure and density as a function of M or M_0 can be obtained from Eq. (11-87b) or (11-89), respectively, with the aid of the equation of state for a perfect gas and the isentropic relation of p and ρ. The relation for temperature is

$$\frac{T}{T_0} = \frac{p}{p_0}\frac{\rho_0}{\rho} = \left(\frac{p}{p_0}\right)^{(\kappa-1)/\kappa} = \left(\frac{\rho}{\rho_0}\right)^{(\kappa-1)}$$

and the pressure and Mach number relation can be obtained as

$$\frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{\kappa/(\kappa-1)} = \left(1 + \frac{\kappa - 1}{2} M^2\right)^{-\kappa/(\kappa-1)} = \left(1 - \frac{\kappa - 1}{2} M_0^2\right)^{\kappa/(\kappa-1)} \tag{11-90}$$

Similarly the density and Mach number relation is

$$\frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{1/(\kappa-1)} = \left(1 + \frac{\kappa - 1}{2} M^2\right)^{-1/(\kappa-1)} = \left(1 - \frac{\kappa - 1}{2} M_0^2\right)^{1/(\kappa-1)} \tag{11-91}$$

The variations of the ratios of static to stagnation pressure, temperature, density, and velocity of sound with the local Mach number are shown in Fig. 11.10 and Table E-4.

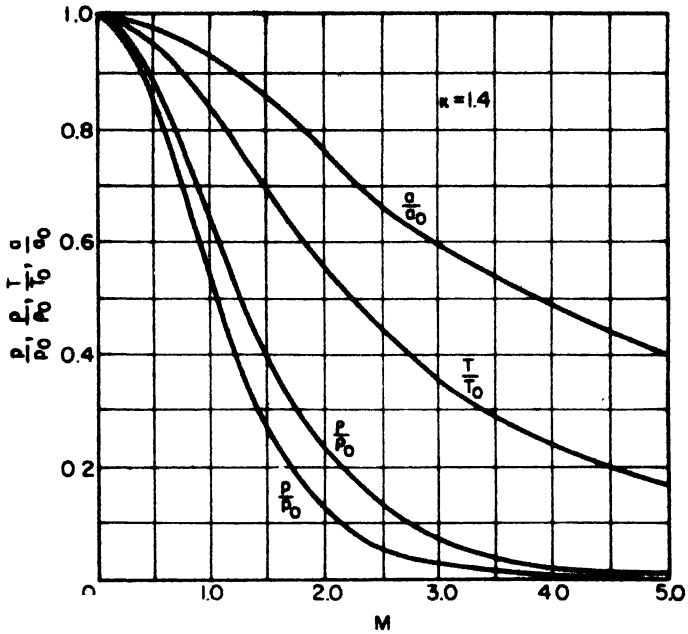


Figure 11.10

In incompressible flow the dynamic pressure [see Eq. (6–10b)] is simply the difference between stagnation and static pressures [See Eq. (6–11)]. However, in compressible flow it depends on both the Mach number and the static pressure and is written in the form

$$\frac{1}{2}\rho q^2 = \frac{1}{2}\rho a^2 M^2 = \frac{1}{2}\rho\left(\frac{\kappa p}{\rho}\right)M^2 = \frac{1}{2}\kappa p M^2 \qquad (11\text{–}92)$$

The dimensionless pressure difference ratio is usually expressed as

$$\frac{p_0 - p}{\rho q^2/2} = \frac{2}{\kappa M^2}\frac{p_0^2 - p}{p} = \frac{2}{\kappa M^2}\left(\frac{p_0}{p} - 1\right) \qquad (11\text{–}93a)$$

Substituting the expression of p_0/p from Eq. (11–90) into Eq. (11–93a), one obtains

$$\frac{p_0 - p}{\frac{1}{2}\rho q^2} = \frac{2}{\kappa M^2}\left[\left(1 + \frac{\kappa - 1}{2}M^2\right)^{\kappa/(\kappa-1)} - 1\right] \qquad (11\text{–}93b)$$

which, expanded by the use of the binomial theorem for $M^2 < 2/(\kappa - 1)$, gives

$$\frac{p_0 - p}{\frac{1}{2}\rho q^2} = 1 + \frac{1}{4}M^2 + \frac{2 - \kappa}{24}M^4 + \frac{(2 - \kappa)(3 - 2\kappa)}{192}M^6 + \cdots \qquad (11\text{–}93c)$$

For air ($\kappa = 1.4$), we obtain

$$\frac{p_0 - p}{\frac{1}{2}\rho q^2} = 1 + \frac{1}{4}M^2 + \frac{1}{40}M^4 + \frac{1}{1600}M^6 + \cdots \qquad (11\text{-}93d)$$

The expansion of four terms above will provide sufficient accuracy for $M^2 < 5.0$. For an incompressible flow, the series in Eq. (11–93d) is equal to unity; hence Eq. (11–93d) reduces to Eq. (6–11). It is seen that the Pitot-static tube for velocity measurement must be calibrated by taking the effect of compressibility into consideration when $M > 0.3$.

Illustrative Example 11.2. A speed indicator of an airplane which is calibrated without taking the effect of compressibility into consideration has registered a speed of 500 mph at 20,000 ft altitude. Determine the true air speed. Also calculate the stagnation temperature and density at the nose of the Pitot tube.

Solution. According to Eq. (2–20) the temperature, pressure, and density at 20,000 ft altitude are computed as follows:

$$T = [1 - 7.03 \times 10^{-6}(20{,}000)](518.7) = 445.8^\circ\text{R}$$

$$p = [1 - 7.03 \times 10^{-6}(20{,}000)]^{5.14}(2116) = 971.1 \text{ lbf/ft}^2$$

$$\rho = [1 - 7.03 \times 10^{-6}(20{,}000)]^{4.14}(0.002378) = 0.001270 \text{ slugs/ft}^3$$

The speed of sound at 20,000 ft altitude is

$$a = \sqrt{\kappa \frac{p}{\rho}} = \sqrt{1.4\left(\frac{971.1}{0.001270}\right)} = 1034.7 \text{ ft/sec}$$

For an incompressible fluid Eq. (11–93d) gives

$$p_0 - p = \rho \frac{q^2}{2} = \frac{1}{2}(0.001270)(1.467 \times 500)^2 = 341.6 \text{ lbf/ft}^2$$

Hence, the indicated stagnation pressure is $p_0 = p + 341.6 = 1312.8$ lbf/ft^2. The Mach number may be found from Eq. (11–90) as follows:

$$\left(1 + \frac{\kappa - 1}{2}M^2\right) = \left(\frac{p_0}{p}\right)^{(\kappa-1)/\kappa}$$

or

$$M^2 = \frac{2}{\kappa - 1}\left[\left(\frac{p_0}{p}\right)^{(\kappa-1)/\kappa} - 1\right] = 5.0\left[\left(\frac{1312.8}{971.1}\right)^{0.286} - 1\right]$$

$$= 0.4494$$

and

$$M = 0.6704$$

Therefore the true air speed is

$$q = 1034.7 \times 0.6704 = 693.7 \text{ ft/sec}$$
$$= 472.8 \text{ mph}$$

The stagnation temperature and density may be obtained from Eqs. (11–87b) and (11–91), respectively, i.e.,

$$T_0 = \left(1 + \frac{\kappa - 1}{2} M^2\right) T = [1 + 0.2(0.6704)^2](445.8)$$
$$= 485.87^\circ\text{R} \quad \text{or} \quad 25.87^\circ\text{F}$$
$$\rho_0 = \rho \left(\frac{T_0}{T}\right)^{1/(\kappa-1)} = 0.001270(1.09)^{2.5}$$
$$= 0.0015748 \text{ slugs/ft}^3$$

11.13. Pressure Coefficient

The pressure coefficient, by virtue of Eq. (11–92), can be expressed by

$$C_p = \frac{p - p_\infty}{\rho_\infty U_\infty^2/2} = \frac{p - p_\infty}{p_\infty} \frac{2}{\kappa M_\infty^2} = \frac{2}{\kappa M_\infty^2} \left(\frac{p}{p_\infty} - 1\right) \qquad (11\text{–}94)$$

where the M_∞ and p_∞ are the free-stream Mach number and pressure, respectively. The pressure ratio p/p_∞ can be expressed in terms of M and M_∞. With the aid of Eq. (11–90) we have

$$\frac{p}{p_\infty} = \frac{p}{p_0} \frac{p_0}{p_\infty} = \left[\frac{1 + [(\kappa - 1)/2] M_\infty^2}{1 + [(\kappa - 1)/2] M^2}\right] \qquad (11\text{–}95)$$

where

$$M^2 = \frac{q^2}{a^2} = \frac{(U_\infty + u)^2 + v^2 + w^2}{a^2}$$

In order to simplify the pressure coefficient expression it is assumed that the velocities around a thin body vary only slightly from the value of the free-stream velocity U_∞. This is the case of small disturbances. In other words, the disturbance velocities u, v, w along the coordinate axes x, y, z are small in comparison to the free-stream velocity U_∞ ($u/U_\infty \ll 1$, $v/U_\infty \ll 1$, $w/U_\infty \ll 1$). According to Eq. (11–80) the local velocity of sound is

$$a^2 = a_\infty^2 - \frac{\kappa - 1}{2}[q^2 - U^2]$$
$$= a_\infty^2 - \frac{\kappa - 1}{2}[2uU_\infty + u^2 + v^2 + w^2] . \qquad (11\text{–}96)$$

Hence,

$$1 + \frac{\kappa - 1}{2} M^2 = 1 + \frac{\kappa - 1}{2} \left[\frac{(U_\infty + u)^2 + v^2 + w^2}{a_\infty^2 - [(\kappa - 1)/2](2uU_\infty + u^2 + v^2 + w^2)} \right]$$

$$= \frac{1 + [(\kappa - 1)/2] M_\infty^2}{1 - [(\kappa - 1)/2a_\infty^2](2uU_\infty + u^2 + v^2 + w^2)} \tag{11-97}$$

Substituting Eq. (11-97) into Eq. (11-95), after using the binomial expansion ($u/a_\infty \ll 1$), there results the pressure relation

$$\frac{p}{p_\infty} = \left[1 - \frac{\kappa - 1}{2a_\infty^2} (2uU_\infty + u^2 + v^2 + w^2) \right]^{\kappa/(\kappa-1)}$$

$$= 1 - \kappa \frac{u}{U_\infty} M_\infty^2 - \frac{\kappa}{2} \left(\frac{u^2 + v^2 + w^2}{U_\infty^2} \right) M_\infty^2$$

$$+ \frac{\kappa}{8} \left(\frac{2uU_\infty + u^2 + v^2 + w^2}{U_\infty^2} \right)^2 M_\infty^4 - \cdots \tag{11-98}$$

Neglecting higher powers of the perturbation velocities in Eq. (11-98) and substituting the result into Eq. (11-94), the pressure coefficient is obtained:

$$C_p = -\left[2\frac{u}{U_\infty} + (1 - M_\infty^2) \frac{u^2}{U_\infty^2} + \frac{v^2 + w^2}{U_\infty^2} \right] \tag{11-99}$$

For two-dimensional flow we need to retain only the first term in Eq. (11-99) and the expression for C_p reduces to

$$C_p = -2\frac{u}{U_\infty} \tag{11-100}$$

For flow over axially symmetric bodies the third term in Eq. (11-99) must also be retained; hence

$$C_p = -\frac{2u}{U_\infty} - \frac{v^2 + w^2}{U_\infty^2} \tag{11-101}$$

It must be borne in mind that for high Mach numbers (in the hypersonic range) the pressure coefficients given in Eqs. (11-100) and (11-101) are not adequate. The similarity rule of hypersonic flow must be applied in order to derive a suitable expression for the desired pressure coefficient.

D. STEADY ONE-DIMENSIONAL FLOW

In Chap. Six we have discussed the dynamics of the frictionless motion of a one-dimensional incompressible fluid. Although the discussion deals with the most elementary phase of fluid dynamics, it provides results which are applicable to many important and practical problems. In the following sections we will deal with one-dimensional compressible flow and, in particular, with flow through stream tubes.

For a one-dimensional incompressible fluid the continuity equation gives the kinetic relation between the velocity and the cross-sectional area, and the pressure can be obtained from the Bernoulli equation. On the other hand the continuity and momentum equations are interdependent in compressible flow, thereby complicating the relation between velocity and cross-sectional area.

11.14. Isentropic Flow Through Tubes

a. Relation for Flow Through Tubes

For a steady flow of gas through a stream tube the continuity equation as obtained in Eq. (11–12) is

$$\rho V A = \text{Constant} \tag{11–12}$$

Equation (11–12) can be applied to flow through tubes with the assumptions that the curvature of the tube is very small and the variable cross-sectional area is sufficiently small for the velocity, pressure, density, and temperature to be considered constant over each cross-section (Fig. 11.1). Taking the logarithmic differentiation of Eq.(11–12) we obtain the relation

$$\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0 \tag{11–102}$$

The differential form of the equation of motion without body forces can be obtained from Eq. (11–16) in the form

$$V\,dV = -\frac{dp}{\rho} = -\frac{dp}{d\rho}\frac{d\rho}{\rho} = -a^2\frac{d\rho}{\rho} \tag{11–103}$$

Eliminating $d\rho/\rho$ from Eqs. (11–102) and (11–103) and introducing $V/a = \mathrm{M}$, there results

$$-(1 - \mathrm{M}^2)\frac{dV}{V} = \frac{dA}{A} \tag{11–104}$$

This important relation governing the flow at any given cross section A is attributed to Hugoniot. The velocity of sound in Eq. (11–104) can be found from Eq. (11–80):

$$a^2 = a_0^2 - \frac{\kappa - 1}{2}V^2 \tag{11–105}$$

For an incompressible flow, Eq. (11–12) reduces to a simple relation between the velocity and the cross-sectional area, i.e.,

$$VA = \text{Constant}$$

The important conclusions which we can draw from the velocity-area relation in Eq. (11–104) are:

(i) If the flow is subsonic ($M^2 < 1$) upstream, the coefficient of dV/V in Eq. (11–104) will be negative. In this case an increase in the cross-sectional area of the tube ($dA > 0$) causes a decrease in the velocity ($dV < 0$). Similarly, a decrease in dA ($dA < 0$) produces an increase in dV ($dV > 0$). These statements are also true for an incompressible flow ($M = 0$).

(ii) In the case of supersonic flow ($M > 1$) upstream, the coefficient of dV/V in Eq. (11–104) will be positive. Hence an increase in the cross-sectional area of the tube in the direction of the flow ($dA > 0$) produces an increase in velocity ($dV > 0$) while a decrease in area ($dA < 0$) produces a decrease in velocity ($dV < 0$). When the velocity increases in a diverging section, an expansion is said to be taking place.

(iii) For $dA = 0$, the cross-sectional area of the tube must be either a maximum, A_M, or a minimum, A_m, (Fig. 11.11a and b). In this case, according to Eq. (11–104), $dV = 0$ and/or $M = 1$. At A_M (Fig. 11.11a) the velocity is maximum for supersonic flow along the forward section of the tube and minimum for subsonic flow. Conversely, at A_m (Fig. 11.11b) the velocity is minimum for supersonic flow along the forward section of the tube and maximum for subsonic flow. It is seen that the critical velocity ($M = 1$) can never occur in an expanding tube (Fig. 11.11a). On the other hand, the flow, subsonic or supersonic upstream, in a convergent tube (Fig. 11.11b) may increase or decrease towards the minimum cross section and reach the critical velocity at the cross section A_m. This cross section is generally known as the critical cross section or sonic throat, A_c. Hence we can conclude that the velocity of sound occurs only at the minimum cross section in a converging tube.

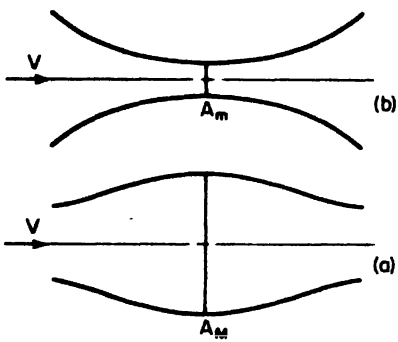


Figure 11.11

The change in pressure and density can be obtained as a function of the change in the cross-sectional area from Eqs. (11–103) and (11–104) as follows:

$$\frac{dp}{p} = \frac{\kappa M^2}{1 - M^2} \frac{dA}{A} \tag{11–106}$$

and

$$\frac{d\rho}{\rho} = \frac{M^2}{1 - M^2}\frac{dA}{A} \tag{11-107}$$

These relations indicate that for subsonic flow the pressure and density increase or decrease with the increase or decrease in the cross-sectional area. For supersonic flow these relations show that the pressure and density increase or decrease with the respective decrease or increase in the cross section. The relations above together with the velocity-area relation are illustrated by the sketches of Fig. 11.12.

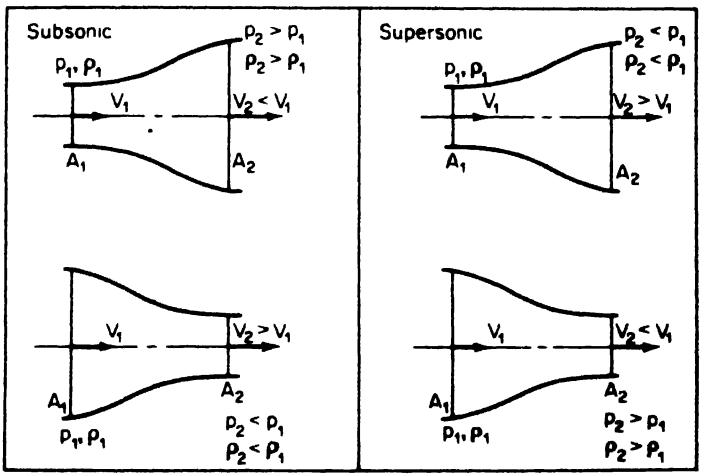


Figure 11.12

b. Mass Flow Through a Converging Nozzle

Figure 11.9 shows a converging nozzle connected with a large plenum chamber in which a gas is at rest. The gas flows out through the minimum cross-sectional area of the nozzle at pressure p which is equal to the surrounding pressure. The mass flow (or mass rate of flow), $\dot{m}$, at any station along the nozzle is given by the equation of continuity:

$$\dot{m} = \rho V A \tag{11-108}$$

Substituting the velocity V from Eq. (11-77) into Eq. (11-108) yields the expression for mass flow:

$$m \quad \rho A \sqrt{\frac{2\kappa}{\kappa - 1}\frac{p_0}{\rho_0}\left[1 - \left(\frac{p}{p_0}\right)^{(\kappa-1)/\kappa}\right]} \tag{11-109a}$$

which reveals that $M = 1$ results in $A = A_c$.

If the variation of the cross-sectional area is known, the corresponding variation of the pressure and Mach number can be determined from Eqs. (11–112) and (11–113). Graphical representation of A/A_c vs. p/p_0 and A/A_c vs. M for $\kappa = 1.4$ are shown in Figs. 11.14 and

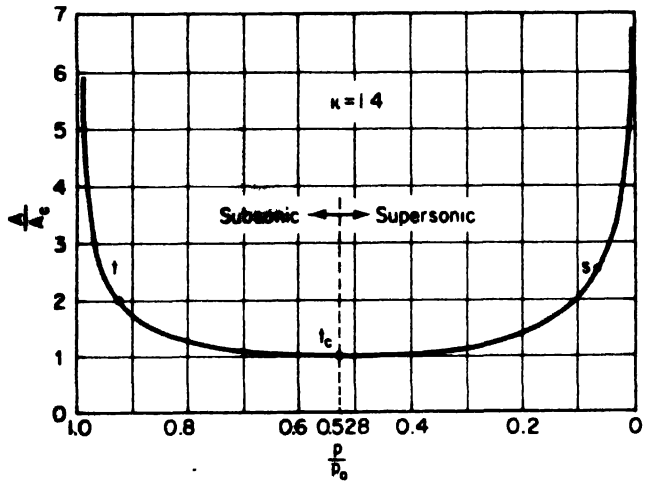


Figure 11.14

11.15, respectively. It is seen that each curve is double valued, with a subsonic and a supersonic branch. The flow begins along the subsonic branch from $p/p_0 = 1.0$ in Fig. 11.14 and $M = 0$ in Fig. 11.15. When the flow is subsonic throughout, the throat area ratio is greater than unity ($A/A_c > 1$). In this case the flow reaches only the point t on both curves and returns along the same branch. On the other hand, if the flow just reaches a sonic velocity at the throat ($A_t = A_c$) which is indicated by t_c on the curves, the flow beyond the throat lies on the supersonic branch and reaches some point s on both curves.

Further illustration of the theoretical pressure distribution along a convergent-divergent nozzle is given in Fig. 11.16b. If the pressures at both the inlet and exit sections are the same, there is on flow through the nozzle as indicated by curve A. As long as the mass flow does not reach the maximum value at the throat, the flow will be subsonic throughout and the exit pressure will remain within A and D. The curve B is the typical example of this case. When the maximum mass flow is reached at the throat ($p_t = p_c$) the flow may be subsonic in the divergent section if the exit pressure is at D ($p_D > p_c$) or supersonic

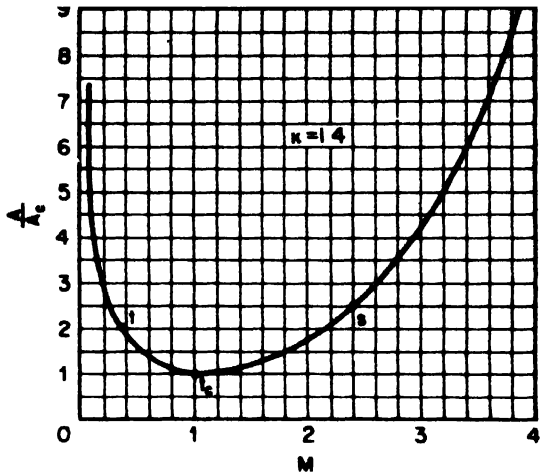


Figure 11.15

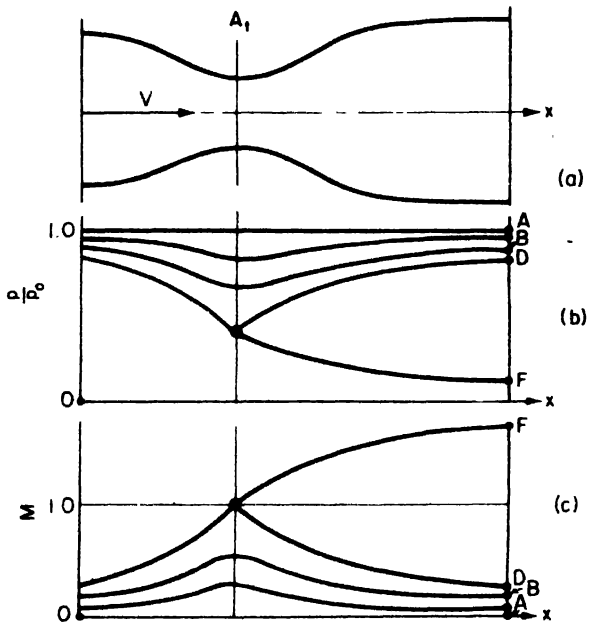


Figure 11.16

if it is at F ($p_F < p_c$). The behavior when the exit pressure lies between D and F or beyond F will be discussed later after studying shock and expansion waves.

The corresponding variation of the Mach number in a convergent-divergent nozzle is shown in Fig. 11.16c. It is seen that when the flow reaches sonic velocity at the throat the flow immediately reverts to subsonic or expands to supersonic, depending on the exit pressure at D or F, respectively. Since the flow immediately beyond the critical throat is supersonic for exit pressures lower than the critical pressure, the exit pressure is never transmitted upstream. For this reason, the maximum mass flow at the critical throat is not influenced by exit pressures lower in value than the critical pressure.

Illustrative Example 11.3. A supersonic wind tunnel is designed for M = 3.0 at the test section (Fig. 11.B). If the air supply in the reservoir is at $p_0 = 20$ lbf/in.^{2} and 80° F, determine the mass flow rate, the area of the test section, and the pressure, temperature, and density at the throat and test sections. The area of the throat section is assumed to be 1 ft^2.

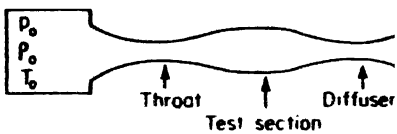


Figure 11.B

Solution. According to the equation of state for a perfect gas the stagnation density is

$$\rho_0 = \frac{p_0}{RT_0} = \frac{20 \times 144}{32.2 \times 53.34 \times 540} = 0.003105 \text{ slugs ft}^{-3}$$

From Eq. (11–8b) the critical values of the pressure, density, and temperature at the throat section are given by

$$p_c = 0.528\, p_0 = 0.528(20) = 10.56 \text{ lbf/in.}^2$$

$$\rho_c = 0.634\, \rho_0 = (0.634)(0.003105) = 0.001969 \text{ slugs/ft}^3$$

$$T_c = 0.833\, T_0 = (0.833)(540) = 450° \text{R}$$

At the test section the pressure, density, temperature, and cross-sectional area may be obtained in terms of the Mach number from Eqs. (11–90), (11–91), (11–87b), and (11–113), respectively (or from Table E-4). They are:

$$p = \frac{p_0}{\{1 + [(\kappa - 1)/2]M^2\}^{\kappa/(\kappa-1)}} = \frac{20}{[1 + 0.2(3^2)]^{3.5}} = 0.5445 \text{ lbf/in.}^2$$

$$\rho = \frac{\rho_0}{\{1 + [(\kappa - 1)/2]M^2\}^{1/(\kappa-1)}} = \frac{0.0031}{(1 + 1.8)^{2.5}} = 0.0002367 \text{ slugs/ft}^3$$

$$T = \frac{T_0}{1 + [(\kappa - 1)/2]M^2} = \frac{540}{2.8} = 193.86° \text{R}$$

$$A = \sqrt{A_c}\frac{1}{M}\left[\frac{2}{\kappa+1}\left(1+\frac{\kappa-1}{2}M^2\right)\right]^{(\kappa+1)/2(\kappa-1)} = \frac{1}{3}[2.333]^3 = 4.235 \text{ ft}^2$$

The velocities at the throat and test sections are, respectively,

$$V_c = Ma_c = M\sqrt{\left(\kappa\frac{p_c}{\rho_c}\right)} = \sqrt{(1.4)\frac{10.57 \times 144}{0.00197}} = 1039.9 \text{ ft/sec}$$

$$V_{test} = Ma = 3.0\sqrt{(1.4)\frac{0.546 \times 144}{0.00236}} = 2042.9 \text{ ft/sec}$$

The mass flow rate is

$$\dot{m} = \rho_c A_c a_c = 0.001969 \times 1.0 \times 1039.9 = 2.048 \text{ slugs/sec}$$

11.15. Flow Through a Constant-Area Tube with Heat Transfer

Isentropic flow through tubes with a variable cross-sectional area which we discussed in the previous section has brought out the essential features of the flow of a compressible fluid. The high speed wind tunnel is one of the many applications of this type of flow. There is another important type of flow which also suffers no frictional losses, but during the flow the gas exchanges heat with its surroundings (see Fig. 11.17). This is a reversible nonadiabatic flow which is also called diabatic flow. Heat exchangers and combustion chambers are two well-known applications of flow through ducts with heat transfer. In order not to complicate our problem, yet bring out the physical features, we shall restrict our discussions to the case of (1) uniform cross section, (2) no friction, (3) a perfect gas, and (4) no change of gas composition.

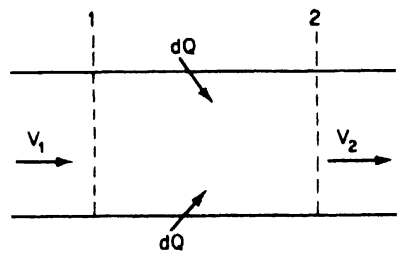


Figure 11.17

a. Fundamental Equations for Flow with Heat Addition

The basic equations for one-dimensional steady flow in a nozzle of constant cross section with consideration of heat addition are obtained from Eqs. (11-12), (11-16), (B-14), and (5-37), respectively,

$$\rho V = \text{Constant} \tag{11-114a}$$

$$\rho V\,dV = -dp \tag{11-114b}$$

The corresponding variation of the Mach number in a convergent-divergent nozzle is shown in Fig. 11.16c. It is seen that when the flow reaches sonic velocity at the throat the flow immediately reverts to subsonic or expands to supersonic, depending on the exit pressure at D or F, respectively. Since the flow immediately beyond the critical throat is supersonic for exit pressures lower than the critical pressure, the exit pressure is never transmitted upstream. For this reason, the maximum mass flow at the critical throat is not influenced by exit pressures lower in value than the critical pressure.

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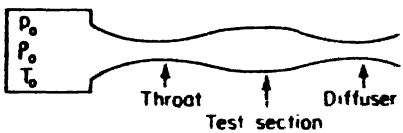


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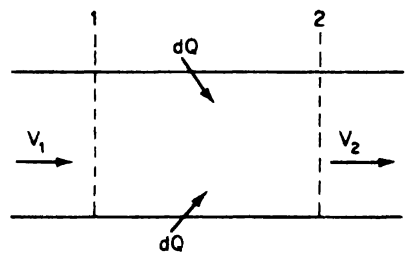


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The basic equations for one-dimensional steady flow in a nozzle of constant cross section with consideration of heat addition are obtained from Eqs. (11-12), (11-16), (B-14), and (5-37), respectively,

$$\rho V = \text{Constant} \tag{11-114a}$$

$$\rho V\,dV = -dp \tag{11-114b}$$

$$dQ = C_v\,dT + p\,d\left(\frac{1}{\rho}\right) \tag{11-114c}$$

$$p = \rho RT \tag{11-114d}$$

The body force in Eq. (11-16) is neglected in Eq. (11-114b) and the term dQ in Eq. (11-114c) represents the heat addition to the system. In the case of reversible adiabatic flow it can easily be shown that Eqs. (11-114c) and (11-114d) reduce to the isentropic relation [see Eq. (B-43)]. After differentiating Eqs. (11-114a) and (11-114d), respectively, we find

$$-\frac{d\rho}{\rho} = \frac{dV}{V} \tag{11-115a}$$

$$\frac{dp}{p} = \frac{R}{p}(\rho dT + T d\rho) = \frac{d\rho}{\rho} + \frac{dT}{T} \tag{11-115b}$$

Equations (11-114b) and (11-114c) can be rewritten in the forms

$$-\frac{dp}{p} = \frac{\rho}{p} V\,dV = \frac{V^2}{RT}\frac{dV}{V} = \kappa M^2 \frac{dV}{V} \tag{11-115c}$$

$$dQ = C_v\,dT + \frac{dp}{\rho} = C_p dT + V\,dV \tag{11-115d}$$

Equations (11-115a)–(11-115d) will be used to obtain the differential relations of the velocity, density, pressure, temperature, and Mach number as a function of the heat addition.

The calculation of these relations may proceed by first obtaining the velocity and temperature relation from Eqs. (11-115a), (11-115b), and (11-115c) as

$$\frac{dT}{T} = \frac{dp}{p} - \frac{d\rho}{\rho} = (1 - \kappa M^2)\frac{dV}{V} \tag{11-116a}$$

The differential relation of the velocity and the heat addition is obtained from Eq. (11-115d):

$$\frac{dV}{V} = \frac{dQ}{V^2} - \frac{C_v\,dT}{V^2} = \frac{1}{M^2}\left(\frac{C_v}{\kappa RT}\right)\frac{dQ}{C_v} - \frac{1}{M^2}\frac{C_v}{\kappa RT}dT$$

$$= \frac{1}{(\kappa - 1)M^2}\left(\frac{dQ}{h} - \frac{dT}{T}\right) \tag{11-116b}$$

Substituting Eq. (11-116a) into Eq. (11-116b) we obtain

$$\frac{dV}{V} = \frac{1}{1 - M^2}\frac{dQ}{h} \tag{11-117a}$$

where h is the enthalpy per unit mass. The differential relations of the density, pressure, and temperature are obtained from Eqs. (11-115a), (11-115c), and (11-116a), respectively, with the aid of Eq. (11-117a):

$$\frac{d\rho}{\rho} = -\frac{1}{1-M^2}\frac{dQ}{h} \tag{11-117b}$$

$$\frac{dp}{p} = -\frac{\kappa M^2}{1-M^2}\frac{dQ}{h} \tag{11-117c}$$

$$\frac{dT}{T} = \frac{1-\kappa M^2}{1-M^2}\frac{dQ}{h} \tag{11-117d}$$

The differential relation of the Mach number and the heat addition can be derived as follows:

$$\frac{dM^2}{M^2} = \frac{d(V^2/a^2)}{V^2/a^2} = \frac{T}{V^2}d\left(\frac{V^2}{T}\right) = 2\frac{dV}{V} - \frac{dT}{T}$$

After substituting Eqs. (11-117a) and (11-117d) into this equation, there results

$$\frac{dM^2}{M^2} = \frac{1+\kappa M^2}{1-M^2}\frac{dQ}{h} = \frac{1+\kappa M^2}{1-M^2}\frac{dS}{C_p} \tag{11-117e}$$

where dS is the change of entropy.

The effect of heat addition on the thermodynamic and flow properties can now be explained from Eqs. (11-117) for flows either in the subsonic range or the supersonic range. When the heat is added to a subsonic flow, we have

$\frac{dV}{V}$ is positive or V increases

$\frac{d\rho}{\rho}$ is negative or ρ decreases

$\frac{dp}{p}$ is negative or p decreases

$\frac{dM^2}{M^2}$ is positive or M increases

$\frac{dT}{T}$ is positive or T increases if $M^2 < \frac{1}{\kappa}$

$\frac{dT}{T}$ is negative or T decreases if $M^2 > \frac{1}{\kappa}$

When $1/\kappa < M^2 < 1$ the heat added to the flow results in an increase in the kinetic energy and the internal energy of the mean flow. Hence, the temperature actually decreases when the heat is being added provided $1/\kappa < M^2 < 1$ (see Fig. 11.18).

On the other hand, when heat is added to a supersonic flow the change of the fluid and the flow properties are:

$\frac{dV}{V}$ is negative or V decreases

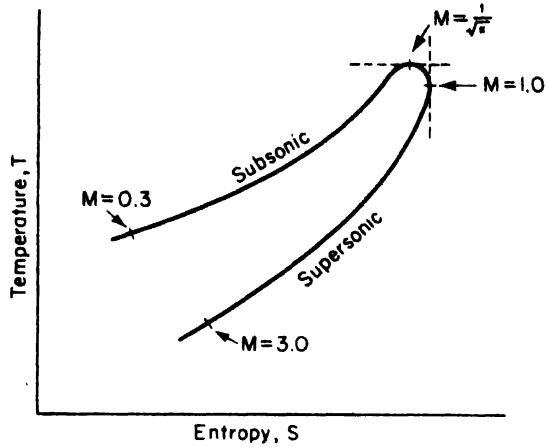


Figure 11.18

$\frac{d\rho}{\rho}$ is positive or ρ increases

$\frac{dp}{p}$ is positive or p increases

$\frac{dT}{T}$ is positive or T increases

$\frac{dM^2}{M^2}$ is negative or M decreases

It must be emphasized that Eqs. (11-117) break down at $M = 1$ for $dQ \neq 0$ and all the variables V, p, ρ, T, etc., have an infinite gradient. This indicates that the Mach number approaches unity with an infinite rate. The effect of the heat addition is to accelerate the flow when it is subsonic and decelerate it when the flow is supersonic. For a flow in a tube of constant cross section a subsonic flow cannot become supersonic but will always approach the sonic state and a supersonic flow will always approach the sonic state' but cannot become subsonic (without external influences).

b. Relations Among the Thermodynamic Properties for a Rayleigh Process

The Rayleigh line is the locus of points of the gas properties during a constant-area flow with a reversible nonadiabatic process. One of the Rayleigh line equations can be obtained from the continuity equation [Eq. (11-114a)] and the momentum equation [Eq. (11-114b)] as follows:

$$p + \frac{C_1^2}{\rho} = \text{Constant} = C_2 \tag{11-118}$$

where $C_1 = \text{Constant} = \rho V$. The Rayleigh line on a diagram showing p vs. $1/\rho$ will be a straight line with a slope of $-C_1^2$ and an intercept on the p-axis of C_2.

Prior to determining the relations of pressure, temperature, density, and entropy in terms of the Mach number, it is advisable to investigate the effect of heat addition on the change of stagnation temperature. From the energy equation [Eq. (11-115d)] we have, after integrating between the conditions indicated at stations 1 and 2 in Fig. 11.17,

$$Q = C_p(T_2 - T_1) + \frac{V_2^2 - V_1^2}{2} = C_p(T_{02} - T_{01}) \tag{11-119}$$

where T_{01} and T_{02} are the stagnation temperatures at the reservoir conditions before station 1 and after station 2, respectively. For an adiabatic flow $T_{02} = T_{01} = T_0$, but in nonadiabatic flow the stagnation temperature is not determined simply by the reservoir conditions.

We may now proceed to determine the pressure ratio for a Rayleigh process. After eliminating dQ/h from Eqs. (11-117c) and (11-117e), there results

$$\frac{dp}{p} = -\frac{\kappa \mathrm{M}^2}{1 + \kappa \mathrm{M}^2} \frac{d\mathrm{M}^2}{\mathrm{M}^2} \tag{11-120a}$$

Integrating Eq. (11-120a) between two state points in the flow, one obtains

$$\ln \frac{p_2}{p_1} = \ln \frac{1 + \kappa \mathrm{M}_1^2}{1 + \kappa \mathrm{M}_2^2}$$

$$\frac{p_2}{p_1} = \frac{1 + \kappa \mathrm{M}_1^2}{1 + \kappa \mathrm{M}_2^2} \tag{11-120b}$$

According to Eq. (11-90) the stagnation pressure in terms of static pressure and the Mach number is given by the relation

$$p_0 = p\left(1 + \frac{\kappa - 1}{2} \mathrm{M}^2\right)^{\kappa/(\kappa-1)} \tag{11-90}$$

Hence the ratio of the stagnation pressures is obtained from Eq. (11-90) with the aid of Eq. (11-120b) as

$$\frac{p_{02}}{p_{01}} = \frac{1 + \kappa \mathrm{M}_1^2}{1 + \kappa \mathrm{M}_2^2} \left[\frac{1 + \dfrac{\kappa - 1}{2} \mathrm{M}_2^2}{1 + \dfrac{\kappa - 1}{2} \mathrm{M}_1^2} \right]^{\kappa/(\kappa-1)} \tag{11-121}$$

In a similar manner the temperature ratio in terms of the Mach number can be obtained from Eqs (11-117d) and (11-117e),

$$\frac{T_2}{T_1} = \frac{M_2^2}{M_1^2}\left(\frac{1 + \kappa M_1^2}{1 + \kappa M_2^2}\right)^2 \tag{11-122}$$

With the stagnation temperature,

$$T_0 = T\left(1 + \frac{\kappa - 1}{2} M^2\right) \tag{11-87b}$$

and the T_2/T_1 ratio of Eq. (11-122), the ratio of the stagnation temperatures is obtained:

$$\frac{T_{02}}{T_{01}} = \frac{M_2^2}{M_1^2}\left(\frac{1 + \kappa M_1^2}{1 + \kappa M_2^2}\right)^2 \left[\frac{1 + \frac{\kappa - 1}{2} M_2^2}{1 + \frac{\kappa - 1}{2} M_1^2}\right] \tag{11-123}$$

The density and velocity ratios are obtained from the equation of state for a perfect gas and Eqs. (11-120b) and (11-122),

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{p_2 T_1}{p_1 T_2} = \frac{M_1^2}{M_2^2}\left(\frac{1 + \kappa M_2^2}{1 + \kappa M_1^2}\right) \tag{11-124}$$

Finally, we can use Eq. (11-117e) to determine the entropy increase for a Rayleigh process:

$$dS = C_p\left(\frac{1 - M^2}{1 + \kappa M^2}\right)\frac{dM^2}{M^2} \tag{11-117e}$$

Integrating Eq. (11-117e) between two state points in the flow,

$$S_2 - S_1 = \frac{\kappa R}{\kappa - 1} \ln\left[\frac{M_2^2}{M_1^2}\left(\frac{1 + \kappa M_1^2}{1 + \kappa M_2^2}\right)^{(\kappa+1)/\kappa}\right] \tag{11-125}$$

where $C_p = \kappa R/(\kappa - 1)$.

Equation (11-125) together with Eq. (11-122) may be used to construct the loci of the Mach number for Rayleigh flow in the T-S diagram (Fig. 11.18). It is seen from Fig. 11.18 that the temperature along the subsonic line increases with an increase in the Mach number up to $M = 1/\sqrt{\kappa}$ then decreases between $1/\sqrt{\kappa} < M < 1$. Along the supersonic line the temperature increases with the decrease of the Mach number and again the heat addition tends to make the Mach number approach unity. Since the maximum entropy occurs at $M = 1$ and the entropy always increases with the heat addition, it is not possible for either subsonic or supersonic flow to pass from one domain into the other. The phenomenon clearly obeys the second law of thermodynamics.

The discussion above is limited to the case for heat addition to the system; however, an opposite effect will be found if the heat is withdrawn (cooling) from the system.

Illustrative Example 11.4. In dealing with constant-area flow with heat

addition, the asterisk quantity is normally used to designate the point on a Rayleigh line which represents a Mach number of unity. By letting $M_1 = 1$, the following relations are obtained directly from Eqs. (11-120b), (11-121), (11-122), (11-123), and (11-124), respectively,

$$\frac{p}{p^*} = \frac{1 + \kappa}{1 + \kappa M^2} \tag{a}$$

$$\frac{p_0}{p_0^*} = \frac{1 + \kappa}{1 + \kappa M^2}\left[\frac{1 + [(\kappa - 1)/2]M^2}{1 + (\kappa - 1)/2}\right]^{\kappa/(\kappa-1)} \tag{b}$$

$$\frac{T}{T^*} = \frac{(1 + \kappa)^2 M^2}{(1 + \kappa M^2)^2} \tag{c}$$

$$\frac{T_0}{T_0^*} = \frac{(1 + \kappa)^2 M^2\{1 + [(\kappa - 1)/2]M^2\}}{[1 + (\kappa - 1)/2](1 + \kappa M^2)^2} \tag{d}$$

$$\frac{\rho}{\rho^*} = \frac{V^*}{V} = \frac{(1 + \kappa M^2)}{(1 + \kappa)M^2} \tag{e}$$

Equations. (a), (b), (c), (d), and (e) are computed and tabulated as a function of M. These values appear as a part of Meteor Report No. 14, MIT, 1949, by A. H. Shapiro, W. R. Hawthorne, and C. M. Edelman.

As an example of using the equations above, consider the flow of air at standard atmospheric conditions through a heated constant-area pipe. If the flow is decelerated from $M_1 = 1.5$ to $M_2 = 1.0$, determine (a) the change in temperature and (b) the heat addition.

Solution. (a) Air at standard atmospheric condition has $T_1 = 519°$ R. From Eq. (c) corresponding to $M_1 = 1.5$ we have

$$\frac{T_1}{T^*} = \frac{(1 + \kappa)^2(1.5)^2}{[1 + (1.5)^2\kappa]^2} = 0.7525$$

Since the final Mach number is unity, it follows that $T_2 = T^*$, i.e.,

$$T_2 = \frac{T_1}{0.7525} = \frac{519}{0.7525} = 689.7° \text{ R}$$

Hence, the increase in temperature due to heat addition is

$$\Delta T = T_2 - T_1 = 170.7° \text{ R}$$

(b) The heat addition is given by

$$\Delta Q = C_p(T_2 - T_1) + \frac{V_2^2 - V_1^2}{2g(778)}$$

The initial velocity is

$$a_1 = \sqrt{\kappa R T_1} = 1{,}117 \text{ ft/sec}$$

$$V_1 = a_1 M_1 = 1{,}677 \text{ ft/sec}$$

According to Eq. (e) we have

$$V_2 = V^* = V_1\left[\frac{1 + \kappa(1.5)^2}{(1+\kappa)(1.5)^2}\right] = (1{,}677)(0.7685) = 1{,}289 \text{ ft/sec}$$

Hence,

$$\Delta Q = 0.24(\Delta T) + \frac{(1289)^2 - (1677)^2}{2 \times 32.2 \times 778} = 40.968 - 22.969 = 18 \text{ BTU/lbf}$$

An alternate method to find heat addition is

$$\triangle Q = C_p(T_{02} - T_{01})$$

From Eq. (11-123) we obtain

$$\frac{T_{02}}{T_{01}} = \frac{1}{(1.5)^2}\left[\frac{1 + (1.5)^2\kappa}{1 + \kappa}\right]^2\left[\frac{1 + 0.2}{1 + 0.2 \times (1.5)^2}\right] = 1.1$$

But Eq. (11-87b) gives

$$T_{01} = T_1\left(1 + \frac{\kappa - 1}{2}\,\mathrm{M}_1^2\right) = (519)(1.45) = 752.6^\circ \text{R}$$

Hence,

$$\Delta Q = 0.24\,[(1.1 - 1)T_{01}] = 0.24(0.1 \times 752.6) = 18.06 \text{ BTU/lbf}$$

Illustrative Example 11.5. Air flows from a reservoir at standard conditions through a convergent-divergent nozzle to a constant cross-section pipe (Fig. 11.C). Assuming the mass flow rate from the reservoir to the Rayleigh flow system is 0.362 slugs/ft²-sec, construct a T-S diagram under Rayleigh flow conditions.

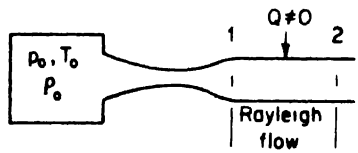


Figure 11.C

Solution. The temperature, pressure, and density of the reservoir at standard conditions are: $T_0 = 517^\circ$ R, $p_0 = 2{,}117$ lbf/ft², $\rho_0 = 0.002378$ slugs/ft³. According to Eqs. (11-42) and (11-87b) the velocity at the beginning of the Rayleigh flow is

$$V_1 = \mathrm{M}_1 a_1 = \mathrm{M}_1\sqrt{\kappa RT_1} = \mathrm{M}_1\sqrt{\frac{\kappa RT_0}{1 + [(\kappa - 1)/2]\mathrm{M}_1^2}} \qquad \text{(a)}$$

From the isentropic relation and Eq. (11-87b) the density at the beginning of the Rayleigh flow is

$$\rho_1 = \rho_0\left(\frac{T_1}{T_0}\right)^{1/(\kappa-1)} = \rho_0\left(1 + \frac{\kappa-1}{2}\,M_1^2\right)^{1/(\kappa-1)} \tag{b}$$

According to the continuity equation we obtain

$$\rho_1 V_1 = 0.362 = \rho_0\left[1 + \frac{\kappa-1}{2}\,M_1^2\right]^{1/(\kappa-1)} M_1\sqrt{\frac{\kappa R T_0}{1 + [(\kappa-1)/2]M_1^2}} \tag{c}$$

from which the Mach number at the beginning of the Rayleigh flow is obtained,

$$M_1\left(\frac{1}{1 + 0.2M_1^2}\right)^3 = 0.01367 \tag{d}$$

Equation (d) gives one of the solutions which is $M_1 = 3.0$. With this value of M_1, the temperature at the beginning of the Rayleigh flow is

$$T_1 = \frac{T_0}{1 + 0.2\,M_1^2} = \frac{517}{1 + 0.2(3)^2} = 184.6^\circ\,\mathrm{R}$$

The temperature and entropy as a function of the Mach number for Rayleigh flow are given by Eqs. (11-122) and (11-125) as follows:

$$T_2 = T_1\left[\left(\frac{M_2}{M_1}\right)^2\left(\frac{1 + \kappa M_1^2}{1 + \kappa M_2^2}\right)^2\right] \tag{11-122}$$

$$\frac{S_2 - S_1}{C_p} = \ln\left(\frac{M_2}{M_1}\right)^2\left(\frac{1 + \kappa M_1^2}{1 + \kappa M_2^2}\right)^{(\kappa+1)/\kappa} \tag{11-125}$$

For $M_1 = 3.0$, a relation between temperature and entropy can be established for various assumed values of M_2. The results computed from Eqs. (11-122) and (11-125) are:

M_2	T_2	$(S_2 - S_1)/C_p$
0.48	499.8	0.3300
0.60	603.9	0.5559
0.80	675.6	0.7342
0.85	677.6	0.7541
0.90	674.9	0.7670
0.95	668.4	0.7742
1.00	658.8	0.7764
1.10	632.6	0.7689
1.20	600.7	0.7494
1.60	462.3	0.6071
2.00	348.4	0.4285
2.50	249.5	0.2059
3.00	184.6	0.0000

The Rayleigh line curve is plotted in Fig. 11.D.

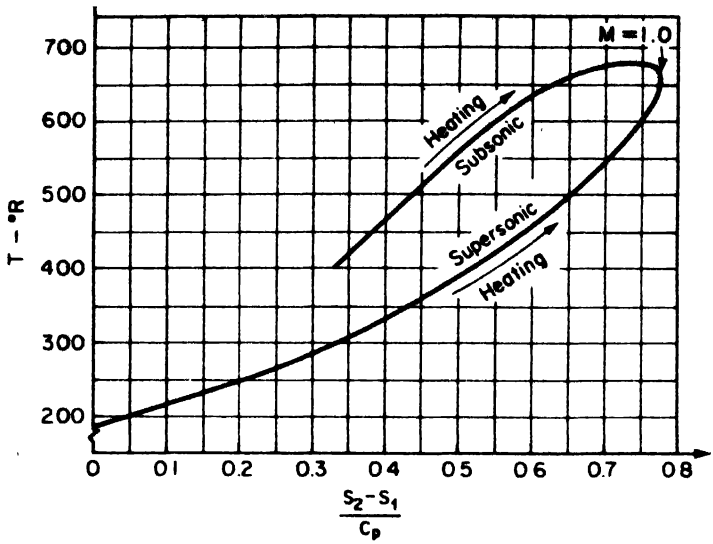


Figure 11.D

E. WAVE PHENOMENA

It was discussed in § 11.7 that a very small pressure change (pressure impulse) propagates through a stationary fluid with the velocity of sound. When a large pressure change in a small volume is produced, as in an explosion, the resulting pressure wave propagates at a velocity much higher than the velocity of sound. As the wave passes any point, the pressure, density, temperature, and entropy increase abruptly from the ambient values and this phenomenon is called a progressing shock wave. The formation of a shock was discussed in § 11.8 and the changes of fluid properties and flow characteristics will be studied in this section. Actually in a real fluid, discontinuities in fluid properties cannot exist for finite periods of time as the effects of viscosity and heat conduction of the fluid tend to smooth out such discontinuities.[3] However, theoretical calculations and experimental measurements show that the thickness of a normal shock wave is of the same order of magnitude as the mean free path of the molecules. For this reason we can neglect the viscosity and heat conduction when we study the shock wave and a mathematical discontinuity is employed to replace the interior details of the shock wave. A further assumption in the study of shock waves is that the perfect gas law governs the behavior of the gas both in front of and behind the shock wave.

11.16. Normal Shock Waves

a. Governing Equations and the Solution

Consider a progressing shock wave moving through a stationary gas with a velocity u_1 in a direction perpendicular to the shock. The pressure, density, temperature, internal energy per unit mass of the gas, and the velocity of sound in front of and behind the shock wave are denoted by p_1, ρ_1, T_1, E_1, and a_1, and p_2, ρ_2, T_2, E_2, and a_2, respectively (Fig. 11.19a). The velocity of the gas behind the shock is u_g which is perpendicular to the shock.

In order to simplify the analysis the coordinate system is so chosen that it moves at the same speed as the shock front. This stationary shock wave with reference to the new coordinate system is shown in Fig. 11.19b. The moving gas now is divided by the stationary shock wave into regions in front and behind the shock, which are denoted by quantities with subscripts 1 and 2, respectively.

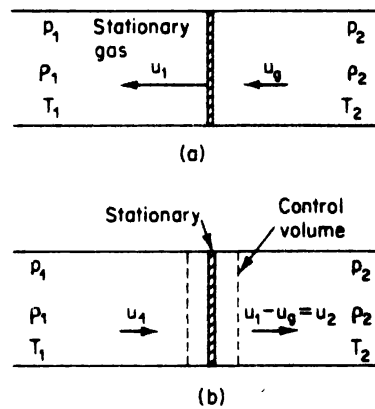


Figure 11.19

As mentioned previously, the process of change across a shock wave is very rapid and the transition region of the shock is indeed so very small that it may be treated for practical purposes as a discontinuity in the flow. Since the shock involves no addition of mass or total energy (only conversion of mechanical energy into heat energy) and no impulsive force, the laws of conservation of mass, momentum, and energy apply here. Hence, with conditions known in front of a normal shock wave, the conditions behind the shock can be determined from the equations based on the conservation laws.

The law of conservation of mass requires that the mass of gas per unit area flowing into the shock region in time dt, $\rho_1 u_1 dt$ be equal to that flowing out, $\rho_2(u_1 - u_g)dt$, i.e.,

$$\rho_1 u_1 = \rho_2(u_1 - u_g) = \rho_2 u_2 \tag{11-126}$$

According to the law of conservation of momentum we have

$$(\rho_1 u_1)u_g = p_2 - p_1 \tag{11-127a}$$

where $p_2 - p_1$ is the net force per unit area acting on the control

volume and $(\rho_1 u_1)u_g$ the rate of change of momentum per unit area of the gas inside the control volume. Equation (11–127a) can be re-written in the form

$$\rho_1 u_1(u_1 - u_2) = p_2 - p_1$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \tag{11–127b}$$

Based on the law of conservation of energy, the energy equation can be derived by equating the increase in kinetic and internal energy per unit area of the gas passing the shock front and the net work done per unit area on the control volume as follows:

$$dt\{\tfrac{1}{2}\rho_1 u_1[(u_2^2 - u_1^2)] + (E_2 - E_1)\rho_1 u_1\} = \{p_1 u_1 - p_2 u_2\}\, dt \tag{11–128a}$$

Substituting $E = (C_\tau/R)(p/\rho)$ for a perfect gas into Eq. (11–128a), the required energy equation becomes

$$\frac{p_1}{\rho_1} + \frac{C_\tau}{R}\frac{p_1}{\rho_1} + \frac{1}{2}u_1^2 = \frac{p_2}{\rho_2} + \frac{C_\tau}{R}\frac{p_2}{\rho_2} + \frac{1}{2}u_2^2$$

or

$$\frac{\kappa}{\kappa - 1}\frac{p_1}{\rho_1} + \frac{1}{2}u_1^2 = \frac{\kappa}{\kappa - 1}\frac{p_2}{\rho_2} + \frac{1}{2}u_2^2 = \frac{\kappa}{\kappa - 1}\frac{p_0}{\rho_0} = \frac{\kappa}{\kappa - 1}RT_0 \tag{11–128b}$$

where the quantities with subscript ^{0} denote the stagnation quantities.

Equations (11–126), (11–127b), and (11–128b) will now be used to determine the net change of various properties across the normal shock wave. First, we have from Eqs. (11–126) and (11–127b),

$$u_1 - u_2 = \frac{p_2}{\rho_2 u_2} = \frac{p_1}{\rho_1 u_1} \tag{11–129a}$$

Multiplying Eq. (11–129a) by $(u_1 + u_2)$ yields

$$u_1^2 - u_2^2 = \left(\frac{p_2}{\rho_2 u_2} - \frac{p_1}{\rho_1 u_1}\right)(u_1 + u_2) = (p_2 - p_1)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \tag{11–129b}$$

Combining Eqs. (11–129b) and (11–128b) we obtain

$$(p_2 - p_1)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) = \frac{2\kappa}{\kappa - 1}\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right)$$

or

$$\frac{p_2}{p_1}\left(\frac{\rho_2}{\rho_1} - \frac{\kappa + 1}{\kappa - 1}\right) = 1 - \frac{\rho_2}{\rho_1}\left(\frac{\kappa + 1}{\kappa - 1}\right) \tag{11–130}$$

Equation (11–130) yields the following important relations:

$$\frac{p_2}{p_1} = \frac{\dfrac{\kappa + 1}{\kappa - 1}\dfrac{\rho_2}{\rho_1} - 1}{\dfrac{\kappa + 1}{\kappa - 1} - \dfrac{\rho_2}{\rho_1}} = \frac{1 + \dfrac{\kappa + 1}{2}\dfrac{\rho_2 - \rho_1}{\rho_1}}{1 - \dfrac{\kappa - 1}{2}\dfrac{\rho_2 - \rho_1}{\rho_1}} \tag{11–131a}$$

and

$$\frac{\rho_2}{\rho_1} = \frac{\dfrac{\kappa+1}{\kappa-1}\dfrac{p_2}{p_1} + 1}{\dfrac{\kappa+1}{\kappa-1} + \dfrac{p_2}{p_1}} = \frac{1 + \dfrac{\kappa+1}{2\kappa}\dfrac{p_2 - p_1}{p_1}}{1 + \dfrac{\kappa-1}{2\kappa}\dfrac{p_2 - p_1}{p_1}} \tag{11-131b}$$

Equations (11-131a) and (11-131b) are known as Rankine-Hugoniot relations, named after the English and French mathematicians who independently discovered them. These relations govern the change of state across a normal shock wave, a hypothetical surface of discontinuity. Since the shock waves produce a dissipation of mechanical energy which is converted into heat energy, the flow across a shock is nonisentropic and the entropy increases.

This sudden decrease in velocity across a normal shock wave can be found by rewriting Eq. (11-128b) as

$$u_1 + \frac{2\kappa}{\kappa-1}\frac{p_1}{\rho_1 u_1} = \frac{2\kappa}{\kappa-1}\frac{p_0}{\rho_0}\frac{1}{u_1}$$

and

$$u_2 + \frac{2\kappa}{\kappa-1}\frac{p_2}{\rho_2 u_2} = \frac{2\kappa}{\kappa-1}\frac{p_0}{\rho_0}\frac{1}{u_2}$$

The difference of the two equations above gives

$$u_1 - u_2 + \frac{2\kappa}{\kappa-1}\left(\frac{p_1}{\rho_1 u_1} - \frac{p_2}{\rho_2 u_2}\right) = \frac{2\kappa}{\kappa-1}\frac{p_0}{\rho_0}\left(\frac{1}{u_1} - \frac{1}{u_2}\right)$$

$$= \frac{2a_0^2}{\kappa-1}\frac{(u_2 - u_1)}{u_1 u_2} \tag{11-132}$$

Substituting Eq. (11-129a) into Eq. (11-132) yields

$$u_1 - u_2 - \frac{2\kappa}{\kappa-1}(u_1 - u_2) = \frac{2a_0^2}{\kappa-1}\frac{(u_2 - u_1)}{u_1 u_2}$$

or

$$1 - \frac{2\kappa}{\kappa-1} = -\frac{2a_0^2}{\kappa-1}\frac{1}{u_1 u_2} \tag{11-133a}$$

Introducing the critical velocity from Eq. (11-82) into Eq. (11-133a) we have finally

$$u_1 u_2 = \frac{2a_0^2}{\kappa+1} = a_c^2 \tag{11-133b}$$

Equation (11-133b) gives the velocity relation across a normal shock wave and is generally known as the Prandtl relation. The importance of this relation will be demonstrated later.

The relation between the Mach numbers in front of and behind the normal shock can be obtained from Eq. (11-87a) when $q = u_1$ and

$a = a_1$ for flow in front of the shock, and $q = u_2$ and $a = a_2$ behind the shock; we have from Eq. (11-87a), respectively,

$$(\kappa - 1) + \frac{2}{M_1^2} = \frac{2}{u_1^2} a_0^2 = (\kappa + 1)\left(\frac{a_c}{u_1}\right)^2 \tag{11-134a}$$

and

$$(\kappa - 1) + \frac{2}{M_2^2} = \frac{2}{u_2^2} a_0^2 = (\kappa + 1)\left(\frac{a_c}{u_2}\right)^2 \tag{11-134b}$$

Multiplying these two equations, we obtain

$$\left(\frac{\kappa - 1}{2} + \frac{1}{M_1^2}\right)\left(\frac{\kappa - 1}{2} + \frac{1}{M_2^2}\right) = \left(\frac{\kappa + 1}{2}\right)^2 \tag{11-135}$$

Hence, the Mach number behind the shock depends only on the Mach number in front of the shock and the ratio of specific heats, regardless of other fluid properties.

It is both useful and convenient to express the changes of pressure, density, and temperature across a normal shock in terms of the Mach number in front of the shock. Rewriting Eq. (11-131a) with the aid of Eqs. (11-126) and (11-133b) we obtain the pressure ratio

$$\frac{p_2}{p_1} = \frac{\dfrac{\kappa + 1}{\kappa - 1}\dfrac{u_1}{u_2} - 1}{\dfrac{\kappa + 1}{\kappa - 1} - \dfrac{u_1}{u_2}} = \frac{\dfrac{\kappa + 1}{\kappa - 1}\left(\dfrac{u_1}{a_c}\right)^2 - 1}{\dfrac{\kappa + 1}{\kappa - 1} - \left(\dfrac{u_1}{a_c}\right)^2} \tag{11-136a}$$

Substituting u_1/a_c from Eq. (11-134a) into Eq. (11-136a), we find

$$\frac{p_2}{p_1} = \frac{\dfrac{\kappa + 1}{\kappa - 1}\dfrac{\kappa + 1}{\kappa - 1 + \dfrac{2}{M_1^2}} - 1}{\dfrac{\kappa + 1}{\kappa - 1} - \dfrac{\kappa + 1}{\kappa - 1 + \dfrac{2}{M_1^2}}} = \frac{2\kappa}{\kappa + 1} M_1^2 - \frac{\kappa - 1}{\kappa + 1} \tag{11-136b}$$

In a like manner the density change across the shock can be obtained from Eqs. (11-131b) and (11-136b), as

$$\frac{\rho_2}{\rho_1} = \frac{\left(\dfrac{\kappa + 1}{\kappa - 1}\right)\left(\dfrac{2\kappa}{\kappa + 1} M_1^2 - \dfrac{\kappa - 1}{\kappa + 1}\right) + 1}{\dfrac{\kappa + 1}{\kappa - 1} + \dfrac{2\kappa}{\kappa + 1} M_1^2 - \dfrac{\kappa - 1}{\kappa + 1}} = \frac{(\kappa + 1)M_1^2}{(\kappa - 1)M_1^2 + 2} \tag{11-137}$$

The expression for temperature change across the normal shock may be obtained from Eqs. (11-136b) and (11-137),

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} = \left(\frac{2\kappa}{\kappa + 1} M_1^2 - \frac{\kappa - 1}{\kappa + 1}\right)\left(\frac{\kappa - 1}{\kappa + 1} M_1^2 + \frac{2}{\kappa + 1}\right)\frac{1}{M_1^2}$$

$$= \frac{1}{(\kappa+1)^2 M_1^2} [2\kappa M_1^2 - (\kappa - 1)][(\kappa - 1)M_1^2 + 2] \qquad (11\text{-}138)$$

Finally the relative increase in pressure across the shock (a measure of the shock strength) can be derived from Eq. (11–136b) as follows:

$$\frac{p_2 - p_1}{p_1} = \frac{2\kappa}{\kappa + 1}(M_1^2 - 1) \qquad (11\text{-}139)$$

It is seen from Eq. (11–133b) that if $u_1 > a_c$ ($a_1 < a_c$ from Bernoulli's equation), then $u_2 < a_c$ ($a_2 > a_c$). It follows that if the flow is supersonic in front of a normal shock wave ($M_1 > 1$), then the flow behind the shock must be subsonic ($M_2 < 1$). In this case, it is apparent that the pressure increases across a normal shock wave. However, it is physically impossible to transform a subsonic flow to a supersonic flow through a normal shock wave. Before this statement can be verified the variation of entropy across a normal shock must be studied.

It should be pointed out that, according to Eqs. (11–136b) and (11–137), the static pressure ratio across a normal shock becomes infinite as the Mach number in front of the shock reaches infinity. However, the density ratio $\rho_2/\rho_1 \to (\kappa + 1)/(\kappa - 1)$ as $M_1 \to \infty$ (for air with $\kappa = 1.405$, $\rho_2/\rho_1 \to 6$ as $M_1 \to \infty$). The relation between the flow parameters in front of and behind a normal shock wave as a function of the initial Mach number is shown graphically in Fig. 11.20. The relation of the pressure across a normal shock as compared to the isentropic case is shown in Fig. 11.21.

b. Entropy Change Across the Shock

The relation between the Mach number in front of and behind a normal shock wave as given in Eq. (11–135) can be written in the form

$$M_2^2 = \frac{(\kappa - 1)M_1^2 + 2}{(2M_1^2 - 1)\kappa + 1} \qquad (11\text{-}140)$$

It can easily be seen from this equation that $M_2 < 1$ if $M_1 > 1$ and $M_2 > 1$ if $M_1 < 1$. We are now to show that the transformation of a subsonic flow in front of a normal shock into a supersonic flow behind the shock is physically impossible although mathematically correct, according to Eq. (11–140).

The entropy relation given in Eq. (B-41) may be applied for the entropy change across a normal shock wave. If the pressure and density ratios in Eqs. (11–136b) and (11–137) are used, the entropy equation becomes

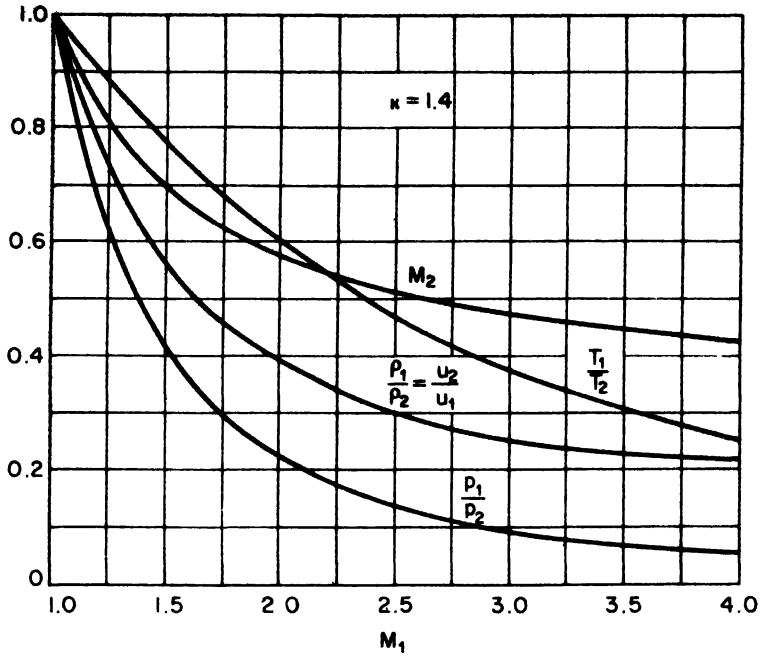


Figure 11.20

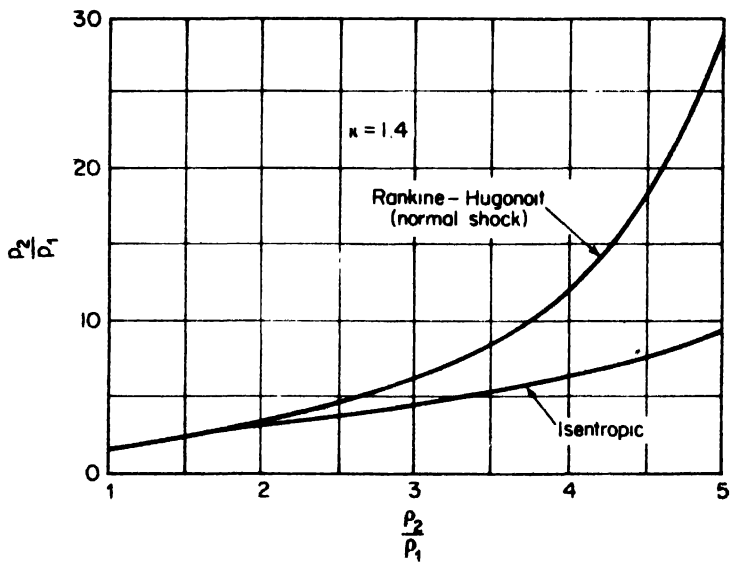


Figure 11.21

$$S_2 - S_1 = C_v \ln \frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^\kappa = \frac{R}{\kappa - 1} \ln \left[\frac{2\kappa}{\kappa + 1} M_1^2 - \frac{\kappa - 1}{\kappa + 1}\right]\left[\frac{(\kappa - 1)M_1^2 + 2}{(\kappa + 1)M_1^2}\right]^\kappa \tag{11-141}$$

Equation (11-141) relates the change of entropy across a normal shock and the Mach number in front of the shock for any given gas. If we put $M_1^2 = 1 + \varepsilon$ where $\varepsilon \ll 1$ the logarithm expressions in Eq. (11-141) can be expanded in a series of powers of ε. In this way a simple expression for entropy change will be obtained for M_1 near unity (either supersonic or subsonic flow initially). Rewriting Eq. (11-141) in the form

$$\begin{aligned} S_2 - S_1 &= \frac{R}{\kappa - 1} \ln \left[\frac{2\kappa}{\kappa + 1}(1 + \varepsilon) - \frac{\kappa - 1}{\kappa + 1}\right]\left[\frac{(\kappa - 1)(1 + \varepsilon) + 2}{(\kappa + 1)(1 + \varepsilon)}\right]^\kappa \\ &= \frac{R}{\kappa - 1}\left[\ln\left(1 + \frac{2\kappa}{\kappa + 1}\varepsilon\right) + \kappa \ln\left(1 + \frac{\kappa - 1}{\kappa + 1}\varepsilon\right) - \kappa \ln(1 + \varepsilon)\right] \end{aligned} \tag{11-142}$$

and expanding in a series of powers of ε:

$$S_2 - S_1 = \frac{2\kappa R}{(\kappa + 1)^2} \frac{\varepsilon^3}{3} + \cdots = \frac{2\kappa R}{(\kappa + 1)^2} \frac{(M_1^2 - 1)^3}{3} + \cdots \tag{11-143}$$

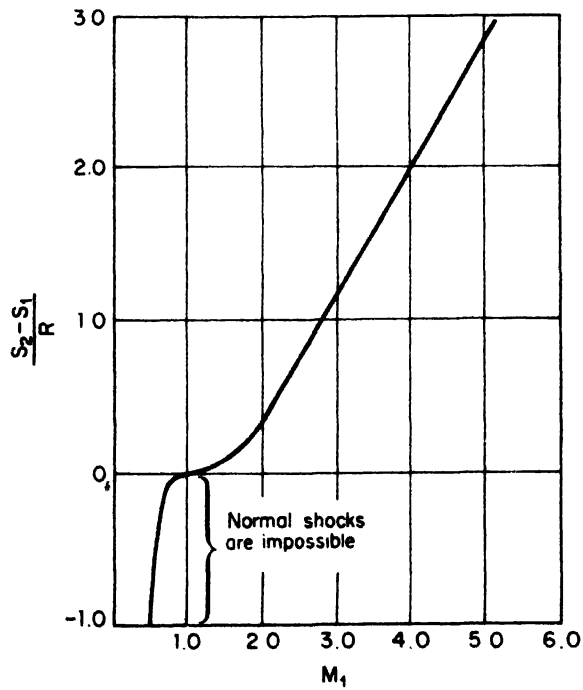


Figure 11.22

According to Eq. (11-143) there is a decrease in entropy across a normal shock if the Mach number in front of it is less than unity. This phenomenon is clearly in contradiction with the second law of thermodynamics when an adiabatic process is considered. It follows then that an abrupt expansion cannot take place in a perfect gas. Furthermore, Eq. (11-143) indicates that the change of state through a shock wave is irreversible as the entropy always increases. Figure 11.22 shows the entropy change in terms of the Mach number before the shock for $\kappa = 1.4$.

11.17. Oblique Shock Waves

a. Governing Equations and Basic Relatives

In Fig. 11.23 is shown a photograph of the flow over a 15° wedge at a free-stream Mach number of 1.9 obtained by the Schlieren method in which the oblique shock wave is clearly indicated at the leading edge of the body. Figure 11.24a is an enlarged view of the leading edge of the body shown in Fig. 11.23. The coordinate system

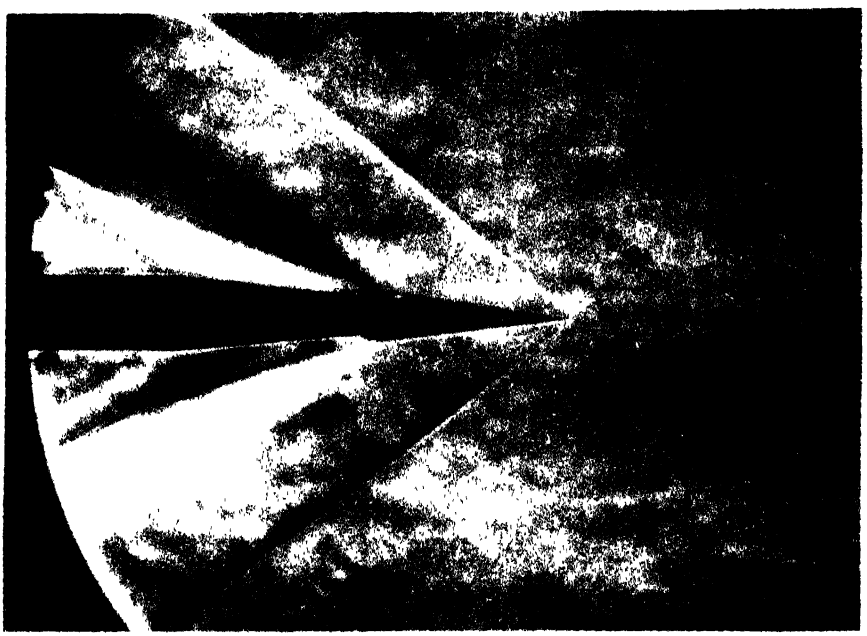


Figure 11.23. Courtesy of Air Force Flight Dynamics Laboratories, Wright-Patterson AFB, Ohio

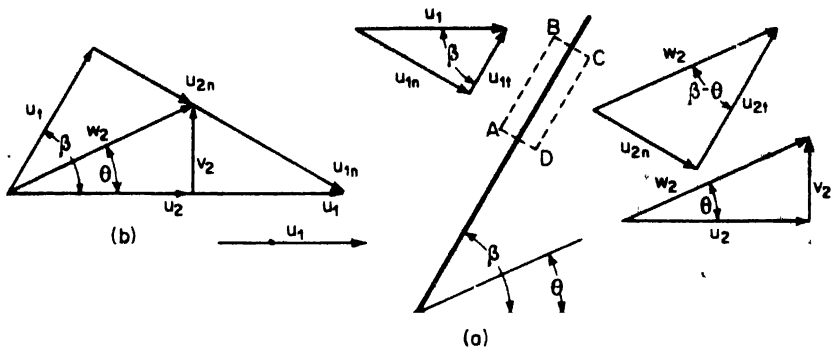


Figure 11.24

now is attached to the body so that it appears that the shock wave is stationary in a moving fluid. It can be seen from Fig. 11.24a that the shock wave makes an acute angle β with respect to the stream in front of the shock, hence it is called an oblique shock wave. When $\beta = \pi/2$ the shock reduces to a normal shock wave. The angle θ is called the deflection angle or the wedge angle.

Let the known flow parameters in front of the oblique shock wave be p_1, ρ_1, u_1 and the unknown flow parameters behind the shock be p_2, ρ_2, w_2 (Fig. 11.24a). Referring to Fig. 11.24b the components of the velocity w_2 may be expressed in terms of w_2 and θ as follows:

$$u_2 = w_2 \cos\theta, \qquad v_2 = w_2 \sin\theta, \qquad = \tan\theta \tag{11-144}$$

Furthermore, the velocity components along paths normal and tangent to the oblique shock wave are related to their respective velocities in front of and behind the shock in the forms

$$u_{1n} = u_1 \sin\beta \qquad u_{1t} = u_1 \cos\beta \tag{11-145a}$$

ahead of the wave, and

$$u_{2n} = w_2 \sin(\beta - \theta) = u_2 \sin\beta - v_2 \cos\beta \tag{11-145b}$$

$$u_{2t} = w_2 \cos(\beta - \theta) = u_2 \cos\beta + v_2 \sin\beta \tag{11-145c}$$

behind the wave. Our present problem is to determine the four unknown parameters, p_2, ρ_2, w_2, and θ or β. In order to establish the four required equations for this case we will follow the discussion given for the normal shock relations and apply them to the control volume *ABCD* at the shock wave. The continuity equation for the normal shock may be used here if the components of velocity u_1 and u_2 are replaced by u_{1n} and u_{2n}, respectively, i.e. (see Fig. 11.24a),

$$\rho_1 u_{1n} = \rho_2 u_{2n} \tag{11-146a}$$

or

$$\rho_1 u_1 \sin\beta = \rho_2 w_2 \sin(\beta - \theta) = \rho_2(u_2 \sin\beta - v_2 \cos\beta) \qquad (11\text{–}146b)$$

Since the momentum is a vector quantity, there will be a momentum equation along the normal to the oblique shock as well as along the tangent to the shock. Along the normal to the shock the momentum equation is obtained after Eq. (11–127b),

$$p_1 + \rho_1 u_{1n}^2 = p_2 + \rho_2 u_{2n}^2 \qquad (11\text{–}147a)$$

or

$$p_1 + \rho_1 u_1^2 \sin^2\beta = p_2 + \rho_2 w_2^2 \sin^2(\beta - \theta)$$
$$= p_2 + \rho_2(u_2 \sin\beta - v_2 \cos\beta)^2 \qquad (11\text{–}147b)$$

The momentum equation along the tangent to the shock requires further explanation. According to Eq. (8–7) the net rate of transport of momentum per unit area is equal to the shearing stress and since there exists no shearing stresses in this case, we have

$$\rho_1 u_{1n} u_{1t} = \rho_2 u_{2n} u_{2t} \qquad (11\text{–}148a)$$

or

$$\rho_1 u_1^2 \sin\beta\cos\beta = \rho_2 w_2^2 \sin(\beta - \theta)\cos(\beta - \theta)$$
$$= \rho_2(u_2 \sin\beta - v_2 \cos\beta)(u_2 \cos\beta + v_2 \sin\beta) \qquad (11\text{–}148b)$$

The energy equation for the oblique shock wave is identical to that for the normal shock except the velocity behind the oblique shock is w_2 instead of u_2. Replacing u_2 by w_2 in Eq. (11–128b), the energy equation becomes

$$\frac{\kappa}{\kappa - 1}\frac{p_1}{\rho_1} + \frac{1}{2}u_1^2 = \frac{\kappa}{\kappa - 1}\frac{p_2}{\rho_2} + \frac{1}{2}w_2^2 = \frac{\kappa}{\kappa - 1}\frac{p_0}{\rho_0} \qquad (11\text{–}149)$$

Equations (11–146)–(11–149) constitute the governing equations which express the properties of the gas behind the shock in terms of the gas properties in front of it. If we combine Eqs. (11–146a) and (11–148a) we find that the tangential velocity components in front of and behind the shock are equal, i. e.,

$$u_{1t} = u_{2t} = u_t \qquad (11\text{–}150)$$

Equation (11–150) satisfies the physical condition that there exists no shearing stress through the shock. Another useful relation can be obtained from Eq. (11–150),

$$u_1 \cos\beta = w_2 \cos(\beta - \theta) = u_2 \cos\beta + v_2 \sin\beta \qquad (11\text{–}151a)$$

from which the wave angle β is expressed as a function of velocity,

$$\tan\beta = \frac{u_1 - u_2}{v_2} \qquad (11\text{–}151b)$$

With the relation of Eq. (11–150) we can rewrite the energy equation in terms of the normal components of the velocities; hence the basic equations reduce to the forms of the equations for a normal shock wave as follows :

$$\rho_1 u_{1n} = \rho_2 u_{2n} \tag{11–152a}$$

$$p_1 + \rho_1 u_{1n}^2 = p_2 + \rho_2 u_{2n}^2 \tag{11–152b}$$

$$\frac{\kappa}{\kappa - 1}\frac{p_1}{\rho_1} + \frac{1}{2}u_{1n}^2 = \frac{\kappa}{\kappa - 1}\frac{p_2}{\rho_2} + \frac{1}{2}u_{2n}^2 = \frac{\kappa}{\kappa - 1}\frac{p_0}{\rho_0} - \frac{u_t^2}{2} \tag{11–152c}$$

It is seen that Eqs. (11–152) are identical to the corresponding equations for a normal shock wave, i. e., Eqs. (11–126), (11–127b), and (11–128b) ; hence the Rankine-Hugoniot relations [Eqs. (11–131a) and (11–131b)] can be used for the oblique shock wave. The other relations for a normal shock may be used if M_1 and M_2 in the normal shock equations are replaced by the following equivalences, respectively,

$$M_{1n} = \frac{u_{1n}}{a_1} = \frac{u_1}{a_1}\sin\beta = M_1 \sin\beta \tag{11–153a}$$

$$M_{2n} = \frac{u_{2n}}{a_2} = \frac{w_2}{a_2}\sin(\beta - \theta) = M_2 \sin(\beta - \theta) \tag{11–153b}$$

Hence, the pressure, density, and temperature ratios across an oblique shock may be written by referring to Eqs. (11–136b), (11–137), and (11–138), respectively,

$$\frac{p_2}{p_1} = \frac{2\kappa}{\kappa + 1} M_1^2 \sin^2\beta - \frac{\kappa - 1}{\kappa + 1} \tag{11–154}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\kappa + 1)M_1^2 \sin^2\beta}{(\kappa - 1)M_1^2 \sin^2\beta + 2} \tag{11–155}$$

and

$$\frac{T_2}{T_1} = \frac{[2\kappa M_1^2 \sin^2\beta - (\kappa - 1)][(\kappa - 1)M_1^2 \sin^2\beta + 2]}{(\kappa + 1)^2 M_1^2 \sin^2\beta} \tag{11–156}$$

Similarly, the relative increase in pressure across an oblique shock is obtained from Eq. (11–139),

$$\frac{p_2 - p_1}{p_1} = \frac{2\kappa}{\kappa + 1}(M_1^2 \sin^2\beta - 1) \tag{11–157}$$

The relation between M_{1n} and M_{2n} of an oblique shock may be written from Eq. (11–135),

$$\left(\frac{\kappa - 1}{2} + \frac{1}{M_1^2 \sin^2\beta}\right)\left[\frac{\kappa - 1}{2} + \frac{1}{M_2^2 \sin^2(\beta - \theta)}\right] = \left(\frac{\kappa + 1}{2}\right)^2 \tag{11–158}$$

Finally the entropy change across an oblique shock corresponding to Eq. (11–141) for the normal shock may be written as

$$S_2 - S_1 = \frac{R}{\kappa - 1} \ln \left[\frac{2\kappa}{\kappa + 1} M_1^2 \sin^2 \beta - \frac{\kappa - 1}{\kappa + 1} \right] \left[\frac{(\kappa - 1)M_1^2 \sin^2 \beta + 2}{(\kappa + 1)M_1^2 \sin^2 \beta} \right]^{\kappa} \tag{11-159}$$

According to the entropy consideration [Eq. (11-141)], it was noted that a normal shock wave could exist only if $M_1 > 1.0$. Following the same analogy the existence of an oblique shock requires that

$$M_1 \sin \beta > 1$$

On the other hand, an oblique shock becomes a normal one when $\beta = \pi/2$. Hence, for a supersonic flow in front of an oblique shock the range of possible wave angles is

$$\sin^{-1} \frac{1}{M_1} < \beta < \frac{\pi}{2} \tag{11-160}$$

b. Relation Between β and θ

For given values of M_1 and β the characteristics of the flow behind an oblique shock wave can be determined from Eqs. (11-154)–(11-159) for any specified gas. In these cases the deflection angle θ has to be determined. On the other hand, for given values of M_1 and θ the wave angle β must first be determined before these equations can be used to determine the flow properties behind the shock. For this reason it is essential that an implicit relation between β and θ be found. This relation can be obtained from the governing equations for an oblique shock.

According to Eq. (11-152c) we may write

$$u_{1n} + \frac{2\kappa}{\kappa - 1} \frac{p_1}{\rho_1 u_{1n}} = \frac{1}{u_{1n}} \left(\frac{2\kappa}{\kappa - 1} \frac{p_0}{\rho_0} - u_t^2 \right) = \frac{1}{u_{1n}} \left(\frac{2a_0^2}{\kappa - 1} - u_t^2 \right)$$

$$u_{2n} + \frac{2\kappa}{\kappa - 1} \frac{p_2}{\rho_2 u_{2n}} = \frac{1}{u_{2n}} \left(\frac{2a_0^2}{\kappa - 1} - u_t^2 \right)$$

The difference of these two equations yields

$$u_{1n} - u_{2n} + \frac{2\kappa}{\kappa - 1} \left(\frac{p_1}{\rho_1 u_{1n}} - \frac{p_2}{\rho_2 u_{2n}} \right) = \left(\frac{2a_0^2}{\kappa - 1} - u_t^2 \right) \left(\frac{u_{2n} - u_{1n}}{u_{1n} u_{2n}} \right) \tag{11-161a}$$

The Bernoulli equation [Eq. (11-80)] may be written in the form

$$\frac{2a_0^2}{\kappa - 1} - u_t^2 = \left(\frac{2a_1^2}{\kappa - 1} + u_1^2 \right) - u_t^2 = \frac{2a_1^2}{\kappa - 1} + u_{1n}^2 \tag{11-161b}$$

and Eq. (11-152b) gives

$$u_{1n} - u_{2n} = \frac{p_2}{\rho_2 u_{2n}} - \frac{p_1}{\rho_1 u_{1n}} \tag{11-161c}$$

Substituting Eqs. (11-161b) and (11-161c) into Eq. (11-161a), we have

$$u_{1n} - u_{2n} - \frac{2\kappa}{\kappa - 1}(u_{1n} - u_{2n}) = -\left(\frac{2a_1^2}{\kappa - 1} + u_{1n}^2\right)\left(\frac{u_{1n} - u_{2n}}{u_{1n}u_{2n}}\right)$$

or

$$a_1^2 = -\frac{\kappa - 1}{2}u_{1n}^2 + \frac{\kappa + 1}{2}u_{1n}u_{2n} \tag{11-162}$$

where the expression for u_{2n} can be obtained from Eqs. (11-145b) and (11-151a) as follows:

$$\begin{aligned} u_{2n} &= w_2 \sin(\beta - \theta) \\ &= \frac{u_1 \cos\beta}{\cos(\beta - \theta)} \sin(\beta - \theta) = u_1 \cos\beta \tan(\beta - \theta) \end{aligned} \tag{11-163}$$

With the expressions for u_{1n} from Eq. (11-145a) and u_{2n} from Eq. (11-163), Eq. (11-162) gives the relation between β and θ in terms of M_1^2 and κ.

$$a_1^2 = -\frac{\kappa - 1}{2}u_1^2 \sin^2\beta + \frac{\kappa + 1}{2}u_1^2 \sin\beta \cos\beta \tan(\beta - \theta)$$

or

$$\begin{aligned} \frac{1}{M_1^2} &= \sin^2\beta - \frac{\kappa + 1}{2}\frac{\sin\beta \sin\theta}{\cos(\beta - \theta)} \\ &= \sin^2\beta - \frac{\kappa + 1}{2}\left(\frac{1}{1 + \cot\beta \cot\theta}\right) \end{aligned} \tag{11-164a}$$

Equation (11-164a) may be expressed in two useful forms:

$$\frac{2}{\kappa + 1}\left(\sin^2\beta - \frac{1}{M_1^2}\right) = \frac{1}{1 + \cot\theta \cot\beta} \tag{11-164b}$$

and

$$\cot\theta = \left(\frac{\kappa + 1}{2}\frac{M_1^2}{M_1^2 \sin^2\beta - 1} - 1\right)\tan\beta \tag{11-164c}$$

The angle $(\beta - \theta)$, between the tangent to the shock and the direction of the velocity w_2, can also be expressed in terms of M_1 and β. From the continuity equation [Eq. (11-152a)] and Eqs. (11-145a) and (11-163),

$$\frac{\rho_1}{\rho_2} = \frac{u_{2n}}{u_{1n}} = \frac{\tan(\beta - \theta)}{\tan\beta} \tag{11-165}$$

Substituting Eq. (11-155) for ρ_1/ρ_2 into Eq. (11-165), we obtain the relation

$$\tan(\beta - \theta) = \frac{2\tan\beta}{\kappa + 1}\left(\frac{1}{M_1^2 \sin^2\beta} + \frac{\kappa - 1}{2}\right) \tag{11-166}$$

The velocity components behind an oblique shock can now be determined in terms of M_1^2 and β by utilizing Eqs. (11–151a), (11–144), and (11-164b):

$$\frac{u_2}{u_1} = \frac{u_2}{u_2 + v_2 \tan\beta} = \frac{1}{1 + \tan\theta \tan\beta} \tag{11-167a}$$

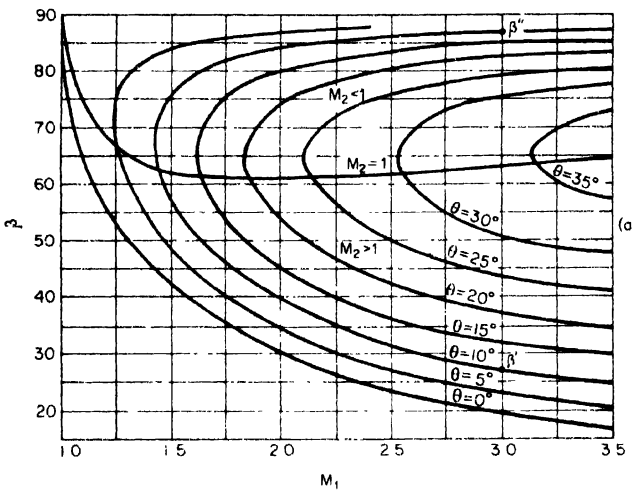
but

$$1 - \frac{1}{1 + \tan\theta \tan\beta} = \frac{1}{1 + \cot\theta \cot\beta}$$

it follows that

$$\frac{u_2}{u_1} = 1 - \frac{1}{1 + \cot\theta \cot\beta} = 1 - \frac{2}{\kappa + 1}\left(\sin^2\beta - \frac{1}{M_1^2}\right) \tag{11-167b}$$

Similarly, we have



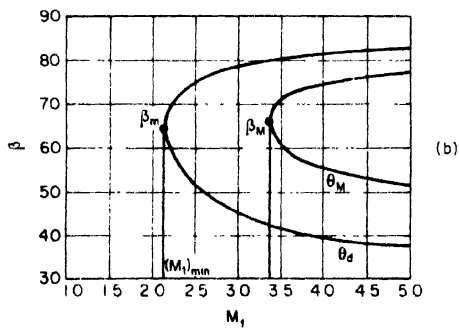


Figure 11.25

$$\frac{v_2}{u_1} = \frac{v_2}{u_2 + v_2 \tan\beta} = \frac{\tan\theta}{1 + \tan\theta\tan\beta}$$

$$= \left(1 - \frac{1}{1 + \tan\theta\tan\beta}\right)\cot\beta = \frac{2}{\kappa + 1}\left(\sin^2\beta - \frac{1}{M_1^2}\right)\cot\beta \tag{11-167c}$$

The relation between β and θ given in Eq. (11-164c) may be used to calculate θ when β and M_1 are given. On the other hand, for given values of M_1 and θ the solution of β cannot be obtained analytically. However, β can be determined graphically as shown in Fig. 11.25a. The relation between M_1 and β is plotted for various values of θ on Fig. 11.25a; hence there exists a network of β vs. M_1 curves, each of which corresponds to a specifie value of θ. For a given value of M_1 it can be seen from Fig. 11.25a that, in general, there are two possible values of β; however, there exists one curve for which only one value of β is found (the vertical line of M_1 tangent to the curve). The latter case corresponds to a maximum value of θ or θ_{max} and the large value of β in the former case gives the strong shock and the lower value of β, the weak shock. These two cases of shock waves are divided by the line of $M_2 = 1.0$ as the portion below this line is in the supersonic region (weak shock) and above it the subsonic region (strong shock).

Illustrative Example 11.6. Determine the wave angles, Mach numbers behind the oblique shocks, and the pressure ratios across the oblique shocks for $M_1 = 3.0$ and $\theta = 10°$.

Solution. For $M_1 = 3.0$ and $\theta = 10°$ we have from Fig. 11.25a, $\beta' = 27° 23'$ and $\beta'' = 86° 25'$. The pressure ratio across the shock and the Mach number behind the shock can be obtained from Eqs. (11-154) and (11-158), respectively. They are

$$\frac{p_2'}{p_1} = 2.055, \quad M_2' = 2.5; \qquad \frac{p_2''}{p_1} = 10.29, \quad M_2'' = 0.49$$

The results with prime superscripts correspond to the weak shock (supersonic flow behind the wave), and the results with double prime superscripts correspond to a strong shock (subsonic flow behind the wave). Hence, for given values of M_1 and θ, the flow behind an oblique shock wave can be either supersonic or subsonic (see shock polar for further discussion).

c. Formation of Detached Shock Waves

Referring to Fig. 11.25a we see that, in general, a vertical line $M_1 =$ Constant intersects the curve $\theta =$ Constant at two points, β' and

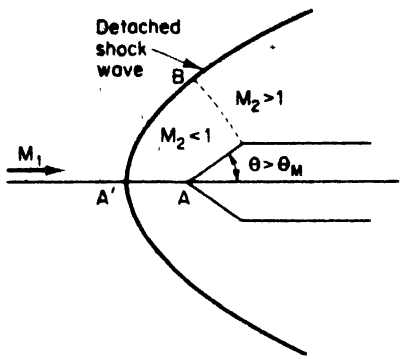


Figure 11.26

β''; however, it is tangent to only one curve, $\theta = \theta_M$ = Constant. The latter phenomenon is shown in Fig. 11.25b where β_M is the intersection of the line M_1 = Constant and $\theta = \theta_M$ = Constant. For a given value of M_1, θ_M represents the maximum deflection angle for which the shock remains attached to the vertex of the body. When $\theta > \theta_M$ for a given M_1 the shock wave will be detached from the vertex of the body. This wave is known as a detached shock wave (Fig. 11.26).

On the other hand, for a given curve $\theta = \theta_d$ = Constant there will, in general, be two points of intersection with any vertical line M_1 = Constant. There is, however, only one line $(M_1)_{max}$ = Constant which would intersect at the point β_m with the given curve $\theta = \theta_d$ = Constant (Fig. 11.25b). For any given value of M_1 smaller than $(M_1)_{max}$, there will be no intersection between the line M_1 = Constant, and the curve $\theta = \theta_d$ = Constant. In this case we again have a detached shock wave.

In order to determine the maximum deflection angle θ_M for a given M_1, we may first determine the angle β_M corresponding to θ_M by differentiating Eq. (11-164c) with respect to β and setting it equal to zero. It yields

$$\frac{d(\cot\theta)}{d\beta} = \sec^2\beta_M\left[\frac{\kappa+1}{2}\frac{M_1^2}{M_1^2\sin^2\beta_M - 1} - 1\right] + \tan\beta_M\left\{\frac{\kappa+1}{2}M_1^2\left[\frac{-2M_1^2\sin\beta_M\cos\beta_M}{(M_1^2\sin^2\beta_M - 1)^2}\right]\right\} = 0$$

from which

$$\frac{\kappa+1}{2}\frac{M_1^2}{M_1^2\sin^2\beta_M - 1}\left[1 - \frac{2M_1^2\sin\beta_M\cos\beta_M}{M_1^2\sin^2\beta_M - 1}\right] - 1 = 0 \qquad (11\text{-}168)$$

Simplifying Eq. (11-168) we obtain a quadratic equation in $\sin^2\beta_M$,

$$\sin^2\beta_M = \frac{1}{\kappa M_1^2}\left[\frac{\kappa+1}{4}M_1^2 - 1 + \sqrt{(\kappa+1) + \frac{\kappa^2-1}{2}M_1^2 + \frac{(\kappa+1)^2}{16}M_1^4}\right] \qquad (11\text{-}169)$$

Substituting the value of β_M determined from Eq. (11-169) into Eq. (11-164c), the maximum deflection angle θ_M is found as follows:

$$\cot\theta_M = \left[\frac{\kappa+1}{2}\frac{M_1^2}{M_1^2\sin^2\beta_M - 1} - 1\right]\tan\beta_M \qquad (11\text{-}170)$$

Since the value of θ_M increases with the increase of the Mach number M_1, it is interesting to find that there exists a maximum value of θ_M corresponding to $M_1 \to \infty$. For $M_1 \to \infty$, Eq. (11-169) gives

$$\sin^2 \beta_M = \frac{\kappa + 1}{2\kappa}$$

With this value substituted for $\sin^2 \beta_M$, Eq. (11-170) yields

$$\cot \theta_M = (\kappa - 1) \tan \beta_M = (\kappa - 1) \sqrt{\frac{\sin^2 \beta_M}{\cos^2 \beta_M}} = \sqrt{(\kappa - 1)(\kappa + 1)} \tag{11-171}$$

and for air, $\theta_M = 45.22°$ (for $M_1 \to \infty$). Above this value of θ_M the shock wave is always detached regardless of the value of M_1. Hence, if the vertex of a body is blunt, the detached shock wave always occurs even in hypersonic flight.

The second condition which leads to the formation of detached shock waves is determined by the minimum value of M_1 for a given deflection angle θ_d. Taking the derivative of Eq. (11-164a) with respect to β and equating it to zero, we have

$$\frac{d(1/M_1^2)}{d\beta} = 2 \sin \beta_m \cos \beta_m + \frac{\kappa + 1}{2} \left[\frac{-\cot \theta_d \csc^2 \beta_m}{(1 + \cot \theta_d \cot \beta_m)^2} \right] = 0$$

from which

$$\sin 2\beta_m (\sin \beta_m + \cot \theta_d \cos \beta_m)^2 = \frac{\kappa + 1}{2} \cot \theta_d \tag{11-172}$$

In terms of the angle $2\beta_m$, Eq. (11-172) results in the form

$$\sin 2\beta_m [1 + \sin 2\theta_d \sin 2\beta_m + \cos 2\theta_d \cos 2\beta_m] = \frac{\kappa + 1}{2} \sin 2\theta_d \tag{11-173}$$

The solution of Eq. (11-173) yields the value of β_m in terms of a given θ_d. With this value of β_m the minimum value $(M_1)_{min}$, below which a detached shock will occur, can be obtained from Eq. (11-164a) as

$$\frac{1}{(M_1^2)_{min}} = \sin^2 \beta_m - \frac{\kappa + 1}{2} \frac{1}{1 + \cot \theta_d \cot \beta_m} \tag{11-174}$$

Experimental observation reveals that a curved shock wave which is detached from the wedge has a shape as sketched in Fig. 11.26. In the immediate vicinity of the point A' the shock wave is normal or nearly normal to the stream, and the flow is subjected to compression producing a subsonic condition behind the wave. The subsonic flow in the region between points A' and A is then accelerated in the downstream direction. Hence, the pressure of the downstream flow decreases with the distance from the region A'-A. The shock becomes weaker and less steep with the increase of distance from A'. Eventually the

shock wave becomes a Mach wave at B (Fig. 11.26) and the flow behind it becomes supersonic. It is seen that the wave characteristics along the detached wave vary from the normal shock wave to the oblique shock wave (tending to a Mach wave far out from the body); thus various wave angles are experienced with a single detached shock wave. This characteristic leads to great difficulty in the quest for analytical solutions.

11.18. Shock Wave Polar

In §11.17 we have discussed that oblique shock waves are completely determined if the values u_1 and β or u_1 and θ are specified. Since

$$\tan\theta = \frac{v_2}{u_2} \tag{11-144}$$

it is equivalent to say that if u_1 and u_2 are specified the wave is also completely determined. However, we still need to find v_2 as a function of u_1 and u_2, and this may be accomplished by elimination of p_2, ρ_2, and β from Eqs. (11-146), (11-147), (11-148), and (11-149).

From Eqs. (11-147b) and (11-146b), we obtain

$$\begin{aligned} p_2 &= p_1 + \rho_1 u_1^2 \sin^2\beta - \rho_2(u_2 \sin\beta - v_2\cos\beta)^2 \\ &= p_1 + \rho_1 u_1 \sin\beta[(u_1 - u_2)\sin\beta + v_2\cos\beta] \end{aligned} \tag{11-175}$$

Substituting the expression for v_2 from Eq. (11-151b) into Eq. (11-175) gives

$$p_2 = p_1 + \rho_1 u_1(u_1 - u_2) \tag{11-176}$$

Now β can be eliminated from Eqs. (11-146b) and (11-151b),

$$\rho_1 u_1 = \rho_2 u_2 - \rho_2 v_2 \cot\beta = \rho_2 u_2 - \rho_2 \frac{v_2^2}{u_1 - u_2}$$

or

$$\rho_1 u_1(u_1 - u_2) = \rho_2[u_2(u_1 - u_2) - v_2^2] \tag{11-177}$$

The energy equation given in Eq. (11-149) can be rewritten in terms of the critical velocity a_c in the form

$$\frac{\kappa}{\kappa - 1}\frac{p_1}{\rho_1} + \frac{1}{2}u_1^2 = \frac{\kappa}{\kappa - 1}\frac{p_2}{\rho_2} + \frac{1}{2}(u_2^2 + v_2^2) = \frac{\kappa + 1}{\kappa - 1}\frac{a_c^2}{2} \tag{11-178}$$

from which

$$\frac{p_1}{\rho_1} = \frac{\kappa + 1}{2\kappa}a_c^2 - \frac{\kappa - 1}{2\kappa}u_1^2 \tag{11-179a}$$

and

$$\frac{p_2}{\rho_2} = \frac{\kappa + 1}{2\kappa}a_c^2 - \frac{\kappa - 1}{2\kappa}(u_2^2 + v_2^2) \tag{11-179b}$$

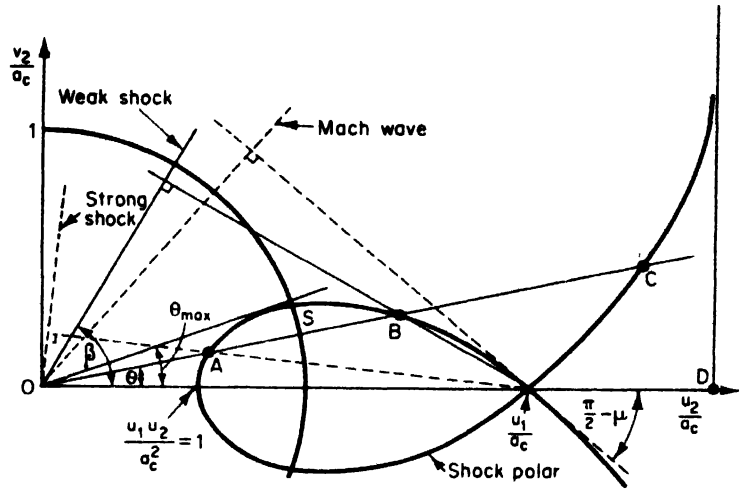


Figure 11.27

Substituting Eqs. (11-179a) and (11-179b) into Eq. (11-178) we obtain the following equation in which p_2 and p_1 are eliminated,

$$\frac{\kappa+1}{2\kappa}a_c^2 - \frac{\kappa-1}{2\kappa}(u_2^2 + v_2^2) = \left[\frac{\kappa+1}{2\kappa}a_c^2 - \frac{\kappa-1}{2\kappa}u_1^2 + u_1(u_1 - u_2)\right]\frac{\rho_1}{\rho_2} \tag{11-180}$$

The density ratio ρ_1/ρ_2 in Eq. (11-180) can be eliminated with the aid of Eq. (11-177), giving

$$(\kappa+1)a_c^2 - (\kappa-1)u_2^2 - (\kappa-1)v_2^2 = (\kappa+1)a_c^2\frac{u_2}{u_1} - (\kappa+1)\frac{a_c^2 v_2^2}{u_1(u_1-u_2)}$$
$$-(\kappa-1)u_1u_2 + (\kappa-1)\frac{u_1 v_2^2}{u_1-u_2} + 2\kappa u_2(u_1-u_2) - 2\kappa v_2^2$$

Finally, v_2^2 is obtained in terms of u_1, u_2, and a_c:

$$v_2^2 = \frac{(u_1-u_2)^2(u_1u_2 - a_c^2)}{\frac{2u_1^2}{\kappa+1} - u_1u_2 + a_c^2} \tag{11-181a}$$

or in nondimensional form,

$$\left(\frac{v_2}{a_c}\right)^2 = \frac{\left(\frac{u_1}{a_c} - \frac{u_2}{a_c}\right)^2\left(\frac{u_1u_2}{a_c^2} - 1\right)}{\frac{2}{\kappa+1}\left(\frac{u_1}{a_c}\right)^2 - \frac{u_1u_2}{a_c^2} + 1} \tag{11-181b}$$

Equation (11-181b) may be used to construct a shock polar diagram. If the values of u_1 and a_c (or u_1/a_c) are given, the relation between u_2/a_c and v_2/a_c can be formed from Eq. (11-181b). A plot of u_2/a_c vs. v_2/a_c for a given value of u_1/a_c is known as the shock polar. The typical results of a single shock polar (u_1/a_c = Constant) are shown

in Fig. 11.27. The effectiveness and simplicity of the shock polar in the interpretation and solution of the characteristics of the flow behind the shock wave can be seen from the following properties:

(a) The intersection of the shock polar with any ray from the origin O specifies the values of u_2/a_c and v_2/a_c for any given deflection angle θ. In other words, a ray from the origin O making an angle θ with the u_2/a_c axis (abscissa) cuts the shock polar at three points, i.e., A, B, and C. The last point C has no physical significance since it implies that $u_2 > u_1$ which represents a decrease in entropy across the shock. Points A and B represent two theoretically possible solutions of the oblique shock. The flow in regions A and B are separated at the point S, the intersection of the sonic circle and the shock polar, where $M_2 = 1$. The subsonic region behind the shock is represented by point A which is associated with the *strong shock* and the supersonic region by point B associated with the *weak shock*. It must be noted that only the condition represented by point B has been found experimentally for flow behind an oblique shock. It is interesting to note that the *weak shock* wave can be obtained by a line drawn through the origin O and perpendicular to the line through the points B and $(u_1/a_c, 0)$. Similarly, the *strong shock* can also be obtained.

(b) The intersections of the shock polar with the u_2/a_c axis give $u_1u_2/a_c^2 = 1$ and $u_1/a_c = u_2/a_c$ which correspond to the normal shock and Mach wave, respectively.

(c) The ray through the origin which is tangent to the shock polar makes the angle θ_{sr} with the u_2/a_c axis. This is the maximum deflection angle, above which the shock wave will be detached from the vertex of the body. In this case the present analysis cannot be applied.

In the construction of the shock polar it must be noted that u_1/a_c is the given parameter instead of $M_1 = u_1/a_1$. Therefore, the relation of u_1/a_c and M_1 is sought. From Bernoulli's equation [Eq. (11–80)] and Eq. (11–82) we have

$$\frac{1}{2}u_1^2 + \frac{a_1^2}{\kappa - 1} = \frac{a_0^2}{\kappa - 1} = \frac{\kappa + 1}{\kappa - 1}\frac{a_c^2}{2}$$

from which the following relations are obtained:

$$\left(\frac{u_1}{a_c}\right)^2 = \frac{[(\kappa + 1)/2]M_1^2}{1 + [(\kappa - 1)/2]M_1^2} \tag{11–182a}$$

and

$$M_1^2 = \frac{\dfrac{2}{\kappa + 1}\left(\dfrac{u_1}{a_c}\right)^2}{1 - \dfrac{\kappa - 1}{\kappa + 1}\left(\dfrac{u_1}{a_c}\right)^2} \tag{11–182b}$$

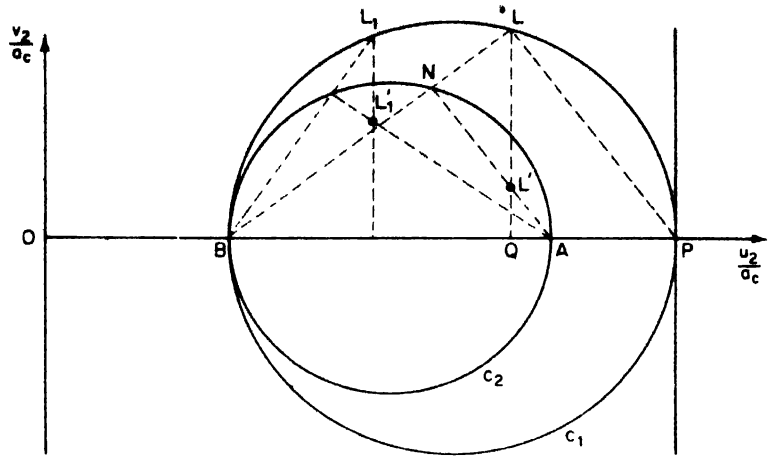


Figure 11.E

According to Eq. (11–182a) when $M_1 \to \infty$ the maximum value for $(u_1/a_c)^2$ is obtained:

$$\left(\frac{u_1}{a_c}\right)^2_{\max} = \frac{\kappa + 1}{\kappa - 1} = 6 \quad \text{for air } (\kappa = 1.4) \tag{11–183}$$

Hence, u_1/a_c for gases reaches a finite value as $M_1 \to \infty$.

Illustrative Example 11.7. Construct a shock polar diagram for a given u_1/a_c.

Solution. If u_1/a_c is given (see Fig. 11.E) we can calculate $\overline{OB} = 1/(u_1/a_c)$ when $v_2/a_c = 0$ and $\overline{OP} = 1/(u_1/a_c) + [2/(\kappa + 1)]\, u_1/a_c$ when $v_2/a_c = \infty$ [see Eq. (11-181b)]. Now two circles c_1 and c_2 may be drawn with diameter $\overline{BP}$ through points B and P and diameter $\overline{BA}$ through points B and A, respectively. From any arbitrary point L on the circle c_1, line $\overline{BL}$ is drawn intersecting circle c_2 at N, and the line connecting points A and N is then perpendicular to $\overline{BL}$ ($\angle BNA$ is opposite to the diameter $\overline{BA}$). Draw line $\overline{LQ}$ perpendicular to the axis $\overline{OP}$. The intersection point L' of lines $\overline{NA}$ and $\overline{LQ}$ is then a point on the required shock polar. Repeat this process for other points on the shock polar.

The geometrical construction above of a shock polar may be proved as follows: If the point L' is a point on the shock polar, the line $\overline{L'Q}$ must be equal to v_2/a_c. From the geometry we have

$$\frac{\overline{L'Q}}{\overline{QA}} = \frac{\overline{BQ}}{\overline{LQ}} \tag{a}$$

therefore,

$$\left(\frac{v_2}{a_c}\right)^2 = \overline{L'Q}^2 = \frac{(\overline{BQ})^2(\overline{QA})^2}{(\overline{LQ})^2} = \frac{(\overline{BQ})^2(\overline{QA})^2}{(\overline{BQ})(\overline{QP})} = (\overline{QA})^2 \frac{\overline{BQ}}{\overline{QP}} \tag{b}$$

But

$$\overline{QA} = \frac{u_1}{a_c} - \frac{u_2}{a_c}, \qquad \overline{BQ} = \overline{OQ} - \overline{OB} = \frac{u_2}{a_c} - \frac{1}{(u_1/a_c)}$$

and

$$\overline{QP} = \overline{OP} - \overline{OQ} = \frac{1}{(u_1/a_c)} + \frac{2}{\kappa + 1}\frac{u_1}{a_c} - \frac{u_2}{a_c} \tag{c}$$

Finally substituting Eq. (c) into Eq. (b) we find:

$$\left(\frac{v_2}{a_c}\right)^2 = \frac{\left(\frac{u_1}{a_c} - \frac{u_2}{a_c}\right)^2 \left(\frac{u_2}{a_c} - \frac{1}{\frac{u_1}{a_c}}\right)}{\frac{1}{\frac{u_1}{a_c}} + \frac{2}{\kappa + 1}\frac{u_1}{a_c} - \frac{u_2}{a_c}} \tag{d}$$

Equation (c) is identical to Eq. (11-181b).

11.19. Compression and Expansion Waves

a. Compression by Turning

In the previous section we have already remarked that deflecting the wall through an inclination angle θ (a sharp concave corner) in a supersonic flow generates an oblique shock of sufficient strength to turn the flow through the deflection angle (see Fig. 11.24a). In crossing this shock wave the flow is compressed and experiences a rapid increase of pressure and consequently a rapid rise in density and temperature as well as a sudden decrease in velocity.

For a weak oblique shock wave and small deflection angle θ, the value of the shock wave angle β is very close to the Mach angle μ which is equal to [see Eq. (11-76)]

$$\tan\beta \doteq \tan\mu = \frac{1}{\sqrt{M_1^2 - 1}} \tag{11-184}$$

With this simple relation of β and M_1, Eq. (11-164a) reduces to the form

$$M_1^2 \sin^2\beta - 1 = \frac{\kappa + 1}{2}\left(\frac{M_1^2 \sin\beta}{\cos\beta}\right)\theta = \frac{\kappa + 1}{2}\left(\frac{M_1^2}{\sqrt{M_1^2 - 1}}\right)\theta \tag{11-185}$$

Hence the approximate expression for the pressure increase across a weak oblique shock is obtained from Eqs. (11-157) and (11-185) as follows:

$$\frac{\Delta p}{p} = \frac{p_2 - p_1}{p_1} = \left(\frac{\kappa M_1^2}{\sqrt{M_1^2 - 1}}\right)\theta \tag{11-186}$$

Equation (11-186) indicates that the strength of the weak oblique shock is directly proportional to the deflection angle.

According to Eqs. (11-159), (11-143), and (11-185), the entropy increase across the weak oblique shock is found to be proportional to the third power of the deflection angle θ

$$\Delta S = S_2 - S_1 \sim \theta^3 \tag{11-187}$$

If we now imagine that the single corner of Fig. 11.28a is replaced by a succession of small corners as illustrated in Fig. 11.28b, i.e., $n\Delta\theta = \theta$, we see that the flow is compressed through successive weak oblique shocks. The flow between consecutive shocks is uniform. The

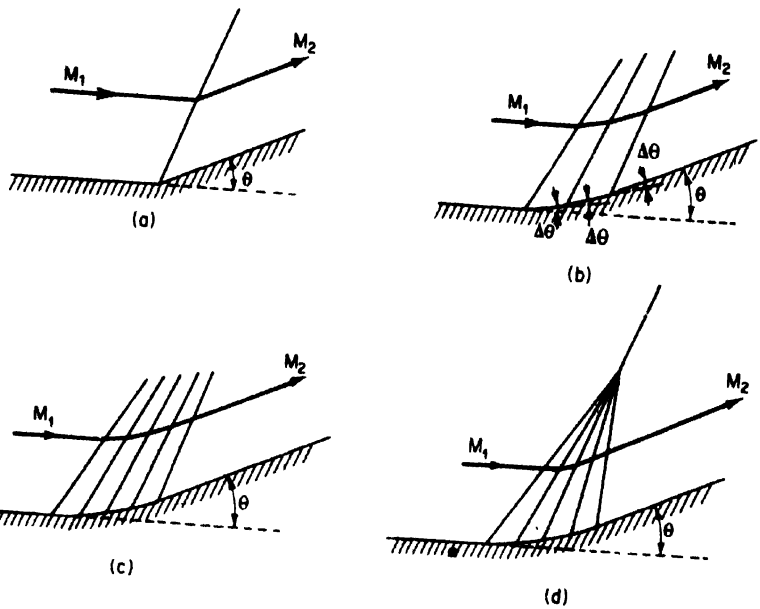


Figure 11.28

pressure and entropy increase across each wave in this case are, respectively,

$$\delta p \sim \Delta\theta$$
$$\delta S \sim (\Delta\theta)^3 \tag{11-188a}$$

and for the complete turn of Fig. 11.28b they are

$$\Delta p \sim n\Delta\theta = \theta$$
$$\Delta S \sim n(\Delta\theta)^3 = \frac{\theta^3}{n^2} \tag{11-188b}$$

If the number of turns in Fig. 11.28b is made infinitely large or the segments vanishingly small, then in the limit we have a smooth curved surface as shown in Fig. 11.28c. In this case the shocks become vanishingly weak and the surface generates an infinite family of Mach waves (only a few are shown), of such a strength as to induce the required curvature of the flow to conform to the boundary. According to Eq. (11–188b) the total entropy change in this case is vanishingly small; hence, the compression is isentropic for flow along a smooth curved surface.

In a compressive flow the Mach number is decreasing through successive Mach waves and the succeeding Mach waves make increasingly larger angles with the stream direction (see Fig. 11.28d). Consequently there is a tendency for Mach waves to converge (coalesce) and form an envelope as shown in Fig. 11.28d. Due to the convergence of the Mach waves the gradients of flow properties become large and the conditions are no longer isentropic; then an oblique shock wave is developed.

b. Expansion by Turning and the Prandtl-Meyer Function

Figure 11.29a shows the case of supersonic flow over a convex corner of the wall, i.e., the wall is deflected away from the oncoming stream. If a turn through a single oblique wave (Fig. 11.29a) is assumed we shall find that the normal component of the velocity after the wave, u_{2n}, is greater than the normal component of velocity in front of the wave u_{1n}. This result can easily be seen from Fig. 11.29a since the two tangential components u_{1t} and u_{2t} must be equal. This method of expansion must be ruled out because it gives a loss of entropy across the wave.

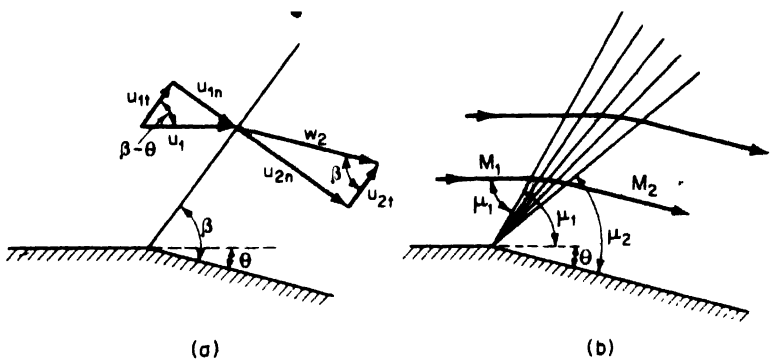


Figure 11.29

In Fig. 11.28d we saw that a family of Mach waves tend to converge and form an envelope. Since the flow through successive Mach lines is isentropic, the reverse motion is thermodynamically possible, namely an overall expansion. Hence, as supersonic flow turns a sharp convex corner, a series of Mach lines emanate from the corner and produce decreasing gradients in flow around the corner. In this case all the Mach waves are centered on the corner, forming an expansion fan (Fig. 11.29b). The flow conditions are constant along each Mach line but vary as a function of angular position measured at the corner (see Fig. 11.29b).

Let us consider now a single Mach wave as in Fig. 11.30. We have stated that flow conditions along any Mach line are constant and the flow is isentropic; therefore, we can find a unique relation

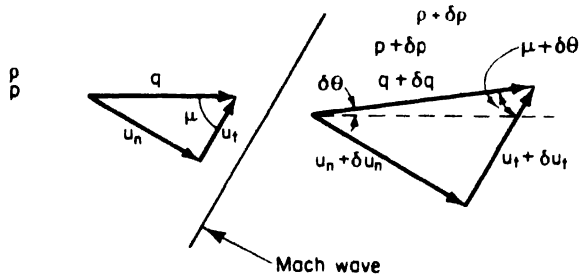


Figure 11.30

between θ and any one of the parameters u, M, or p for a given set of initial conditions. From the diagram of Fig. 11.30 for the direction parallel to the wave we can write the continuity equation and the momentum equation, respectively,

$$\rho u_n = (\rho + \delta\rho)(u_n + \delta u_n) \tag{11-189}$$

$$\rho u_n u_t = (\rho + \delta\rho)(u_n + \delta u_n)(u_t + \delta u_t) \tag{11-190}$$

The solution of Eqs. (11-189) an (11-190) gives

$$\delta u_t = 0$$

i.e., the velocity component parallel to the wave must remain unchanged. The trigonometric relation in Fig. 11.30 gives

$$u_t = q \cos \mu = (q + \delta q) \cos (\mu - \delta\theta)$$

Since $\delta\theta$ is small, the relation above reduces to

$$d\theta = -\frac{dq}{q} \cot \mu = -\frac{dq}{q} \sqrt{M^2 - 1} \tag{11-191}$$

The relation between q and M can be found from the expressions

$$q = a\mathrm{M}$$

and

$$\frac{q^2}{2} + \frac{a^2}{\kappa - 1} = \frac{a_0^2}{\kappa - 1} \tag{11-80}$$

as follows:

$$\frac{dq}{q} = \frac{1}{1 + \dfrac{\kappa - 1}{2}\mathrm{M}^2} \frac{d\mathrm{M}}{\mathrm{M}} \tag{11-192}$$

Substituting Eq. (11-192) into Eq. (11-191) and integrating, we have

$$-\theta + \text{Constant} = \int \frac{\sqrt{\mathrm{M}^2 - 1}}{1 + \dfrac{\kappa - 1}{2}\mathrm{M}^2} \frac{d\mathrm{M}}{\mathrm{M}} = \nu(\mathrm{M}) \tag{11-193}$$

where the integral $\nu(\mathrm{M})$ can be evaluated as follows:

$$\nu(\mathrm{M}) = \int \frac{\sqrt{\mathrm{M}^2 - 1}}{1 + \dfrac{\kappa - 1}{2}\mathrm{M}^2} \frac{d\mathrm{M}}{\mathrm{M}} = \frac{1}{2} \int \frac{\sqrt{\mathrm{M}^2 - 1}}{1 + \dfrac{\kappa - 1}{2}\mathrm{M}^2} \frac{d\mathrm{M}^2}{\mathrm{M}^2}$$

Simplifying,

$$\nu(\mathrm{M}) = \frac{1}{2} \int \frac{1}{\sqrt{\mathrm{M}^2 - 1}} \left[\frac{\dfrac{\kappa + 1}{2}}{1 + \dfrac{\kappa - 1}{2}\mathrm{M}^2} - \frac{1}{\mathrm{M}^2} \right] d\mathrm{M}^2$$

$$= \int \left[\frac{Z\,dZ}{Z\left(1 + \dfrac{\kappa - 1}{2} + \dfrac{\kappa - 1}{2}Z^2\right)} - \frac{Z\,dZ}{Z(1 + Z^2)} \right]$$

where $Z^2 = \mathrm{M}^2 - 1$. Finally, after performing the simple integration, we have

$$\nu(\mathrm{M}) = \frac{\dfrac{2}{\kappa - 1}}{\sqrt{1 + 2/(\kappa - 1)}} \tan^{-1} \frac{Z}{\sqrt{1 + 2/(\kappa - 1)}} - \tan^{-1} Z$$

$$= \sqrt{\frac{\kappa + 1}{\kappa - 1}} \tan^{-1} \sqrt{\frac{\kappa - 1}{\kappa + 1}(\mathrm{M}^2 - 1)} - \tan^{-1} \sqrt{\mathrm{M}^2 - 1} \tag{11-194}$$

The function $\nu(\mathrm{M})$ is generally called the *Prandtl-Meyer function.* The constant of integration has been arbitrarily chosen so that $\nu(1) = 0$. Tabulated values of M and ν (in degrees instead of radians) as computed from Eq. (11-194) are given in Table E-6.

It will be seen from Eq. (11-194) that ν reaches a maximum when $\mathrm{M} \to \infty$, and this maximum is

$$\nu_{max} = \frac{\pi}{2}\left(\sqrt{\frac{\kappa+1}{\kappa-1}} - 1\right)$$

$$= 130.3° \quad \text{for } \kappa = 1.4 \tag{11-195}$$

Physically the flow starts from $M = 1$ and expands into a vacuum, i.e., $M \to \infty$ (see Fig. 11.31).

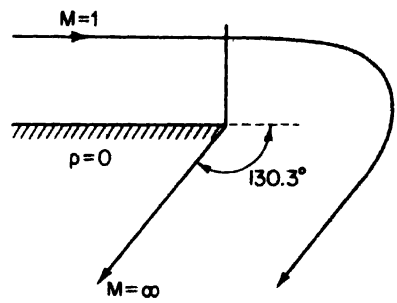


Fig. 11.31.

Equation (11-194) may be used to determine the flow conditions for a uniform supersonic flow expanded around a corner (Fig. 11.29b). If the original flow corresponds to $M_1 = 1$, Eq. (11-194) gives the local Mach number $M_2 = M$ after expansion corresponding to the angle of turn θ. If the original flow corresponds to $M_1 > 1$, Eq. (11-193) must be used to determine the proper constant.

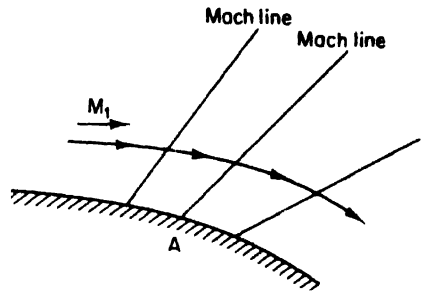


Fig. 11.32

Equation (11-194) may be used also to determine the flow conditions of a supersonic flow expanded continuously over a curved surface (Fig. 11.32). If the original Mach number is given by M_1, the local Mach number at any point A is determined from the angle of turn between the original flow direction and the tangent to the surface at the point A.

Important applications of the Prandtl-Meyer function are made in the area of supersonic airfoils, nozzle flows, ballistics, etc. For example, the local flow at the rear shoulder of a projectile is a Prandtl-Meyer expansion (similar expansion at the shoulder of a wedge is shown in Fig. 11.23). In this connection the method of characteristics has been developed for the investigation of supersonic flow fields. The readers may refer to advanced texts on supersonic flow for additional information on the characteristics method.

Illustrative Example 11.8. Using a table of the Prandtl-Meyer function, find the Mach number behind the oblique wave if a uniform flow of $M_1^* = 2.41$ turns through (a) a compression of 10°, (b) an expansion of 10°.

Solution. Let us rewrite Eq. (11-193) for computation purposes by considering a positive angle, $+\theta$, for a compression turn and a negative angle, $-\theta$, for an expansion turn. After determining the constants of integration, the equations are

$$\nu = \nu_1 + |\theta - \theta_1| \quad \text{for expansion}$$
$$\nu = \nu_1 - |\theta - \theta_1| \quad \text{for compression}$$

Now if we assume a uniform flow of $M_1 = 2.41$ either expanded or compressed through a turn of 10°, the local Mach number after the expansion or compression can be found from Table E-6. From Table E-6 $\nu_1 = 37°$ corresponds to $M_1 = 2.41$. If the flow is turned through an expansion of 10°, the final values of ν_2 and M_2 are obtained as $\nu_2 = 47°$ and $M_2 = 2.861$. If the flow is turned through a compression of 10°, then $\nu_2 = 27°$ and $M_2 = 2.041$ are the final values.

11.20. *Application of the Shock-Expansion Theory*

a. Loss of Pressure Energy Through a Shock Wave

It has been discussed in § 11.16b that entropy always increases across a shock wave while mechanical energy is dissipated and transformed into heat. This results in a loss of pressure energy which is usually expressed by a decrease in the stagnation pressure. Let the pressure, density, and temperature in the upstream reservoir of a convergent-divergent nozzle (Fig. 11.33) be denoted by p_{01}, ρ_{01}, and T_{01}, respectively. If a normal shock occurs in the divergent section of the nozzle, the stagnation pressure in the imaginary reservoir at the exit of the nozzle would be smaller than p_{01}. The pressure, density, and temperature in the imaginary reservoir are designated as p_{02}, ρ_{02}, and T_{02}, respectively.

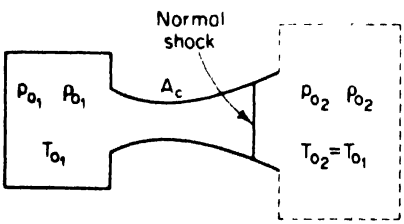


Figure 11.33

In order to determine the ratio of the stagnation pressures, p_{02}/p_{01}, the condition that total enthalpy remains constant across a shock wave must be applied, i.e.,

$$\frac{p_{01}}{\rho_{01}} = \frac{p_{02}}{\rho_{02}} = RT_{02} = RT_{01} \tag{11-196}$$

Since the process is isentropic both in front of and behind the shock wave, we have the relations

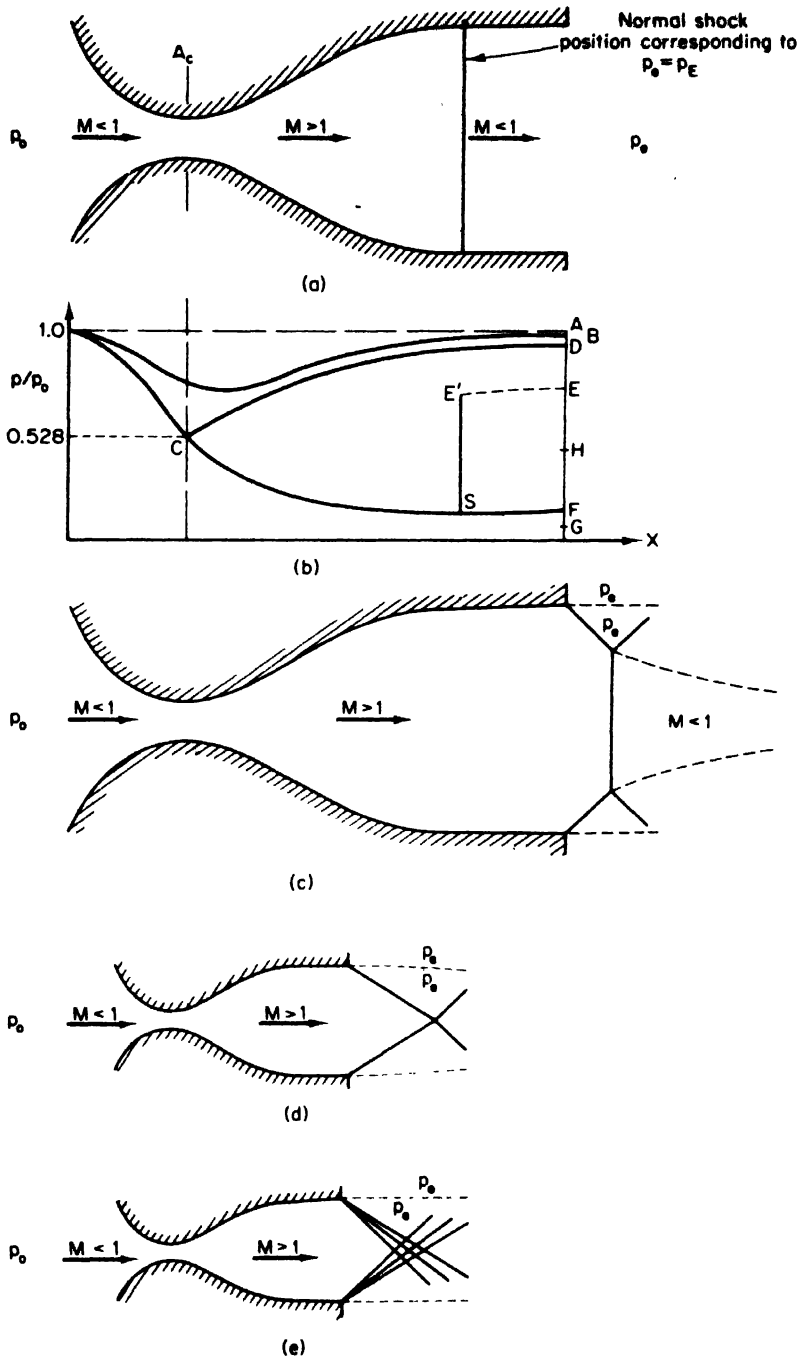

Normal shock
position corresponding to
$p_e = p_E$
A_c
p_b
M < 1
M > 1
M < 1
p_e
(a)
1.0
p/p_0
0.528
A
B
D
E'
E
C
H
S
F
G
X
(b)
p_e
p_e
p_0
M < 1
M > 1
M < 1
(c)
p_e
p_e
p_0
M < 1
M > 1
(d)
p_e
p_e
p_0
M < 1
M > 1
(e)

Figure 11.34

$\mathbf{X}$	external or body force per unit mass, Eq. (11-13)
x, y, z	Cartesian coordinates
β	shock wave angle, Eq. (11-145a) and Fig. 11.24
Γ	circulation, Eq. (11-30)
θ	deflection angle (or wedge angle), Eq. (11-144) and Fig. 11.24
κ	ratio of specific heats, Eq. (11-5)
μ	(1) coefficient of viscosity, Eq. (11-1) (2) Mach angle, Eq. (11-76)
$\nu(\mathrm{M})$	the Prandtl-Meyer function, Eq. (11-194)
ρ	density, Eq. (11-1)
ϕ	velocity potential, Eq. (11-14)
ϕ_f	force potential, Eq. (11-14)
ψ	stream function, Eq. (11-7b)
$\boldsymbol{\Omega}$	vorticity vector, Eq. (11-13)
$(\ )_1, (\ )_2$	quantities in front and behind an oblique shock wave, respectively, § 11.16
$(\ )_c$	critical condition, Eqs. (11-83)-(11-86)
$(\ )_0$	reservoir quantity, Eqs. (11-24) and (11-78)
$(\ )_\infty$	free-stream condition, Eq. (11-2)
$(\ )^*$	nondimensional quantities, Eq. (11-2) with the exception of those specified in illustrative example 11.4

PROBLEMS

11.1. Express Eq. (11-14) in Cartesian notation and then put the resulting equations in the form of Eqs. (7-2a), (7-2b), and (7-2c).

11.2. Verify that Eqs. (11-21) and (11-29) are identical. Since Eq. (11-21) is obtained on the basis of an adiabatic process, and Eq. (11-29) on the basis of an isentropic process, explain why they are identical.

11.3. The speed of sound in air may be expressed by the equation

$$a = C\sqrt{T}$$

Find the numerical value and the units of the constant C when the speed of sound is given in

(a) miles per hour,
(b) feet per second,
(c) meters per second.

11.4. Compute the speed of sound in miles per hour and meters per second at an altitude of 30,000 ft.

11.5. Derive the expression for the speed of sound

$$a^2 = \frac{\kappa p}{\rho}$$

from the one-dimensional momentum and energy equations [Eqs. (11-16) and (11-21) with $\phi_f = 0$].

11.6. Show that

$$M^2 = -\frac{dp/p}{dq/q}$$

may be derived from the equation of motion.

11.7. Show that

$$p_1 = p_\infty [A + B \sin(x - a_\infty t) + C \cos D(x - a_\infty t)]$$

is a solution to the one-dimensional acoustic wave equation [Eq. (11-50)] where A, B, C, and D are arbitrary constants.

11.8. Show that the acoustic wave equation for a source of disturbance with spherical symmetry (spherical form with its center at the point of origin) is

$$\frac{\partial^2}{\partial t^2}(\omega p_1) = a_\infty^2 \frac{\partial^2}{\partial \omega^2}(\omega p_1) \tag{a}$$

where p_1 is the pressure change due to disturbances, and ω is the radial coordinate in spherical coordinates. Verify that the solution of Eq. (a) is

$$p_1 = \frac{1}{\omega}[f(\omega - a_\infty t) + g(\omega + a_\infty t)] \tag{b}$$

11.9. Illustrate the propagation of a plane wave of high intensity by the following expression for the density change:

$$\rho = \sin\left[x - \left(a_\infty + \frac{\kappa + 1}{2}u\right)t\right]$$

11.10. Suppose a supersonic airplane is flying at $M = 3.0$ at an altitude of 15,000 ft. Compute the time for an observer on the ground to hear the disturbance from the instant when the airplane is vertically overhead. Use average temperature between the ground and 15,000 ft for the computation.

11.11. Consider a flow system which consists of an infinite reservoir and a convergant nozzle as shown in Fig. 11.9. Find the velocity at the exit which results from the expansion of air at standard temperature, 59°F, to the condition of absolute vacuum. If the exit pressure is $0.7p_0$, calculate the exit Mach number.

11.12. Show that, for a perfect gas, the maximum velocity at the exit of a reservoir (Fig. 11.9) which may be reached is given by

$$q_{max} = \sqrt{2C_p T_0}$$

and the corresponding h and M relation is

$$h = q^2 \frac{1}{(\kappa - 1)M^2}$$

where h is the enthalpy per unit mass.

11.13. Show that for an isentropic flow

(a) $$a_c^2 = \left(\frac{2}{\kappa + 1}\right)a^2 + \left(\frac{\kappa - 1}{\kappa + 1}\right)q^2$$

(b) $$a_c = a_0\sqrt{\frac{2}{\kappa - 1}\left(1 - \frac{T_c}{T_0}\right)}$$

11.14. Show that for an isentropic flow

$$M_c = \sqrt{\frac{(\kappa + 1)M^2}{(\kappa - 1)M^2 + 2}}$$

where $M_c = q/a_c$.

11.15. Plot $p_0 - p$ vs. the velocity q (also indicate M) at standard sea-level conditions for both compressible and incompressible flows.

11.16. A jet transport is flying at 30,000 ft altitude, and its speed indicator reads 600 mph. If the indicator is calibrated without taking the effect of compressibility into consideration, find the true air speed.

11.17. Find the stagnation temperature, density, and Mach number in Prob. 11.16.

11.18. A diffuser (Fig. 11.B) is a device in which a pressure increase may be gained at the expense of velocity. A certain diffuser has an area ratio of 2:1. If the air enters the diffuser with a Mach number of 0.8, a pressure of 60 lbf/in.^{2}, and a temperature of 150°F, compute:

(a) the exit velocity
(b) the exit pressure
(c) the exit temperature
(d) the exit Mach number

Assume the flow is isentropic.

11.19. Calculate the exit velocity and the mass flow of nitrogen which is discharged from a reservoir ($p_0 = 50$ lbf/in.^{2} and $T_0 = 60$°F) through a converging nozzle of $2\frac{1}{2}$ in. throat diameter to standard atmospheric conditions. Repeat the calculation for air.

11.20. From the first law of thermodynamics (see Appendix B) and the Bernoulli equation derive the following temperature and velocity relation for an isentropic flow through a de Laval nozzle:

$$\frac{dT}{T} = (1 - \kappa)M^2\frac{dV}{V}$$

11.21. Air flows from a reservoir ($p_0 = 100$ lbf/in.^{2} and $T_0 = 100$°F) through a de Laval nozzle with a throat diameter of 3.0 in. and a maximum Mach number of 0.75. Calculate the mass flow rate, nozzle diameter, velocity, pressure, and temperature at the exit where $M = 0.50$.

11.22. Air flows from a reservoir ($p_0 = 100$ lbf/in.^{2} and $T_0 = 250$°F) through a de Laval nozzle and is discharged at standard atmospheric pressure. If the mass flow rate is 0.03 slugs/sec, determine the Mach number, the velocity, and the cross-sectional area at the exit and the pressure, velocity, and the cross-sectional area at the throat.

11.23. Show that in flow through a tube of constant cross-sectional area with heat addition, (a) the Mach number at the point of maximum temperature is equal to $1/\sqrt{\kappa}$ and (b) the Mach number at the point of maximum entropy is equal to unity.

11.24. Air at standard atmospheric conditions enters a constant-area tube at $M = 0.25$. If heat is added to the tube, and the flow is accelerated to $M = 0.80$, determine the amount of heat added and the percentage change in pressure.

11.25. Air enters a constant-area pipe with a velocity of 500 ft/sec, a temperature of 150°F, and a pressure of 60 lbf/in^2. If 80 BTU/lbf of heat is added to the pipe, find

(a) the final stagnation temperature
(b) the final temperature
(c) the final pressure
(d) the final Mach number
(e) the final velocity

11.26. In a heated constant-area pipe, the final stagnation temperature of air is increased from the initial value of 100°F to 500°F. If the inlet Mach number is 0.4, find the final Mach number and the percentage change in pressure.

11.27. Construct a temperature-entropy diagram for air undergoing the Rayleigh flow when the mass flow rate is 0.25 slugs/ft^2-sec, and the reservoir is at standard conditions.

11.28. Verify the following Rankine-Hugoniot relation for normal shocks

$$\frac{p_2 - p_1}{\rho_2 - \rho_1} = \kappa\left(\frac{p_1 + p_2}{\rho_1 + \rho_2}\right)$$

11.29. Show that the expression given in Prob. 11.28 may be written in the form

$$\frac{1}{2}(p_1 + p_2)\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = -\frac{1}{(\kappa - 1)}\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right)$$

11.30. The Rankine-Hugoniot relation for normal shocks is given by Eq. (11-131a); show that

(a) for weak shocks

$$\frac{\Delta\rho}{\rho_1} \approx \frac{1}{\kappa}\frac{\Delta p}{p_1}$$

$$\frac{\Delta T}{T_1} \approx \left(\frac{\kappa - 1}{\kappa}\right)\frac{\Delta p}{p_1}$$

(b) for strong shocks

$$\frac{\rho_2}{\rho_1} \longrightarrow \left(\frac{\kappa + 1}{\kappa - 1}\right)$$

$$\frac{T_2}{T_1} \longrightarrow \left(\frac{\kappa - 1}{\kappa + 1}\right)\frac{p_2}{p_1}$$

11.31. Show that the Mach number behind a normal shock wave may be written in the form

$$M_2 = \left[\left(\frac{\kappa + 1}{2\kappa}\right)\frac{p_1}{p_2} + \left(\frac{\kappa - 1}{2\kappa}\right)\right]^{1/2}$$

11.32. The Mach number behind a normal shock wave in air is 0.8. What is the pressure ratio, temperature ratio, density ratio, and the entropy increase across this shock?

11.33. Show that the product of the normal velocities, in front of and behind an oblique shock wave, is given by the relation

$$u_{1n}u_{2n} = a_c^2 - \left(\frac{\kappa - 1}{\kappa + 1}\right)u_t^2$$

11.34. A wedge with a total angle of 20° is introduced into a flow having a Mach number of 2.5. What wave angles can the shock exhibit? Determine the pressure and density ratios across the possible oblique wave and the Mach number behind the wave. *Hint*: Wave angles may be obtained from the oblique shock chart in Appendix E for the first approximation. Equation (11-164c) should be used for calculating a more accurate result.

11.35. Show that the pressure difference across an oblique shock wave may be expressed in the form

$$\frac{p_2 - p_1}{\frac{1}{2}\rho_1 u_1^2} = \frac{4}{\kappa - 1}\left(\sin^2\beta - \frac{1}{M^2}\right)$$

11.36. Derive Eq. (11-169) from Eq. (11-168) by showing all the detailed steps.

11.37. Derive Eq. (11-173) from Eq. (11-172) by showing all the detailed steps.

11.38. Compute the maximum deflection angles for which the oblique shock remains attached to the wedge when $M_1 = 2.0$ and $M_1 = 3.0$.

11.39. Compute the minimum values of Mach number, M_1, for which the oblique shock remains attached to the wedge for deflection angles of $\theta_d = 15°$, $\theta_d = 25°$, and $\theta_d = 40°$

11.40. Construct a shock polar diagram for $M_1 = 2.5$. Find the values of the wave angles and the Mach numbers behind the oblique shocks. Also determine θ_{max}. Check the graphical solution with the corresponding equations.

11.41. Using Table E-6 for the Prandtl-Meyer function, find the Mach number behind the oblique wave if a uniform flow of $M_1 = 2.75$ turns through (a) a compression of 10°, (b) an expansion of 15°. Check the answer with Eq. (11-194).

11.42. Show that the Prandtl-Meyer flow around a corner is irrotational (see Fig. 11.29b).

11.43. If a normal shock occurs in the divergent section of a convergent-divergent nozzle (Fig. 11.33), determine the ratios of the stagnation pressures, stagnation densities, and stagnation temperatures, as well as the increase of

entropy across the normal shock for $M_1 = 2.25$. $\kappa = 1.4$ may be assumed.

11.44. If a normal shock occurs at the exit of a de Laval nozzle and the flow ($\kappa = 1.4$) in front of the shock has a Mach number equal to 2.3, find the ratio of the exit pressure to the stagnation pressure of the upstream reservoir.

12

INTRODUCTION TO VISCOUS COMPRESSIBLE FLOW

In the preceding chapter we have discussed the flow of a compressible fluid without taking into consideration the effect of the viscosity of the fluid. Since the influence of viscosity of the fluid on the flow is generally restricted to a thin layer over a surface of the body (see Chap. Nine), the general flow pattern under consideration in Chap. Eleven is influenced very little by the viscosity; hence our analysis there is valid. On the other hand, however, the influence of the viscosity of the fluid allows us to explain many flow phenomena such as skin friction, form drag of a body, separation of flow, etc. For viscous incompressible flow these phenomena have been discussed in Chaps. Eight, Nine, and Ten.

If the flow is compressible, heat due to friction as well as temperature change due to compression must be taken into account. In addition, it is necessary to consider the effects of the variation of viscosity with temperature. This dependence of temperature with viscosity and density makes the calculations of problems in viscous compressible flow much more difficult than that for the case of incompressible flow.

The general governing equations of continuity, momentum, energy, and state for the flow of viscous compressible fluids have been derived and explained in Chap. Five. The significance of the four important controlling parameters in viscous compressible fluids was discussed in the preceding chapter. In this chapter we shall discuss

some of the simpler problems in viscous compressible flow. The simplicity of the problems will enable us to demonstrate the solution of the equations in viscous compressible flow and to exhibit the effects of compressibility, viscosity, and heat conduction on flow problems.

In § 12.1 plane Couette flow is discussed. The effects of the Mach number and Prandtl number on the velocity distribution and coefficient of friction are shown. Temperature distributions and heat flux at the wall are illustrated.

In § 12.2 steady flow through a constant-area pipe with wall friction is presented. The friction parameter and entropy change as a function of the Mach number are derived. The effects of the friction on thermodynamic properties are shown.

In § 12.3 laminar boundary layer equations in compressible flows are derived from the Navier-Stokes equations and the general energy equation from Chap. Five.

In § 12.4 velocity and temperature relations in the laminar compressible boundary layer are derived for the case of $\mathrm{Pr} = 1$ with and without the pressure gradient.

In § 12.5 the momentum integral theorem for the boundary layer in compressible fluids is derived. The integral energy equation is also given.

In § 12.6 the application of the momentum integral equation to laminar boundary layer problems is discussed and a numerical example is given to illustrate the method.

12.1. *Plane Couette Flow—An Exact Solution*

In § 8.3 the case of steady incompressible flow in a two-dimensional channel has been discussed. It may be recalled that a Couette flow is the flow between two parallel flat plates, the lower plate is at rest and the upper plate is moving with a velocity U parallel to the fixed plate. The problem now is to calculate the compressible flow of the gas in the channel.

a. Basic Equations

For a viscous compressible fluid in steady two-dimensional flow the Navier-Stokes equations in Cartesian coordinates with negligible body forces can be reduced from Eqs. (5-17) to yield

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left\{\mu\left[2\frac{\partial u}{\partial x} - \frac{2}{3}(\nabla \cdot \mathbf{q})\right]\right\} + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \tag{12-1a}$$

$$\rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial y}\left\{\mu\left[2\frac{\partial v}{\partial y}-\frac{2}{3}(\nabla\cdot\mathbf{q})\right]\right\}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] \tag{12-1b}$$

The continuity equation is [see Eq. (5-9a)]

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0 \tag{12-2}$$

The energy equation is reduced to [see Eq. (5-33)]

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)-\frac{2}{3}\mu(\nabla\cdot\mathbf{q})^2+2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]+\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2=\rho u\frac{\partial C_pT}{\partial x}+\rho v\frac{\partial C_pT}{\partial y}-\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right) \tag{12-3}$$

The equation of state for a perfect gas is

$$p=\rho RT \tag{5-37}$$

Let x be the direction of the flow, y the direction normal to the flow, and the width of the plates parallel to the z-direction be large compared to the distance h between the plates (Fig. 12.1). For this simple configuration, we have

$$v=0, \qquad w=0, \qquad \frac{\partial}{\partial z}=0, \qquad p=\text{Constant} \tag{12-4a}$$

The boundary conditions are then (see Fig. 12.1)

$$\begin{aligned} &y=0: \quad u=0, \quad v=0, \quad q=q_w, \quad (\tau_{yx})=\tau_w \\ &y=h: \quad u=U, \quad v=0, \quad p=p_\infty \end{aligned} \tag{12-4b}$$

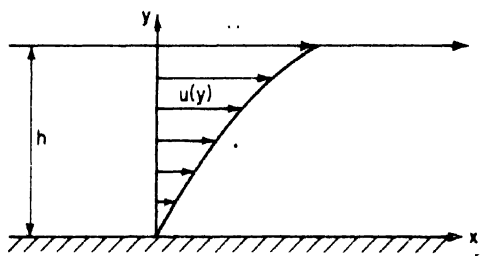


Figure 12.1

According to Eqs. (12-2) and (12-4) the velocity component u is independent of x. Hence, Eqs. (12-1a) and (12-1b) reduce to

$$\frac{d}{dy}\left(\mu\frac{du}{dy}\right)=0 \tag{12-5a}$$

which yields, after integration,

$$\mu \frac{du}{dy} = \text{Constant} \tag{12-5b}$$

The component of the velocity u is found,

$$u = \tau_w \int_0^y \frac{dy}{\mu} \tag{12-6}$$

where the constant of integration τ_w is determined from the condition at the fixed plate as

$$\tau_w = \mu_w \left(\frac{du}{dy}\right)_w = \mu \frac{du}{dy} \tag{12-7}$$

The energy equation under the present flow conditions reduces to ($T \sim 1/\rho \sim u$),

$$\frac{d}{dy}\left(k \frac{dT}{dy}\right) = -\mu \left(\frac{du}{dy}\right)^2 \tag{12-8a}$$

Substituting Eq. (12-7) into Eq. (12-8a), we obtain

$$\frac{d}{dy}\left(k \frac{dT}{dy}\right) = -\frac{\tau_w^2}{\mu} \tag{12-8b}$$

Integrating Eq. (12-8b) yields

$$k\left(\frac{dT}{dy}\right) = -\tau_w^2 \int_0^y \frac{dy}{\mu} + C = -\tau_w u + C = -\mu u \frac{du}{dy} + C \tag{12-8c}$$

where the constant of integration is determined from the condition at the fixed plate, i.e.,

$$C = k_w \left(\frac{dT}{dy}\right)_w = -q_w \tag{12-8d}$$

The negative sign is conventional, to show that q_w (the flow of heat through the wall per unit time) is positive when $(dT/dy)_w$ is negative. With the relation in Eq. (12-8d) we can rewrite Eq. (12-8c) in the form

$$k\left(\frac{dT}{dy}\right) + \mu u \frac{du}{dy} = -q_w \tag{12-9}$$

From either the simple kinetic theory of gases or empirical data, the coefficient of viscosity μ can often be expressed with sufficient accuracy as a power of the absolute temperature,

$$\frac{\mu}{\mu_\infty} = \left(\frac{T}{T_\infty}\right)^m \tag{12-10}$$

For air at ordinary temperature, $m = 0.76$ is generally used. As the temperature increases, m decreases toward $\frac{1}{2}$. The Prandtl number Pr is very nearly constant (of order unity) for all common gases. Since C_p is also nearly constant for a fairly wide range of temperatures around

ordinary temperatures, the coefficient of heat conductivity k is directly proportional to μ. Based on these arguments Eq. (12–9) may be written as

$$\mu \frac{d}{dy}\left(\frac{C_p}{\mathrm{Pr}} T + \frac{1}{2}u^2\right) = -q_w \tag{12–11}$$

The integration of Eq. (12–11) results in

$$C_p T + \frac{1}{2}\,\mathrm{Pr}\, u^2 = -\mathrm{Pr}\, q_w \int_0^y \frac{dy}{\mu} + C \tag{12–12}$$

where $C = C_p T_w$ and T_w corresponds to the temperature at the fixed plate. Substituting Eq. (12–6) into Eq. (12–12), one finds

$$C_p T + \frac{1}{2}\mathrm{Pr}\, u^2 = -\mathrm{Pr}\frac{q_w}{\tau_w} u + C \tag{12–13}$$

b. Temperature and Velocity Distributions in Couette Flow

If the temperature of the moving plate is denoted by T_∞ the constant of integration in Eq. (12–13) may be determined from the boundary condition at the upper plate, i.e.,

$$C = C_p T_\infty + \frac{1}{2}\mathrm{Pr}\, U^2 + \mathrm{Pr}\frac{q_w}{\tau_w} U$$

With this value of C, Eq. (12–13) becomes

$$C_p(T - T_\infty) = \mathrm{Pr}\frac{q_w}{\tau_w}(U - u) + \frac{1}{2}\,\mathrm{Pr}\,(U^2 - u^2) \tag{12–14}$$

Dividing Eq. (12–14) by $C_p T_\infty$ and remembering $u^2/C_p T_\infty = (\kappa - 1)\,\mathrm{M}_\infty^2$, the temperature distribution in Couette flow is obtained:

$$\frac{T}{T_\infty} = 1 + \mathrm{Pr}\frac{q_w}{\tau_w}\frac{(\kappa - 1)}{U}\mathrm{M}_\infty^2\left(1 - \frac{u}{U}\right) + \mathrm{Pr}\frac{(\kappa - 1)}{2} M_\infty^2\left(1 - \frac{u^2}{U^2}\right) \tag{12–15}$$

The velocity distribution in Couette flow may be obtained from Eq. (12–7) as follows:

$$\tau_w y = \int_0^u \mu\, du \tag{12–16}$$

With the relation of μ in Eq. (12–10) and T in Eq. (12–15), Eq. (12–16) may be written as

$$\frac{\tau_w y}{\mu_\infty} = \int_0^u \left[1 + \mathrm{Pr}\frac{q_w}{\tau_w}\left(\frac{\kappa - 1}{U}\right)\mathrm{M}_\infty^2\left(1 - \frac{u}{U}\right) + \mathrm{Pr}\frac{(\kappa - 1)}{2}\mathrm{M}_\infty^2\left(1 - \frac{u^2}{U^2}\right)\right]^m du \tag{12–17}$$

For an arbitrary value of m the integrals in Eq. (12–17) have to be evaluated numerically. If $m = 1$, Eq. (12–17) yields

$$\frac{\tau_w y}{\mu_\infty U} = \frac{u}{U} + \Pr \frac{q_w}{\tau_w}\left(\frac{\kappa-1}{U}\right) M_\infty^2 \left[\frac{u}{U} - \frac{1}{2}\left(\frac{u}{U}\right)^2\right] + \Pr \frac{(\kappa-1)}{2} M_\infty^2 \left[\frac{u}{U} - \frac{1}{3}\left(\frac{u}{U}\right)^3\right] \tag{12-18}$$

For the case of an adiabatic wall, $q_w = 0$, and Eq. (12-18) reduces to

$$\frac{\tau_w y}{\mu_\infty U} = \frac{u}{U} + \Pr M_\infty^2 \frac{(\kappa-1)}{2}\left[\frac{u}{U} - \frac{1}{3}\left(\frac{u}{U}\right)^3\right] \tag{12-19}$$

The shearing stress τ_w can be obtained from Eq. (12-19) by letting $u = U$ and $y = h$ in Eq. (12-19). The result may be written as

$$\frac{\tau_w h}{\mu_\infty U} = 1 + \Pr M_\infty^2 \frac{(\kappa-1)}{3} \tag{12-20}$$

Dividing Eq. (12-19) by Eq. (12-20), we obtain

$$\frac{y}{h} = \frac{1}{1 + \Pr M_\infty^2 (\kappa-1)/3}\left[\frac{u}{U} + \Pr M_\infty^2 \frac{(\kappa-1)}{2}\left\{\frac{u}{U} - \frac{1}{3}\left(\frac{u}{U}\right)^3\right\}\right] \tag{12-21}$$

When $M_\infty^2 \to \infty$, we have

$$\frac{y}{h} = \frac{3}{2}\frac{u}{U}\left[1 - \frac{1}{3}\left(\frac{u}{U}\right)^2\right] \tag{12-22}$$

The velocity distribution of the Couette flow is plotted in Fig. 12.2 as a function of the distance from the fixed wall for various Mach numbers and Prandtl numbers. It is seen that the effect of the Mach number on the Couette flow is to decrease the velocity gradient at the stationary wall and to increase it at the moving wall. Since the Prandtl number appears paired with the Mach number (both increase heat transfer to the fluid), it has the same influence on the velocity distribution as the Mach number. A comparison of the velocity distributions for $m = 1.0$ and $m = 0.76$ [see Eq. (12-10)] is also depicted in Fig. 12.2.

The temperature distribution of the Couette flow can be easily calculated from Eq. (12-15) once the velocity distribution has been determined. Typical temperature profiles in Couette flow ($\Pr M_\infty^2 = 2$) for an adiabatic wall ($q_w = 0$), a heated wall ($q_w < 0$), and a cooled wall ($q_w > 0$) are shown in Fig. 12.3. It is interesting to note that the temperature remains unchanged for an incompressible fluid. Also, the temperature gradient becomes zero for a compressible flow with an adiabatic wall. This special temperature at the fixed insulated wall (adiabatic wall) is called the recovery temperature, and is denoted by T_r. According to Eq. (12-15) we can find T_r by putting $q_w = 0$ and $u = 0$ as follows:

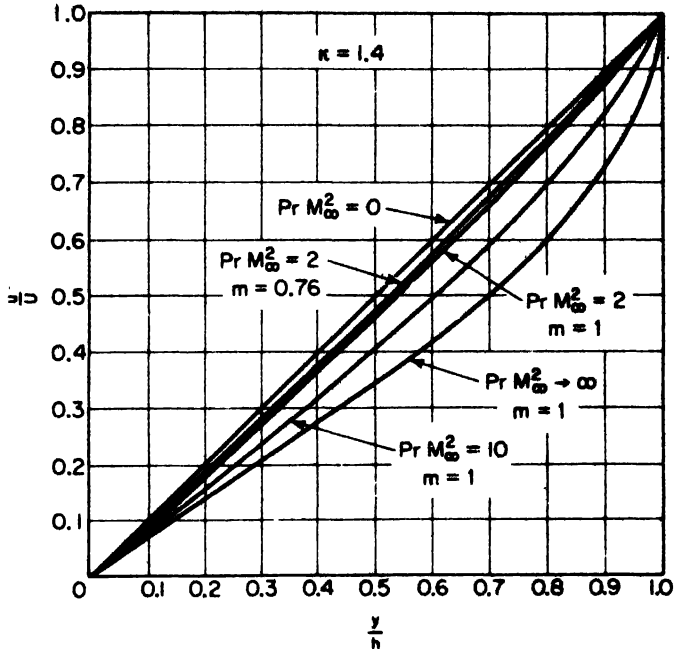


Figure 12.2

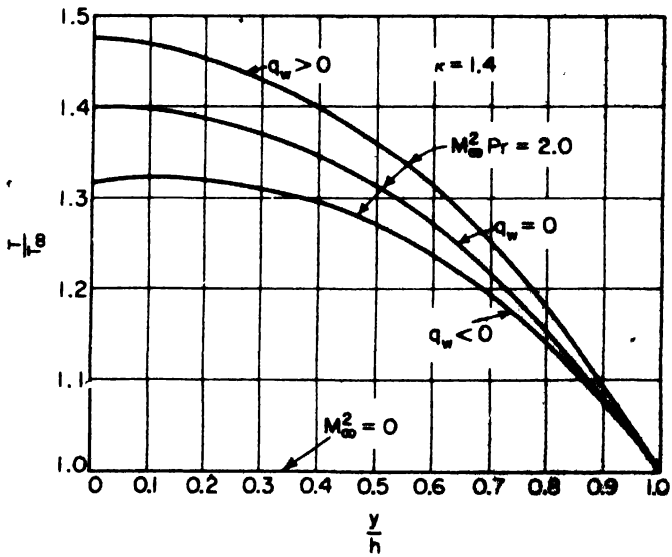


Figure 12.3

$$\frac{T_r}{T_\infty} = 1 + \text{Pr}\, \text{M}_\infty^2 \frac{(\kappa - 1)}{2} \tag{12-23}$$

The coefficient of friction at the fixed wall ($q_w = 0$) may be obtained from Eq. (12–20) [see Eq. (8–36)]:

$$C_f = \frac{\tau_w}{\rho_\infty (U^2/2)} = \frac{1 + \text{Pr}\, \text{M}_\infty^2 \dfrac{\kappa - 1}{3}}{\text{Re}/2} \tag{12-24}$$

where $\text{Re} = Uh/\nu_\infty$. The shearing stress of Eq. (12–20) reduces to the first term of Eq. (8–29) when $\text{M}_\infty \to 0$. It must be noted that the velocity gradient for a compressible fluid varies from the stationary wall to the moving wall (Fig. 12.2), and the shearing stress given in Eq. (12–20) is a constant in Couette flow. The constancy of the shearing stress in Couette flow can be shown from the velocity gradient by

$$\tau_w = \mu_w \left(\frac{du}{dy}\right)_w = \mu_\infty \left(\frac{du}{dy}\right)_\infty$$

According to this expression we need only to show that

$$\frac{\mu_w}{\mu_\infty} = \frac{(du/dy)_\infty}{(du/dy)_w} \tag{12-25}$$

The viscosity is given by Eq. (12–10), and with the aid of Eq. (12–15) it may be written as ($q_w = 0$)

$$\frac{\mu_w}{\mu_\infty} = \frac{T_w}{T_\infty} = 1 + \text{Pr}\, \text{M}_\infty^2 \frac{(\kappa - 1)}{2} \tag{12-26}$$

Differentiating Eq. (12–19) with respect to y and imposing the limits $y = 0$ and $y = \text{h}$, we have

$$\left(\frac{du}{dy}\right)_w = \frac{\tau_w}{\mu_\infty} \frac{1}{1 + \text{Pr}\, \text{M}_\infty^2 \dfrac{\kappa - 1}{2}} \tag{12-27a}$$

and

$$\left(\frac{du}{dy}\right)_\infty = \frac{\tau_w}{\mu_\infty} \tag{12-27b}$$

The ratio of the velocity gradients in Eqs. (12–27) gives

$$\frac{\left(\dfrac{du}{dy}\right)_\infty}{\left(\dfrac{du}{dy}\right)_w} = 1 + \text{Pr}\, \text{M}_\infty^2 \frac{(\kappa - 1)}{2} \tag{12-28}$$

Hence Eq. (12–25) is verified and the shearing stress is constant in Couette flow.

The increase of the skin-friction coefficient at the fixed wall ($q_w = 0$) with the increase of the Mach number and Prandtl number is depicted in Fig. 12.4.

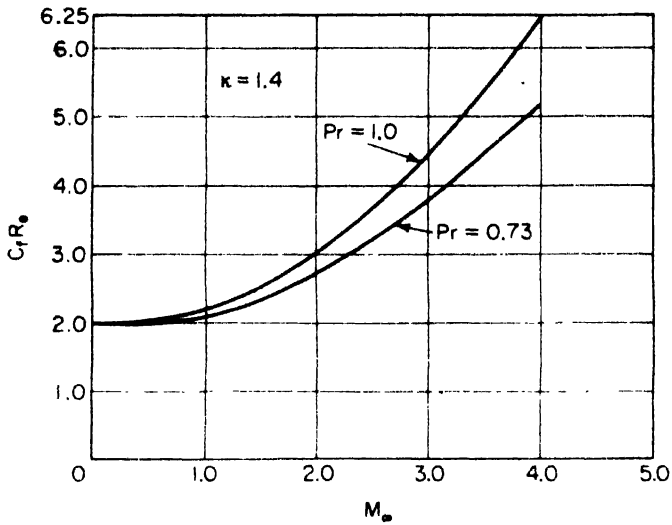


Figure 12.4

12.2. *Steady Flow Through a Constant-Area Pipe*

In § 11.15 we discussed inviscid compressible flow through a constant-area tube with heat transfer. In actual practice the problem of the compressible flow of a gas through a pipe of constant cross section with skin friction can be very important. Since the formal solution of this problem is very complicated we shall limit our discussion to flow through a constant-area pipe in which there is no work done and no heat transfer to or from the pipe (Fig. 12.5).

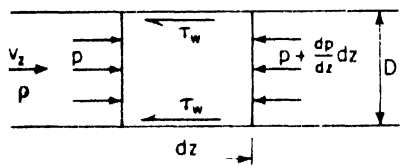


Figure 12.5

a. Basic Equations

Let z be the direction of flow along the axis of the pipe and r denote the radial direction measured outwards from the z-axis. The velocity components in axial, radial, and tangential directions are, respectively, v_z, v_r, and v_θ. Under the present flow conditions we have

$$v_z = v_z(z), \qquad v_\theta = v_r = 0, \qquad \frac{\partial}{\partial r} = \frac{\partial}{\partial \theta} = 0 \tag{12-29}$$

According to these conditions the continuity equation [Eq. (5-40)] reduces to

$$\frac{d(\rho v_z)}{dz} = 0 \tag{12-30a}$$

or

$$\rho v_z = \dot{m} = \text{Constant} \tag{12-30b}$$

The Navier-Stokes equation for the axial direction [Eq. (5-52c)] without body forces becomes

$$\rho v_z \frac{dv_z}{dz} = -\frac{dp}{dz} - 4C_f \frac{\rho v_z^2}{2D} \tag{12-31}$$

where C_f is the skin-friction coefficient and D the diameter of the pipe. The second term on the right-hand side of Eq. (12-31) represents the pressure drop due to friction which was given by Eqs. (8-53) and (8-54) in the Poiseuille flow. The simple friction term will be used to replace the complicated viscous terms in Eq. (5-52c).

For steady adiabatic flow without work done the energy equation can be reduced from Eq. (11-20) to the form

$$\frac{d}{dz}\left(C_p T + \frac{v_z^2}{2}\right) = 0 \tag{12-32a}$$

or

$$C_p \frac{dT}{dz} + v_z \frac{dv_z}{dz} = 0 \tag{12-32b}$$

where C_p is considered to be constant.

The equation of state for a perfect gas is

$$p = \rho RT \tag{5-37}$$

or expressing it in the form as given in Eq. (11-116a),

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \tag{12-33}$$

b. Relations for Perfect Gases and the Friction Parameter

The variation of fluid properties along the axial direction of a constant-area pipe may be obtained from the continuity equation, the energy equation, and the equation of state for a perfect gas. From the definition of the Mach number, we have

$$\mathrm{M}^2 = \frac{v_z^2}{\kappa RT}$$

Differentiating the expression logarithmically, there results

$$\frac{d\mathrm{M}}{\mathrm{M}} = \frac{dv_z}{v_z} - \frac{1}{2}\frac{dT}{T} \tag{12-34}$$

The energy equation, Eq. (12-32b), can be written as*

$$\frac{\kappa R}{\kappa - 1}\,dT + v_z\,dv_z = 0 \tag{12-32c}$$

or in the form given in Eq. (11-116a),

$$\frac{dT}{T} = (1 - \kappa)\mathrm{M}^2\frac{dv_z}{v_z} \tag{12-35}$$

Combining Eqs. (12-30a) and (12-33) the following relation is obtained:

$$\frac{dp}{p} = \frac{dT}{T} - \frac{dv_z}{v_z} \tag{12-36}$$

The velocity change along the pipe is obtained by eliminating the temperature terms in Eqs. (12-34) and (12-35) as follows:

$$\frac{dv_z}{v_z} = \frac{\dfrac{d\mathrm{M}}{\mathrm{M}}}{\dfrac{\kappa - 1}{2}\mathrm{M}^2 + 1} = -\frac{d\rho}{\rho} \tag{12-37a}$$

Equation (12-37a) may be written in the form

$$\frac{dv_z}{v_z} = \frac{1}{2}\left[\frac{1}{\mathrm{M}^2} - \frac{1}{\mathrm{M}^2 + \dfrac{2}{\kappa - 1}}\right]d\mathrm{M}^2 \tag{12-37b}$$

Integrating,

$$\frac{v_z}{v_{z_1}} = \frac{\mathrm{M}}{\mathrm{M}_1}\left[\frac{1 + \dfrac{\kappa - 1}{2}\mathrm{M}_1^2}{1 + \dfrac{\kappa - 1}{2}\mathrm{M}^2}\right]^{1/2} \tag{12-37c}$$

Hence the density change is easily found from Eqs. (12-30a) and (12-37b),

$$\frac{\rho}{\rho_1} = \frac{\mathrm{M}_1}{\mathrm{M}}\left[\frac{1 + \dfrac{\kappa - 1}{2}\mathrm{M}^2}{1 + \dfrac{\kappa - 1}{2}\mathrm{M}_1^2}\right]^{1/2} \tag{12-38}$$

Substituting Eq. (12-37a) into Eq. (12-35) the variation of temperature along the pipe is obtained:

$$\frac{dT}{T} = \frac{(1 - \kappa)\mathrm{M}\,d\mathrm{M}}{\dfrac{\kappa - 1}{2}\mathrm{M}^2 + 1} = -\frac{d\mathrm{M}^2}{\mathrm{M}^2 + \dfrac{2}{\kappa - 1}} \tag{12-39}$$

or in integrated form,

$$\frac{T}{T_1} = \frac{1 + \frac{\kappa - 1}{2} M_1^2}{1 + \frac{\kappa - 1}{2} M^2} \tag{12-40}$$

Finally, the pressure change is obtained from Eq. (12-33) with the aid of Eqs. (12-37a) and (12-39),

$$\frac{dp}{p} = \frac{[-\frac{1}{M} + (1 - \kappa)M]\, dM}{[(\kappa - 1)/2]\, M^2 + 1} = -\left[\frac{1}{M} + \frac{M}{M^2 + 2/(\kappa - 1)}\right] dM \tag{12-41}$$

Integrating, we find

$$\frac{p}{p_1} = \frac{M_1}{M}\left[\frac{1 + \frac{\kappa - 1}{2} M_1^2}{1 + \frac{\kappa - 1}{2} M^2}\right]^{1/2} \tag{12-42}$$

We can now proceed to calculate the friction parameter from Eq. (12-31) which may be written in the form

$$dp = -4C_f \frac{\rho v_z^2}{2D} dz - \rho v_z\, dv_z \tag{12-43a}$$

or

$$\frac{dp}{p} = -4C_f \frac{v_z^2}{2D} \frac{\rho}{p} dz - \frac{\rho}{p} v_z\, dv_z \tag{12-43b}$$

Substituting the relations $a^2 = \kappa(p/\rho)$ and $M = v_z/a$ into Eq. (12-43b), there results

$$4C_f \frac{dz}{2D} \kappa M^2 = -\frac{dp}{p} - \kappa M^2 \frac{dv_z}{v_z} \tag{12-43c}$$

Combining Eqs. (12-37a), (12-41), and (12-43c), we obtain

$$4C_f \frac{dz}{D} = \frac{2(1 - M^2)\, dM}{\kappa M^2 \left(\frac{\kappa - 1}{2} M^2 + 1\right)}$$

$$= \frac{2}{\kappa}\left[\frac{1}{M^3} - \frac{\frac{\kappa + 1}{2}}{M} + \frac{\frac{\kappa + 1}{2} M}{M^2 + \frac{2}{\kappa - 1}}\right] dM \tag{12-44}$$

The integration of Eq. (12-44) yields

$$4C_f \frac{z}{D} = -\frac{1}{\kappa} \frac{1}{M^2} - \frac{\kappa + 1}{2\kappa} \ln M^2 + \frac{\kappa + 1}{2\kappa} \ln\left(M^2 + \frac{2}{\kappa - 1}\right) + C \tag{12-45a}$$

or

$$4C_f \frac{z - z_1}{D} = \frac{1}{\kappa}\left(\frac{1}{M_1^2} - \frac{1}{M^2}\right) + \frac{\kappa + 1}{2\kappa} \ln \frac{M_1^2}{M^2} \frac{1 + \frac{\kappa - 1}{2} M^2}{1 + \frac{\kappa - 1}{2} M_1^2}$$

(12–45b)

where M_1 is the initial Mach number and M the Mach number at any section in the flow. The friction parameter $4C_f(L_{max}/D)$ is obtained by putting M = 1.0 in Eq. (12–45b).

$$4C_f \frac{L_{max}}{D} = \frac{1}{\kappa}\left(\frac{1}{M_1^2} - 1\right) + \frac{\kappa + 1}{2\kappa} \ln \frac{[(\kappa + 1)/2]M_1^2}{1 + [(\kappa - 1)/2]M_1^2} \quad (12–46)$$

Equation (12–46) is true because no entropy increase is possible beyond that required for M = 1. This will be discussed further in the next section. Figure 12.6 shows the variation of the friction parameter with the Mach number.

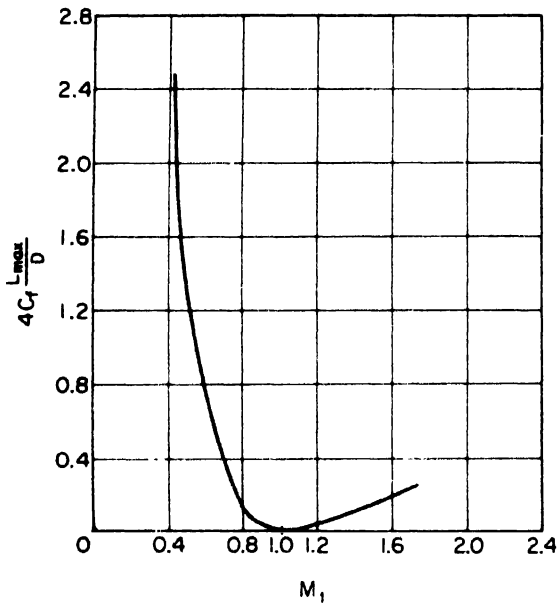


Figure 12.6

Illustrative Example 12.1. Air flows through a pipe (inside diameter = 3 in.) with an initial pressure of 14.7 psi, temperature of 59°F, and Mach number of 3.5. If the frictional coefficient C_f is 0.001, determine the maximum possible length this pipe may have, and values of p, v_z, T, and L at M = 2.5.

Solution. From Eq. (12–46),

$$\frac{4(0.001)}{3/12}L_{\max} = \frac{1}{1.4}\left[\frac{1}{(3.5)^2} - 1\right] + \frac{2.4}{2.8}\ln\left[\frac{(2.4/2)(3.5)^2}{1 + 0.2(3.5)^2}\right]$$

from which the maximum possible length is

$$L_{\max} = \frac{0.5864}{0.016} = 36.6 \text{ ft}$$

The length from the upstream section at M = 3.5 to the section where M = 2.5 may be obtained from Eq. (12–45b).

$$\frac{4(0.001)}{3/12}L = \frac{1}{1.4}\left[\frac{1}{(3.5)^2} - \frac{1}{(2.5)^2}\right] + \frac{2.4}{2.8}\ln\left\{\frac{(3.5)^2}{(2.5)^2}\left[\frac{1 + 0.2(2.5)^2}{1 + 0.2(3.5)^2}\right]\right\} = 0.1544$$

or

$$L = \frac{0.1544}{0.016} = 9.65 \text{ ft}$$

The values of p, v_z, and T at the section where M = 2.5 may be obtained from Eqs. (12–42), (12–40), and (12–37c), respectively.

$$p = (14.7)\left(\frac{3.5}{2.5}\right)\left[\frac{1 + 0.2(3.5)^2}{1 + 0.2(2.5)^2}\right]^{1/2} = 25.5 \text{ lbf/in}^2$$

$$v_z = (3.5)(1117)\left(\frac{2.5}{3.5}\right)\left[\frac{1 + 0.2(3.5)^2}{1 + 0.2(2.5)^2}\right]^{1/2} = 3455 \text{ ft/sec}$$

$$T = (519)\left[\frac{1 + 0.2(3.5)^2}{1 + 0.2(2.5)^2}\right] = 795°\text{R}$$

In routine computation the reader may find the values above directly from tables (*The Dynamics and Thermodynamics of Compressible Fluid Flow, Vol. I*, Ronald Press, New York, 1955, by A. H. Shapiro). However, the tabulated values are given in p/p^*, ρ/ρ^*, and T/T^* where the asterisk quantities refer to the condition where M=1.0. These quantities are calculated from Eqs. (12–42), (12–40), and (12–37c), respectively, by putting $M_1 = 1.0$, i.e.,

$$\frac{p}{p^*} = \frac{1}{M}\left[\frac{\frac{\kappa + 1}{2}}{1 + \frac{\kappa - 1}{2}M^2}\right]^{1/2}$$

$$\frac{\rho}{\rho^*} = \frac{v_z^*}{v_z} = \frac{1}{M}\left[\frac{1 + \frac{\kappa - 1}{2}M^2}{\frac{\kappa + 1}{2}}\right]^{1/2}$$

$$\frac{T}{T^*} = \frac{\frac{\kappa + 1}{2}}{1 + \frac{\kappa - 1}{2}M^2}$$

For $M_1 = 3.5$ the table gives

$$\frac{p_1}{p^*} = 0.1685 \quad \text{or} \quad p^* = \frac{14.7}{0.1685} = 87.2 \text{ lbf/in.}^2$$

$$\frac{v_{z_1}}{v_z^*} = 2.0642 \quad \text{or} \quad v_z^* = \frac{3.5 \times 1117}{2.0642} = 1892 \text{ ft/sec}$$

$$\frac{T_1}{T^*} = 0.3478 \quad \text{or} \quad T^* = \frac{519}{0.3478} = 1491°\text{R}$$

For $M_1 = 2.5$ the table gives

$$\frac{p_2}{p^*} = 0.2921 \quad \text{or} \quad p_2 = 2545 \text{ lbf/in.}^2$$

$$\frac{v_{z_2}}{v_z^*} = 1.826 \quad \text{or} \quad v_{z_2} = 3455 \text{ ft/sec}$$

$$\frac{T_2}{T^*} = 0.5333 \quad \text{or} \quad T_2 = 795 \ °R$$

Hence these values are in agreement with the previous calculations.

c. The Fanno Line

The flow through a constant-area conduit (steady and adiabatic flow with friction) which is under consideration is generally referred to as Fanno flow. The equations given in § 12.2b which represent the variation of fluid properties in Fanno flow are called Fanno equations.

If we rewrite the continuity equation, Eq (12–30a), with the aid of Eq. (12–30b) we obtain

$$\frac{dv_z}{dz} = -\frac{\dot{m}}{\rho^2}\frac{d\rho}{dz} \tag{12–47}$$

Substituting Eq. (12–47) into Eq. (12–32b) results in

$$C_p \frac{dT}{dz} - \frac{\dot{m}^2}{\rho^3}\frac{d\rho}{dz} = 0 \tag{12–48}$$

Equation (12–48) may be integrated to give the relation between the enthalpy and the density of the fluid, i.e.,

$$h = h_1 - \frac{\dot{m}^2}{2}\left(\frac{1}{\rho^2} - \frac{1}{\rho_1^2}\right) \tag{12–49}$$

which is the equation of the Fanno line. The enthalpy and density relations are depicted by a family of curves or Fanno lines in Fig. 12.7.

For an adiabatic but irreversible thermodynamic process there is an entropy increase due to friction. From the thermodynamic relation (Appendix B) the change of entropy may be written in the form

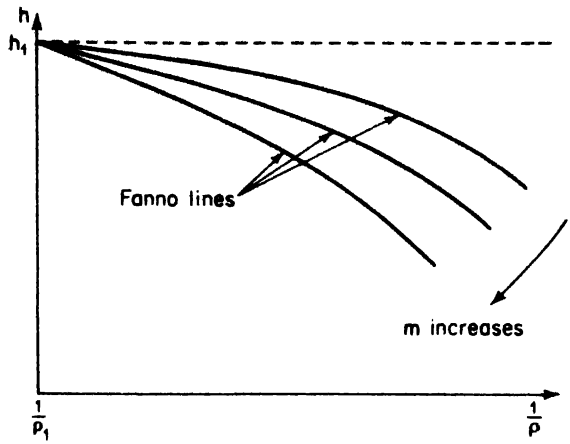


Figure 12.7

$$T\,dS = dh - \frac{dp}{\rho} \tag{12-50a}$$

or

$$dS = R\left[\frac{\kappa}{\kappa - 1}\frac{dT}{T} - \frac{dp}{p}\right. \tag{12-50b}$$

Substituting Eqs. (12-39) and (12-41) into Eq. (12-50b) for dT/T and dp/p, respectively, we obtain a relation between the change of entropy and the Mach number,

$$dS = \frac{R}{\frac{\kappa - 1}{2}\mathrm{M}^2 + 1}\left[-\kappa\mathrm{M} + \frac{1}{\mathrm{M}} - (1 - \kappa)\mathrm{M}\right] d\mathrm{M} \tag{12-51a}$$

or

$$\frac{dS}{d\mathrm{M}} = \frac{R(1 - \mathrm{M}^2)}{\mathrm{M}\left(\frac{\kappa - 1}{2}\mathrm{M}^2 + 1\right)} \tag{12-51b}$$

Equation (12-51b) gives indication that the entropy will be maximum ($dS/d\mathrm{M} = 0$) when $\mathrm{M} = 1.0$. If we rewrite Eq. (12-51b) in the form

$$\frac{dS}{R} = \left|\frac{1}{\mathrm{M}} - \frac{\frac{\kappa + 1}{2}\mathrm{M}}{\frac{\kappa - 1}{2}\mathrm{M}^2 + 1}\right| d\mathrm{M} \tag{12-51c}$$

and integrate, we obtain the entropy expression as a function of Mach number,

$$\frac{S}{R} = \ln \mathrm{M} - \frac{1}{2}\left(\frac{\kappa + 1}{\kappa - 1}\right)\ln\left(\mathrm{M}^2 + \frac{2}{\kappa - 1}\right) + C \tag{12-52}$$

or the change of entropy may be written as

$$\frac{S - S_1}{R} = \ln \frac{M}{M^{1}}^{|(\varepsilon+1)/2(\varepsilon-1)} \tag{12-53}$$

Equation (12–53) together with Eq. (12–40) may be used to construct a T–S diagram (or h–S diagram) for various Mach numbers. The general shape of this curve (Fanno line) for a given mass flow is shown in Fig. 12.8. The upper branch of the Fanno line (Fig. 12.8)

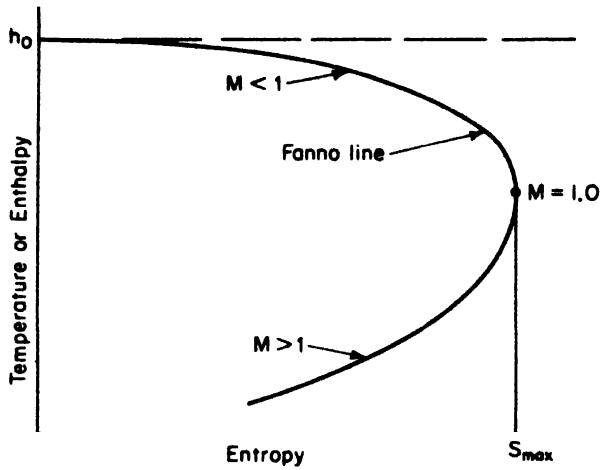


Figure 12.8

indicates that the flow is subsonic; and as the entropy increases, the flow velocity increases. When the value of the entropy reaches its maximum, the Mach number of the flow becomes unity as indicated in Eq. (12–51b). On the other hand, the supersonic flow is represented by the lower portion of the Fanno line. Again, the entropy becomes maximum at $M = 1$; however, in this case, the Mach number decreases as the entropy increases. It is seen that the effect of friction is to decelerate a flow when it is supersonic and accelerate it when subsonic. If there is enough friction in the pipe, the flow Mach number in both cases will approach the limiting condition of unity. Furthermore, if friction continues to exist after the entropy becomes a maximum the flow will be choked (a shock wave occurs before choking if the flow is originally supersonic).

d. Friction and Thermodynamic Properties

The effects of friction on the thermodynamic properties of the flow through a constant-area pipe can now be illustrated from the results

obtained in the preceding two sections. According to Eqs. (12–43c), (12–36), and (12–35) the pressure change along a Fanno line in terms of the friction function is obtained as follows:

$$\frac{dp}{p} = -4C_f\frac{dz}{D}\frac{\kappa M^2}{2} - \kappa M^2\frac{dv_z}{v_z} \tag{12–43c}$$

$$\frac{dv_z}{v_z} = \frac{dT}{T} - \frac{dp}{p} \tag{12–36}$$

$$\frac{dT}{T} = -(\kappa - 1)M^2\frac{dv_z}{v_z} \tag{12–35}$$

Combining these three equations, the pressure change along a Fanno line in terms of the friction function is found,

$$\frac{dp}{p} = -\frac{\kappa M^2[1 + (\kappa - 1)M^2]}{2(1 - M^2)}\left(4C_f\frac{dz}{D}\right) \tag{12–54}$$

The velocity and density changes can be obtained from Eqs. (12–54) and (12–43c) as

$$\frac{dv_z}{v_z} = -\frac{d\rho}{\rho} = \frac{\kappa M^2}{2(1 - M^2)}\left(4C_f\frac{dz}{D}\right) \tag{12–55}$$

From Eqs. (12–35) and (12–55), the temperature change is obtained:

$$\frac{dT}{T} = -\frac{\kappa(\kappa - 1)M^4}{2(1 - M^2)}\left(4C_f\frac{dz}{D}\right) \tag{12–56}$$

Finally, the entropy change is found by the substitution of Eqs. (12–54) and (12–56) into Eq. (12–50b):

$$\frac{dS}{R} = \frac{\kappa M^2}{2}\left(4C_f\frac{dz}{D}\right) \tag{12–57}$$

In accordance with the expressions given in Eqs. (12–54)–(12–57), the effect of friction on the thermodynamic and flow properties can be summarized in the following table:

Property	Effect of Friction	
	Initially Subsonic Flow	Initially Supersonic Flow
Pressure	decreases	increases
Density	decreases	increases
Temperature	decreases	increases
Entropy	increases	increases
Velocity	increases	decreases

Illustrative Example 12.2. Air flows from a reservoir at standard conditions through a convergent-divergent nozzle to a constant cross-section

pipe. Assuming the mass flow rate from the reservoir to the Fanno flow system is 0.362 slugs/ft²-sec, construct a T–S diagram under Fanno flow conditions.

Solution. According to illustrative example 11.5, the temperature and Mach number at the beginning of the Fanno flow are

$$T_1 = 184.5°R \qquad \text{and} \qquad M_1 = 3.0$$

According to Eqs. (12–40) and (12–53) we have

$$T_2 = T_1\left[\frac{1 + \frac{\kappa - 1}{2}M_1^2}{1 + \frac{\kappa - 1}{2}M_2^2}\right] = 184.5\left[\frac{2.8}{1 + 0.2M_2^2}\right]°R$$

$$\frac{S_2 - S_1}{C_p} = \frac{\kappa - 1}{\kappa}\ln\frac{M_2}{M_1}\left[\frac{1 + \frac{\kappa - 1}{2}M_1^2}{1 + \frac{\kappa - 1}{2}M_2^2}\right]^{(\kappa+1)/2(\kappa-1)}$$

$$= 0.286\ln\left\{\frac{M_2}{3}\left[\frac{2.8}{1 + 0.2M_2^2}\right]^3\right\}$$

The values of temperature and entropy for various values of M_2 are tabulated as follows:

M_2	T_2	$(S_2 - S_1)/C_p$
0.2	512.9°R	0.1020
0.3	507.9	0.2094
0.6	482.3	0.3631
0.8	458.3	0.4016
1.0	430.8	0.4124
1.2	401.4	0.4038
1.4	371.4	0.3813
1.6	341.9	0.3486
1.8	313.7	0.3084
2.2	262.7	0.2136
2.6	219.8	0.1086
3.0	184.6	0.0000

The Fanno line curve is plotted in Fig. 12.A.

12.3. *Laminar Boundary Layer Equations in Compressible Flow*

The concept of boundary layer flow of viscous fluids in the vicinity of a solid wall has been discussed in § 9. 1. The mathematical procedure for reducing the Navier–Stokes equations to the boundary layer

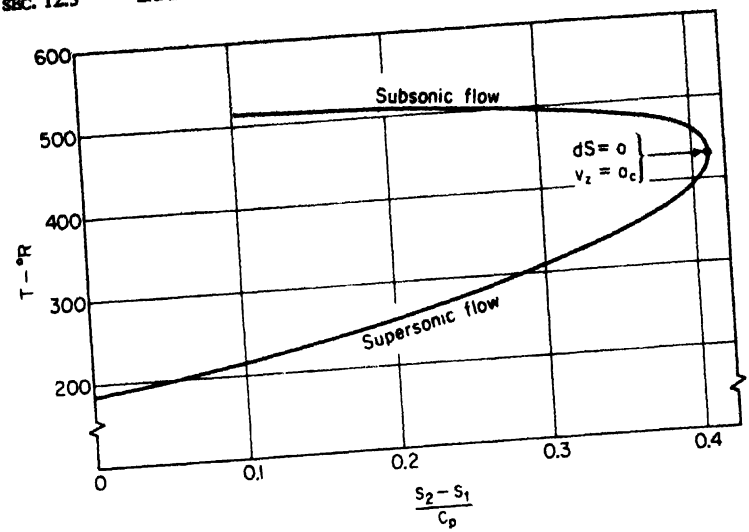


Figure 12.A

equations in two-dimensional incompressible flow was given in §9.2. In incompressible laminar boundary layers the relative orders of magnitude of u, v, x, and y were seen to be 1, δ, 1, and δ, respectively.

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{12-59}$$

The energy equation is obtained from Eqs. (5-33) and (5-23b) as

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \left[2\mu\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2 + \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right] = \rho\frac{\partial(C_p T)}{\partial t} + \rho u\frac{\partial(C_p T)}{\partial x} + \rho v\frac{\partial(C_p T)}{\partial y} - \frac{\partial p}{\partial t} - u\frac{\partial p}{\partial x} - v\frac{\partial p}{\partial y} \tag{12-60}$$

Since the Prandtl number $\mu C_p/k$ of a gas is of the order of unity and C_p is almost constant for usual gases at moderate temperatures, μ and k can be considered as quantities of the same order. With these orders of magnitude in mind and following the same procedure as that of § 9. 2, Eqs. (12-58)-(12-60) reduce to, respectively,

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{12-61a}$$

$$\frac{\partial p}{\partial y} = 0 \tag{12-61b}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{12-62}$$

$$\rho\frac{\partial(C_p T)}{\partial t} + \rho u\frac{\partial(C_p T)}{\partial x} + \rho v\frac{\partial(C_p T)}{\partial y} = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} \tag{12-63}$$

Equations (12-61a)-(12-63) are called the boundary layer equations for a two-dimensional compressible flow. The boundary conditions for a steady flow are:

$$\begin{aligned} y = 0: &\quad u = v = 0, \quad T = T_w(x), \quad p = p_1(x) \\ y = \infty: &\quad u = U(x), \quad T = T_1(x), \quad p = p_1(x) \end{aligned} \tag{12-64}$$

where the subscripts 1 and w indicate the conditions at the outer edge of the boundary layer and the wall, respectively.

The equations above are derived without consideration of the curvature of the wall. However, if the radius of curvature of the wall is large compared with the thickness of the boundary layer, the equations developed here hold also for curved walls.

12.4. Velocity and Temperature Relation in Laminar Boundary Layers

a. Boundary Layer with Pressure Gradient

Let us consider the case of two-dimensional steady flow in a laminar boundary layer (Fig. 9. 3). The system of boundary layer equations [Eq. (12–61a)–(12–63)] in this case becomes

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{12–65}$$

$$\frac{\partial p}{\partial y} = 0 \tag{12–61b}$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{12–66}$$

$$\rho u \frac{\partial(C_p T)}{\partial x} + \rho v \frac{\partial(C_p T)}{\partial y} = u \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{12–67}$$

In general, the solution of the system of equations above together with the equation of state for a perfect gas for given values of $U(x)$ and $T_w(x)$ is very involved. However, in the particular case when Pr $= 1$, the integration of the differential equations of the boundary layer becomes much simpler. If we assume that the Prandtl number is equal to unity, i.e.,

$$\mu C_p = k \tag{12–68a}$$

and that the temperature T is a function of the velocity component u only,

$$C_p T = f(u) \tag{12–68b}$$

then Eq. (12–67) can be rewritten as

$$\left(\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y}\right) \frac{df(u)}{du} = u \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left[\mu \frac{\partial u}{\partial y} \frac{df(u)}{du}\right] + \mu\left(\frac{\partial u}{\partial y}\right)^2$$

$$= u \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \frac{df(u)}{du} + \mu\left(\frac{\partial u}{\partial y}\right)^2 \left[\frac{d^2 f(u)}{du^2} + 1\right] \tag{12–69}$$

It is seen that Eq. (12–69) reduces to Eq. (12–65) when

$$\frac{df(u)}{du} = -u \tag{12–70a}$$

and

$$\frac{d^2f(u)}{du^2} = -1 \tag{12-70b}$$

According to these conditions Eq. (12-68) becomes

$$C_pT = \text{Constant} - \frac{u^2}{2} \tag{12-71a}$$

The constant of integration in Eq. (12-71) must be the stagnation enthalpy of the free stream [see Eq. (11-21)], i.e.,

$$C_pT_0 = C_pT_1 + \frac{U^2}{2} \tag{12-71b}$$

Hence we can rewrite Eq. (12-71a) as

$$C_pT = C_pT_0 - \frac{u^2}{2} \tag{12-72}$$

which reveals that the sum of the enthalpy and the kinetic energy is a constant throughout the boundary layer. The heat transfer into or from the wall is [Eq. (12-8d)]

$$-q_w \sim \left(\frac{\partial T}{\partial y}\right)_w \sim \left(u\frac{\partial u}{\partial y}\right)_{y=0} = 0 \tag{12-73}$$

This is the condition of the so-called adiabatic wall. Therefore, if $\text{Pr} = 1$ and $q_w = 0$ the stagnation temperature is equal to the wall temperature

$$T_0 = T_w \tag{12-74}$$

b. Boundary Layer with Zero Pressure Gradient

When the pressure gradient along the wall is zero (flow along a flat plate as shown in Fig. 9.4), the system of boundary layer equations, Eqs. (12-65)-(12-67), can be further simplified by letting $\partial p/\partial x = 0$. These equations are

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{12-75}$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{12-76}$$

$$\rho u\frac{\partial(C_pT)}{\partial x} + \rho v\frac{\partial(C_pT)}{\partial y} = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + u\left(\frac{\partial u}{\partial y}\right)^2 \tag{12-77}$$

For the Prandtl number equal to one and the temperature and velocity relation as given in Eq. (12-68b), we can rewrite Eq. (12-77) as

$$\left(\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y}\right)\frac{df(u)}{du} = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)\frac{df(u)}{du} + \mu\left(\frac{\partial u}{\partial y}\right)^2\left[\frac{d^2f(u)}{du^2} + 1\right] \tag{12-78}$$

Equation (12–78) reduces now to Eq. (12–65) only if

$$\frac{d^2 f(u)}{du^2} = -1 \tag{12–70b}$$

which can be integrated to give

$$C_p T = f(u) = C_1 + C_2 u - \frac{u^2}{2} \tag{12–79}$$

where C_1 and C_2 are integration constants. The boundary conditions from which the two constants are to be determined are:

$$\begin{aligned} y = 0: \quad & u = v = 0, \quad && T = T_w = \text{Constant} \\ y = \infty: \quad & u = U_\infty, && T = T_\infty = \text{Constant} \end{aligned} \tag{12–80}$$

Therefore, after applying conditions (12–80) to Eq. (12–79),

$$C_p T_w = C_1 \tag{12–81}$$

and

$$C_p T_\infty = C_1 + C_2 U_\infty - \frac{U_\infty^2}{2} \tag{12–82}$$

Solving for C_1 and C_2 and substitutiug these values into Eq. (12–79), we obtain[1]

$$\begin{aligned} \frac{T}{T_\infty} = \frac{T_w}{T_\infty} &= \left(1 - \frac{T_w}{T_\infty}\right)\frac{u}{U_\infty} + \frac{1}{2C_p T_\infty}(U_\infty - u)u \\ &= \frac{T_w}{T_\infty} + \left(1 - \frac{T_w}{T_\infty}\right)\frac{u}{U_\infty} + \frac{\kappa - 1}{2}\mathrm{M}_\infty^2\left(1 - \frac{u}{U_\infty}\right)\frac{u}{U_\infty} \end{aligned} \tag{12–83}$$

where $C_p T_\infty = a_\infty^2/(\kappa - 1)$ and $\mathrm{M}_\infty = U_\infty/a_\infty$, the free-stream Mach number. Figure 12.9 shows the temperature and velocity relation for various Mach numbers.

The shearing stress at the wall and the heat conduction to the wall per unit area per unit time are defined, respectively, as [Eqs. (12–7) and (12–8d)]

$$\tau_w = \left(\mu \frac{\partial u}{\partial y}\right)_w \tag{12–7}$$

$$q_w = \left(k \frac{\partial T}{\partial y}\right)_w \tag{12–8d}$$

Differentiating Eq. (12–83) and substituting the result into Eq. (12–8d), we have

$$q_w = k_w T_\infty \left[\left(1 - \frac{T_w}{T_\infty}\right)\frac{1}{U_\infty}\left(\frac{\partial u}{\partial y}\right)_w + \frac{\kappa - 1}{2}\mathrm{M}_\infty^2 \frac{1}{U_\infty}\left(\frac{\partial u}{\partial y}\right)_w\right] \tag{12–84}$$

With the aid of Eqs. (12–68a) and (12–7) the relation between the heat flux and shearing stress at the wall for $\mathrm{Pr} = 1$ is obtained:

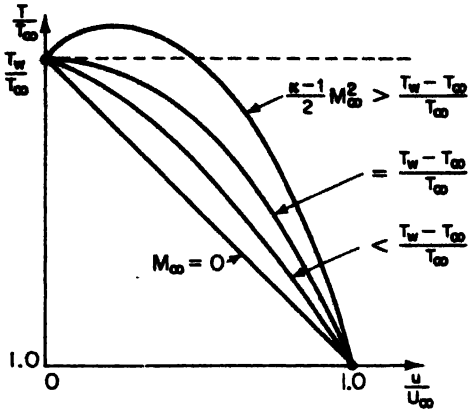


Figure 12.9

$$q_w = \frac{C_p T_\infty \tau_w}{U_\infty}\left[\left(1 - \frac{T_w}{T_\infty}\right) + \frac{\kappa - 1}{2}M_\infty^2\right] \tag{12-85}$$

If the stagnation temperature

$$\frac{T_0}{T_\infty} = 1 + \frac{\kappa - 1}{2}M_\infty^2 \tag{11-87b}$$

is introduced into Eq. (12–85) then q_w can be written in the form

$$q_w = \frac{C_p \tau_w}{U_\infty}[T_0 - T_w] \tag{12-86}$$

Equation (12–86) indicates clearly that due to dynamic heating of the fluid, the reference temperature which is used to calculate the temperature drop for heat conduction is actually the stagnation temperature and not the static temperature T_∞. Equation (12–86) also shows that an adiabatic wall exists ($q_w = 0$) when $T_0 = T_w$, that heat is conducted into the wall ($q_w > 0$) when $T_0 > T_w$, and that heat is conducted to the fluid ($q_w < 0$) when $T_0 < T_w$. The last phenomenon is generally referred to as cooling the wall. The important relation between q_w and τ_w represented by Eq. (12–86) is called the generalized Reynolds analogy with heat transfer and skin friction. For an incompressible fluid ($M = 0$) it can easily be shown that the following relation between coefficients of heat transfer and friction exists,

$$C_h = \frac{C_f}{2} \tag{12-87}$$

where the heat transfer coefficient is defined as

$$C_h = \frac{q_w}{\rho U_\infty C_p (T_\infty - T_w)} \tag{12-88}$$

12.5. Integral Theorem for Boundary Layers

As mentioned in § 12.4, the solution of the system of boundary layer equations, Eqs. (12–65)–(12–67) and Eq. (5–37), is in general very difficult. In § 9.5 we have discussed the momentum integral equation for incompressible fluids which was obtained by integrating the Prandtl boundary layer equations with respect to y. In a like manner the momentum integral equation for compressible fluids can be derived.

Let us integrate first the dynamic equation, Eq. (12–65), with respect to y from $y = 0$ to $y = \delta(x)$, the outer edge of the boundary layer (Fig. 12.10a). The result is

$$\int_0^\delta \rho u \frac{\partial u}{\partial x} dy + \int_0^\delta \rho v \frac{\partial u}{\partial y} dy = -\int_0^\delta \frac{\partial p}{\partial x} dy - \tau_w \tag{12–89}$$

where $\tau_{yx} = 0$ at $y = \delta$ is implied. The second term on the left-hand side of Eq. (12–89) can be integrated by parts.

$$\begin{aligned} \int_0^\delta \rho v \frac{\partial u}{\partial y} dy &= \int_0^\delta \frac{\partial(\rho u v)}{\partial y} dy - \int_0^\delta u \frac{\partial(\rho v)}{\partial y} dy \\ &= [\rho u v]_0^\delta - \int_0^\delta u \frac{\partial(\rho v)}{\partial y} dy \\ &= U \int_0^\delta \frac{\partial(\rho v)}{\partial y} dy - \int_0^\delta u \frac{\partial(\rho v)}{\partial y} dy \end{aligned} \tag{12–90a}$$

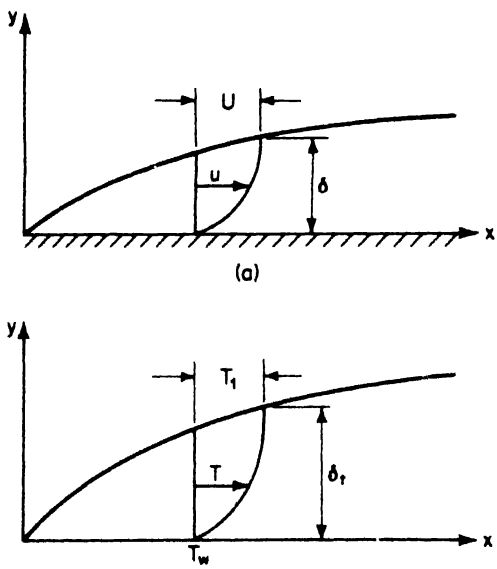


Figure 12.10

With the aid of the continuity equation, Eq. (12-66), Eq. (12-90a) becomes

$$\int_0^\delta \rho v \frac{\partial u}{\partial y} dy = -U \int_0^\delta \frac{\partial(\rho u)}{\partial x} dy + \int_0^\delta u \frac{\partial(\rho u)}{\partial x} dy \qquad (12\text{-}90b)$$

Substituting Eq. (12-90b) into Eq. (12-89) and making use of the relation given in Eq. (9-18), we obtain

$$\int_0^\delta \rho u \frac{\partial u}{\partial x} dy - U \int_0^\delta \frac{\partial(\rho u)}{\partial x} dy + \int_0^\delta u \frac{\partial(\rho u)}{\partial x} dy = \frac{dU}{dx} \int_0^\delta \rho_1 U \, dy - \tau_w \qquad (12\text{-}91)$$

But

$$U \int_0^\delta \frac{\partial(\rho u)}{\partial x} dy = \int_0^\delta \frac{\partial}{\partial x}(\rho u U) dy - \frac{dU}{dx} \int_0^\delta \rho u \, dy \qquad (12\text{-}92a)$$

$$\int_0^\delta u \frac{\partial(\rho u)}{\partial x} dy = \int_0^\delta \frac{\partial}{\partial x}(\rho u^2) dy - \int_0^\delta \rho u \frac{\partial u}{\partial x} dy \qquad (12\text{-}92b)$$

and the first term on the right-hand side of Eqs. (12-92a) and (12-92b) can, according to the Leibnitz rule, be written in the forms, respectively,

$$\int_0^\delta \frac{\partial}{\partial x}(\rho u U) dy = \frac{d}{dx} \int_0^\delta \rho u U \, dy - \rho_1 U^2 \frac{d\delta}{dx} \qquad (12\text{-}92c)$$

$$\int_0^\delta \frac{\partial}{\partial x}(\rho u^2) dy = \frac{d}{dx} \int_0^\delta \rho u^2 \, dy - \rho_1 U^2 \frac{d\delta}{dx} \qquad (12\text{-}92d)$$

Hence, Eq. (12-91) becomes

$$\frac{d}{dx} \int_0^\delta \rho u (U - u) dy + \frac{dU}{dx} \int_0^\delta (\rho_1 U - \rho u) dy = \tau_w \qquad (12\text{-}93)$$

This is the momentum integral equation of the boundary layer in compressible fluids.

The energy integral equation can be obtained in a similar manner by integrating Eq. (12-67) with respect to y from $y = 0$ to $y = \delta_t$, the thickness of the thermal boundary layer (Fig. 12.10b). The integral energy equation becomes

$$\int_0^{\delta_t} \rho u \frac{\partial(C_p T)}{\partial x} dy + \int_0^{\delta_t} \rho v \frac{\partial(C_p T)}{\partial y} dy = \int_0^{\delta_t} u \frac{\partial p}{\partial x} + \int_0^{\delta_t} \mu \left(\frac{\partial u}{\partial y}\right)^2 - q_w \qquad (12\text{-}94)$$

The second term on the left-hand side of Eq. (12-94) can be written as

$$\int_0^{\delta_t} \rho v \frac{\partial(C_p T)}{\partial y} dy = \int_0^{\delta_t} \frac{\partial}{\partial y}(\rho v C_p T)\, dy - \int_0^{\delta_t} C_p T \frac{\partial(\rho v)}{\partial y} dy$$

$$= C_p T_1 \int_0^{\delta_t} \frac{\partial(\rho v)}{\partial y} dy - \int_0^{\delta_t} C_p T \frac{\partial(\rho v)}{\partial y} dy$$

$$= -\int_0^{\delta_t} \frac{\partial}{\partial x}(C_p T_1 \rho u)\, dy + \frac{d(C_p T_1)}{dx} \int_0^{\delta_t} \rho u\, dy$$

$$+ \int_0^{\delta_t} C_p T \frac{\partial(\rho u)}{\partial x} dy \qquad (12\text{–}95a)$$

However, in the free stream the energy equation is

$$\tfrac{1}{2}(U^2 + v_1^2) + C_p T_1 = \text{Constant} \qquad (11\text{–}21)$$

Differentiating Eq. (11–21) with respect to x, we have

$$U \frac{dU}{dx} + v_1 \frac{dv_1}{dx} = -\frac{d(C_p T_1)}{dx} \qquad (12\text{–}95b)$$

But in the free stream since the flow is irrotational, it follows that

$$\frac{\partial v_1}{\partial x} = \frac{\partial U}{\partial y} = 0 \qquad (12\text{–}95c)$$

With the aid of Eqs. (9–18) and (12-95c), Eq. (12–95b) becomes

$$\frac{\partial p}{\partial x} = \rho_1 \frac{\partial(C_p T_1)}{\partial x} \qquad (12\text{-}95d)^*$$

After substituting Eqs. (12–95a) and (12–95d) into Eq. (12–94), the result may be written as

$$\int_0^{\delta_t} \rho u \frac{\partial(C_p T)}{\partial x} dy - \int_0^{\delta_t} \frac{\partial}{\partial x}(C_p T_1 \rho u)\, dy + \frac{d(C_p T_1)}{dx} \int_0^{\delta_t} \rho u\, dy$$

$$+ \int_0^{\delta_t} C_p T_1 \frac{\partial(\rho u)}{\partial x} dy = \int_0^{\delta_t} \rho_1 u \frac{\partial(C_p T_1)}{\partial x} dy + \int_0^{\delta_t} \mu \left(\frac{\partial u}{\partial y}\right)^2 dy - q_w$$

and simplifying, we find

$$\frac{d}{dx} \int_0^{\delta_t} \rho u C_p (T_1 - T)\, dy + \frac{d(C_p T_1)}{dx} \int_0^{\delta_t} u(\rho_1 - \rho)\, dy + \int_0^{\delta_t} \mu \left(\frac{\partial u}{\partial y}\right)^2 dy = q_w \qquad (12\text{–}96)$$

Equation (12–96) is the integral relation of the energy equation in compressible fluids.

An approximate solution of the boundary layer problem can now be made by assuming *profiles* for both the velocity and the temperature in the boundary layer in the forms

*This equation could have been derived immediately from Eq. (12–67).

$$\frac{u}{U} = f\left(\frac{y}{\delta}\right) \quad \text{for} \quad 0 < y < \delta$$
$$= 1 \quad \text{for} \quad y \geq \delta \tag{12-97}$$

and

$$\frac{T}{T_1} = g\left(\frac{y}{\delta_t}\right) \quad \text{for} \quad 0 < y < \delta_t$$
$$= 1 \quad \text{for} \quad y \geq \delta_t \tag{12-98}$$

The functions f and g are chosen to satisfy the following boundary conditions as nearly as possible:

(a) At the outer edges of the boundary layer

$$y = \delta: \quad \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial^3 u}{\partial y^3} = \cdots = 0 \tag{12-99a}$$

and

$$y = \delta_t: \quad \frac{\partial T}{\partial y} = \frac{\partial^2 T}{\partial y^2} = \frac{\partial^3 T}{\partial y^3} = \cdots = 0 \tag{12-99b}$$

(b) At the wall, $y = 0$:

$$u = 0, \quad \frac{\partial p}{\partial x} = \left[\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)\right]_w \tag{12-99c}$$

and

$$T = T_w, \quad \left[\frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right)\right]_w = -\left[\mu\left(\frac{\partial u}{\partial y}\right)^2\right]_w \tag{12-99d}$$

When the proper functions f and g are chosen they can be substituted into Eqs. (12–93) and (12–96), respectively, yielding two simultaneous first-order partial differential equations for δ and δ_t. By solving these equations, the values of the boundary layer thicknesses δ and δ_t together with τ_w and q_w can be obtained. It might be mentioned that, in general, the thermal boundary layer thickness δ_t is greater than the gas boundary layer thickness δ.

12.6: *Application of the Momentum Integral Equation to Boundary Layers*

The momentum integral equation, Eq. (12–93), may be rewritten in nondimensional form as

$$\frac{d}{dx}\int_0^\delta \frac{\rho u}{\rho_1 U}\left(1 - \frac{u}{U}\right) dy + \left(\frac{1}{\rho_1}\frac{d\rho_1}{dx} + \frac{2}{U}\frac{dU}{dx}\right)\int_0^\delta \frac{\rho u}{\rho_1 U}\left(1 - \frac{u}{U}\right) dy$$
$$+ \frac{1}{U}\frac{dU}{dx}\int_0^\delta \left(1 - \frac{\rho u}{\rho_1 U}\right) dy = \frac{\tau_w}{\rho_1 U^2} \tag{12-100}$$

If we introduce the displacement and momentum thicknesses, respectively,

$$\delta^* = \int_0^\delta \left(1 - \frac{\rho u}{\rho_1 U}\right) dy \tag{12-101}$$

and

$$\theta = \int_0^\delta \frac{\rho u}{\rho_1 U}\left(1 - \frac{u}{U}\right) dy \tag{12-102}$$

and take into account that

$$-\rho_1 U \frac{dU}{dx} = \frac{dp}{dx} = a_1^2 \frac{d\rho_1}{dx} \tag{12-103}$$

a convenient form of the momentum integral equation is obtained,

$$\frac{d\theta}{dx} + \left(2 - \mathrm{M}_1^2 + \frac{\delta^*}{\theta}\right)\frac{\theta}{U}\frac{dU}{dx} = \frac{\tau_w}{\rho_1 U^2} \tag{12-104}$$

where $\mathrm{M}_1 = U/a_1$

In evaluating δ^*, θ, and τ_w we still need the expressions for ρ and μ as a function of the absolute temperature T. Since the pressure remains constant across the boundary layer, it follows from the equation of state for a perfect gas that

$$\frac{\rho}{\rho_1} = \frac{T_1}{T} \tag{12-105}$$

The relation between μ and T is given in Eq. (12-10) as follows:

$$\frac{\mu}{\mu_1} = \left(\frac{T}{T_1}\right)^m \tag{12-106}$$

For the Prandtl number $\mathrm{Pr} = 1$ and $dp/dx = 0$, the temperature profile in the boundary layer can be obtained directly from the corresponding velocity profile. This relation is given in Eq. (12-83).

As an example, let us consider the problem of a uniform flow passing over an insulated flat plate. If $\mathrm{Pr} = 1$ is assumed the temperature and velocity relation can be expressed by

$$\frac{T}{T_\infty} = \frac{T_w}{T_\infty} + \left(1 - \frac{T_w}{T_\infty}\right)\frac{u}{U_\infty} + \frac{\kappa - 1}{2}\mathrm{M}_\infty^2\left(1 - \frac{u}{U_\infty}\right)\frac{u}{U_\infty} \tag{12-83}$$

For an adiabatic wall, we have [see Eq. (12-86)]

$$\frac{T_w}{T_\infty} = \frac{T_0}{T_\infty} = 1 + \frac{\kappa - 1}{2}\mathrm{M}_\infty^2$$

Substituting this wall temperature relation into Eq. (12-83), we find

$$\frac{T}{T_\infty} = 1 + \frac{\kappa - 1}{2}\mathrm{M}_\infty^2\left(1 - \frac{u^2}{U_\infty^2}\right) \tag{12-107}$$

According to Eqs. (12–105) and (12–106) the density and viscosity relations are, respectively,

$$\frac{\rho}{\rho_\infty} = \frac{T_\infty}{T} = \frac{1}{1 + \alpha - \alpha(u^2/U_\infty^2)} = \frac{1}{\alpha_1}\left[\frac{1}{1 - k\left(\frac{u}{U_\infty}\right)^2}\right] \tag{12–108}$$

$$\frac{\mu_w}{\mu_\infty} = \left(\frac{T_w}{T_\infty}\right)^m = \alpha_1^m \tag{12–109}$$

where

$$\alpha = \frac{\kappa - 1}{2}\mathrm{M}_\infty^2, \qquad \alpha_1 = 1 + \frac{\kappa - 1}{2}\mathrm{M}_\infty^2 \qquad \text{and} \qquad k = \frac{\alpha}{\alpha_1}$$

In order to simplify the calculation, a linear velocity profile is assumed, i.e.,

$$\frac{u}{U_\infty} = \left(\frac{y}{\delta}\right) = \eta \tag{12–110}$$

The momentum thickness may be calculated from Eq. (12–102) with the aid of Eqs. (12–108) and (12–110), and the thickness θ is given by

$$\begin{aligned}\theta &= \frac{\delta}{\alpha_1}\int_0^1 \frac{\eta(1-\eta)}{1 - k\eta^2}\,d\eta \\ &= \frac{\delta}{\alpha_1}\left[\frac{1}{k}\int_0^{\sqrt{k}} \frac{z\,dz}{1 - z^2} - \frac{1}{k^{3/2}}\int_0^{\sqrt{k}} \frac{z^2}{1 - z^2}\,dz\right] \\ &= \frac{\delta}{\alpha_1 k}\left[1 - \frac{1}{2}\ln(1-k) - \frac{1}{2\sqrt{k}}\ln\frac{1+\sqrt{k}}{1-\sqrt{k}}\right] = \frac{\delta}{\alpha_1}k_1\end{aligned} \tag{12–111}$$

where $z = \sqrt{k}\ \eta$ and

$$k_1 = \frac{1}{k}\left[1 - \frac{1}{2}\ln(1-k) - \frac{1}{2\sqrt{k}}\ln\frac{1+\sqrt{k}}{1-\sqrt{k}}\right] \tag{12–112}$$

Substituting Eqs. (12–111) and (12–109) into the momentum integral equation, Eq. (12–104), yields the thickness of the boundary layer,

$$\frac{\delta}{x} = \frac{1}{\sqrt{\mathrm{Re}_x}}\sqrt{\frac{2}{k_1}}(\alpha_1)^{(m+1)/2} \tag{12–113}$$

where $\mathrm{Re}_x = U_\infty x/\nu_\infty$. The total local skin-friction coefficient is

$$C_f = \frac{2\tau_w}{\rho_\infty(U^2/2)} = \frac{2\sqrt{2k_1}}{\sqrt{\mathrm{Re}_x}}\alpha_1^{(m-1)/2} \tag{12–114}$$

von Kármán and Tsien[2] found an exact solution of the boundary layer equations, Eqs. (12–65), for uniform flow passing over a flat plate of constant wall temperature. In this analysis the Prandtl number is assumed equal to unity and the temperature distributions are

calculated from the expression given in Eq. (12-83). In order to compare the coefficient of friction versus the free-stream Mach number given in Eq. (12-114) with that of von Kármán-Tsien's result, both cases are based on an insulated wall and the exponent $m = 0.76$. It can be seen from Fig. 12.11 that the linear profile of velocity approximation is amazingly good in the range of high Mach numbers. The reason for this good agreement is because the velocity distribution in the boundary layer approaches a linear profile when the Mach number reaches a value of 6 and beyond. This can be seen from Fig. 12.12a which was based on von Kármán-Tsien's result for an insulated plate. Based on the velocity distributions given in Fig. 12.12a the temperature distributions versus the Mach number in the boundary layer are shown in Fig. 12.12b.

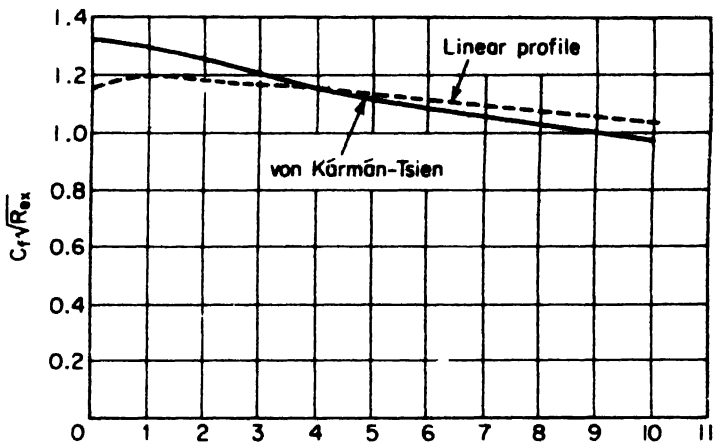


Figure 12.11

NOMENCLATURE FOR CHAPTER TWELVE

C_h	coefficient of heat transfer, Eq. (12-88)
C_f	coefficient of skin friction, Eqs. (12-24) and (12-114)
$4C_f\,(L_{max}/D)$	friction parameter in Fanno flow, Eq. (12-46)
C_p	specific heat at constant pressure, Eq. (12-3)
D	diameter of the pipe, Eq. (12-31)
$f(u)$	enthalpy or temperature function, Eq. (12-68b)
h	(1) distance between two plates, Eq. (12-4b) and Fig. 12.1 (2) enthalpy per unit mass, Eq. (12-50a)
k	thermal conductivity, Eq. (12-3)

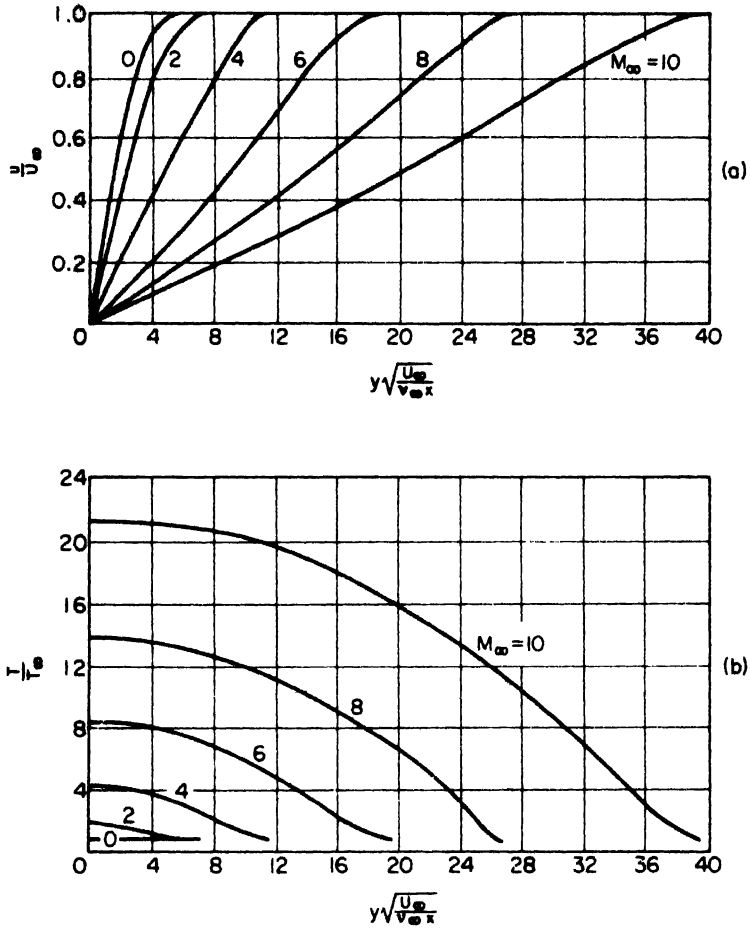


Figure 12.12

L_{max}	maximum length in Fanno flow, Eq. (12-46)
m	an exponent, Eq. (12-10)
$\dot{m}$	mass flow, Eq. (12-30b)
M	Mach number, Eq. (12-34)
M_∞	(1) Mach number of the plate motion, § 12.1b (2) Mach number of the free stream, Eq. (12-83)
p	pressure, Eq. (12-1)
Pr	Prandtl number, Eq. (12-4)
q	heat flux, Eq. (12-46)
q_w	heat flow to the wall, Eqs. (12-4b) and (12-8d)
$\mathbf{q}$	velocity vector, Eq. (12-1)
R	gas constant, Eq. (12-50b)

Re_x	Reynolds number, Eq. (12-113)
r, θ, z	cylindrical coordinates
S	entropy per unit mass, Eq. (12-50a)
T	temperature, Eq. (12-32a)
T_∞	(1) temperature at the moving plate, § 12.1b (2) free-stream temperature, Eq. (12-80)
T_r	recovery temperature, Eq. (12-23)
t	time, Eq. (12-58a)
U	(1) constant velocity of the moving plate, § 12.1b (2) velocity in the potential flow, Eq. (12-64)
U_∞	free stream velocity, Eq. (12-80)
u, v, w	velocity in Cartesian coordinates
v_r, v_θ, v_z	velocity in cylindrical coordinates
x, y, z	Cartesian coordinates
δ	boundary layer thickness, Eq. (12-89)
δ_t	thermal boundary layer thickness, Eq. (12-96)
δ^*	displacement thickness, Eq. (12-101)
η	nondimensional distance parameter, Eq. (12-110)
θ	momentum thickness, Eq. (12-102)
κ	ratio of specific heats, Eq. (12-15)
μ	coefficient of viscosity, Eq. (12-1)
$\tau_w = (\tau_{yx})_w$	wall shearing stress, Eq. (12-7)
$(\ )_1$	quantity in the free stream, Eq. (12-64)
$(\ )_0$	stagnation quantity, Eq. (12-72)
$(\ )_w$	quantity at the wall, Eq. (12-7)

PROBLEMS

12.1. Calculate and plot the velocity distribution for Couette flow using the following viscosity relation:

$$\frac{\mu}{\mu_\infty} = \left(\frac{T}{T_\infty}\right)^{1/2}$$

Compare the result with that shown in Fig. 12.2 for $m = 1$ and $m = 0.76$. $q_w = 0$ and $\kappa = 1.4$ may be assumed.

12.2. Calculate and plot the temperature distribution for Couette flow using the viscosity relation

$$\frac{\mu}{\mu_\infty} = \left(\frac{T}{T_\infty}\right)^{1/2}$$

for

(a) $q_w/\tau_w u = 1.0$,
(b) $q_w/\tau_w u = 0$, and
(c) $q_w/\tau_w u = -0.3$.

Assume $\mathrm{Pr}\,\mathrm{M}_\infty^2 = 3.0$ and $\kappa = 1.4$.

12.3. Compute the coefficient of skin friction for Couette flow using the following viscosity relation:

$$\frac{\mu}{\mu_\infty} = \left(\frac{T}{T_\infty}\right)^{3/4}$$

Plot the coefficient $C_f \mathrm{Re}$ vs. the Mach number for $\mathrm{Pr} = 1.0$ and $\kappa = 1.4$ and compare the result with that shown in Fig. 12.4.

12.4. Air flows through a constant cross-section pipe (adiabatic wall) with an initial pressure of 25 psi and an initial temperature of 65°F. If it experiences an increase in the Mach number from 0.25 to 0.75 as a result of friction, determine the pressure, temperature, and velocity at the section where $\mathrm{M} = 0.75$.

12.5. Air having a Mach number of 3.0 is decelerated in an insulated pipe (inside diameter = 3 in.) to $\mathrm{M} = 1.5$. If the frictional coefficient C_f is equal to 0.002, find the length of the pipe between the two sections.

12.6. Air flows in a 4 in. pipe (adiabatic wall) at the rate of 500 lbf/min. The pressure drops from the initial section at 50 psi to a latter section at 40 psi. If the temperature at the initial section is 500°F, find the Mach number at the section where the pressure is 40 psi.

12.7. What is the temperature and the entropy change at a section where the pressure is 40 psi in Prob. 12.4?

12.8. Construct a temperature-entropy diagram for air undergoing Fanno flow for a mass flow $\dot{m} = 0.25$ slugs/ft^2-sec which is discharged from a reservoir at standard conditions.

12.9. For a Poiseuille flow through a channel with flat walls the velocity distribution is given by

$$u(y) = u_{\max}\left[1 - 4\left(\frac{y}{h}\right)^2\right]$$

If the temperature distribution is assumed to be a function of y only, and μ and k are assumed to be constants, verify that the maximum temperature at the center of the channel is

$$T_{\max} - T_w = \frac{1}{3}\frac{\mu u_{\max}^2}{k}$$

The result above may be obtained from the solution of Eq. (12-63) under the present conditions.

12.10. Verify that the following equation may be derived from Eqs. (12-65) and (12-67):

$$\rho u \frac{\partial(C_p T_0)}{\partial x} + \rho v \frac{\partial(C_p T_0)}{\partial y} = \frac{\partial}{\partial y}\left[\mu \frac{\partial(C_p T_0)}{\partial y}\right] + \left(\frac{1}{\mathrm{Pr}} - 1\right)\frac{\partial}{\partial y}\left[\mu \frac{\partial(C_p T_0)}{\partial y}\right]$$

where $C_p T_0$ is the stagnation enthalpy.

12.11. The velocity profile in the boundary layer of a flat plate is assumed to have the form

$$\frac{u}{U_\infty} = a\eta + b\eta^3 + c\eta^4$$

Plot the temperature distribution, T/T_∞ [Eq. (12-83)] vs. η at $M_\infty^2 = 0$, 1.0, 2.0, and 5.0 for the cases:

(a) $T_w/T_\infty = \frac{1}{2}$
(b) $T_w/T_\infty = 1.0$
(c) $T_w/T_\infty = \frac{5}{4}$

12.12. Show that for an incompressible fluid the following relation between coefficients of heat transfer and friction exists:

$$C_h = \frac{C_f}{2}$$

where

$$C_h = \frac{q}{\rho U_\infty C_p (T_\infty - T_w)}$$

12.13. Derive the momentum integral equation of the boundary layer for compressible fluids [Eq. (12-93)] by the momentum law (see § 9.5b).

12.14. Calculate the coefficient of friction factor $C_f\sqrt{\mathrm{Re}_x}$ for flow over a flat plate under the following conditions:

(a) $\frac{u}{U_\infty} = 2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4$
(b) $\mathrm{Pr} = 1.0, \quad \kappa = 1.4$
(c) $\frac{\mu}{\mu_\infty} = \left(\frac{T}{T_\infty}\right)^{1/2}$
(d) adiabatic wall

Plot $C_f\sqrt{\mathrm{Re}_x}$ vs. M_∞.

REFERENCES

A. Selected Reference Books

1. General Fluid Mechanics

1. Birkhoff, G., *Hydrodynamics.* Princeton, N. J.: Princeton University Press, 1963.
2. Carafoli, E., *High Speed Aerodynamics (Compressible Flow).* New York: Pergamon Press, 1957.
3. Courant, R., and K. O. Fredrichs, *Supersonic Flow and Shock Waves.* New York: Interscience Publishers, 1948.
4. Duncan, J. W., A. S. Thom, and A. D. Young, *An Elementary Treatise of the Mechanics of Fluid.* London: Edward Arnold Publishers, Ltd. 1960.
5. Durand, W. F., "Fluid Mechanics, Vol. 1." Div. B., *Aerodynamic Theory.* Berlin: Julius Springer, 1934.
6. Ferri, A., *Elements of Aerodynamics* of *Supersonic Flows.* New York: The Macmillan Company, 1949.
7. Goldstein, S., ed., *Modern Development in Fluid Dynamics.* 2 Vols. New York: Oxford University Press, 1938.
8. Howarth, L., ed., *Modern Developments in Fluid Dynamics, High Speed Flow.* 2 Vols. New York: Oxford University Press, Inc., 1953.
9. Kármán, Th. von, *Aerodynamics.* Ithaca, N. Y: Cornell University Press, 1954.

10. Kármán, Th. von, and J. M. Burgers, "General Aerodynamic Theory (Perfect Fluids)," Div. E. Vol. II., *Aerodynamic Theory*. Berlin: Julius Springer, 1954.

11. Kuethe, A. M., and J. D. Schetzer, *Foundations of Aerodynamics*. New York: John Wiley & Sons, Inc., 1959.

12. Lamb, H., *Hydrodynamics*. New York: Dover Publications, Inc. 1945.

13. Landau, L. D., and E, M. Lifshitz, *Fluid Mechanics*. New York: Pergamon Press, 1959.

14. Liepmann, H. W., and A. Roshko, *Elements of Gas Dynamics*. New York: John Wiley & Sons, Inc., 1956.

15. McLeod, E. B., Jr., *Introduction to Fluid Dynamics*. New York: The Macmillan Company, 1963.

16. Milne-Thompson, L. M., *Theoretical Hydrodynamics*. New York: The Macmillan Company, 1955.

17. Oswatitsch, K., *Gas Dynamics*. New York: Academic Press, 1956.

18. Owczarek, J. A., *Fundamentals of Gas Dynamics*. Scranton, Pa: International Textbook, 1964.

19. Pai, S. I., *Viscous Flow Theory—Laminar Flow*. Princeton, N. J.: D. Van Nostrand Co., Inc., 1956.

20. Pai, S. I., *Viscous Flow Theory—Turbulent Flow*. Princeton, N. J.: D. Van Nostrand Co., Inc., 1957.

21. Pai, S. I., *Introduction to the Theory of Compressible Flow*. Princeton, N. J.: D. Van Nostrand Co., Inc., 1959.

22. Prandtl, L., *Essentials of Fluid Dynamics*. New York: Hafner Publishing Co., Inc., 1952.

23. Prandtl, L., "The Mechanics of Viscous Fluids" Div. G. Vol. III., *Aerodynamic Theory*. Berlin: Julius Springer, 1934.

24. Prandtl, L., and O. G. Tietjens, *Fundamentals of Hydro- and Aero-Mechanics and Applied Hydro- and Aero-Mechanics*. New York: McGraw-Hill Book Company Inc., 1934.

25. Rosenhead, L., *Laminar Boundary Layers*. New York: Oxford University Press, 1963.

26. Schlichting H., *Boundary Layer Theory*. New York: McGraw-Hill Book Company, Inc., 1960.

27. Sedov, L. I., *Two-Dimensional Motion of an Ideal Fluid*. Moscow: Oborongiz, 1939.

28. Shapiro, A. H., *The Dynamics and Thermodynamics of Compressible Fluid Flow*, 2 Vols., New York: The Ronald Press Company, 1958.

29. Streeter, V. L., ed., *Handbook of Fluid Dynamics*. New York: McGraw-Hill Book Company, Inc., 1961.

30. Thwaites, B., ed., *Incompressible Aerodynamics.* New York: Oxford University Press, 1960.

31. Taylor, G. I., and J. W. Maccoll, "The Mechanics of Compressible Fluids," Div. H. *Aerodynamic Theory.* Vol. III. Berlin: Julius Springer, 1934.

II. Special Subjects

32. Aris, R., *Vectors, Tensors, and Basic Equations of Fluid Mechanics.* Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1962.

33. Birkhoff, G., and E. H. Zarantonello, *Jet, Wakes and Cavities.* New York: Academic Press, 1957.

34. Chernyi, G. G., *Introduction to Hypersonic Flow.* New York: Academic Press, 1959.

35. Chow, K. T., *Open-Channel Hydraulics.* New York: McGraw-Hill Book Company, Inc., 1959.

36. Dorrance, W. H., *Viscous Hypersonic Flow.* New York: McGraw-Hill Book Company, Inc., 1962.

37. Dryden, H. L, "Transition from Laminar to Turbulent Flow." Sec. A., Vol. V., *High Speed Aerodynamics and Jet Propulsion.* Princeton, N. J.: Priceton University Press, 1959.

38. Hinze, J. O., *Turbulence,* New York: McGraw-Hill Book Company, Inc., 1959.

39. Kármán, Th. von, *From Low-Speed Aerodynamics to Astronautics.* New York: The Macmillan Company, 1963.

40. Langlois, W. E., *Slow Viscous Flow.* New York: The Macmillan Company, 1964.

41. Lagerstrom, P. A., "Laminar Flow Theory," Sec. B., Vol. VI, *High Speed Aerodynamics and Jet Propulsion.* Princeton, N. J.: Princeton University Press, 1964.

42. Lin, C. C., *Hydrodynamic Stability,* New York: Cambridge University Press, 1955.

43. Meksyn, D., *New Method in Laminar Boundary-Layer Theory.* New York: Pergamon Press, 1961.

44. Moore, F. K., "Hypersonic Boundary Layer Theory," Sec. E., Vol. IV, *High Speed Aerodynamics and Jet Propulsion.* Princeton, N. J.: Princeton University Press, 1964.

45. Pankhurst, R. C., and D. W. Holder, *Wind Tunnel Techniques.* London: Pitman, 1952.

46. Pope, A., *Wind Tunnel Testing.* New York: John Wiley & Sons, Inc. 1954.

47. Radzimovsky, E. I., *Lubrication of Bearings.* New York: The Ronald Press Company, 1959.

48. Schubauer, G. D., and C. M. Tchen, "Turbulent Flow," Sec. B., Vol. V., *High Speed Aerodynamics and Jet Propulsion.* Princeton, N. J.: Princeton University Press, 1959.

49. Sedov, L. I., *Similarity and Dimensional Methods in Mechanics.* New York: Academic Press, 1959.

50. Sommerfeld, A., *Mechanics of Deformable Bodies.* New York: Academic Press, 1950.

51. Stoker, J. J., *Water Waves.* New York: Interscience Publishers, 1957.

52. Tsien, H. S., "Equations of Gas Dynamics," Sec. A., Vol. III, *High Speed of Aerodynamics and Jet Propulsion.* Princeton, N. J.: Princeton University Press, 1958.

53. Truitt, R. W., *Hypersonic Aerodynamics.* New York: The Ronald Press Company, 1959.

54. Van Dyke, M., *Perturbation Methods in Fluid Mechanics.* New York: Academic Press, 1964.

55. Vavra, M. H., *Aero-thermodynamics and Flow in Turbomachines.* New York: John Wiley & Sons, Inc., 1960.

56. Yuan, S. W., "Cooling by Protective Fluid Films," Sec. G., Vol. V, *High Speed Aerodynamics and Jet Propulsion.* Princeton, N. J.: Princeton University Press, 1959.

B. Cited References

Chapter Eight

8.1 Reynolds, O., "On the Theory of Lubrication, etc.," *Phil. Trans., Royal Soc. of London,* 157, 1886.

8.2 Stokes, G. G., "On the Effect of the Internal Friction of Fluids on the Motion of Pendulums," *Cambridge Phil. Trans.* 9, 1851.

8.3 Peirce and Foster, *A Short Table of Integrals* (4th ed.). Boston, Mass: Ginn & Co., 1957, pp. 128-132.

Chapter Nine

9.1 Howarth, L., "On the Solution of the Laminar Boundary Layer Equations," *Proc. Royal Soc. of London,* London: A164, 547 (1938).

9.2 Nikuradse, J., *Laminare Reibungsschichten an der Langssangestromen Platte,* Monograph, Zentrale f. wiss. Berichtswegung, Berlin (1942).

9.3 Liepmann, H. W., and S. Dhawan, "Direct Measurements of Local Skin Friction in Low-Speed and High-Speed Flow," *Proc. First U. S. Nat. Congr. Appl. Mech.,* 869 (1951).

11.3 Morduchow, M., and P. A. Libby, "On a Complete Solution of the One-Dimensional Flow Equations of a Viscous, Heat-Conducting Compressible Gas," *J. Aeronautical Sci.* 16, No. 11. (1949) 674–684.

Chapter Twelve

12.1 Kármán, Th. von, "The Problem of Resistance in Compressible Fluids," V. Convengo della Foundazione Allesandro Volta (Tema, Le Alto Velocita in Avaizione), Realo Accodemia D'Italia, Rome.

12.2 Kármán, Th. von, and H. S. Tsien, "Boundary Layer in Compressible Fluid," *Jour. Aero. Sci.*, 5, No. 6, (1938) 227–232.

APPENDIXES

A

SUMMARY OF VECTOR ANALYSIS

In the study of fluid mechanics, a great saving in complexity of notation may be accomplished by using the notation of vector analysis. In addition to this valuable shorthand, the vector analysis presentation gives a simpler, clearer, and more suggestive exposition of the physical ideas involved. Thus, one may directly express the proper mathematical relations in vector form when he understands a physical situation; or conversely, if a set of mathematical relations is expressed in vector form, he may readily attribute physical significance to these relations. It is the purpose of this appendix to equip the reader with vector notation and a summary of the properties of vectors. The material presented in this appendix is strictly confined to the application contained in this text.

A.1. Definitions

Scalar. A quantity which has magnitude only. That is, it can be completely specified by a single number. Typical scalar quantities are mass, density, and temperature. The magnitude of a scalar (a real number) will change when the units expressing the scalar are changed, but the physical entity remains the same. A scalar is defined to be a tensor of order zero.

Vector. A quantity which is specified by both magnitude and direc-

tion. Force, velocity, and displacement are typical vector quantities. A vector will be denoted by the use of boldface symbols, e. g., A. The magnitude of a vector A ($|\mathbf{A}|$) is designated simply by the corresponding letter A which is a scalar. The graphical representations of a line segment and an element of area as a vector are shown in Fig. A.1a and A.1b, respectively. A vector is defined to be a tensor of order one.

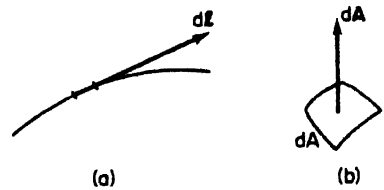


Figure A.1

Scalar Field. If at every point in a region a physical scalar function has a defined value, the region is called a scalar field. The mass density of a body and temperature of a rod are examples of scalar functions.

Vector Field. If at every point in a region a physical vector function has a defined value, the region is called a vector field. Examples of vector functions are the force due to gravity and wind velocity.

Unit Vector. A vector whose magnitude is unity. Thus a unit vector ν whose direction is parallel to the vector A can be written

$$\nu A = \mathbf{A} \tag{A-1}$$

A.2. Components

a. Components of a Vector

Let **i**, **j**, **k** represent the unit vectors in the x, y, and z directions, respectively, in a Cartesian coordinate system. If **q** is chosen to be a vector in space, and u, v, and w are the projections of **q** on the x, y, and z axes, respectively, then

$$\mathbf{q} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w \tag{A-2}$$

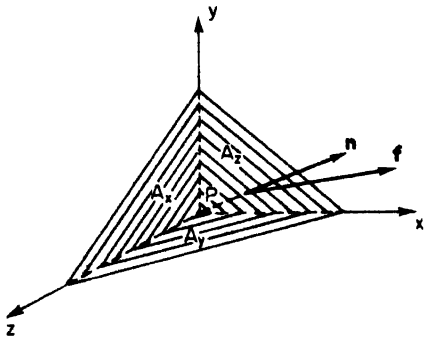
where u, v, and w are the components of **q**, and $\mathbf{i}u$, $\mathbf{j}v$, $\mathbf{k}w$ are the corresponding component vectors. The magnitude of **q** is

$$|\mathbf{q}| = \mathbf{q} = \sqrt{u^2 + v^2 + w^2} \tag{A-3}$$

And the direction cosines of **q** are

$$\begin{aligned} l &= \cos\alpha = \frac{u}{|\mathbf{q}|} \\ m &= \cos\beta = \frac{v}{|\mathbf{q}|} \\ n &= \cos\gamma = \frac{w}{|\mathbf{q}|} \end{aligned} \tag{A-4}$$

when $|\mathbf{q}| \neq 0$; otherwise, we have the null vector.



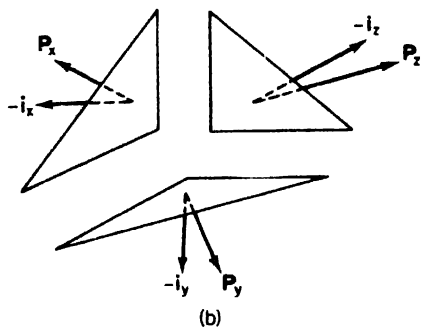


Figure A.2

b. Components of the Stress

Consider a small tetrahedron with three of its faces parallel to the coordinate planes through $P(x_0, y_0, z_0)$ and the slant face with normal $\mathbf{n}$. If the elementary surface force acting on the slant face ΔA is denoted by the vector $\mathbf{f}$, the average stress on the slant face or the stress at $P(x_0, y_0, z_0)$ on a plane with normal $\mathbf{n}$ is

$$\boldsymbol{\sigma}(n) = \lim_{\Delta A \to 0} \frac{\mathbf{f}}{\Delta A} \qquad \text{(A-5)}$$

Referring to Fig. A.2 it is easy to see that the force $\mathbf{f}$ can be resolved into three components as follows:

$$\begin{aligned} \mathbf{f} &= \mathbf{i}f_x + \mathbf{j}f_y + \mathbf{k}f_z \\ &= \mathbf{i}(\mathbf{P}_x \cdot d\mathbf{A}) + \mathbf{j}(\mathbf{P}_y \cdot d\mathbf{A}) \\ &\quad + \mathbf{k}(\mathbf{P}_z \cdot d\mathbf{A}) \end{aligned} \qquad \text{(A-6)}$$

where $d\mathbf{A}$ is the element of the vectorial area of the slant face. The resultant stresses $\mathbf{P}_x$, $\mathbf{P}_y$, and $\mathbf{P}_z$, on the $x = x_0$, $y = y_0$, and $z = z_0$ faces, respectively, have the following components (see Figs. A.2 and A.3):

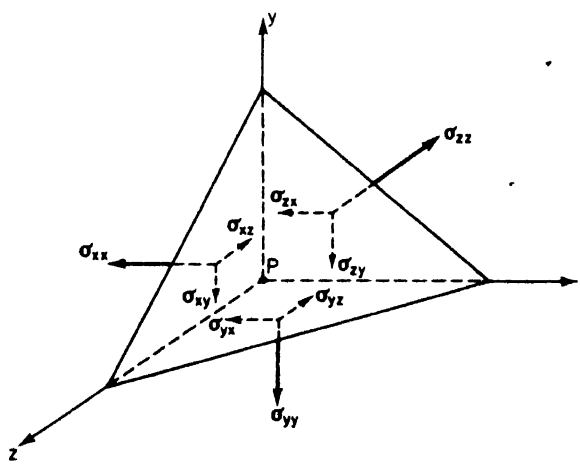


Figure A.3

$$\begin{aligned}\mathbf{P}_x &= \mathbf{i}\sigma_{xx} + \mathbf{j}\sigma_{xy} + \mathbf{k}\sigma_{xz}\\ \mathbf{P}_y &= \mathbf{i}\sigma_{yx} + \mathbf{j}\sigma_{yy} + \mathbf{k}\sigma_{yz}\\ \mathbf{P}_z &= \mathbf{i}\sigma_{zx} + \mathbf{j}\sigma_{zy} + \mathbf{k}\sigma_{zz}\end{aligned} \tag{A-7}$$

Substituting Eq. (A-7) into Eq. (A-6), we have

$$\begin{aligned}f_x &= (\mathbf{i}\sigma_{xx} + \mathbf{j}\sigma_{xy} + \mathbf{k}\sigma_{xz})\cdot(\mathbf{i}\,dA_x + \mathbf{j}\,dA_y + \mathbf{k}\,dA_z)\\ &= \sigma_{xx}\,dA_x + \sigma_{xy}\,dA_y + \sigma_{xz}\,dA_z\end{aligned} \tag{A-8a}$$

Similarly,

$$f_y = \sigma_{yx}\,dA_x + \sigma_{yy}\,dA_y + \sigma_{yz}\,dA_z \tag{A-8b}$$

$$f_z = \sigma_{zx}\,dA_x + \sigma_{zy}\,dA_y + \sigma_{zz}\,dA_z \tag{A-8c}$$

Recalling illustrative example 4.1, the condition for equilibrium gives

$$\sigma_{xy} = \sigma_{yx}, \qquad \sigma_{yz} = \sigma_{zy}, \quad \text{and} \quad \sigma_{zx} = \sigma_{xz}$$

Hence, the total stress $\boldsymbol{\sigma}(n)$ (f on the surface $d\mathbf{A}$) is obtained from Eqs. (A-6) and (A-8) as

$$\begin{aligned}\boldsymbol{\sigma}(n) = \frac{\mathbf{f}}{dA} &= \mathbf{i}(\sigma_{xx}\mathbf{i}\cdot\mathbf{n} + \sigma_{xy}\mathbf{j}\cdot\mathbf{n} + \sigma_{xz}\mathbf{k}\cdot\mathbf{n})\\ &\quad + \mathbf{j}(\sigma_{yx}\mathbf{i}\cdot\mathbf{n} + \sigma_{yy}\mathbf{j}\cdot\mathbf{n} + \sigma_{yz}\mathbf{k}\cdot\mathbf{n})\\ &\quad + \mathbf{k}(\sigma_{zx}\mathbf{i}\cdot\mathbf{n} + \sigma_{zy}\mathbf{j}\cdot\mathbf{n} + \sigma_{zz}\mathbf{k}\cdot\mathbf{n})\end{aligned} \tag{A-9}$$

since $dA_x = \mathbf{i}\cdot d\mathbf{A} = (\mathbf{i}\cdot\mathbf{n})\,dA$, $dA_y = (\mathbf{j}\cdot\mathbf{n})\,dA$, and $dA_z = (\mathbf{k}\cdot d\mathbf{A})$

Finally,

$$\begin{aligned}\boldsymbol{\sigma}(n) &= [(\sigma_{xx}\mathbf{ii} + \sigma_{xy}\mathbf{ij} + \sigma_{xz}\mathbf{ik}) + (\sigma_{yx}\mathbf{ji} + \sigma_{yy}\mathbf{jj} + \sigma_{yz}\mathbf{jk})\\ &\quad + (\sigma_{zx}\mathbf{ki} + \sigma_{zy}\mathbf{kj} + \sigma_{zz}\mathbf{kk})]\cdot\mathbf{n}\\ &= \phi\cdot\mathbf{n}\end{aligned} \tag{A-10}$$

where ϕ is called the stress tensor

$$\begin{aligned}\phi = \sigma_{xx}\mathbf{ii} + \sigma_{xy}\mathbf{ij} + \sigma_{xz}\mathbf{ik} + \sigma_{yx}\mathbf{ji} + \sigma_{yy}\mathbf{jj}\\ + \sigma_{yz}\mathbf{jk} + \sigma_{zx}\mathbf{ki} + \sigma_{zy}\mathbf{kj} + \sigma_{zz}\mathbf{kk}\end{aligned} \tag{A-11}$$

The quantities of the form ii, ij, etc., have no interpretation except that when "dotted" into a vector, they yield another vector. It follows that when the stress tensor, ϕ, is dotted into the unit vector $\mathbf{n}$, it results in the vector $\boldsymbol{\sigma}(n)$, which is the stress vector on the plane whose normal is $\mathbf{n}$.

Thus we see that stress is not just a vector, with three components in three-dimensional space, but has nine components in three-dimensional space. Such a quantity is called a tensor of order two, which is completely specified by a magnitude and two directions. The force in question has its own direction and the area on which the force is exerted gives the second direction which is normal to the surface.

A.3. Vector Algebra

a. Addition and Subtraction of Vectors

Vectors may be added or subtracted by corresponding operations on the components as follows.

$$\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z \quad \text{and} \quad \mathbf{B} = \mathbf{i}B_x + \mathbf{j}B_y + \mathbf{k}B_z \tag{A-12}$$

$$\mathbf{A} + \mathbf{B} = \mathbf{i}(A_x + B_x) + \mathbf{j}(A_y + B_y) + \mathbf{k}(A_z + B_z) \tag{A-13}$$

$$\mathbf{A} - \mathbf{B} = \mathbf{i}(A_x - B_x) + \mathbf{j}(A_y - B_y) + \mathbf{k}(A_z - B_z) \tag{A-14}$$

b. Scalar Multiplication of Two Vectors (Dot Product)

This is defined as

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}|\cdot|\mathbf{B}|\cos(\mathbf{A},\mathbf{B}) = AB\cos\theta \tag{A-15}$$

where θ is the angle between the vector A and B. It follows from the definition of the dot product that

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}, \qquad \mathbf{A}\cdot\mathbf{A} = A^2 \tag{A-16}$$

$$\mathbf{A}\cdot(\mathbf{B} + \mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C} \tag{A-17}$$

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = \cos 0^\circ = 1, \qquad \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = \cos\frac{\pi}{2} = 0 \tag{A-18}$$

c. Vector Multiplication of Vectors (Cross Product)

This is defined by

$$\mathbf{A}\times\mathbf{B} = \mathbf{n}AB\sin\theta \tag{A-19}$$

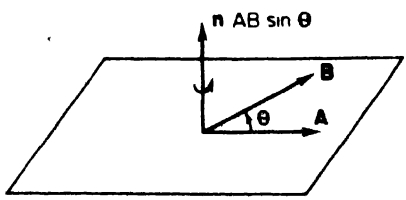


Figure A.4

where θ is the angle measured from A to B and $\mathbf{n}$ is a unit vector normal to the plane consisting of A and B (Fig. A.4). The positive sense of $\mathbf{n}$ is defined as that direction in which a right-hand screw along $\mathbf{n}$ would advance if rotated in the positive direction of θ.

From the definition (A-19) the following results are established

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{A-20}$$

$$(\mathbf{A} + \mathbf{B})\times\mathbf{C} = \mathbf{A}\times\mathbf{C} + \mathbf{B}\times\mathbf{C} \tag{A-21}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} \tag{A-22}$$

$$\left.\begin{aligned} &\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0 \\ &\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j} \end{aligned}\right\} \tag{A-23}$$

d. The Triple Scalar Product

The dot product $(\mathbf{A} \times \mathbf{B})\cdot\mathbf{C}$ yields a scalar. The dot product of $\mathbf{A} \times \mathbf{B}$ in Eq. (A–20) and a vector $\mathbf{C} = \mathbf{i}C_x + \mathbf{j}C_y + \mathbf{k}C_z$ gives

$$(\mathbf{A} \times \mathbf{B})\cdot\mathbf{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \mathbf{A}\cdot(\mathbf{B} \times \mathbf{C}) = \mathbf{B}\cdot(\mathbf{C} \times \mathbf{A}) \tag{A-24}$$

Equation (A–24) represents the volume of the parallelepiped having A, B, C as three of its edges.

A.4. Vector Calculus

a. The Gradient

In Cartesian coordinates the gradient

$$\text{grad} \equiv \nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{A-25}$$

defines the vector differential operator ∇(read del). The gradient of a scalar function $\phi\ (x, y, z)$, $\nabla\phi$ or grad ϕ is

$$\nabla\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z} \tag{A-26}$$

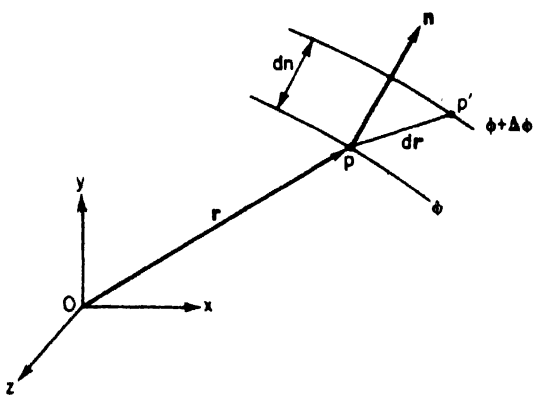


Figure A.5

Let P and P' be points lying on the surfaces $\phi = \text{const.}$ and $\phi + \Delta\phi = \text{const.}$, respectively, (Fig. A.5).

The infinitesimal distance between the two surfaces is given by

$$dn = \mathbf{n} \cdot d\mathbf{r} \tag{A-27}$$

where $\mathbf{n}$ is the unit normal vector. The increment $\Delta\phi$ in the notation of differentials becomes

$$d\phi = \frac{\partial \phi}{\partial n} dn = \frac{\partial \phi}{\partial n} \mathbf{n} \cdot d\mathbf{r} \tag{A-28}$$

From Eq. (A-26) we can write

$$\begin{aligned} d\phi &= \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \\ &= \left(\mathbf{i} \frac{\partial \phi}{\partial x} + \mathbf{j} \frac{\partial \phi}{\partial y} + \mathbf{k} \frac{\partial \phi}{\partial z} \right) \cdot (\mathbf{i}\, dx + \mathbf{j}\, dy + \mathbf{k}\, dz) \\ &= (\nabla \phi) \cdot d\mathbf{r} \end{aligned} \tag{A-29}$$

This result is very useful and is sometimes employed to define $\nabla\phi$. Equating Eqs. (A-28) and (A-29) we obtain

$$\nabla \phi = \frac{\partial \phi}{\partial n} \mathbf{n} \tag{A-30}$$

Thus, the gradient $\nabla\phi$ extends in the direction in which the derivative $(\partial\phi/\partial n)$ is a maximum, and $\nabla\phi$ is equal to that maximum.

b. Divergence and Curl of a Vector

(i) The divergence of a vector function $\mathbf{q}$, $\nabla \cdot \mathbf{q}$ or div $\mathbf{q}$ is

$$\text{div } \mathbf{q} = \nabla \cdot \mathbf{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{A-31}$$

where $\mathbf{q} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$.

Consider at time t, a rectangular parallelepiped element of fluid volume, $\delta\tau = \delta x\, \delta y\, \delta z$, moving in the fluid. At time $t + \delta t$ the same element will become a slightly oblique parallelepiped and to the first order, the projections of the edges on the x, y, z axes are, respectively,

$$\left(1 + \frac{\partial u}{\partial x} \delta t\right) \delta x, \qquad \left(1 + \frac{\partial v}{\partial y} \delta t\right) \delta y, \qquad \left(1 + \frac{\partial w}{\partial z} \delta t\right) \delta z$$

Hence, to the first order in δt the volume becomes

$$\begin{aligned} \delta\tau + \frac{d}{dt}(\delta\tau)\delta t &= \left[\left(1 + \frac{\partial u}{\partial x}\delta t\right)\delta x\right]\left[\left(1 + \frac{\partial v}{\partial y}\delta t\right)\delta y\right]\left[\left(1 + \frac{\partial w}{\partial z}\delta t\right)\delta z\right] \\ &= \delta x\, \delta y\, \delta z + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\delta t\, \delta x\, \delta y\, \delta z \end{aligned} \tag{A-32}$$

Substituting Eq. (A-31) into Eq. (A-32), we obtain

$$\nabla\cdot\mathbf{q} = \frac{1}{\delta\tau}\left[\frac{d}{dt}(\delta\tau)\right] \tag{A-33}$$

Thus, in the fluid, the divergence of the velocity vector signifies the rate of volume flow with respect to the original volume. This is zero for an incompressible fluid.

(ii) The curl or rotation of the vector function $\mathbf{q}$, $\nabla \times \mathbf{q}$, is

$$\text{curl}\,\mathbf{q} \equiv \text{rot}\,\mathbf{q} \equiv \nabla \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{A-34}$$

The derivation of Eq. (A-34) is given in §3.5.

c. *Vector Identities*

(i) Expansion Relations

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi \tag{A-35a}$$

$$\nabla\cdot(\phi\mathbf{A}) = (\nabla\phi)\cdot\mathbf{A} + \phi(\nabla\cdot\mathbf{A}) \tag{A-35b}$$

$$\nabla \times (\phi\mathbf{A}) = (\nabla\phi) \times \mathbf{A} + \phi(\nabla \times \mathbf{A}) \tag{A-35c}$$

$$\nabla\cdot(\mathbf{A} \times \mathbf{B}) = \mathbf{B}\cdot(\nabla \times \mathbf{A}) - \mathbf{A}\cdot(\nabla \times \mathbf{B}) \tag{A-35d}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) \tag{A-35e}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B}) \tag{A-35f}$$

where

$$(\mathbf{A}\cdot\nabla)\mathbf{B} = A_x\frac{\partial\mathbf{B}}{\partial x} + A_y\frac{\partial\mathbf{B}}{\partial y} + A_z\frac{\partial\mathbf{B}}{\partial z} \tag{A-35g}$$

(ii) Second-order Differential Functions

$$\nabla\cdot\nabla\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \tag{A-36a}$$

$$(\nabla\cdot\nabla)\mathbf{A} = \nabla^2\mathbf{A} = \frac{\partial^2\mathbf{A}}{\partial x^2} + \frac{\partial^2\mathbf{A}}{\partial y^2} + \frac{\partial^2\mathbf{A}}{\partial z^2} \tag{A-36b}$$

$$\nabla \times \nabla\phi = 0 \tag{A-36c}$$

$$\nabla\cdot(\nabla \times \mathbf{A}) = 0 \tag{A-36d}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} \tag{A-36e}$$

or

$$\nabla(\nabla\cdot\mathbf{A}) = \nabla\times(\nabla\times\mathbf{A}) + \nabla^2\mathbf{A}$$

d. The Divergence Theorem Gauss's Theorem

Let A denote a surface bounding a volume τ, and $\mathbf{n}$ the unit vector normal to the surface, drawn outward. Then

$$\int_\tau \nabla\cdot\mathbf{q}\, d\tau = \int_A \mathbf{n}\cdot\mathbf{q}\, dA \tag{A-37a}$$

The following theorems are deducible from the divergence theorem

$$\int_\tau \nabla\phi\, d\tau = \int_A \mathbf{n}\phi\, dA \tag{A-37b}$$

$$\int_\tau \nabla\times\mathbf{q}\, d\tau = \int_A \mathbf{n}\times\mathbf{q}\, dA \tag{A-37c}$$

e. Stoke's Theorem (see §7.3b)

Let A be a surface bounded by a contour C, and $\mathbf{n}$ be the unit vector normal to the surface, in the sense which is positive when the right-hand rule with respect to the direction of traversing the curve C is observed. Then

$$\int_C \mathbf{q}\cdot d\mathbf{r} = \int_A (\nabla\times\mathbf{q})\cdot\mathbf{n}\, dA = \int_A (\nabla\times\mathbf{q})\cdot d\mathbf{A} \tag{A-38}$$

A.5. Curvilinear Coordinates

Let x_1, x_2, x_3 be a set of othogonal curvilinear coordinates, i.e., each of these has a level-surface passing through an arbitrary point P (Fig. A.6). Let $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ be unit vectors along x_1, x_2, x_3, respectively. Since the directions of these unit vectors vary from point to point, they are not constant vectors. The relations between curvilinear coordinates and the rectangular Cartesian coordinates may be written

$$\begin{aligned} x &= x(x_1, x_2, x_3) \\ y &= y(x_1, x_2, x_3) \\ z &= z(x_1, x_2, x_3) \end{aligned} \tag{A-39}$$

But with x, y, z completely defined and continuous in the first derivative, we have for $J\left(\frac{x, y, z}{x_1, x_2, x_3}\right) > 0$,

$$\begin{aligned} x_1 &= x_1(x, y, z) \\ x_2 &= x_2(x, y, z) \\ x_3 &= x_3(x, y, z) \end{aligned} \tag{A-40}$$

The element of arc ds in Cartesian coordinates is

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 \quad \text{(A–41)}$$

Differentiating Eq. (A–39) and substituting the results into Eq. (A–41) we obtain

$$(ds)^2 = (h_1\,dx_1)^2 + (h_2\,dx_2)^2 + (h_3\,dx_3)^2 \quad \text{(A–42)}$$

where

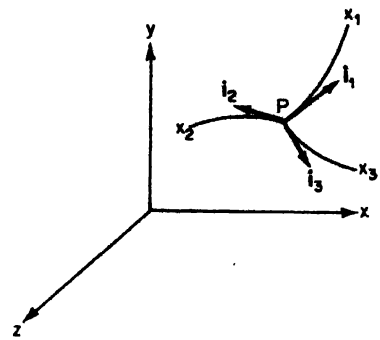
$$h_1^2 = \left(\frac{\partial x}{\partial x_1}\right)^2 + \left(\frac{\partial y}{\partial x_1}\right)^2 + \left(\frac{\partial z}{\partial x_1}\right)^2$$

$$h_2^2 = \left(\frac{\partial x}{\partial x_2}\right)^2 + \left(\frac{\partial y}{\partial x_2}\right)^2 + \left(\frac{\partial z}{\partial x_2}\right)^2 \quad \text{(A–43)}$$

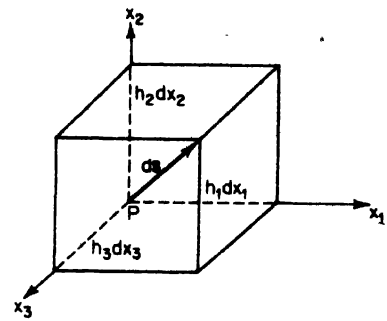
$$h_3^2 = \left(\frac{\partial x}{\partial x_3}\right)^2 + \left(\frac{\partial y}{\partial x_3}\right)^2 + \left(\frac{\partial z}{\partial x_3}\right)^2$$

The expressions for the gradient, divergence, and curl in terms of x_1, x_2, x_3 coordinates are obtained as follows:

Let $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$ be unit vectors in the directions of positive x_1, x_2, and x_3, ϕ be an arbitrary scalar, and **A**, **B** be arbitrary vectors with components in the directions x_1, x_2, x_3 denoted by A_1, B_1; A_2, B_2; and A_3, B_3; respectively. Then



(a)



(b)

Figure A.6

(i)

$$\nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial x_1}\mathbf{i}_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial x_2}\mathbf{i}_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial x_3}\mathbf{i}_3 \quad \text{(A–44)}$$

(ii)

$$\nabla\cdot\mathbf{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3A_1) + \frac{\partial}{\partial x_2}(h_3h_1A_2) + \frac{\partial}{\partial x_3}(h_1h_2A_3)\right] \quad \text{(A–45)}$$

(iii)

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial x_3}\right)\right] \quad \text{(A–46)}$$

(iv)

$$\nabla\times\mathbf{A} = \frac{1}{h_1h_2h_3}\left\{h_1\left[\frac{\partial(h_3A_3)}{\partial x_2} - \frac{\partial(h_2A_2)}{\partial x_3}\right]\mathbf{i}_1 + h_2\left[\frac{\partial(h_1A_1)}{\partial x_3} - \frac{\partial(h_3A_3)}{\partial x_1}\right]\mathbf{i}\right.$$
$$\left. + h_3\left[\frac{\partial(h_2A_2)}{\partial x_1} - \frac{\partial(h_1A_1)}{\partial x_2}\right]\mathbf{i}_3\right\}$$

$$= \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1\mathbf{i}_1 & h_2\mathbf{i}_2 & h_3\mathbf{i}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} \tag{A-47}$$

(v)

$$\begin{aligned} \nabla^2\mathbf{A} = {} & \left\{\frac{1}{h_1}\frac{\partial}{\partial x_1}(\nabla\cdot\mathbf{A}) + \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial x_3}\left\{\frac{h_2}{h_3 h_1}\left(\frac{\partial h_1 A_1}{\partial x_3} - \frac{\partial h_3 A_3}{\partial x_1}\right)\right\}\right.\right. \\ & \left.\left. - \frac{\partial}{\partial x_2}\left\{\frac{h_3}{h_1 h_2}\left(\frac{\partial h_2 A_2}{\partial x_1} - \frac{\partial h_1 A_1}{\partial x_2}\right)\right\}\right]\right\}\mathbf{i}_1 \\ & + \left\{\frac{1}{h_2}\frac{\partial}{\partial x_2}(\nabla\cdot\mathbf{A}) + \frac{1}{h_3 h_1}\left[\frac{\partial}{\partial x_1}\left\{\frac{h_3}{h_1 h_2}\left(\frac{\partial h_2 A_2}{\partial x_1} - \frac{\partial h_1 A_1}{\partial x_2}\right)\right\}\right.\right. \\ & \left.\left. - \frac{\partial}{\partial x_3}\left\{\frac{h_1}{h_2 h_3}\left(\frac{\partial h_3 A_3}{\partial x_2} - \frac{\partial h_2 A_2}{\partial x_3}\right)\right\}\right]\right\}\mathbf{i}_2 \\ & + \left\{\frac{1}{h_3}\frac{\partial}{\partial x_3}(\nabla\cdot\mathbf{A}) + \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial x_2}\left\{\frac{h_1}{h_2 h_3}\left(\frac{\partial h_3 A_3}{\partial x_2} - \frac{\partial h_2 A_2}{\partial x_3}\right)\right\}\right.\right. \\ & \left.\left. - \frac{\partial}{\partial x_1}\left\{\frac{h_2}{h_3 h_1}\left(\frac{\partial h_1 A_1}{\partial x_3} - \frac{\partial h_3 A_3}{\partial x_1}\right)\right\}\right]\right\}\mathbf{i}_3 \\ = {} & \nabla(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A}) \end{aligned} \tag{A-48}$$

(vi)

$$\begin{aligned} (\mathbf{A}\cdot\nabla)\mathbf{B} = {} & \frac{1}{h_1}\left\{\left(A_1\frac{\partial B_1}{\partial x_1} + A_2\frac{\partial B_2}{\partial x_1} + A_3\frac{\partial B_3}{\partial x_1}\right)\right. \\ & \left. - \frac{A_2}{h_2}\left(\frac{\partial h_2 B_2}{\partial x_1} - \frac{\partial h_1 B_1}{\partial x_2}\right) + \frac{A_3}{h_3}\left(\frac{\partial h_1 B_1}{\partial x_3} - \frac{\partial h_3 B_3}{\partial x_1}\right)\right\}\mathbf{i}_1 \\ & + \frac{1}{h_2}\left\{\left(A_1\frac{\partial B_1}{\partial x_2} + A_2\frac{\partial B_2}{\partial x_2} + A_3\frac{\partial B_3}{\partial x_2}\right)\right. \\ & \left. - \frac{A_3}{h_3}\left(\frac{\partial h_3 B_3}{\partial x_2} - \frac{\partial h_2 B_2}{\partial x_3}\right) + \frac{A_1}{h_1}\left(\frac{\partial h_2 B_2}{\partial x_1} - \frac{\partial h_1 B_1}{\partial x_2}\right)\right\}\mathbf{i}_2 \\ & + \frac{1}{h_3}\left\{\left(A_1\frac{\partial B_1}{\partial x_3} + A_2\frac{\partial B_2}{\partial x_3} + A_3\frac{\partial B_3}{\partial x_3}\right)\right. \\ & \left. - \frac{A_1}{h_1}\left(\frac{\partial h_1 B_1}{\partial x_3} - \frac{\partial h_3 B_3}{\partial x_1}\right) + \frac{A_2}{h_2}\left(\frac{\partial h_3 B_3}{\partial x_2} - \frac{\partial h_2 B_2}{\partial x_3}\right)\right\}\mathbf{i}_3 \\ = {} & \nabla_B(\mathbf{B}\cdot\mathbf{A}) - \mathbf{A}\times(\nabla\times\mathbf{B}) \end{aligned} \tag{A-49}$$

where

$$\nabla_B(\mathbf{B}\cdot\mathbf{A}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + (\mathbf{A}\cdot\nabla)\mathbf{B}$$

The application of the results above to cylindrical and spherical coordinates may be made with the following relations:

(i) Cylindrical coordinates (Fig. A.7)

$$(ds)^2 = (dz)^2 + (r\,d\theta)^2 + (dr)^2$$
$$= (h_1\,dx_1)^2 + (h_2\,dx_2)^2 + (h_3\,dx_3)^2$$

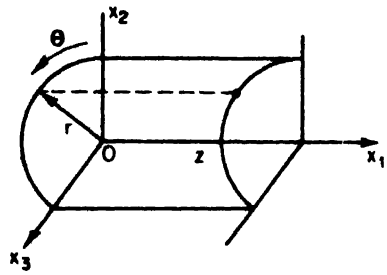


Figure A.7

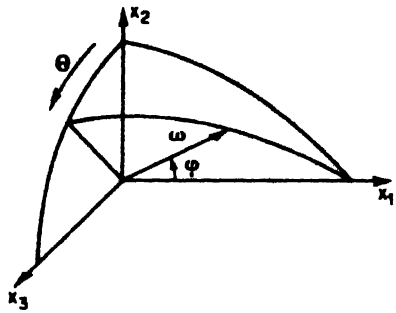


Figure A.8

Hence,

$$h_1 = 1, \qquad h_2 = r, \qquad h_3 = 1;$$
$$dx_1 = dz, \qquad dx_2 = d\theta, \qquad dx_3 = dr \tag{A-50}$$

(ii) Spherical coordinates (Fig. A.8)

$$(ds)^2 = (d\omega)^2 + (\omega\,d\varphi)^2 + (\omega \sin\varphi\,d\theta)^2$$
$$= (h_1\,dx_1)^2 + (h_2\,dx_2)^2 + (h_3\,dx_3)^2$$

Hence,

$$h_1 = 1, \qquad h_2 = \omega, \qquad h_3 = \omega \sin\varphi$$
$$dx_1 = d\omega, \qquad dx_2 = d\varphi, \qquad dx_3 = d\theta \tag{A-51}$$

The expressions for the rate-of-strain components in terms of x_1, x_2, x_3 coordinates are obtained as follows:

$$\varepsilon_{x_1x_1} = \frac{1}{h_1}\frac{\partial q_{x_1}}{\partial x_1} + \frac{q_{x_2}}{h_1h_2}\frac{\partial h_1}{\partial x_2} + \frac{q_{x_3}}{h_3h_1}\frac{\partial h_1}{\partial x_3} \tag{A-52}$$

$$\varepsilon_{x_2x_2} = \frac{1}{h_2}\frac{\partial q_{x_2}}{\partial x_2} + \frac{q_{x_3}}{h_2h_3}\frac{\partial h_2}{\partial x_3} + \frac{q_{x_1}}{h_1h_2}\frac{\partial h_2}{\partial x_1} \tag{A-53}$$

$$\varepsilon_{x_3x_3} = \frac{1}{h_3}\frac{\partial q_{x_3}}{\partial x_3} + \frac{q_{x_1}}{h_3h_1}\frac{\partial h_3}{\partial x_1} + \frac{q_{x_2}}{h_2h_3}\frac{\partial h_3}{\partial x_2} \tag{A-54}$$

$$\gamma_{x_1x_2} = \frac{h_2}{h_1}\frac{\partial}{\partial x_1}\left(\frac{q_{x_2}}{h_2}\right) + \frac{h_1}{h_2}\frac{\partial}{\partial x_2}\left(\frac{q_{x_1}}{h_1}\right) \tag{A-55}$$

$$\gamma_{x_2x_3} = \frac{h_3}{h_2}\frac{\partial}{\partial x_2}\left(\frac{q_{x_3}}{h_3}\right) + \frac{h_2}{h_3}\frac{\partial}{\partial x_3}\left(\frac{q_{x_2}}{h_2}\right) \tag{A-56}$$

$$\gamma_{x_3x_1} = \frac{h_1}{h_3}\frac{\partial}{\partial x_3}\left(\frac{q_{x_1}}{h_1}\right) + \frac{h_3}{h_1}\frac{\partial}{\partial x_1}\left(\frac{q_{x_3}}{h_3}\right) \tag{A-57}$$

where q_{x_1}, q_{x_2}, and q_{x_3} are the velocity components in the direction of x_1, x_2, and x_3, respectively.

REFERENCES

1. Aris, R., *Vectors, Tensors, and the Basic Equations of Fluid Mechanics.* Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1962.
2. Phillips, H. B., *Vector Analysis*, 3rd ed. New York: John Wiley & Sons, Inc., 1934.
3. Schwartz, M., S. Green, and W. A. Rutledge, *Vector Analysis.* New York: Harper & Row, Publishers, 1960.
4. Weatherburn, C. E., *Advanced Vector Analysis.* London: G. Bell & Sons, Ltd., 1928.

B

THERMODYNAMIC RELATIONS

It was mentioned in Chapter Eleven that when the fluid is compressible, the laws of mechanics alone are insufficient for a complete description of the physical quantities involved. In addition, thermodynamics is necessary to completely determine these quantities.

In §1.1e we have stated the *zero*th law of thermodynamics which permits us to establish a scale of the degree of heat, i.e., temperature. Thermodynamics deals with systems which have, in addition to mechanical parameters, a thermal parameter, the temperature. Hence, the science of thermodynamics deals with the various forms of energy, and, in particular, with the laws that govern their transformation. The system that we shall consider here is a simple homogeneous system; i.e., a system composed of a single gas or liquid whose thermodynamic properties are uniform throughout in all respects.

The concepts of thermodynamics are presented here, in summary form, to supplement the discussion of compressible flow theory discussed in Chapters Five, Eleven, and Twelve. The thermodynamic units are given in Appendix C.

B.1. *Equation of State of a Perfect Gas*

Consider a simple, homogeneous system composed of a single gas with definite values for its pressure p, absolute temperature T, and

volume τ, for a given mass. It is an experimental fact that specifying any two thermodynamic variables uniquely determines a third thermodynamic variable. The relationship between three thermodynamic variables can be written in the form

$$F(p, \tau, T) = 0 \tag{B-1}$$

which is commonly called an *equation of state.*

For a given mass of the gas the volume τ can be expressed as a function of the pressure and the absolute temperature in the form

$$\tau = \tau(p, T) \tag{B-2}$$

The change in τ resulting from a simultaneous change in both p and T is then given by partial differentiation.

$$d\tau = \left(\frac{\partial \tau}{\partial p}\right)_T dp + \left(\frac{\partial \tau}{\partial T}\right)_p dT \tag{B-3}$$

But according to Boyle's (or Mariotte's) law and Charles's (or Gay-Lüssac's) law, we have respectively

$$\tau = \frac{C_1}{p} \qquad (T \text{ Constant}) \tag{B-4}$$

$$\tau = C_2 T \qquad (p \text{ Constant}) \tag{B-5}$$

where C_1, and C_2 are constants. Then

$$\left(\frac{\partial \tau}{\partial p}\right)_T = -\frac{C_1}{p^2} = -\frac{\tau}{p} \tag{B-6}$$

$$\left(\frac{\partial \tau}{\partial T}\right)_p = C_2 = \frac{\tau}{T} \tag{B-7}$$

Substituting Eqs. (B-6) and (B-7) into Eq. (B-3), one obtains

$$d\tau = -\frac{\tau}{p} dp + \frac{\tau}{T} dT$$

or

$$\frac{d\tau}{\tau} + \frac{dp}{p} = \frac{dT}{T} \tag{B-8}$$

Integrating Eq. (B-8) yields

$$p\tau = \text{Constant } T \tag{B-9}$$

If two specimens of a certain gas are compared at the same pressure and temperature, the volume and thus the constant in Eq. (B-9) must be proportional to the mass M of the specimen. Equation (B-9) can thus be written as

$$p\tau = MRT \tag{B-10a}$$

or

$$\frac{p}{\rho} = RT \tag{B-10b}$$

where R is a constant which is a charateristic of the gas. The gas which obeys the law expressed by Eq. (B-10) is called perfect gas and the equation is called the equation of state for a perfect gas.

According to Avogadro's law, if a specimen of gas has a mass equal to the molecular weight (i.e., one mole of the gas), then at a pressure of one atmosphere and a temperature of 492°R the volume is 358.7 ft³. This implies that if m is the molecular weight, mR must be the same for any gas which obeys Eq. (B-10). The product of mR is called the universal gas constant and has a value which depends only upon the units used. In the English system of units we have

$$\mathscr{R} = mR = \frac{p\left(\frac{\tau}{M}m\right)}{T} = \frac{(14.7 \times 144)(358.7)}{492}$$

$$= 1545 \frac{\text{ft–lbf}}{\text{lb mole °}R} \tag{B-11a}$$

For air, $m = 28.96$ lbm; the gas constant is

$$R = \frac{1545}{28.96} = 53.35 \frac{\text{ft–lbf}}{\text{lbm °}R} \quad \text{or} \quad 1715 \frac{\text{ft}^2}{\text{sec}^2\,\text{°}R} \tag{B-11b}$$

It is assumed that the equation of state is valid when the gas is in motion as well as at rest. This assumption has been found to be in good agreement with experimental results.

B.2. *The First Law of Thermodynamics*

a. Analytical Formulation

The principle of conservation of energy, which is generally referred to as the first law of thermodynamics, is that energy can never be created or destroyed but can only be changed in form. Experiments have demonstrated conclusively that heat is a form of energy which can be expressed in terms of mechanical units through the mechanical heat equivalent; conversely, mechanical energy can be measured in terms of heat units.

If a small quantity of heat is added to a simple homogeneous system, this energy, by the first law, can be converted in the system to other forms of energy, such as an increase in the kinetic energy of the molecules (the internal energy), the kinetic energy of the system proper, and external mechanical work done by the system. It is assumed that the system is stationary (zero velocity) or nearly sta-

tionary, so that change in kinetic energy of the system need not be considered. Under these conditions the first law of thermodynamics states that a small quantity of heat added to any simple system is equal to the variation of the kinetic energy of the molecules plus the external mechanical work done by the system. This can be written analytically as

$$dQ = dE + dW \tag{B-12}$$

where dQ is the (infinitesimal) heat added to a unit mass of the system, dE the increase in internal energy per unit mass of the system, and dW the external mechanical work done per unit mass by the system on its environment.

Let A be the surface bounding the volume V of the mass considered. The mechanical work done by the pressure exerted on the surface becomes

$$\begin{aligned} dW &= \left[p \int_A dA\, \mathbf{n}\right] \cdot d\mathbf{r} \\ &= \left(p \int_A dA\right) dn = pA\, dn = p\, dV \end{aligned} \tag{B-13}$$

where V denotes the volume per unit mass. Substituting Eq. (B-13) into Eq. (B-12), we obtain

$$dQ = dE + p\, dV \tag{B-14}$$

b. Reversible and Irreversible Processes

When a small quantity of heat is added to a system, a very complicated transient state of affairs is produced, where the temperature and pressure vary from point to point throughout the system, until the system reaches a new equilibrium condition. It is seen that only the initial and final conditions of the thermodynamic state are characterized by a single uniform temperature. Consequently, any system which undergoes any physical change cannot in general be treated as a homogeneous system, i.e., a system has a uniform temperature throughout and is in thermal equilibrium. However, if the changes which occur in any finite period of time are infinitesimal, the system will be continuously in a state of equilibrium and may be treated as a homogeneous system. The great theoretical advantage of such infinitely slow processes is that they are reversible, i.e., they can be carried out in both directions.

Consider a reversible process. The change of internal energy for each infinitesimal portion of this process is given in Eq. (B-12). Hence,

the total change in the internal energy of the system from a state 1 to a state 2 is

$$\Delta E = \int_1^2 dE = Q - W \tag{B-14a}$$

If Eq. (B-14a) is applied to any cyclic process, a very important result will be obtained. In this case the energy contained in the system must be the same at the initial and final state, and experiment shows that $Q = W$. Equation (B-14a) thus becomes

$$\oint dE = 0 \tag{B-15}$$

for any reversible cyclic process. This means the internal energy of the system, E, must be a function of the state of the system (a characteristic property of the gas) and is determined as soon as the volume and temperature (or pressure and temperature) are known.

If we consider a real process such as a gas expanding in a cylinder, we see that due to pressure waves and internal friction, this real process would perform less work than a corresponding reversible process. For example, the effect of irreversibility is thus seen as a decrease in the efficiency of a motor. In the application of the first law of thermodynamics to a real process the mechanical work in Eq. (B-12) may be written in the form

$$dW = p\,dV = dW_e + dQ_i \tag{B-16}$$

where dW_e is the external mechanical work done and dQ_i the dissipation of mechanical work into heat due to friction, i.e., the heat developed internally because of the irreversibility of the process. Denoting by dQ_e the heat added externally to the unit mass of the system Eq. (B-12) can be written,

$$dQ_e = dE + dW_e = dE + dW - dQ_i \tag{B-17}$$

A process for which the heat added externally to the unit mass of the system is zero, i.e., $dQ_e = 0$, is called an adiabatic process. It can be seen that an adiabatic process is not necessarily reversible.

B.3. *Internal Energy and Enthalpy. Specific Heats*

In the previous discussion we have shown that both the internal energy and the temperature are functions of state; hence, the following quantity

$$h = E + pV \tag{B-18}$$

is also a function of state. The quantity h is called *enthalpy per unit mass of the system*, and it is also referred to as the *total heat content* (heat introduced into the system). Differentiating Eq. (B–18), one obtains

$$dh = dE + p\,dV + V\,dp = dQ + V\,dp \tag{B–19}$$

If the process of constant pressure is maintained, the increase in enthalpy may be obtained by integrating Eq. (B–19) from 0 to T,

$$\Delta h = \int_0^{Q(T)} dQ = Q \tag{B–20}$$

Equation (B–20) clearly indicates that the increase in enthalpy is equal to the total heat content at a given pressure.

The term specific heat C is defined as the quantity of heat needed to raise a unit mass of the system by a unit temperature, i.e.,

$$C = \frac{dQ}{dT} \tag{B–21}$$

The value of the specific heat depends on the process by which the heat is added. Of particular importance to our discussion are the specific heats determined at constant volume C_τ and at constant pressure C_p. Thus

$$C_\tau = \left(\frac{dQ}{dT}\right)_\tau \tag{B–22}$$

$$C_p = \left(\frac{dQ}{dT}\right)_p \tag{B–23}$$

where the subscript τ and p define the process at constant volume and pressure, respectively.

For a simple system the internal energy of a real gas depends upon V and T; hence Eq. (B–14) can be written

$$dE = dQ - p\,dV = \left(\frac{\partial E}{\partial V}\right)_T dV + \left(\frac{\partial E}{\partial T}\right)_V dT \tag{B–24}$$

Combining Eq. (B–24) with Eqs. (B–22) and (B–23), we obtain

$$C_\tau = \left(\frac{\partial E}{\partial T}\right)_V \tag{B–25}$$

$$C_p = \left(\frac{\partial E}{\partial T}\right)_V + \left[\left(\frac{\partial E}{\partial V}\right)_T + p\right]\left(\frac{\partial V}{\partial T}\right)_p \tag{B–26}$$

With p and T as the dependent variables of the system the change of enthalpy in Eq. (B–19) may be written as

$$dh = dQ + V\,dp = \left(\frac{\partial h}{\partial T}\right)_p dT + \left(\frac{\partial h}{\partial p}\right)_T dp \tag{B–27}$$

Equation (B–27) together with Eqs. (B–22) and (B–23) give

$$C_p = \left(\frac{\partial h}{\partial T}\right)_p \tag{B-28}$$

$$C_\tau = \left(\frac{\partial h}{\partial T}\right)_p + \left[\left(\frac{\partial h}{\partial p}\right)_T - V\right]\left(\frac{\partial p}{\partial T}\right)_V \tag{B-29}$$

For a perfect gas, Joule's and Gay Lüssac's experiments showed that

$$\left(\frac{\partial E}{\partial V}\right)_T = 0 \tag{B-30}$$

hence,

$$E = E(T) \tag{B-31}$$

and

$$h = E(T) + pV = E(T) + RT = h(T) \tag{B-32}$$

Thus, Eqs. (B-26) and (B-29) become

$$C_p = C_\tau + p\left(\frac{\partial V}{\partial T}\right)_p = C_\tau + R \tag{B-33a}$$

and

$$C_\tau = C_p - V\left(\frac{\partial p}{\partial T}\right)_V = C_p - R \tag{B-33b}$$

Consequently, for a perfect gas the following important relations are established:

$$C_p - C_\tau = R \tag{B-34}$$

$$E(T) = \int_0^T C_\tau \, dT = C_\tau T \tag{B-35}$$

$$h(T) = \int_0^T C_p \, dT = C_p T \tag{B-36}$$

where C_p and C_τ are assumed to be constant. (This need not be assumed.)

B.4. *Entropy and the Second Law of Thermodynamics*

According to Eqs. (B-19) and (B-36) we have, for a perfect gas, the following relation

$$dQ = C_p \, dT - V \, dp \tag{B-37}$$

Since

$$\frac{\partial C_p}{\partial p} = 0 \neq \frac{\partial(-V)}{\partial T}$$

the term dQ is not an exact differential. If the integrating factor $1/T$ is multiplied on both sides of Eq. (B–37), an exact differential, denoted by dS, is obtained

$$dS = \frac{dQ}{T} = C_p \frac{dT}{T} - V\frac{dp}{T}$$

$$= C_p \frac{dT}{T} - R\frac{dp}{p} \tag{B–38}$$

Integrating Eq. (B–38) from the initial values T_0 and p_0 to the final values T and p, we obtain

$$S - S_0 = C_p \ln \frac{T}{T_0} - R \ln \frac{p}{p_0} \tag{B–39}$$

Thus, the function S of the system, which depends only on the initial and final states of the gas, must be a function of the state of the system (a characteristic property of the gas). The function S is called specific entropy (entropy per unit mass).

If we substitute

$$\frac{R}{C_p} = \frac{C_p - C_\tau}{C_p} = \frac{\kappa - 1}{\kappa}$$

where $\kappa = (C_p/C_\tau)$ is called the ratio of specific heats, into Eq. (B–39), the expression for the entropy change becomes

$$S - S_0 = C_p \ln \left[\frac{T}{T_0}\left(\frac{p_0}{p}\right)^{\frac{\kappa-1}{\kappa}}\right] \tag{B–40}$$

Eliminating the temperature ratio of Eq. (B–40) from Eq. (B–10b), we finally obtain

$$S - S_0 = C_\tau \ln \left[\frac{p}{p_0}\left(\frac{\rho}{\rho_0}\right)^{\kappa}\right] \tag{B–41}$$

or

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\kappa} e^{\frac{S-S_0}{C_\tau}} \tag{B–42}$$

If there is no change in the entropy of the system during the process, Eq. (B–42) reduces to

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\kappa} \tag{B–43}$$

Such a process is referred to as an isentropic process.

Referring to Eq. (B–17) the expression for the entropy change can be written

$$dS = \frac{dQ}{T} = \frac{dQ_e}{T} + \frac{dQ_i}{T} \tag{B–44}$$

If the heat added from external sources is equal to zero, $dQ_e = 0$, Eq. (B–44) becomes

$$dS = \frac{dQ_i}{T} > 0 \tag{B–45}$$

Hence, the isentropic process is not equivalent to an adiabatic process but a reversible adiabatic process. Furthermore, since the loss of mechanical work into heat always corresponds to a positive heat of dissipation ($dQ_i > 0$), it is evident that entropy always increases in an adiabatic process ($dQ_e = 0$).

We can now come to the statement of the second law of thermodynamics which is, like the first law, the result of observation and experiment. The law may be stated in the following manner:

(i) Heat cannot be transferred from a body of lower temperature to one of higher temperature without other simultaneous changes occurring in the two systems or their environments.

(ii) Heat from a single source cannot be transformed into mechanical work without other simultaneous changes in the system or the environment.

(iii) The transfer of mechanical work into heat by friction is irreversible.

(iv) In an isolated system the entropy cannot decrease.

The last statement can easily be seen from Eq. (B–45). If the process is reversible, the change of entropy is equal to zero. On the other hand the entropy is always increased if the process is irreversible.

B.5. *Thermodynamic Process of Perfect Gases*

a. The Constant Volume Process

When the volume of the gas is held constant, the term pdV in Eq. (B–14) becomes zero, and the heat added per unit mass must equal the change in internal energy per unit mass. Thus

$$Q = E = C_v(T_2 - T_1) \tag{B–46}$$

b. The Constant Pressure Process

Since heat addition to a unit mass of gas is performed at constant pressure, the work done per unit mass can be calculated as

$$W = \int_{V_1}^{V_2} p\,dV = p\,(V_2 - V)_1 \tag{B–47}$$

The heat added to a unit mass of gas is

$$Q = C_p(T_2 - T_1) \tag{B-48}$$

c. The Constant Temperature Process

If the temperature of a gas is held constant while the pressure is decreased, the internal energy per unit mass of the gas remains constant. The heat added per unit mass must be equal to the work done by the gas; hence

$$W = Q = \int_{V_1}^{V_2} p\,dV = RT\int_{V_1}^{V_2}\frac{dV}{V}$$

$$= RT\ln\frac{V_2}{V_1} \tag{B-49}$$

d. The Adiabatic Process

For a process in which there is no heat added to the gas, Eq. (B-12) becomes

$$W = -E = C_v(T_1 - T_2)$$

If the process is reversible adiabatic, it is called isentropic, and the following relation exists:

$$pV^\kappa = \text{Constant} \tag{B-50}$$

e. Polytropic Process

In general, we can have a relation between the pressure and specific volume for each of the preceding processes of the form

$$pV^n = \frac{p}{\rho^n} = \text{Constant} \tag{B-51}$$

where n is a given constant for each process. Thus

$n = 0$ for p = Constant (isobaric process)

$n = 1$ for T = Constant (isothermal process)

$n = \kappa$ for S = Constant (isentropic process)

$n = \infty$ for V = Constant (isometric process)

Equation (B-51) is generally referred to as the polytropic equation of state for a perfect gas. The graphical representation of the processes above by a system of two independent variables p and V, or T and S, are shown in Figs. B.1 and B.2, respectively.

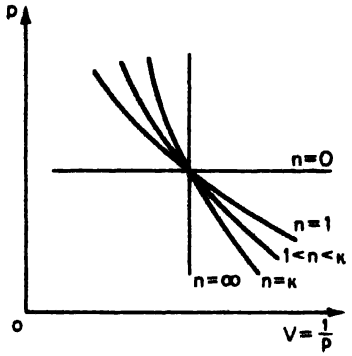


Figure B.1

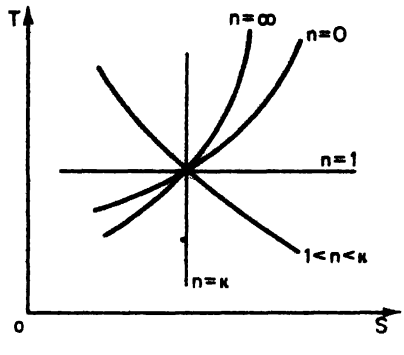


Figure B.2

REFERENCES

1. Epstein, P. S., *Textbook of Thermodynamics.* New York: John Wiley & Sons, Inc., 1937.

2. Lay, J. E., *Thermodynamics.* Columbia, Ohio: Charles E. Merrill Books, Inc., 1963.

3. Sommerfeld, A., *Thermodynamics and Statistical Mechanics,* Vol. V. New York: Academic Press, 1964.

C

DIMENSIONAL METHOD IN FLUID MECHANICS

C.1. *Dimensions and Unit Systems*

The relation between a physical quantity, its measure, and the unit of measurement must be thoroughly understood before we can proceed to discuss the dimensional method in fluid mechanics. Any direct observation of a physical nature can ordinarily be connected with the magnitude of the measured quantity. For example, when we refer to the velocity of the earth in its orbit, we say 97,702 ft/sec. Hence, we see that the quantity has dimensions. In other words, when the magnitude of the numerical results depends on the units chosen, we say that the quantity has dimensions.

Units are either fundamental (or primary) or derived (or secondary). The nature of the fundamental units is somewhat arbitrary. If certain of the measurable properties of physical quantities are chosen as fundamental (they are independent of one another), then the units of measurement of all the remaining quantities can be expressed in terms of these fundamental quantities. Hence, the latter units are called derived units.

Length and time are regarded as fundamental quantities in all dimensional systems in mechanics. In some systems mass is regarded as a fundamental quantity and force the derived quantity. In other dimensional systems force is chosen as a fundamental quantity and mass

the derived quantity. At the present time, the physical and engineering systems of units of measurement are indeed widespread. The following systems of units are commonly used in mechanics.

a. British Units

The fundamental units in the *British* (or *English*) system are the foot (length), the pound (mass or force), and the second (time). The foot was derived from the length of the human foot (or the king's foot to be exact). The second was defined as 1/86,400 of a mean solar day but since 1960 the definition has been changed to 1/31,556,925.9747 of the tropical year 1900. A standard one-pound mass of platinum is kept at the Standards Office in Westminster, London, England. The standard of force can be taken as the weight of the standard platinum mass at some prescribed point on earth.

In order to relate these units, Newton's second law of motion may be used, which becomes

$$F \sim Ma$$

or

$$F = \frac{1}{g_c} Ma \tag{C-1}$$

where g_c is the constant of proportionality which has magnitude and dimensions depending on the units chosen for mass, force, length, and time. It should be noted that g_c is in no way related to the standard acceleration of gravity, $g = 32.174$ ft/sec^2. There are three basic systems of units used in mechanics in British units:

(i) *British Absolute Units.* The fundamental units in this system are mass (pound mass), length (feet), and time (second). The unit of force is the poundal (the force required to accelerate 1 lbm at the rate of 1 ft/sec^2) which is derived from Eq. (C-1) as follows:

$$[1 \text{ poundal}] = \frac{1}{g_c}[1 \text{ lbm}][1 \text{ ft/sec}^2] \tag{C-2}$$

where the symbol [ ] is used to denote the dimensions of a quantity. The constant g_c is then

$$g_c = 1 \frac{\text{lbm-ft}}{\text{poundal-sec}^2} \tag{C-3}$$

(ii) *British Gravitational Units.* Force (pound force), length (feet), and time (second) are taken as the fundamental units. The unit of mass (slug) is derived from Eq. (C-1) which becomes

$$[1 \text{ slug}] = g_c \frac{[1 \text{ lbf}]}{[1 \text{ ft/sec}^2]} \tag{C-4}$$

Hence, the slug is defined as the mass which will be accelerated at the rate of 1 ft/sec² when acted upon by a force of one pound force. The constant g_c in this system is

$$g_c = 1 \frac{\text{slug-ft}}{\text{lbf-sec}^2} \tag{C-5}$$

In this text the British gravitational units are principally used.

(iii) *Engineering Units.* The fundamental units are force (lbf), mass (lbm), length (ft), and time (sec). The value of g_c is determined from Eq. (C-1) as

$$[1 \text{ lbf}] = \frac{1}{g_c}[1 \text{ lbm}][32.174 \text{ ft/sec}]$$

or

$$g_c = 32.174 \frac{\text{lbm-ft}}{\text{lbf-sec}^2} \tag{C-6}$$

The relations between different units of force and mass can be obtained from Eqs. (C-3) and (C-6), and Eqs. (C-5) and (C-6), respectively. They are

$$\begin{aligned} 1 \text{ lbf} &= 32.174 \text{ poundals} \\ 1 \text{ slug} &= 32.174 \text{ lbm} \end{aligned} \tag{C-7}$$

Example Determine the weight of the standard pound body at a point on the earth where $g = 30$ ft/sec².

Solution.

$$F = \frac{1}{g_c} ma$$

or

$$W = \frac{1}{g_c} mg$$

using g_c in Eq. (C-6)

$$W = \frac{1}{32.174 \text{ lbm-ft/lbf-sec}^2}(1 \text{ lbm})(30 \text{ ft/sec}^2) = 0.9324 \text{ lbf}$$

Hence, we see that when the standard mass is taken to a point on the earth which is not the standard location, it does *not* weigh one pound force. We also note that while g_c is a constant (not an acceleration), the gravitational acceleration, g, can vary with position on the earth's surface.

b. Metric System

Although the *British* system is most commonly used in engineering applications in the United States, international units are based on the *metric* system. The fundamental units of the *metric* system are the meter (length), the kilogram (mass or force), and the second (time) or the centimeter (10^{-2}m), the gram (10^{-3}kg), and the second. A standard of one kilogram (1 kg) mass of platinum-iridium is kept at the International Bureau of Weights and Measures at Sèvres, France. When force is chosen as a fundamental unit, the standard of force can be taken as the weight of the platinum-iridium mass at a locality of standard acceleration of 9.81 meters/sec^2. The meter was redefined in 1960 as 1,650,763.73 times the wave length of the orange-red line in the spectrum of krypton 86. The second has the same definition as that in the *British* units. There are four basic systems of units used in mechanics in metric units:

(i) *Cgs, Absolute.* In the cgs absolute system the fundamental units are mass (gram), length (centimeter), and time (second). The unit of force is called the dyne which is derived from Newton's Second Law of Motion. It is defined as the force which, when acting on a mass of 1 gm, imparts to it an acceleration of 1 cm/sec^2. According to Eq. (C–1) we have

$$[1 \text{ dyne}] = \frac{1}{g_c}[1 \text{ gm}][1 \text{ cm/sec}^2]$$

or

$$g_c = 1 \frac{\text{gm-cm}}{\text{dyne-sec}^2} \tag{C–8}$$

(ii) *Cgs, Gravitational.* The fundamental units are force (gram force), length (centimeter), and time (second). The unit of mass in grams can be derived from Eq. (C–1) as

$$[1 \text{ gm}] = g_c[1 \text{ gm-force}]/[981 \text{ cm/sec}^2]$$

It is clear that a 1 gm-force acting on a 1 gm-mass produces a standard acceleration of 981 cm/sec^2. The constant g_c in this system is

$$g_c = 981 \frac{\text{gm-cm}}{\text{(gm-force)-sec}^2} \tag{C–9}$$

By comparison of Eqs. (C–8) and (C–9) we obtain

$$1 \text{ gm-force} = 981 \text{ dynes} \tag{C–10}$$

(iii) *Mks, Absolute.* The fundamental units are mass (kilogram), length (meter), and time (second). The derived unit of force is the

newton which is defined as the force acting on a 1 kg mass which produces an acceleration of 1 m/sec². According to this definition we have from Eq. (C-1)

$$[1 \text{ newton}] = \frac{1}{g_c}[1 \text{ kg}][1 \text{ m/sec}^2]$$

and

$$g_c = 1\frac{\text{kg-m}}{\text{newton-sec}^2} \tag{C-11}$$

(iv) *Mks, Gravitational.* The fundamental units are force (kilogram-force), length (meter), and time (second). The derived unit of mass is obtained from Eq. (C-1) as

$$[1 \text{ kg}] = g_c \frac{[1 \text{ kg-force}]}{[9.81 \text{ m/sec}^2]}$$

and

$$g_c = 9.81\frac{\text{kg-m}}{\text{(kg-force)-sec}^2} \tag{C-12}$$

Combining Eqs. (C-11) and (C-12) we obtain the relation

$$1 \text{ kg-force} = 9.81 \text{ newtons} \tag{C-13}$$

The different unit systems which we have just discussed are summarized in Table C-1. Table C-2 presents selected dimensional equivalents and some important physical and mathematical constants. Physical dimensions of quantities occurring in mechanics based on mass, length, and time as fundamental units are given in Table C-3.

C.2. *Dimensional Analysis*

In § 8.1 we have reduced the Navier-Stokes equations to nondimensional form and obtained the nondimensional parameters, the Froude number and the Reynolds number, from the resulting equations. In a similar manner we obtained (§ 11.1) the nondimensional parameters, the Mach number, the Reynolds number, the Prandtl number, and the ratio of the specific heats, from the nondimensional form of the energy equation in a viscous compressible flow. The significance of the Reynolds number and the Froude number can be seen from the requirement of dynamic similarity of an incompressible viscous fluid. This was discussed in § 8.1. For the complete similarity of flows of viscous compressible fluids, the flows must have the same Reynolds numbers, Froude numbers, Mach numbers, Prandtl numbers, and ratio of specific heats.

Table C-1. MECHANICAL UNIT SYSTEMS

	British Absolute	British Gravitational	Engineering	Cgs Absolute	Cgs Gravitational	Mks Absolute	Mks Gravitational
Length	ft	ft	ft	cm	cm	m	m
Time	sec	sec	sec	sec	sec	sec	sec
Mass (Fundamental)	lbm		lbm	gm		kg	
Mass (Derived)		slug			gm		kg
Force (Fundamental)		lbf	lbf		gm-force		kg-force
Force (Derived)	poundal			dyne		newton	
g_c	$1\frac{\text{lbm-ft}}{\text{poundal-sec}^2}$	$1\frac{\text{slug-ft}}{\text{lbf-sec}^2}$	$32.174\frac{\text{lbm-ft}}{\text{lbf-sec}^2}$	$1\frac{\text{gm-cm}}{\text{dyne-sec}^2}$	$981\frac{\text{gm-cm}}{\text{gm-force-sec}^2}$	$1\frac{\text{kg-m}}{\text{newton-sec}^2}$	$9.81\frac{\text{kg-m}}{\text{kg-force-sec}^2}$

Table C-2. SELECTED DIMENSIONAL EQUIVALENTS AND CONSTANTS*

Length	1 m = 10^{-3} km = 3.280 ft = 39.37 in.
	1 cm = 10^{-2} m = 2.540 in. = 0.0328 ft
	1 mm = 10^{-3} m
	1 micron (μ) = 10^{-6} m
	1 Angstrom (Å) = 10^{-10} m
	1 mile = 5280 ft
Time	1 hr = 3600 sec = 60 min.
	1 millisec = 10^{-3} sec
	1 microsec (μsec) = 10^{-6} sec
	1 nanosec (nsec) = 10^{-9} sec
Mass	1 kg = 1000 gm = 2.2046 lbm = 6.8521 × 10^{-2} slugs
	1 slug = 1 lbf-sec^2/ft = 32.174 lbm
Force	1 newton = 1 kg-m/sec^2
	1 dyne = 1 gm-cm/sec^2
	1 lbf = 4.448 × 10^5 dynes = 4.448 newtons
Energy	1 joule = 1 kg-m^2/sec^2
	1 Btu = 778.16 ft-lbf = 1.055 × 10^{10} ergs = 0.252 kcal
	1 cal = 4.186 joules
	1 kcal = 4186 joules
	1 erg = 1 gm-cm^2/sec^2
	1 ev = 1.602 × 10^{-19} joules
Power	1 watt = 1 kg-m^2/sec^3 = 1 joule/sec
	1 hp = 550 ft-lbf/sec
	1 hp = 2545 Btu/hr = 746 watts
	1 kw = 1000 watts = 3413 Btu/hr
Pressure	1 atm = 14.696 lbf/in? = 760 torr
	1 mm Hg = 0.01934 lbf/in? = 1 torr
	1 dyne/cm^2 = 145.04 × 10^{-7} lbf/in?
	1 bar = 14.504 lbf/in? = 10^6 dynes/cm^2
	1 micron (μ) = 10^{-6} mm Hg = 10^{-3} mm Hg
Volume	1 gal = 0.13368 ft^3
	1 liter = 1000.028 cm^3
Temperature	1K° = 1C° = 1.8F° = 1.8R°
	0C° corresponds to 32F°, 273.16K°, and 491.69R°
Avogadro's number	N_0 = 6.025 × 10^{23}/gmmole
Universal gas constant	$\mathscr{R}$ = 1545.33 ft-lbf/lbmole-°R
	= 8.317 joule/gmmole-°K
	= 1.986 Btu/lbmole-°R
	= 1.986 cal/gmmole-°K
Newton constant	g_c = 32.174 ft-lbm/lbf-sec^2 = 981 gm-cm/(gm-force)-sec^2
Joule's constant	J_c = 4.1840 × 10^7 erg cal^{-1}
	= 778.16 ft-lbfBtu^{-1}
Mathematical constants	e = 2.71828, $\log_e 10$ = 2.30259, π = 3.14159

*W. C. Reynolds, *Thermodynamics*. New York: McGraw-Hill Book Company, 1965.

Table C-3. DIMENSIONS OF QUANTITIES IN FLUID MECHANICS

(Let M, L, T, and θ denote the mass, length, time, and temperature, respectively.)

Quantity	Dimensions	Practical Units	
		British Gravitational units	Cgs Absolute units
Absolute temperature	θ	°R	°K
Acceleration	LT^{-2}	ft/sec^2	cm/sec^2
Angular velocity	T^{-1}	rad/sec	rad/sec
Area	L^2	ft^2	cm^2
Body force per unit mass	LT^{-2}	lbf/slug	dyne/gm
Boundary layer thickness	L	ft	cm
Circulation	L^2T^{-1}	ft^2/sec	cm^2/sec
Displacement thickness	L	ft	cm
Dissipation function	$ML^{-1}T^{-3}$	lbf/ft^2-sec	dyne/cm^2-sec
Enthalpy per unit mass	L^2T^{-2}	ft-lbf/slug	dyne-cm/gm
Entropy	$ML^2T^{-2}\theta^{-1}$	ft-lbf/°R	dyne-cm/°K
Euler derivative D/Dt	T^{-1}	1/sec	1/sec
Force	MLT^{-2}	lbf	dyne
Gas constant	$L^2T^{-2}\theta^{-1}$	ft-lbf/slug-°R	cm-dyne/gm-°K
Heat transfer	ML^2T^{-3}	ft-lbf/sec	dyne-cm/sec
Internal energy per unit mass	L^2T^{-2}	ft-lbf/slug	dyne-cm/gm
Kinematic viscosity	L^2T^{-1}	ft^2/sec	cm^2/sec
Laplace operator ∇^2	L^{-2}	1/ft^2	1/cm^2
Length	L	ft	cm
Mass	M	slug	gm
Mass density	ML^{-3}	slug/ft^3	gm/cm^3
Mass flow rate	MT^{-1}	slug/sec	gm/sec
Moment of inertia	ML^2	lbf-ft^3/sec^2	dyne-cm^3/sec^2
Momentum	MLT^{-1}	slug-ft/sec	gm-cm/sec
Momentum thickness	L	ft	cm
Rate of strain	T^{-1}	1/sec	1/sec
Shearing stress	$ML^{-1}T^{-2}$	lbf/ft^2	dyne/cm^2
Specific heat	$L^2T^{-2}\theta^{-1}$	ft-lbf/°R-slug	dyne-cm/°K-gm
Stream function	L^2T^{-1}	ft^2/sec	cm^2/sec
Stress	$ML^{-1}T^{-1}$	lbf/ft^2	dyne/cm^2
Thermal conductivity	$MLT^{-3}\theta^{-1}$	ft-lbf/ft-sec-°R	dyne-cm/cm-sec-°K
Time	T	sec	sec
Torque	ML^2T^{-2}	lbf-ft	dyne-cm
Velocity	LT^{-1}	ft/sec	cm/sec
Velocity potential	L^2T^{-1}	ft^2/sec	cm^2/sec
Viscosity coefficient	$ML^{-1}T^{-1}$	lbf-sec/ft^2	dyne-sec/cm^2
Volume	L^3	ft^3	cm^3
Vorticity	T^{-1}	rad/sec	rad/sec
Weight	MLT^{-2}	lbf	dyne
Work	ML^2T^{-2}	lbf-ft	dyne-cm

*1 poise = 1 dyne-sec/cm^2

The method of forming the nondimensional parameters which we have just discussed is sometimes called *inspection analysis.* Inspection analysis requires a knowledge of the basic differential equations. This procedure should always be used when the basic differential equations for a problem are available. An alternate method, with which the nondimensional parameters may be formed from the physical quantities occurring in the flow problem, is known as *dimensional analysis.* It is derived on the basis of the dimensions in which each of the quantities involved in a phenomenon or action is expressed, and hence, must not depend on the units chosen for the calculations. There are, generally, three accepted methods of dimensional analysis due to Buckingham, Rayleigh, and Bridgeman. We shall discuss Buckingham's Pi theorem here as it is the simplest one among the three methods.

a. Statement of Assumptions

The procedure of dimensional analysis makes use of the following assumptions:

(i) It is possible to select m independent fundamental units (in mechanics, $m = 3$, i.e., length, time, mass or force).

(ii) There exist n quantities involved in a phenomenon whose dimensional formulae may be expressed in terms of m fundamental units.

(iii) The dimensional quantity Q_0 can be related to the independent dimensional quantities $Q_1, Q_2, \ldots, Q_{n-1}$ by

$$Q_0 = F(Q_1, Q_2, \ldots, Q_{n-1}) = KQ_1^{y_1} Q_2^{y_2} \ldots Q_{n-1}^{y_{n-1}} \qquad \text{(C-14)}$$

where K is a nondimensional constant, and $y_1, y_2, \ldots, y_{n-1}$ are integer exponents.

(iv) Equation (C-14) is independent of the type of units chosen and is dimensionally homogeneous, i.e., the quantities occurring on both sides of the equation must have the same dimension.

Example C.1. Consider the problem of a freely falling body near the surface of the earth. If x, w, g, and t represent the distance measured from the initial height, the weight of the body, the gravitational acceleration (assumed constant), and the time, respectively, find a relation for x as a function of w, g, and t.

Solution. Using the fundamental units of force F, length L, and time T, we note that the four physical quantities, $Q_0 = x$, $Q_1 = w$, $Q_2 = g$, and $Q_3 = t$, involve three fundamental units; hence, $m = 3$ and $n = 4$ in assumptions i and ii. By assumption iii we assume a relation of the form:

$$x = F(w, g, t) = Kw^{y_1} g^{y_2} t^{y_3} \qquad \text{(C-15)}$$

where K is an arbitrary nondimensional constant.

Let [ ] denote "dimensions of a quantity." Then the relation above can be written (using assumption iv) as

$$[x] = [w]^{y_1}[g]^{y_2}[t]^{y_3}$$

or

$$F^0L^1T^0 = (F)^{y_1}(LT^{-2})^{y_2}(T)^{y_3} = F^{y_1}L^{y_2}T^{-2y_2+y_3}$$

Equating like exponents, we obtain

$$\begin{aligned} F: &\quad 0 = y_1 \\ L: &\quad 1 = y_2 \\ T: &\quad 0 = -2y_2 + y_3 \quad \text{or} \quad y_3 = 2y_2 = 2 \end{aligned}$$

Therefore, Eq. (C-15) becomes

$$x = Kw^0g^1t^2$$

or

$$x = Kgt^2$$

According to the elementary mechanics we have $x = \frac{1}{2}gt^2$. The constant K in this case is $\frac{1}{2}$ which cannot be obtained from dimensional analysis.

Example C.2. Consider the problem of computing the drag force on a body moving through a fluid. Let D, ρ, μ, l, and V be the drag force, mass density of the fluid, viscosity coefficient of the fluid, body reference length, and body velocity, respectively.

Solution. For this problem $m = 3$, $n = 5$, $Q_0 = D$, $Q_1 = \rho$, $Q_2 = \mu$, $Q_3 = l$, and $Q_4 = V$. Thus, according to Eq. (C-14), we have

$$D = F(\rho, \mu, l, V) = K\rho^{y_1}\mu^{y_2}l^{y_3}V^{y_4} \tag{C-16}$$

or

$$\begin{aligned} [D] &= [\rho]^{y_1}[\mu]^{y_2}[l]^{y_3}[V]^{y_4} \\ F^1L^0T^0 &= (FL^{-4}T^2)^{y_1}(FL^{-2}T)^{y_2}(L)^{y_3}(LT^{-1})^{y_4} \\ &= F^{y_1+y_2}L^{-4y_1-2y_2+y_3+y_4}T^{2y_1+y_2-y_4} \end{aligned}$$

Equating like exponents, we obtain

$$\begin{aligned} F: &\quad 1 = y_1 + y_2 \\ L: &\quad 0 = -4y_1 - 2y_2 + y_3 + y_4 \\ T: &\quad 0 = 2y_1 + y_2 - y_4 \end{aligned}$$

In this case we have three equations and four unknowns. Hence, we can only solve for three of the unknowns in terms of the fourth unknown (a one-parameter family of solutions exists). For example, solving for y_1, y_3, and y_4 in terms of y_2, one obtains

$$y_1 = 1 - y_2$$
$$y_3 = 2 - y_2$$
$$y_4 = 2 - y_2$$

The required solution is

$$D = K\rho^{1-y_2}\mu^{y_2}l^{2-y_2}V^{2-y_2}$$

or

$$D = (2K)\left(\frac{\rho Vl}{\mu}\right)^{-y_2}\left(\frac{\rho}{2}V^2\right)l^2$$

If the Reynolds number is denoted by $\mathrm{Re} = \rho Vl/\mu$, dynamic pressure by $q = \frac{1}{2}\rho V^2$, and area by $S = l^2$, we have

$$D = \frac{2K}{(\mathrm{Re})^{y_2}} \cdot qS = C_D qS \qquad \text{where } C_D = \frac{2K}{(\mathrm{Re})^{y_2}},$$

Theoretical considerations show that for laminar flow

$$2K = 1.328 \quad \text{and} \quad y_2 = \frac{1}{2}$$

b. Pi Theorem

It is seen from the preceding examples that m fundamental units and n physical quantities lead to a system of m linear algebraic equations with n unknowns of the form

$$\begin{aligned} a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n &= b_1 \\ a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{m1}y_1 + a_{m2}y_2 + \cdots + a_{mn}y_n &= b_m \end{aligned} \tag{C-17}$$

or, in matrix form,

$$Ay = b \tag{C-18}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A is referred to as the coefficient matrix of order $m \times n$, and y and b are of order $n \times 1$ and $m \times 1$, respectively.

The matrix A in Eq. (C–18) is rectangular and the largest determinant that can be formed will have the order n or m, whichever is smaller. If any matrix C has at least one determinant of order r which is different from zero and no nonzero determinant of order greater than r, then the matrix C is said to be of rank r, i.e.,

$$R(C) = r \tag{C–19}$$

In order to determine the condition for the solution of the linear system of Eq. (C–17) it is convenient to define the rank of the augmented matrix B. The matrix B is defined as

$$B = A^b = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \tag{C–20}$$

For the solution of the linear system in Eq. (C–17) three possible cases arise:

(i) $R(A) < R(B)$: In this case no solution exists.

(ii) $R(A) = R(B) = r = n$: In this case a unique solution exists.

(iii) $R(A) = R(B) = r < n$: In this case an infinite number of solutions with $n - r$ arbitrary unknowns exist.

Example C.2 falls in case (iii) where

$$R(A) = R(B) = 3 < n = 4 \quad \text{and} \quad (n - r) = (4 - 3) = 1,$$

i.e., an arbitrary unknown exists.

The mathematical reasoning above leads to the following Pi theorem due to Buckingham.

Let n quantities $Q_1, Q_2, \ldots, Q_n$ be involved in a phenomenon, and their dimensional formulae be described by m ($<n$) fundamental units. Let the rank of the augmented matrix B be $R(B) = r \leq m$. Then the relation

$$F_1(Q_1, Q_2, \ldots, Q_n) = 0 \tag{C–21}$$

is equivalent to the relation

$$F_2(\pi_1, \pi_2, \ldots, \pi_{n-r}) = 0 \tag{C–22}$$

where $\pi_1, \pi_2, \ldots \pi_{n-r}$ are dimensionless power products of $Q_1, Q_2, \ldots, Q_n$ taken $r + 1$ at a time.

Thus, the Pi theorem allows one to take n quantities and find the minimum number of nondimensional parameters, $\pi_1, \pi_2, \ldots, \pi_{n-r}$ associated with these n quantities.

C.3. Determination of Minimum Number of π-Terms

In order to apply the Buckingham π-Theorem to a given physical problem the following procedure should be used:

Step 1. Given n quantities involving m fundamental units, set up the augmented matrix B by constructing a table with the quantities on the horizontal axis and the fundamental units on the vertical axis. Under each quantity list a column of numbers which represent the powers of each fundamental unit that makes up its dimensions. For example,

	ρ	p	d	Q
F	1	1	0	0
L	−4	−2	1	3
T	2	0	0	−1

where ρ, p, d, and Q are the mass density, pressure, diameter, and volume flow rate, respectively. The resulting array of numbers represents the augmented matrix B in Buckingham's π-Theorem, i.e.,

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -4 & -2 & 1 & 3 \\ 2 & 0 & 0 & -1 \end{bmatrix}$$

The matrix B is sometimes referred to as the dimensional matrix.

Step 2. Having constructed matrix B, find its rank. From step 1, since

$$\begin{vmatrix} 1 & 1 & 0 \\ -4 & -2 & 1 \\ 2 & 0 & 0 \end{vmatrix} = 1 \neq 0$$

and no larger nonzero determinant exists, then

$$R(B) = 3 = r$$

Step 3. Define π_1 as a power product of r of the n quantities raised to arbitrary integer exponents and any one of the excluded $(n - r)$ quantities, i.e.,

$$\pi_1 = Q_1^{y_{11}} Q_2^{y_{12}} \cdots Q_r^{y_{1r}} Q_{r+1}$$

Step 4. Define $\pi_2, \pi_3, \ldots, \pi_{n-r}$ as power products of the same r quantities used in step 3 raised to arbitrary integer exponents but a different excluded quantity for each π-term, i.e.,

$$\pi_2 = Q_1^{y_{21}} Q_2^{y_{22}} \cdots Q_r^{y_{2r}} Q_{r+2}$$

$$\pi_3 = Q_1^{y_{31}} Q_2^{y_{32}} \cdots Q_r^{y_{3r}} Q_{r+3}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\pi_{n-r} = Q_1^{y_{n-r,1}} Q_2^{y_{n-r,2}} \cdots Q_r^{y_{n-r,r}} Q_n$$

Step 5. Carry out dimensionsional analysis on each π-term to evaluate the exponents.

Example C.3. Rework Example C.1 using the π-theorem.

Solution. Step 1. With F, L, and T as the fundamental units, the dimensional matrix of the quantities w, g, t, and x is

	w	g	t	x
F	1	0	0	0
L	0	1	0	1
T	0	−2	1	0

where

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{bmatrix}$$

Step 2. Since

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \end{vmatrix} = 2 \neq 0$$

and no larger nonzero determinant exists, then

$$R(B) = 3 = r$$

Step 3. Arbitrarily select x, w, and g as the $r = m = 3$ base quantities. The number $n - r$ of independent dimensional products that can be formed by the four quantities is therefore 1, i.e.,

$$\pi_1 = x^{y_{11}} w^{y_{12}} g^{y_{13}} t$$

Step 4. Dimensional analysis gives

$$[\pi_1] = [x]^{y_{11}} [w]^{y_{12}} [g]^{y_{13}} [t]$$

or

$$F^0 L^0 T^0 = (L)^{y_{11}} (F)^{y_{12}} (LT^{-2})^{y_{13}} (T)$$

which results in

$$y_{11} = -\frac{1}{2}, \qquad y_{12} = 0, \qquad \text{and} \qquad y_{13} = \frac{1}{2}$$

Hence,

$$\pi_1 = \sqrt{g \frac{t^2}{x}}$$

Example C.4. Rework Example C.2 using the π-theorem.

Solution. Step 1. With F, L, and T as the fundamental units, the dimensional matrix of the quantities D, ρ, μ, l, and V is

	D	ρ	μ	l	V
F	1	1	1	0	0
L	0	−4	−2	1	1
T	0	2	1	0	−1

where

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -4 & -2 & 1 & 1 \\ 0 & 2 & 1 & 0 & -1 \end{bmatrix}$$

Step 2. Since

$$\begin{vmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = -1 \neq 0$$

and no larger nonzero determinant exists, then

$$R(B) = 3 = r$$

Step 3. Select l, V, and ρ as the $r = 3$ base quantities. By the π-Theorem $(n - r) = (5 - 3) = 2$, π-terms exist.

$$\pi_1 = l^{y_{11}} V^{y_{12}} \rho^{y_{13}} D$$

$$\pi_2 = l^{y_{21}} V^{y_{22}} \rho^{y_{23}} \mu$$

Step 4. Dimensional analysis gives

$$y_{11} = -2, \quad y_{12} = -2, \quad \text{and} \quad y_{13} = -1$$

$$y_{21} = -1, \quad y_{22} = -1, \quad \text{and} \quad y_{23} = -1$$

Hence,

$$\pi_1 = \frac{D}{\rho V^2 L^2}$$

$$\pi_2 = \frac{\mu}{\rho V l}$$

C.4. *Dimensional Analysis in Fluid Mechanics*

If we consider flow systems which have geometrical similarity, then the force exerted by a fluid stream flowing past a submerged body (neglecting the buoyancy) will depend on the physical properties of the fluid (ρ, μ, K, C_p, C_τ, k) on the velocity of the stream (v), and on the linear dimension of the body (l). In the following cases we shall apply dimensional analysis to problems of fluid mechanics.

a. Inviscid Incompressible Fluid

In accordance with the procedure of dimensional analysis we assume that the dynamic force F is a function of ρ, v, l or

$$[F] = c[l]^{y_1}[v]^{y_2}[\rho]^{y_3} \tag{C-23}$$

where c, y_1, y_2, and y_3 are constants. By the principle of dimensional homogeneity the relation given in Eq. (C-23) can be written as

$$MLT^{-2} = (L)^{y_1}(LT^{-1})^{y_2}(ML^{-3})^{y_3} \tag{C-24}$$

Equating like exponents, we obtain

$$\begin{aligned} L: &\quad 1 = y_1 + y_2 - 3y_3 \\ M: &\quad 1 = y_3 \\ T: &\quad -2 = -y_2 \end{aligned} \tag{C-25}$$

The simultaneous solution of Eqs. (C-25) gives $y_1 = 2$, $y_2 = 2$, and $y_3 = 1$. Hence Eq. (C-23) becomes

$$F = c\rho l^2 v^2 \tag{C-26a}$$

or

$$\frac{F}{\rho l^2 v^2} = c \tag{C-26b}$$

Equation (C-26b) is known as a force coefficient which is the only parameter in flows of an inviscid incompressible fluid.

b. Viscous Incompressible Fluid

In this case we assume that the dynamic force is given by

$$F = c[l]^{y_1}[v]^{y_2}[\rho]^{y_3}[\mu]^{y_4} \tag{C-27}$$

where μ is the measure of the viscosity of the fluid. In a like manner, the relation in Eq. (C-27) can be written

$$MLT^{-2} = (L)^{y_1}(LT^{-1})^{y_2}(ML^{-3})^{y_3}(ML^{-1}T^{-1})^{y_4} \tag{C-28}$$

Equating the like exponents, one obtains

$$\begin{aligned} L: &\quad 1 = y_1 + y_2 - 3y_3 - y_4 \\ M: &\quad 1 = y_3 + y_4 \\ T: &\quad -2 = -y_2 - y_4 \end{aligned} \tag{C-29}$$

Since there are four unknowns in the three equations, Eq. (C-29), there is one degree of indeterminacy in the solution. The solutions in terms of y_4 are

$$y_1 = 2 - y_4, \qquad y_2 = 2 - y_4, \qquad y_3 = 1 - y_4 \tag{C-30}$$

Hence Eq. (C-27) becomes

$$F = c\rho v^2 l^2 \left(\frac{\mu}{\rho v l}\right)^{y_4} = c\rho v^2 l^2 \left(\frac{\rho v l}{\mu}\right)^{n} \tag{C-31}$$

or

$$\frac{F}{\rho v^2 l^2} = f(\mathrm{Re}) \tag{C-32}$$

where $\mathrm{Re} = \rho(vl/\mu)$ is the Reynolds number, and the function $f(\mathrm{Re})$ cannot be determined by dimensional analysis. It is seen that for a viscous incompressible fluid, the force coefficient is a certain function of the Reynolds number. This is shown in both Eqs. (9–40) and (10–45) for laminar and turbulent flows, respectively.

c. *Inviscid Compressible Fluid*

For an inviscid compressible fluid it is assumed that the dynamic force depends only on ρ, v, l, the bulk modulus K, and the specific heats C_p and C_τ of the fluid, or

$$[F] = c[l]^{y_1}[v]^{y_2}[\rho]^{y_3}[K]^{y_4}[C_p]^{y_5}[C_\tau]^{y_6} \tag{C-33}$$

Dimensionally, Eq. (C-33) can be written

$$MLT^{-2} = (L)^{y_1}(LT^{-1})^{y_2}(ML^{-3})^{y_3}(ML^{-1}T^{-2})^{y_4}(L^2T^{-2}\theta^{-1})^{y_5}(L^2T^{-2}\theta^{-1})^{y_6} \tag{C-34}$$

where θ is the fundamental unit for temperature. The indicial equations of the relation above are

$$\begin{aligned} L:&\quad 1 = y_1 + y_2 - 3y_3 - y_4 + 2y_5 + 2y_6 \\ M:&\quad 1 = y_3 + y_4 \\ T:&\quad -2 = -y_2 - 2y_4 - 2y_5 - 2y_6 \\ \theta:&\quad 0 = -y_5 - y_6 \end{aligned} \tag{C-35}$$

The solutions of the simultaneous equations above in terms of y_4 and y_6 are $y_1 = 2$, $y_2 = 2 - 2y_4$, $y_3 = 1 - y_4$, $y_6 = -y_5$, and Eq. (C-33) becomes

$$\begin{aligned} F &= c\rho v^2 l^2 \left(\frac{\rho v^2}{K}\right)^{-y_4} \left(\frac{C_p}{C_\tau}\right)^{y_5} \\ &= c\rho v^2 l^2 \left(\frac{\rho v^2}{K}\right)^{n} \left(\frac{C_p}{C_\tau}\right)^{m} \end{aligned} \tag{C-36}$$

where $\rho(v^2/K) = (v^2/a^2) = \mathrm{M}_a$ is the Mach number, and $C_p/C_\tau = \kappa$ is the ratio of the specific heats. n and m are numerical values which

cannot be determined by dimensional analysis. Equation (C-36) can be written as

$$\frac{F}{\rho v^2 l^2} = f(\mathrm{M}_a, \kappa)$$

Hence, for an inviscid compressible fluid the force coefficient is a certain function of the Mach number and the ratio of the specific heats.

d. Viscous Compressible Fluid

In this case we will assume that the dynamic force depends on l, v, ρ, K, C_p, C_τ, μ, and the conductivity coefficient, k, of the fluid, or

$$[F] = c[l]^{y_1}[v]^{y_2}[\rho]^{y_3}[K]^{y_4}[C_p]^{y_5}[C_\tau]^{y_6}[\mu]^{y_7}[k]^{y_8} \tag{C-37}$$

Dimensionally,

$$\begin{aligned} MLT^{-2} = {} & (L)^{y_1}(LT^{-1})^{y_2}(ML^{-3})^{y_3}(ML^{-1}T^{-2})^{y_4} \\ & \times (L^2T^{-2}\theta^{-1})^{y_5+y_6}(ML^{-1}T^{-1})^{y_7}(MLT^{-3}\theta^{-1}) \end{aligned} \tag{C-38}$$

The indicial equations of the relation above are

$$\begin{aligned} L: \quad & 1 = y_1 + y_2 - 3y_3 - y_4 + 2y_5 + 2y_6 - y_7 + y_8 \\ M: \quad & 1 = y_3 + y_4 + y_7 + y_8 \\ T: \quad & -2 = -y_2 - 2y_4 - 2y_5 - 2y_6 - y_7 - 3y_8 \\ \theta: \quad & 0 = -y_5 - y_6 - y_8 \end{aligned} \tag{C-39}$$

There are four degrees of indeterminacy, and the solutions in terms of y_4, y_5, y_7, and y_8 are $y_1 = 2 - y_7 - y_8$, $y_2 = 2 - 2y_4 - y_7 - y_8$, $y_3 = 1 - y_4 - y_7 - y_8$, $y_6 = -y_5 - y_8$, and Eq. (C-37) becomes

$$\begin{aligned} F &= c\rho v^2 l^2 \left(\frac{\rho v^2}{K}\right)^{-y_4} \left(\frac{C_p}{C_\tau}\right)^{y_5} \left(\frac{\mu}{\rho v l}\right)^{y_7} \left(\frac{k}{C_\tau \rho v l}\right)^{y_8} \\ &= c\rho v^2 l^2 (M_a)^n (\kappa)^m \left(\frac{\rho v l}{\mu}\right)^r \left(\frac{\mu C_p}{k}\right)^s \end{aligned} \tag{C-40}$$

where $n = -y_4$, $m = y_5 + y_8$, $r = -(y_7 + y_8)$ and $s = -y_8$. Thus, Eq. (C-40) is equivalent to

$$\frac{F}{\rho v^2 l^2} = f(\mathrm{M}_a, \kappa, \mathrm{Re}, \mathrm{Pr}) \tag{C-41}$$

where $\mathrm{Pr} = \mu(C_p/k)$ is the Prandtl number. Equation (C-41) shows that the force coefficient in a viscous compressible fluid is a certain function of the Reynolds number, Mach number, Prandtl number, and the ratio of the specific heats. This is in complete agreement with the results obtained from the similarity principle (see §11.1).

REFERENCES

1. Birkhoff, G., *Hydrodynamics*. New York: Dover Publications, Inc., 1955.

2. Huntley, H. E., *Dimensional Analysis*. Dover Publications, Inc., New York, 1966.

3. *National Bureau of Standards Technical News Bulletin*, Vol. 44, No. 12, December 1960.

4. Sadov, L. I., *Similarity and Dimensional Methods in Mechanics*. New York: Academic Press, 1959.

D

FUNDAMENTAL EQUATIONS OF THE FLOW OF INVISCID FLUIDS

The purpose of this appendix is to provide a basic knowledge of fundamental equations of the flow of inviscid fluids to the reader whose interests center only in inviscid incompressible and compressible fluids. Without affecting the continuity of the text, the reader can omit Chapters Four and Five and proceed to Chapters Six, Seven, and Eleven after learning the fundamental equations in this appendix.

D.1. The Equation of Continuity—Conservation of Mass

For a compressible fluid the equation of continuity is given in Eq. (5–8) as follows,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q}) = 0 \tag{5-8}$$

In Cartesian coodinates Eq. (5–8) may be written

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{5-9a}$$

In cylindrical coordinates Eq. (5–8) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) + \rho\frac{v_r}{r} = 0 \qquad (5\text{-}40)$$

In spherical coordinates, the equation of continuity takes the form

$$\frac{\partial \rho}{\partial t} + \frac{1}{\omega^2}\frac{\partial}{\partial \omega}(\rho\omega^2 v_\omega) + \frac{1}{\omega \sin\varphi}\frac{\partial}{\partial \varphi}(\rho v_\varphi \sin\varphi) + \frac{1}{\omega \sin\varphi}\frac{\partial}{\partial \theta}(\rho v_\theta) = 0 \qquad (5\text{-}60)$$

For an incompressible fluid where ρ = Constant, Eqs. (5–8), (5–9a), (5–40), and (5–60) reduce to Eqs. (5–9b), (7–1), (5–41), and (5–61), respectively.

The equation of continuity, based on the conservation of mass, was derived in §5.1, and the general orthogonal curvilinear coordinates were discussed in Appendix A.5.

D.2. Equations of Motion (Euler's Equations)—Conservation of Momentum

Consider a fluid mass in motion which at time t occupies a volume τ bounded by the surface A (Fig. D.1). Let $d\tau$ be an element of volume and $\rho\, d\tau$ the elementary mass moving with velocity $\mathbf{q}$. Then $-\rho\, d\tau(D\mathbf{q}/Dt)$ is the force of inertia (D'Alembert's principle, see §2.7) acting upon the elementary mass. The resultant of the inertia forces or the time rate of change of linear momentum is

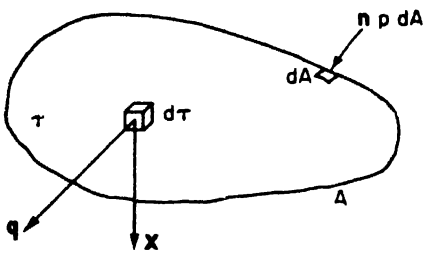


Figure D.1

$$\mathbf{F}_I = -\iiint \frac{D\mathbf{q}}{Dt}\rho\, d\tau \qquad (D\text{-}1)$$

If the external or body force per unit mass acting upon the elementary volume is denoted by X, then $\mathbf{X}\rho\, d\tau$ will be the elementary body force. The resultant of the body forces is

$$\mathbf{F}_B = \iiint \mathbf{X}\rho\, d\tau \qquad (D\text{-}2)$$

The elementary surface force acting on an element of surface dA is expressed by $-\mathbf{n}p\, dA$ where $\mathbf{n}$ is the unit vector normal to the elementary surface, directed outward. According to Gauss's theorem (Appendix A), the resultant of the surface forces is

$$\mathbf{F}_s = -\int\int \mathbf{n}p\, dA = -\int\int\int \nabla p\, d\tau \tag{D-3}$$

By Newton's law of motion the total force acting on a fluid mass enclosed in an arbitrary volume fixed in space is equal to the time rate of change of linear momentum. This is the law of conservation of momentum. Adding the resultants of the body and surface forces in Eqs. (D–2) and (D–3), respectively, to the resultant of the inertia forces in Eq. (D–1), we obtain

$$\int\int\int \left\{-\rho\frac{D\mathbf{q}}{Dt} + \rho\mathbf{X} - \nabla p\right\} d\tau = 0 \tag{D-4}$$

Equation (D–4) is true for all τ no matter how small, and hence the general equation of motion for an inviscid fluid is obtained.

$$= \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q} = \mathbf{X} - \frac{1}{\rho}\nabla p \tag{D-5}$$

Equation (D–5) which is identical to Eq. (5–20) is valid for both incompressible and compressible flows. It is usually called Euler's equation. In Cartesian coordinates Eq. (D–5) may be written

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= X_x - \frac{1}{\rho}\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= X_y - \frac{1}{\rho}\frac{\partial p}{\partial y} \\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= X_z - \frac{1}{\rho}\frac{\partial p}{\partial z}
\end{aligned} \tag{D-6}$$

a. Equations of Motion in Cylindrical Coordinates

In cylindrical coordinates (r, θ, z) shown in Fig. 5.3, we re-examine the inertia term in Eq. (D–4). Referring to Fig. D.2, the components of the acceleration vector, after neglecting higher order terms, can be expressed as follows:

$$\frac{D\mathbf{q}}{Dt} = \mathbf{i}_r\left(\frac{Dv_r}{Dt} - v_\theta\frac{d\theta}{dt}\right) + \mathbf{i}_\theta\left(\frac{Dv_\theta}{Dt} + v_r\frac{d\theta}{dt}\right) + \mathbf{i}_z\frac{Dv_z}{Dt} \tag{D-7}$$

where v_r, v_θ, and v_z are the velocity components in the r, θ, and z directions, respectively, and $\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_z$ are the unit vectors in the direction indicated by the respective subscripts. Substituting

$$v_\theta = r\frac{d\theta}{dt}$$

into Eq. (D–7), we have the resultant of the inertia forces in cylindrical coordinates

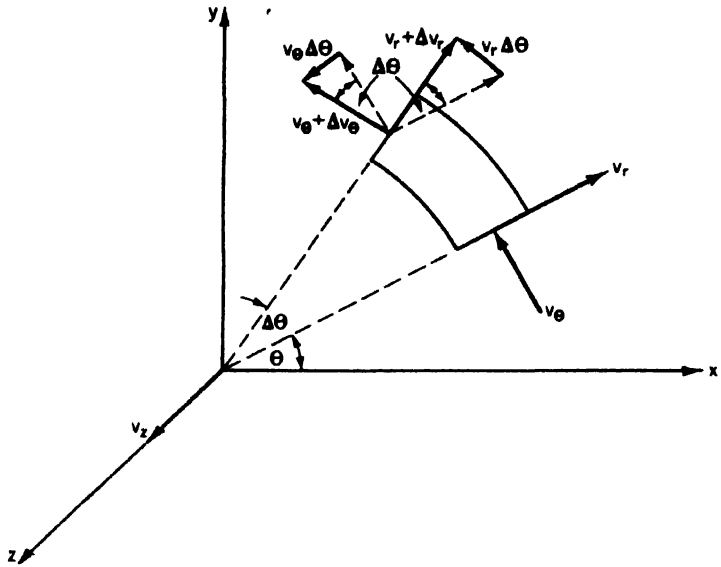


Figure D.2

$$\iiint \rho \frac{Dq}{Dt} d\tau = \iiint \left[\mathbf{i}_r \left(\frac{Dv_r}{Dt} - \frac{v_\theta^2}{r} \right) + \mathbf{i}_\theta \left(\frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r} \right) + \mathbf{i}_z \frac{Dv_z}{Dt} \right] \rho \, d\tau \tag{D-8}$$

The resultants of the surface forces and the body forces are, respectively,

$$\mathbf{F}_s = -\iiint \nabla p \, d\tau = -\iiint \left(\mathbf{i}_r \frac{\partial p}{\partial r} + \frac{\mathbf{i}_\theta}{r} \frac{\partial p}{\partial \theta} + \mathbf{i}_z \frac{\partial p}{\partial z} \right) d\tau \tag{D-9}$$

$$\mathbf{F}_B = \iiint (\mathbf{i}_r X_r + \mathbf{i}_\theta X_\theta + \mathbf{i}_z X_z) \, d\tau \tag{D-10}$$

Combining Eqs. (D-8), (D-9), and (D-10), the equations of motion in cylindrical coordinates are obtained as follows:

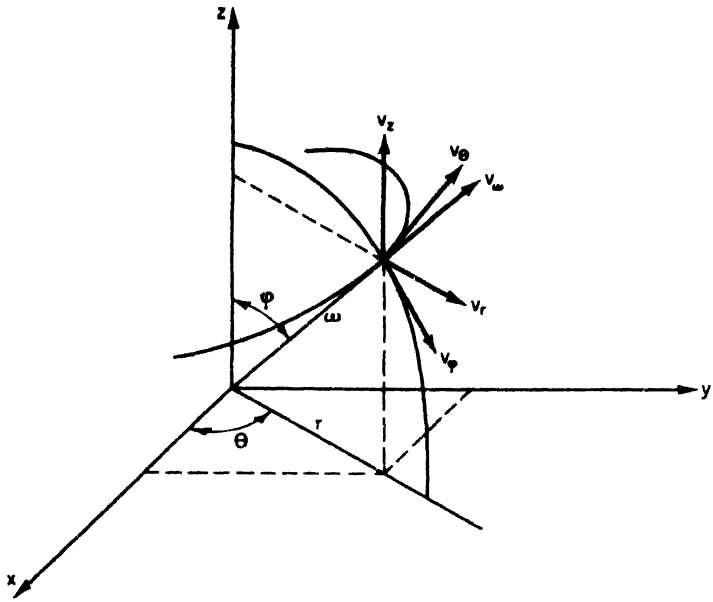
$$\rho \left(\frac{Dv_r}{Dt} - \frac{v_\theta^2}{r} \right) = \rho X_r - \frac{\partial p}{\partial r} \tag{D-11a}$$

$$\rho \left(\frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r} \right) = \rho X_\theta - \frac{1}{r} \frac{\partial p}{\partial \theta} \tag{D-11b}$$

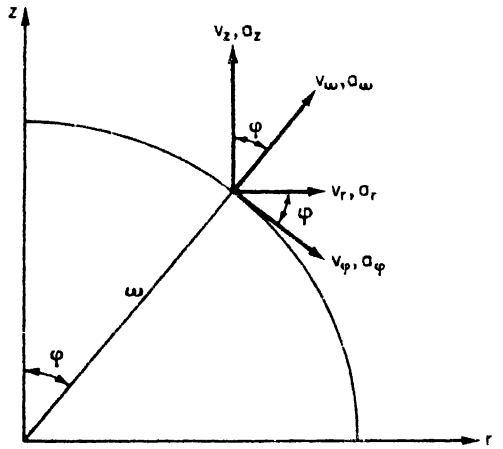
$$\rho \frac{Dv_z}{Dt} = \rho X_z - \frac{\partial p}{\partial z} \tag{D-11c}$$

b. Equations of Motion in Spherical Coordinates

The acceleration terms in the equations of motion in spherical coordinates $(\omega, \varphi, \theta)$ shown in Fig. D.3 can be calculated with the aid



(a)



(b)

Figure D.3

of the velocity and acceleration components previously found in cylindrical coordinates (r, θ, z). The relations between cylindrical and spherical coordinates are as follows:

$$r = \omega \sin \varphi, \qquad \theta = \theta, \qquad z = \omega \cos \varphi \tag{D-12}$$

The components of velocity and acceleration in cylindrical coordinates given in Eq. (D–7) in terms of the spherical system are

$$\begin{aligned}
v_r &= \frac{dr}{dt} = \omega \cos \varphi \frac{d\varphi}{dt} + \sin \varphi \frac{d\omega}{dt} \\
v_\theta &= r\frac{d\theta}{dt} = \omega \sin \varphi \frac{d\theta}{dt} \\
v_z &= \frac{dz}{dt} = -\omega \sin \varphi \frac{d\varphi}{dt} + \cos \varphi \frac{d\omega}{dt} \\
a_r &= \frac{Dv_r}{Dt} - \frac{v_\theta^2}{r} = \frac{Dv_r}{Dt} - \frac{v_\theta^2}{\omega \sin \varphi} \\
a_\theta &= \frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r} = \frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{\omega \sin \varphi} \\
a_z &= \frac{Dv_z}{Dt}
\end{aligned} \tag{D-13}$$

The components of velocity and acceleration in spherical coordinates can be obtained with the aid of Eq. (D–13) in the following procedures

$$v_\theta = \omega \sin \varphi \frac{d\theta}{dt}$$

$$\begin{aligned}
v_\omega &= v_r \sin \varphi + v_z \cos \varphi = \omega \cos \varphi \sin \varphi \frac{d\varphi}{dt} + \sin^2 \varphi \frac{d\omega}{dt} \\
&\quad - \omega \sin \varphi \cos \varphi \frac{d\varphi}{dt} + \cos^2 \varphi \frac{d\omega}{dt} = \frac{d\omega}{dt}
\end{aligned}$$

$$v_\varphi = -v_z \sin \varphi + v_r \cos \varphi = \omega \frac{d\varphi}{dt}$$

$$\begin{aligned}
a_\theta &= \frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{\omega \sin \varphi} = \frac{Dv_\theta}{Dt} + \left(\omega \cos \varphi \frac{d\varphi}{dt} + \sin \varphi \frac{d\omega}{dt}\right) \frac{v_\theta}{\omega \sin \varphi} \\
&= \frac{Dv_\theta}{Dt} + v_\theta \cot \theta \frac{d\varphi}{dt} + \frac{v_\theta}{\omega} \frac{d\omega}{dt} \\
&= \frac{Dv_\theta}{Dt} + \frac{v_\theta v_\omega}{\omega} + \frac{v_\theta v_\varphi}{\omega} \cot \varphi
\end{aligned}$$

$$\begin{aligned}
a_\omega &= a_r \sin \varphi + a_z \cos \varphi = \left(\frac{Dv_r}{Dt} - \frac{v_\theta^2}{\omega \sin \varphi}\right) \sin \varphi + \frac{Dv_z}{Dt} \cos \varphi \\
&= \left[\cos \varphi \frac{d\varphi}{dt} \frac{d\omega}{dt} - \omega \sin \varphi \left(\frac{d\varphi}{dt}\right)^2 + \omega \cos \varphi \frac{d^2\varphi}{dt^2}\right. \\
&\quad \left. + \cos \varphi \frac{d\varphi}{dt} \frac{d\omega}{dt} + \sin \varphi \frac{d^2\omega}{dt^2}\right] \sin \varphi - \frac{v_\theta^2}{\omega}
\end{aligned}$$

$$+\left[-\sin\varphi\frac{d\omega}{dt}\frac{d\varphi}{dt}-\omega\cos\varphi\left(\frac{d\varphi}{dt}\right)^2-\omega\sin\varphi\frac{d^2\varphi}{dt^2}\right.$$

$$\left.+\cos\varphi\frac{d^2\omega}{dt^2}-\sin\varphi\frac{d\omega}{dt}\frac{d\varphi}{dt}\right]\cos\varphi=\frac{Dv_\omega}{Dt}-\frac{v_\varphi^2+v_\theta^2}{\omega}$$

$$a_\varphi=-a_z\sin\varphi+a_r\cos\varphi=-\frac{Dv_z}{Dt}\sin\varphi+\left(\frac{Dv_r}{Dt}-\frac{v_\theta^2}{\omega\sin\varphi}\right)\cos\varphi$$

$$=-\sin\varphi\frac{D}{Dt}\left(-\omega\sin\varphi\frac{d\varphi}{dt}+\cos\varphi\frac{d\omega}{dt}\right)-\frac{v_\theta^2}{\omega}\cot\varphi$$

$$+\cos\varphi\frac{D}{Dt}\left(\omega\cos\varphi\frac{d\varphi}{dt}+\sin\varphi\frac{d\omega}{dt}\right)$$

$$=\frac{Dv_\varphi}{Dt}+\frac{v_\omega v_\varphi}{\omega}-\frac{v_\theta^2\cot\varphi}{\omega} \tag{D-14}$$

Hence the resultant of the inertia forces in spherical coordinates can be written

$$\mathbf{F}_I=-\int\int\int\rho\frac{D\mathbf{q}}{Dt}\,d\tau=-\int\int\int\left[\mathbf{i}_\omega\left(\frac{Dv_\omega}{Dt}-\frac{v_\varphi^2+v_\theta^2}{\omega}\right)\right.$$

$$\left.+\mathbf{i}_\varphi\left(\frac{Dv_\varphi}{Dt}+\frac{v_\omega v_\varphi}{\omega}-\frac{v_\theta^2\cot\varphi}{\omega}\right)+\mathbf{i}_\theta\left(\frac{Dv_\theta}{Dt}+\frac{v_\theta v_\omega}{\omega}+\frac{v_\theta v_\varphi}{\omega}\cot\varphi\right)\right]\rho\,d\tau \tag{D-15}$$

where $\mathbf{i}_\omega$, $\mathbf{i}_\varphi$ and $\mathbf{i}_\theta$ are the unit vectors in the directions indicated by their respective subscripts.

The resultants of the surface forces and the body forces are, respectively,

$$\mathbf{F}_S=-\int\int\int\nabla p\,d\tau=-\int\int\int\left(\mathbf{i}_\omega\frac{\partial p}{\partial\omega}+\frac{\mathbf{i}_\varphi}{\omega}\frac{\partial p}{\partial\varphi}+\frac{\mathbf{i}_\omega}{\omega\sin\varphi}\frac{\partial p}{\partial\theta}\right)d\tau \tag{D-16}$$

$$\mathbf{F}_B=\int\int\int(\mathbf{i}_\omega X_\omega+\mathbf{i}_\varphi X_\varphi+\mathbf{i}_\theta X_\theta)\,d\tau \tag{D-17}$$

Combining Eqs. (D-15), (D-16), and (D-17), the equations of motion in spherical coordinates are obtained as follows:

$$\rho\left(\frac{Dv_\omega}{Dt}-\frac{v_\varphi^2+v_\theta^2}{\omega}\right)=\rho X_\omega-\frac{\partial p}{\partial\omega} \tag{D-18a}$$

$$\rho\left(\frac{Dv_\varphi}{Dt}+\frac{v_\omega v_\varphi}{\omega}-\frac{v_\theta^2}{\omega}\cot\varphi\right)=\rho X_\varphi-\frac{\partial p}{\partial\varphi} \tag{D-18b}$$

$$\rho\left(\frac{Dv_\theta}{Dt}+\frac{v_\theta v_\omega}{\omega}+\frac{v_\varphi v_\theta}{\omega}\cot\varphi\right)=\rho X_\theta-\frac{1}{\omega\sin\varphi}\frac{\partial p}{\partial\theta} \tag{D-18c}$$

Equations (*D*-18a), (D-18b), and (D-18c) are identical to Eqs. (5-79a), (5-79b), and (5-79c), respectively.

D.3. The Energy Equation—Conservation of Energy

According to the First Law of Thermodynamics (Appendix B), the energy added to a closed system increases the internal energy per unit mass of the fluid. The energy is added both by heat and by work done on the fluid. Let Q be the heat added per unit mass of the gas and E the internal energy per unit mass of the gas. The rate of work done by the surface force on a unit volume of the gas is

$$-p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=-p(\nabla\cdot\mathbf{q}) \tag{D-19}$$

The mathematical formulation of the First Law of Thermodynamics in terms of the time variation of energy may be written:

$$\rho\frac{dQ}{dt}-p(\nabla\cdot\mathbf{q})=\rho\frac{DE}{Dt} \tag{D-20}$$

where Q is a function of the time only. From Eq. (5-8), we have

$$\frac{p}{\rho}\nabla\cdot\mathbf{q}=-\frac{p}{\rho^2}\frac{D\rho}{Dt}=\frac{D}{Dt}\left(\frac{p}{\rho}\right)-\frac{1}{\rho}\frac{Dp}{Dt} \tag{D-21}$$

Substituting Eq. (D-21) into Eq. (D-20), the desired energy equation becomes

$$\rho\frac{dQ}{dt}=\rho\frac{Dh}{Dt}-\frac{Dp}{Dt} \tag{D-22}$$

Where $h=E+(p/\rho)$ is the enthalpy of the gas per unit mass and Q is to be specified.

Now let us consider the mathematical formulation of the law of heat conduction. According to the Fourier heat-conduction law the heat flow across an element of surface in Δt seconds is proportional to the greatest rate of decrease of the temperature T, that is,

$$-k\Delta A\,\Delta t\frac{\partial T}{\partial n} \tag{D-23}$$

where k is a constant of proportionality known as the thermal conductivity of a substance in Btu per unit time per unit length per degree R. The negative sign in Eq. (D-23) indicates that heat flows from points of higher temperature to those of lower temperature.

The total amount of heat flowing out in Δt seconds from an arbitrary volume τ bounded by a closed surface A is

$$-\Delta t\int\int k\frac{\partial T}{\partial n}\,dA=-\Delta t\int\int k(\nabla T)\cdot d\mathbf{A} \tag{D-24}$$

Applying the divergence theorem to the right-hand member of Eq. (D-24) yields

$$-\Delta t \int\int k(\nabla T)\cdot d\mathbf{A} = -\Delta t \int\int\int \nabla\cdot[k(\nabla T)]\, d\tau \qquad \text{(D-25)}$$

Hence the heat added to the system per unit volume, $\rho\Delta Q$, becomes

$$\rho\Delta Q = \Delta t\nabla\cdot[k(\nabla T)]$$

or, in the limit as $t \to 0$

$$\rho\frac{dQ}{dt} = \nabla\cdot[k(\nabla T)] \qquad \text{(D-26)}$$

In Cartesian coordinates, Eq. (D-26) becomes

$$\rho\frac{dQ}{dt} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) \qquad \text{(D-27)}$$

Finally, we have the thermal energy equation when direct heating from chemical reaction and radiation heating is absent.

$$\nabla\cdot(k\nabla T) = \rho\frac{Dh}{Dt} - \frac{Dp}{Dt} \qquad \text{(D-28)}$$

In Cartesian coordinates, Eq. (D-28) has the form

$$\begin{aligned}\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\\ = \rho\frac{\partial(C_v T)}{\partial t} + \rho u\frac{\partial(C_p T)}{\partial x} + \rho v\frac{\partial(C_p T)}{\partial y} + \rho w\frac{\partial(C_p T)}{\partial z}\\ - \left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right)\end{aligned} \qquad \text{(D-29)}$$

For gases with ordinary densities it has been shown by the kinetic theory of gases and by experiments that both μ and k are functions of the temperature only.

In cylindrical coordinates the energy equation (D-28) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(k\frac{\partial T}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) = \rho\frac{Dh}{Dt} - \frac{Dp}{Dt} \qquad \text{(D-30)}$$

where

$$\nabla T = \mathbf{i}_r\frac{\partial T}{\partial r} + \frac{\mathbf{i}_\theta}{r}\frac{\partial T}{\partial\theta} + \mathbf{i}_z\frac{\partial T}{\partial z}$$

$$\nabla\cdot(k\nabla T) = \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(k\frac{\partial T}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta} + v_z\frac{\partial}{\partial z}$$

In spherical coordinates, the energy equation (D-28) becomes

$$\begin{aligned}\frac{1}{\omega^2}\frac{\partial}{\partial\omega}\left(\omega^2 k\frac{\partial T}{\partial\omega}\right) + \frac{1}{\omega^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(k\frac{\partial T}{\partial\varphi}\sin\varphi\right) + \frac{1}{\omega^2\sin^2\varphi}\frac{\partial}{\partial\theta}\left(k\frac{\partial T}{\partial\theta}\right)\\ = \rho\frac{Dh}{Dt} - \frac{Dp}{Dt}\end{aligned} \qquad \text{(D-31)}$$

where

$$\nabla T = \mathbf{i}_\omega \frac{\partial T}{\partial \omega} + \mathbf{i}_\varphi \frac{1}{\omega}\frac{\partial T}{\partial \theta} + \mathbf{i}_\theta \frac{1}{\omega \sin \varphi}\frac{\partial T}{\partial \theta}$$

$$\nabla \cdot (k\nabla T) = \frac{1}{\omega^2}\frac{\partial}{\partial \omega}\left(\omega^2 k \frac{\partial T}{\partial \omega}\right) + \frac{1}{\omega^2 \sin \varphi}\frac{\partial}{\partial \varphi}\left(k \frac{\partial T}{\partial \varphi} \sin \varphi\right)$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_\omega \frac{\partial}{\partial \omega} + \frac{v_\varphi}{\omega}\frac{\partial}{\partial \varphi} + \frac{v_\theta}{\omega \sin \varphi}\frac{\partial}{\partial \theta}$$

D.4. The Equation of State—Perfect Gases

The equation of state of a substance is a relation between its pressure, specific volume, and temperature. For every homogeneous substance—solid, liquid, or gas—there exists an equation of state. The exact form of the function connecting these three scalar quantities is, in general, extremely complicated. It is often necessary to express this complicated relation in graphical form.

Experimental data of the thermal properties of gases show that for low pressure the equation of state approaches the same form for all gases, namely,

$$\frac{p}{\rho} = RT \tag{D-32}$$

where R is the gas constant, which, in the system adopted in the present work, is for air (see Table C-2 for other unit systems)

$$R = 1715 \frac{\text{ft}^2}{\text{sec}^2 - {}^\circ R}$$

Any gas (air, oxygen, or helium) which obeys the law of the equation of state, Eq. (D-32), is called a perfect gas.

Further discussion on the equation of state for a perfect gas is given in Appendix B.

E

TABLES AND CHARTS

Table E-1 APPROXIMATE PHYSICAL PROPERTIES OF SOME COMMON FLUIDS AT STANDARD ATMOSPHERIC PRESSURE

Substance	Density slugs/ft³ 1 atm	Specific gravity	Specific weight lbf/ft³	Viscosity lbf-sec/ft² × 10⁵	Thermal conductivity Btu/ft-hr °F
Gases	$T = 32°F$	Air = 1		$T = 32°F$	$T = 32°F$
Air	0.0025085	1.0000	0.08071	0.0356630	0.0140
Argon	0.0034609	1.3796	0.11135	0.0437645	0.00915
Helium	0.0003462	0.13804	0.01114	0.0388368	0.08180
Hydrogen	0.0001744	0.06952	0 005611	0.0174348	0.09660
Nitrogen	0.0024265	0.96724	0.07807	0.0356422*	0.01400
Oxygen	0.0027727	1.10527	0.08921	0.0394632	0.01420
Liquids		Water = 1			$T = 68°F$
Benzene	1.7436439	0.8986	56.1	1.9042560	0.091506**
Carbon Tet.	3.0956673	1.5954	99.6	2.774952	0.05979
Glycerin	2.4429664	1.2590	78.6	25,285.68	0.1650
Mercury	26.387766	13.5992	849.0	3.518280	4.7448***
Water	1.9403866	1.0000	62.43	3.741696	0.3430

*$T = 51.8°F$ **$T = 72.5°F$ ***$T = 32°F$

Table E-2 PROPERTIES OF THE STANDARD ATMOSPHERE†

h (ft)	T (°F)	a (ft/sec)	p (lbf/ft²)	ρ (slugs/ft³)	$\mu \times 10^7$ (slugs/ft-sec)
0	59.00	1117	2116.2	0.002378	3.719
1,000	57.44	1113	2040.9	0.002310	3.699
2,000	51.87	1109	1967.7	0.002242	3.679
3,000	48.31	1105	1896.7	0.002177	3.659
4,000	44.74	1102	1827.7	0.002112	3.639
5,000	41.18	1098	1760.8	0.002049	3.618
6,000	37.62	1094	1696.0	0.001988	3.598
7,000	34.05	1090	1633.0	0.001928	3.577
8,000	30.49	1086	1571.9	0.001869	3.557
9,000	26.92	1082	1512.8	0.001812	3.536
10,000	23.36	1078	1455.4	0.001756	3.515
11,000	19.80	1074	1399.8	0.001702	3.495
12,000	16.23	1070	1345.9	0.001649	3.474
13,000	12.67	1066	1293.7	0.001597	3.453
14,000	9.10	1062	1243.2	0.001546	3.432
15,000	5.54	1058	1194.3	0.001497	3.411
16,000	1.98	1054	1147.0	0.001448	3.390
17,000	−1.59	1050	1101.1	0.001401	3.369
18,000	−5.15	1046	1056.9	0.001355	3.347
19,000	−8.72	1041	1014.0	0.001311	3.326
20,000	−12.28	1037	972.6	0.001267	3.305
21,000	−15.84	1033	932.5	0.001225	3.283
22,000	−19.41	1029	893.8	0.001183	3.262
23,000	−22.97	1025	856.4	0.001143	3.240
24,000	−26.54	1021	820.3	0.001104	3.218
25,000	−30.10	1017	785.3	0.001066	3.196
26,000	−33.66	1012	751.7	0.001029	3.174
27,000	−37.23	1008	719.2	0.000993	3.153
28,000	−40.79	1004	687.9	0.000957	3.130
29,000	−44.36	999	657.6	0.000923	3.108
30,000	−47.92	995	628.5	0.000890	3.086
31,000	−51.48	991	600.4	0.000858	3.064
32,000	−55.05	987	573.3	0.000826	3.041
33,000	−58.61	982	547.3	0.000796	3.019
34,000	−62.18	978	522.2	0.000766	2.997
35,000	−65.74	973	498.0	0.000737	2.974
40,000	−67.6	971	391.8	0.0005857	2.961
45,000	−67.6	971	308.0	0.0004605	2.961
50,000	−67.6	971	242.2	0.0003622	2.961
60,000	−67.6	971	150.9	0.0002240	2.961
70,000	−67.6	971	93.5	0.0001389	2.961
80,000	−67.6	971	58.0	0.0000861	2.961
90,000	−67.6	971	36.0	0.0000535	2.961
100,000	−67.6	971	22.4	0.0000331	2.961
150,000	113.5	1174	3.003	0.00000305	4.032
200,000	159.4	1220	0.6645	0.00000062	4.277
250,000	−8.2	1042	0.1139	0.00000015	3.333

†Data taken from NACA TN 1428. Courtesy of the National Aeronautics and Space Administration.

Table E-3 PHYSICAL PROPERTIES OF WATER†

Temp °F	Specific weight γ lbf/ft³	Density ρ slugs/ft³	Viscosity μ lbf-sec/ft² $10^5\,\mu =$	Kinematic viscosity ν ft²/sec $10^5\,\nu =$	Surface tension σ_s lbf/ft $100\,\sigma_s =$	Vapor pressure head p_v/γ ft	Bulk modulus of elasticity K lbf/in² $10^{-3}\,K =$
32	62.42	1.940	3.746	1.931	0.518	0.20	293
40	62.43	1.940	3.229	1.664	0.514	0.28	294
50	62.41	1.940	2.735	1.410	0.509	0.41	305
60	62.37	1.938	2.359	1.217	0.504	0.59	311
70	62.30	1.936	2.050	1.059	0.500	0.84	320
80	62.22	1.934	1.799	0.930	0.492	1.17	322
90	62.11	1.931	1.595	0.826	0.486	1.61	323
100	62.00	1.927	1.424	0.739	0.480	2.19	327
110	61.86	1.923	1.284	0.667	0.473	2.95	331
120	61.71	1.918	1.168	0.609	0.465	3.91	333
130	61.55	1.913	1.069	0.558	0.460	5.13	334
140	61.38	1.908	0.981	0.514	0.454	6.67	330
150	61.20	1.902	0.905	0.476	0.447	8.58	328
160	61.00	1.896	0.838	0.442	0.441	10.95	326
170	60.80	1.890	0.780	0.413	0.433	13.83	322
180	60.58	1.883	0.726	0.385	0.426	17.33	318
190	60.36	1.876	0.678	0.362	0.419	21.55	313
200	60.12	1.868	0.637	0.341	0.412	26.59	308
212	59.83	1.860	0.593	0.319	0.404	33.90	300

†This table was compiled primarily from *A.S.C.E. Manual of Engineering Practice*, No. 25, Hydraulic Models, 1942.

Table E-4 ONE-DIMENSIONAL ISENTROPIC RELATION†

M	A/A^*	p/p_0	ρ/ρ_0	T/T_0	M	A/A^*	p/p_0	ρ/ρ_0	T/T_0
0.00		1.000	1.000	1.000	0.78	1.05	0.669	0.750	0.891
0.01	57.87	0.9999	0.9999	0.9999	0.80	1.04	0.656	0.740	0.886
0.02	28.94	0.9997	0.9999	0.9999	0.82	1.03	0.643	0.729	0.881
0.04	14.48	0.999	0.999	0.9996	0.84	1.02	0.630	0.719	0.876
0.06	9.67	0.997	0.998	0.999	0.86	1.02	0.617	0.708	0.871
0.08	7.26	0.996	0.997	0.999	0.88	1.01	0.604	0.698	0.865
0.10	5.82	0.993	0.995	0.998	0.90	1.01	0.591	0.687	0.860
0.12	4.86	0.990	0.993	0.997	0.92	1.01	0.578	0.676	0.855
0.14	4.18	0.986	0.990	0.996	0.94	1.00	0.566	0.666	0.850
0.16	3.67	0.982	0.987	0.995	0.96	1.00	0.553	0.655	0.844
0.18	3.28	0.978	0.984	0.994	0.98	1.00	0.541	0.645	0.839
0.20	2.96	0.973	0.980	0.992	1.00	1.00	0.528	0.632	0.833
0.22	2.71	0.967	0.976	0.990	1.02	1.00	0.516	0.623	0.828
0.24	2.50	0.961	0.972	0.989	1.04	1.00	0.504	0.613	0.822
0.26	2.32	0.954	0.967	0.987	1.06	1.00	0.492	0.602	0.817
0.28	2.17	0.947	0.962	0.985	1.08	1.01	0.480	0.592	0.810
0.30	2.04	0.939	0.956	0.982	1.10	1.01	0.468	0.582	0.805
0.32	1.92	0.932	0.951	0.980	1.12	1.01	0.457	0.571	0.799
0.34	1.82	0.923	0.944	0.977	1.14	1.02	0.445	0.561	0.794
0.36	1.74	0.914	0.938	0.975	1.16	1.02	0.434	0.551	0.788
0.38	1.66	0.905	0.931	0.972	1.18	1.02	0.423	0.541	0.782
0.40	1.59	0.896	0.924	0.969	1.20	1.03	0.412	0.531	0.776
0.42	1.53	0.886	0.917	0.966	1.22	1.04	0.402	0.521	0.771
0.44	1.47	0.876	0.909	0.963	1.24	1.04	0.391	0.512	0.765
0.46	1.42	0.865	0.902	0.959	1.26	1.05	0.381	0.502	0.759
0.48	1.38	0.854	0.893	0.956	1.28	1.06	0.371	0.492	0.753
0.50	1.34	0.843	0.885	0.952	1.30	1.07	0.361	0.483	0.747
0.52	1.30	0.832	0.877	0.949	1.32	1.08	0.351	0.474	0.742
0.54	1.27	0.820	0.868	0.945	1.34	1.08	0.342	0.464	0.736
0.56	1.24	0.808	0.859	0.941	1.36	1.09	0.332	0.455	0.730
0.58	1.21	0.796	0.850	0.937	1.38	1.10	0.323	0.446	0.724
0.60	1.19	0.784	0.840	0.933	1.40	1.11	0.314	0.437	0.718
0.62	1.17	0.772	0.831	0.929	1.42	1.13	0.305	0.429	0.713
0.64	1.16	0.759	0.821	0.924	1.44	1.14	0.297	0.420	0.707
0.66	1.13	0.747	0.812	0.920	1.46	1.15	0.289	0.412	0.701
0.68	1.12	0.734	0.802	0.915	1.48	1.16	0.280	0.403	0.695
0.70	1.09	0.721	0.792	0.911	1.50	1.18	0.272	0.395	0.690
0.72	1.08	0.708	0.781	0.906	1.52	1.19	0.265	0.387	0.684
0.74	1.07	0.695	5.771	0.901	1.54	1.20	0.257	0.379	0.678
0.76	1.06	0.682	0.761	0.896	1.56	1.22	0.250	0.371	0.672

Table E-4 (cont.)

M	A/A^*	p/p_0	ρ/ρ_0	T/T_0	M	A/A^*	p/p_0	ρ/ρ_0	T/T_0
1.58	1.23	0.242	0.363	0.667	2.30	2.19	0.080	0.165	0.486
1.60	1.25	0.235	0.356	0.661	2.32	2.23	0.078	0.161	0.482
1.62	1.27	0.228	0.348	0.656	2.34	2.27	0.075	0.157	0.477
1.64	1.28	0.222	0.341	0.650	2.36	2.32	0.073	0.154	0.473
1.66	1.30	0.215	0.334	0.645	2.38	2.36	0.071	0.150	0.469
1.68	1.32	0.209	0.327	0.639	2.40	2.40	0.068	0.147	0.465
1.70	1.34	0.203	0.320	0.634	2.42	2.45	0.066	0.144	0.461
1.72	1.36	0.197	0.313	0.628	2.44	2.49	0.064	0.141	0.456
1.74	1.38	0.191	0.306	0.623	2.46	2.54	0.062	0.138	0.452
1.76	1.40	0.185	0.300	0.617	2.48	2.59	0.060	0.135	0.448
1.78	1.42	0.179	0.293	0.612	2.50	2.64	0.059	0.132	0.444
1.80	1.44	0.174	0.287	0.607	2.52	2.69	0.057	0.129	0.441
1.82	1.46	0.169	0.281	0.602	2.54	2.74	0.055	0.126	0.437
1.84	1.48	0.164	0.275	0.596	2.56	2.79	0.053	0.123	0.433
1.86	1.51	0.159	0.269	0.591	2.58	2.84	0.052	0.121	0.429
1.88	1.53	0.154	0.263	0.586	2.60	2.90	0.050	0.118	0.425
1.90	1.56	0.149	0.257	0.581	2.62	2.95	0.049	0.115	0.421
1.92	1.58	0.145	0.251	0.576	2.64	3.01	0.047	0.113	0.418
1.94	1.61	0.140	0.246	0.571	2.66	3.06	0.046	0.110	0.414
1.96	1.63	0.136	0.240	0.566	2.68	3.12	0.044	0.108	0.410
1.98	1.66	0.132	0.235	0.561	2.70	3.18	0.043	0.106	0.407
2.00	1.69	0.128	0.230	0.556	2.72	3.24	0.042	0.103	0.403
2.02	1.72	0.124	0.225	0.551	2.74	3.31	0.040	0.101	0.400
2.04	1.75	0.120	0.220	0.546	2.76	3.37	0.039	0.099	0.396
2.06	1.78	0.116	0.215	0.541	2.78	3.43	9.038	0.097	0.393
2.08	1.81	0.113	0.210	0.536	2.80	3.50	0.037	0.095	0.389
2.10	1.84	0.109	0.206	0.531	2.82	3.57	0.036	0.093	0.386
2.12	1.87	0.106	0.201	0.526	2.84	3.64	0.035	0.091	0.383
2.14	1.90	0.103	0.197	0.522	2.86	3.71	0.034	0.089	0.379
2.16	1.94	0.100	0.192	0.517	2.88	3.78	0.033	0.087	0.376
2.18	1.97	0.097	0.188	0.513	2.90	3.85	0.032	0.085	0.373
2.20	2.01	0.094	0.184	0.508	2.92	3.92	0.031	0.083	0.370
2.22	2.04	0.091	0.180	0.504	2.94	4.00	0.030	0.081	0.366
2.24	2.08	0.088	0.176	0.499	2.96	4.08	0.029	0.080	0.363
2.26	2.12	0.085	0.172	0.495	2.98	4.15	0.028	0.078	0.360
2.28	2.15	0.083	0.168	0.490	3.00	4.23	0.027	0.076	0.357

†For a perfect gas with constant specific heat. $\kappa = 1.4$.

Set up from I. H. Shames, *Mechanics of Fluids*. New York: McGraw-Hill Book Company, 1962.

Table E-5 ONE-DIMENSIONAL NORMAL-SHOCK RELATION†

M_1	M_2	p_2/p_1	T_2/T_1	$(p_0)_2/(p_0)_1$	M_1	M_2	p_2/p_1	T_2/T_1	$(p_0)_2/(p_0)_1$
1.00	1.000	1.000	1.000	1.000	1.80	0.617	3.613	1.532	0.813
1.02	0.980	1.047	1.013	1.000	1.82	0.612	3.698	1.547	0.804
1.04	0.962	1.095	1.026	1.000	1.84	0.608	3.783	1.562	0.795
1.06	0.944	1.144	1.039	1.000	1.86	0.604	3.869	1.577	0.786
1.08	0.928	1.194	1.052	0.999	1.88	0.600	3.957	1.592	0.777
1.10	0.912	1.245	1.065	0.999	1.90	0.596	4.045	1.608	0.767
1.12	0.896	1.297	1.078	0.998	1.92	0.592	4.134	1.624	0.758
1.14	0.882	1.350	1.090	0.997	1.94	0.588	4.224	1.639	0.749
1.16	0.868	1.403	1.103	0.996	1.96	0.584	4.315	1.655	0.740
1.18	0.855	1.458	1.115	0.995	1.98	0.581	4.407	1.671	0.730
1.20	0.842	1.513	1.128	0.993	2.00	0.577	4.500	1.688	0.721
1.22	0.830	1.570	1.140	0.991	2.02	0.574	4.594	1.704	0.711
1.24	0.818	1.627	1.153	0.988	2.04	0.571	4.689	1.720	0.702
1.26	0.807	1.686	1.166	0.986	2.06	0.567	4.784	1.737	0.693
1.28	0.796	1.745	1.178	0.983	2.08	0.564	4.881	1.754	0.683
1.30	0.786	1.805	1.191	0.979	2.10	0.561	4.978	1.770	0.674
1.32	0.776	1.866	1.204	0.976	2.12	0.558	5.077	1.787	0.665
1.34	0.766	1.928	1.216	0.972	2.14	0.555	5.176	1.805	0.656
1.36	0.757	1.991	1.229	0.968	2.16	0.553	5.277	1.822	0.646
1.38	0.748	2.055	1.242	0.963	2.18	0.550	5.378	1.839	0.637
1.40	0.740	2.120	1.255	0.958	2.20	0.547	5.480	1.857	0.628
1.42	0.731	2.186	1.268	0.953	2.22	0.544	5.583	1.875	0.619
1.44	0.723	2.253	1.281	0.948	2.24	0.542	6.687	1.892	0.610
1.46	0.716	2.320	1.294	0.942	2.26	0.539	5.792	1.910	0.601
1.48	0.708	2.389	1.307	0.936	2.28	0.537	5.898	1.929	0.592
1.50	0.701	2.458	1.320	0.930	2.30	0.534	6.005	1.947	0.583
1.52	0.694	2.529	1.334	0.923	2.32	0.532	6.113	1.965	0.575
1.54	0.687	2.600	1.347	0.917	2.34	0.530	6.222	1.984	0.566
1.56	0.681	2.673	1.361	0.910	2.36	0.527	6.331	2.003	0.557
1.58	0.675	2.746	1.374	0.903	2.38	0.525	6.442	2.021	0.549
1.60	0.668	2.820	1.388	0.895	2.40	0.523	6.553	2.040	0.540
1.62	0.663	2.895	1.402	0.888	2.42	0.521	6.666	2.060	0.532
1.64	0.657	2.971	1.416	0.880	2.44	0.519	6.779	2.079	0.523
1.66	0.651	3.048	1.430	0.872	2.46	0.517	6.894	2.098	0.515
1.68	0.646	3.126	1.444	0.864	2.48	0.515	7.009	2.118	0.507
1.70	0.641	3.205	1.458	0.856	2.50	0.513	7.125	2.138	0.499
1.72	0.635	3.285	1.473	0.847	2.52	0.511	7.242	2.157	0.491
1.74	0.631	3.366	1.487	0.839	2.54	0.509	7.360	2.177	0.483
1.76	0.626	3.447	1.502	0.830	2.56	0.507	7.479	2.198	0.475
1.78	0.621	3.530	1.517	0.821	2.58	0.506	7.599	2.218	0.468

Table E-5 (cont.) ONE-DIMENSIONAL NORMAL-SHOCK RELATION

M_1	M_2	p_2/p_1	T_2/T_1	$(p_0)_2/(p_0)_1$	M_1	M_2	p_2/p_1	T_2/T_1	$(p_0)_2/(p_0)_1$
2.60	0.504	7.720	2.238	0.460	2.84	0.485	9.243	2.496	0.376
2.62	0.502	7.842	2.260	0.453	2.86	0.484	9.376	2.518	0.370
2.64	0.500	7.965	2.280	0.445	2.88	0.483	9.510	2.541	0.364
2.66	0.499	8.088	2.301	0.438	2.90	0.481	9.645	2.563	0.358
2.68	0.497	8.213	2.322	0.431	2.92	0.480	9.781	2.586	0.352
2.70	0.496	8.338	2.343	0.424	2.94	0.479	9.918	2.609	0.346
2.72	0.494	8.465	2.364	0.417	2.96	0.478	10.055	2.632	0.340
2.74	0.493	8.592	2.396	0.410	2.98	0.476	10.194	2.656	0.334
2.76	0.491	8.721	2.407	0.403	3.00	0.475	10.333	2.679	0.328
2.78	0.490	8.850	2.429	0.396					
2.80	0.488	8.980	2.451	0.389					
2.82	0.487	9.111	2.473	0.383					

†For a perfect gas with $\kappa=1.4$.

Set up from I. H. Shames *Mechanics of Fluids*. New York: McGraw-Hill Book Company, 1962.

Table E-6 MACH NUMBER AND MACH ANGLE VERSUS PRANDTL-MEYER FUNCTION†

ν (deg)	M	μ (deg)	ν (deg)	M	μ (deg)
0.0	1.000	90.000	22.5	1.862	32.488
0.5	1.051	72.099	23.0	1.879	32.148
1.0	1.082	67.574	23.5	1.897	31.814
1.5	1.108	64.451	24.0	1.915	31.486
2.0	1.133	61.997	24.5	1.932	31.164
2.5	1.155	59.950	25.0	1.950	39.847
3.0	1.177	58.180	25.5	1.968	30.536
3.5	1.198	56.614	26.0	1.986	30.229
4.0	1.218	55.205	26.5	2.004	29.928
4.5	1.237	53.920	27.0	2.023	29.632
5.0	1.256	52.738	27.5	2.041	29.340
5.5	1.275	51.642	28.0	2.059	29.052
6.0	1.294	50.619	28.5	2.078	28.769
6.5	1.312	49.658	29.0	2.096	28.491
7.0	1.330	48.753	29.5	2.115	28.216
7.5	1.348	47.896	30.0	2.134	27.945
8.0	1.366	47.082	30.5	2.153	27.678
8.5	1.383	46.306	31.0	2.172	27.415
9.0	1.400	45.566	31.5	2.191	27.155
9.5	1.418	44.857	32.0	2.210	26.899
10.0	1.435	44.177	32.5	2.230	26.646
10.5	1.452	43.523	33.0	2.249	26.397
11.0	1.469	42.894	33.5	2.269	26.151
11.5	1.486	42.287	34.0	2.289	25.908
12.0	1.503	41.701	34.5	2.309	25.668
12.5	1.520	41.134	35.0	2.329	25.430
13.0	1.537	40.585	35.5	2.349	25.196
13.5	1.554	40.053	36.0	2.369	24.965
14.0	1.571	39.537	36.5	2.390	24.736
14.5	1.588	39.035	37.0	2.410	24.510
15.0	1.605	38.547	37.5	2.431	24.287
15.5	1.622	38.073	38.0	2.452	24.066
16.0	1.639	37.611	38.5	2.473	23.847
16.5	1.655	37.160	39.0	2.495	23.631
17.0	1.672	36.721	39.5	2.516	23.418
17.5	1.689	36.293	40.0	2.538	23.206
18.0	1.706	35.874	40.5	2.560	22.997
18.5	1.724	35.465	41.0	2.582	22.790
19.0	1.741	35.065	41.5	2.604	22.585
19.5	1.758	34.673	42.0	2.626	22.382
20.0	1.775	34.290	42.5	2.649	22.182
20.5	1.792	33.915	43.0	2.671	21.983
21.0	1.810	33.548	43.5	2.694	21.786
21.5	1.827	33.188	44.0	2.718	21.591
22.0	1.844	32.834	44.5	2.741	21.398

Table E-6 (cont.)

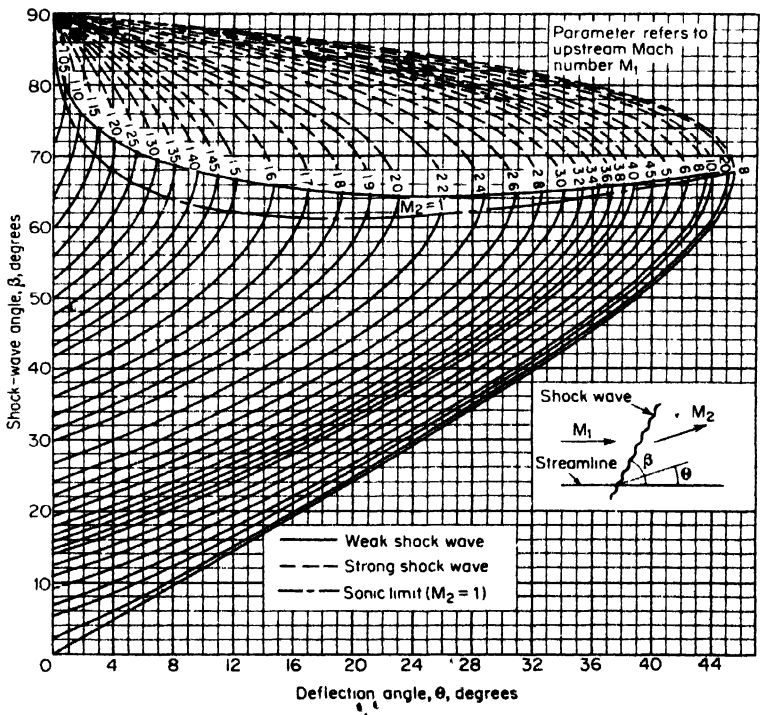
ν (deg)	M	μ (deg)	ν (deg)	M	μ (deg)
45.0	2.764	21.207	67.5	4.133	14.002
45.5	2.788	21.017	68.0	4.173	13.865
46.0	2.812	20.830	68.5	4.214	13.729
46.5	2.836	20.644	69.0	4.255	13.593
47.0	2.861	20.459	69.5	4.297	13.459
47.5	2.886	20.277	70.0	4.339	13.325
48.0	2.910	20.096	70.5	4.382	13.191
48.5	2.936	19.916	71.0	4.426	13.059
49.0	2.961	19.738	71.5	4.470	12.927
49.5	2.987	19.561	72.0	4.515	12.795
50.0	3.013	19.386	72.5	4.561	12.665
50.5	3.039	19.213	73.0	4.608	12.535
51.0	3.065	19.041	73.5	4.655	12.406
51.5	3.092	18.870	74.0	4.703	12.277
52.0	3.119	18.701	74.5	4.752	12.149
52.5	3.146	18.532	75.0	4.801	12.021
53.0	3.174	18.366	75.5	4.852	11.894
53.5	3.202	18.200	76.0	4.903	11.768
54.0	3.230	18.036	76.5	4.955	11.642
54.5	3.258	17.873	77.0	5.009	11.517
55.0	3.287	17.711	77.5	5.063	11.392
55.5	3.316	17.551	78.0	5.118	11.268
56.0	3.346	17.391	78.5	5.174	11.145
56.5	3.375	17.233	79.0	5.231	11.022
57.0	3.406	17.076	79.5	5.289	10.899
57.5	3.436	16.920	80.0	5.348	10.777
58.0	3.467	16.765	80.5	5.408	10.656
58.5	3.498	16.611	81.0	5.470	10.535
59.0	3.530	16.458	81.5	5.532	10.414
59.5	3.562	16.306	82.0	5.596	10.294
60.0	3.594	16.155	82.5	5.661	10.175
60.5	3.627	16.005	83.0	5.727	10.056
61.0	3.660	15.856	83.5	5.795	9.937
61.5	3.694	15.708	84.0	5.864	9.819
62.0	3.728	15.561	84.5	5.935	9.701
62.5	3.762	15.415	85.0	6.006	9.584
63.0	3.797	15.270	85.5	6.080	9.467
63.5	3.832	15.126	86.0	6.155	9.350
64.0	3.868	14.983	86.5	6.232	9.234
64.5	3.904	14.840	87.0	6.310	9.119
65.0	3.941	14.698	87.5	6.390	9.003
65.5	3.979	14.557	88.0	6.472	8.888
66.0	4.016	14.417	88.5	6.556	8.774
66.5	4.055	14.278	89.0	6.642	8.660
67.0	4.094	14.140	89.5	6.729	8.546

Table E-6 (cont.)

ν (deg)	M	μ (deg)	ν (deg)	M	μ (deg)
90.0	6.819	8.433	97.5	8.480	6.772
90.5	6.911	8.320	98.0	8.618	6.664
91.0	7.005	8.207	98.5	8.759	6.556
91.5	7.102	8.095	99.0	8.905	6.448
92.0	7.201	7.983	99.5	9.055	6.340
92.5	7.302	7.871	100.0	9.210	6.233
93.0	7.406	7.760	100.5	9.371	6.126
93.5	7.513	7.649	101.0	9.536	6.019
94.0	7.623	7.538	101.5	9.708	5.913
94.5	7.735	7.428	102.0	9.885	5.806
95.0	7.851	7.318			
95.5	7.970	7.208			
96.0	8.092	7.099			
96.5	8.218	6.989			
97.0	8.347	6.881			

†Numerical values taken from *Publication No.* 26, Jet Propulsion Laboratory, California Institute of Technology.

Chart E-1 OBLIQUE SHOCK CHART*



*From NACA Report 135 (Courtesy of National Aeronautics and Space Administration).

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